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Weitzenbock 354

November 23, 1982

Weitzenbock formula (as explained by Taubes)

$\mathcal{D} = e^i \nabla_i$. Do the calculation at P where the frame $e^i \in T^*|_U$ is flat at P: $\nabla_j e^i|_P = 0$. Then

$$\mathcal{D}^*\mathcal{D} = e^i \nabla_i e^j \nabla_j = e^i e^j \nabla_i \nabla_j \qquad (\text{at } P)$$

$$= -\nabla_i \nabla_i + \tfrac{1}{2}[e^i, e^j] \nabla_i \nabla_j \qquad e^i e^j + e^j e^i = -2\delta_{ij}$$

$$= -\nabla_i \nabla_i + \underbrace{\tfrac{1}{4}[e^i, e^j] R_{ij}}$$

this is a 0-th order operator which can be worked out using linear algebra for the particular Clifford module you are concerned with. e.g. get the scalar curvature for spinors alone, Ricci curvature + curvature for $d+d^*$.

Observation of Nov. 28. Look at the deRham complex on flat space. Then $d = e(dx^\mu)\partial_\mu$ so $d^* = (-\partial_\mu) e(dx^\mu)^* = -i(\langle dx^\mu |)\partial_\mu$ and hence

$$d + d^* = \big(e(dx^\mu) - i(\langle dx^\mu|)\big)\partial_\mu .$$

Hence if $e^\mu = e(dx^\mu) - i(\langle dx^\mu|)$ on the exterior algebra we have

$$e^\mu e^\nu + e^\nu e^\mu = -2\delta^{\mu\nu}.$$

Thus the good way to write the Dirac operator is

$$\not{D} = \sum e^\mu \nabla_\mu$$

where e^μ belong to the Clifford algebra with relations $v.v = -\|v\|^2$. Also the e^μ are skew-adjoint, so if we write

$$\not{D} = \sum \underbrace{(ie^\mu)}_{\gamma^\mu} \frac{1}{i} \nabla_\mu$$

the γ^μ are hermitian with $\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2\delta^{\mu\nu}$.

November 22, 1982

On Loday's conjecture. We want to compute the homology of the Lie algebra $\mathfrak{gl}_n(A) = \mathfrak{gl}_n(k) \otimes_k A$ where A is a k-algebra $(\mathrm{char}(k)=0)$. This homology is computed using a chain complex $C_\cdot(\mathfrak{gl}_n(A))$ such that

$$C_p(\mathfrak{gl}_n(A)) = \Lambda^p(\mathfrak{gl}_n(A)).$$

The first step is to show that the homology can be computed using the subcomplex of chains which are invariant under $\mathfrak{gl}_n(k)$. In general given a Lie algebra $\mathfrak{g}$, there is an action of $\mathfrak{g}$ on $\Lambda^p\mathfrak{g}$ and this action is compatible with the differential in the complex $C_\cdot(\mathfrak{g})$. Moreover the induced action of $\mathfrak{g}$ on $H_\cdot(\mathfrak{g})$ is trivial. Now look at the action of $\mathfrak{gl}_n(k)$ on $C_\cdot(\mathfrak{gl}_n A)$. $\Lambda^p(\mathfrak{gl}_n A)$ is a quotient of $(\mathfrak{gl}_n A)^{\otimes p} = (\mathfrak{gl}_n(k)\otimes A)^{\otimes p}$ $= \mathfrak{gl}_n(k)^{\otimes p} \otimes A^{\otimes p}$. In char. 0 one knows that $\mathfrak{gl}_n(k)^{\otimes p}$ is a semi-simple $\mathfrak{gl}_n(k)$-module. Thus as a $\mathfrak{gl}_n(k)$-module the complex $C_\cdot(\mathfrak{gl}_n A)$ is semi-simple. Then breaking it up according to the different irreducibles of $\mathfrak{gl}_n(k)$, one sees that the subcomplex which is isotypical for an <u>non-trivial</u> repn. is acyclic, since the action on the homology is trivial. Thus the subcomplex of invariants has the same homology.

We can think of $C_\cdot(\mathfrak{g}_A)^{\mathfrak{g}}$ as being ~~the~~ chains on the + construction.

Notation: $\mathfrak{g} = \mathfrak{gl}_n(k)$, $\mathfrak{g}_A = \mathfrak{gl}_n(A) = \mathfrak{g}\otimes A$.

Now I wish to compute explicitly $\Lambda^p(\mathfrak{g}\otimes A)^{\mathfrak{g}}$ for $n \geq p$ using invariant theory for the general linear group. First review the theorem on invariants. If $V = k^n$, then $\mathfrak{g} = \mathrm{End}(V) = \mathrm{Hom}(V,V)$, so $\mathfrak{g}^{\otimes p} = \mathrm{Hom}(V^{\otimes p}, V^{\otimes p})$.

~~this that a~~ There is an obvious map

$$\mathbb{C}[\Sigma_p] \longrightarrow \mathrm{Hom}(V^{\otimes p}, V^{\otimes p}) = \mathfrak{g}^{\otimes p}$$

given by associating to a permutation its action on $V^{\otimes p}$. Thus

$$\sigma \longmapsto (v_1 \otimes \cdots \otimes v_p \longmapsto v_{\sigma^{-1}1} \otimes v_{\sigma^{-1}2} \otimes \cdots \otimes v_{\sigma^{-1}p}).$$

(If $\dim V \geq p$, then taking the v_i to be part of a basis for V shows the above map is injective). The theorem on invariants says the above map is surjective onto the invariants for $\mathfrak{g}$ acting on $\mathfrak{g}^{\otimes p}$, hence for $n \geq p$

$$\mathbb{C}[\Sigma_p] \xrightarrow{\sim} (\mathfrak{g}^{\otimes p})^{\mathfrak{g}}.$$

(Proof: Let $\mathcal{A}$ be the image of $\mathbb{C}[\Sigma_p]$ in the ring $\mathrm{End}(V^{\otimes p}) = \mathfrak{g}^{\otimes p}$. The centralizer of $\mathbb{C}[\Sigma_p]$ is the subspace of symmetric tensors, for conjugating by a permutation in the ring $\mathrm{End}(V^{\otimes p})$ is the same as permuting the factors of $\mathfrak{g}^{\otimes p}$. An element $\alpha \in \mathfrak{g}$ acts on $\mathfrak{g}^{\otimes p}$ via

$$\alpha \otimes 1 \otimes \cdots \otimes 1 + 1 \otimes \alpha \otimes 1 \otimes \cdots \otimes 1 + \cdots + 1 \otimes \cdots \otimes 1 \otimes \alpha$$

and $e^{t\alpha}$ acts as $e^{t\alpha} \otimes \cdots \otimes e^{t\alpha}$. Thus the image of $U(\mathfrak{g})$ in $\mathfrak{g}^{\otimes p}$ contains all endos. $\beta \otimes \cdots \otimes \beta$ with $\beta \in \mathrm{Aut}(V)$, hence also $\beta \in \mathrm{End}(V)$. But these elements span the symmetric tensors. Thus if $\mathcal{B}$ = Image of $U(\mathfrak{g})$ in $\mathfrak{g}^{\otimes p}$, we have $\mathcal{B}$ = centralizer of $\mathcal{A}$. Now because $V^{\otimes p}$ is a semi-simple $\mathcal{A}$ module, the double commutator thm. says that $\mathcal{A}$ = centralizer of $\mathcal{B}$, or that $\mathcal{A}$ = invariants of $\mathfrak{g}$ in $\mathfrak{g}^{\otimes p}$.)

~~In general~~ ───────────

Review: I want to determine $[\Lambda^p(\mathfrak{g} \otimes A)]_{\mathfrak{g}}$ where $\mathfrak{g} = \mathfrak{gl}_n$. I use coinvariants so that one has functorality for Lie homomorphisms.

We start out by describing the $\mathfrak{g}$-invariants in

$$(\mathfrak{g}\otimes A)^{\otimes p} = \mathfrak{g}^{\otimes p}\otimes A^{\otimes p}$$

and we have obtained for $n \geq p$ an isomorphism

$$\mathbb{C}[\Sigma_p] \xrightarrow{\sim} (\mathfrak{g}^{\otimes p})^{\mathfrak{g}} = \mathrm{Hom}_{\mathfrak{g}}(V^{\otimes p}, V^{\otimes p})$$

where to a $\sigma\in\Sigma_p$ is associated the obvious perm. of $V^{\otimes p}$. Next notice that the action of Σ_p on $\mathfrak{g}^{\otimes p}$ corresponds to conjugation of Σ_p on $\mathbb{C}[\Sigma_p]$. So we have obtained an isom. of Σ_p-modules

$$\left[(\mathfrak{g}\otimes A)^{\otimes p}\right]_{\mathfrak{g}} = \mathbb{C}(\Sigma_p)\otimes A^{\otimes p}$$

where Σ_p acts by conjugation on $\mathbb{C}(\Sigma_p)$ and by permuting the factors of $A^{\otimes p}$.

To obtain Λ^p one tensors over Σ_p with the sgn character, and we get

$$\left[\Lambda^p(\mathfrak{g}\otimes A)\right]_{\mathfrak{g}} = \left[\mathbb{C}(\Sigma_p)\otimes A^{\otimes p}\right]\underset{\Sigma_p}{\otimes} \mathrm{sgn}$$

But now this can be decomposed relative to the orbits of Σ_p acting on itself by conjugation. These orbits are conjugacy classes and are described by partitions with a cyclic permutations for each block. So the above is clearly

$$\bigoplus_{\substack{p=q_1+\cdots \\ q_1\geq q_2\geq\cdots}} \bigotimes \left(A^{\otimes q_j}\underset{\mathbb{Z}/q_j\mathbb{Z}}{\otimes}\mathrm{sgn}\right)$$

(This isn't quite correct because the centralizer of an elts involves symm. gp. when the cycle lengths q_j coincide.)

The next step will be to explain this in terms of a product on

$$\lim_{n\to\infty}\left[\Lambda(\mathfrak{g}_n\otimes A)\right]_{\mathfrak{g}_n}.$$

Ultimately it will be necessary to be very careful about the difference between invariants and coinvariants. For the moment lets consider $\Lambda(\mathfrak{g}\otimes A)$ as an algebra in the obvious way, and then the invariants will be an algebra. We have maps

$$\begin{tikzcd} \mathbb{C}[\Sigma_p]\otimes A^{\otimes p} \arrow[r, hook, "\sim"] & \left[(\mathfrak{g}\otimes A)^{\otimes p}\right]^{\mathfrak{g}} \arrow[r, two heads] & \left[\Lambda^p(\mathfrak{g}\otimes A)\right]^{\mathfrak{g}} \end{tikzcd}$$

and it seems clearly compatible with the obvious product.

November 25, 1982

Finish letter to Loday: $\mathfrak{g} = \mathfrak{gl}_n$. I take the algebra $\Lambda(\mathfrak{g}\otimes A)$ and form the subalgebra of invariants $\Lambda(\mathfrak{g}\otimes A)^{\mathfrak{g}}$. Because of the surjection

$$\mathbb{C}[\Sigma_p] \twoheadrightarrow (\mathfrak{g}^{\otimes p})^{\mathfrak{g}}$$

I get a surjection

$$\left(\mathbb{C}[\Sigma_p]\otimes A^{\otimes p}\right)\underset{\Sigma_p}{\otimes} \mathrm{sgn} \twoheadrightarrow (\mathfrak{g}^{\otimes p}\otimes A^{\otimes p})^{\mathfrak{g}} \underset{\Sigma_p}{\otimes} \mathrm{sgn} = \Lambda^p(\mathfrak{g}\otimes A)^{\mathfrak{g}}$$

$\Vert$

degree p part of the anti-comm. alg. generated by

$$\bigoplus_p \left(A^{\otimes p} \underset{\mathbb{Z}_p}{\otimes} \mathrm{sgn}\right) = C_\cdot(A)$$

Thus I get an algebra epimorphism

$$\tilde{S}[C_\cdot(A)] \twoheadrightarrow \Lambda(\mathfrak{g}\otimes A)^{\mathfrak{g}}$$

which is an isomorphism in degrees $p\le n$.

The above only uses the ~~structure~~ of A as a vector space. It might be better to work dually, except something might go wrong.

The reason to work dually is that the algebra map

$$\Lambda(\mathfrak{g}^*\otimes A^*)^{\mathfrak{g}} \longrightarrow S[C_\cdot(A)^*]$$

if it were to exist would be very restricted, since the latter is free; hence you would get control over the cohomology of $\mathfrak{g}\otimes A$ for fixed n. By duality we get

$$\left[(\mathfrak{g}^*)^{\otimes p}\right]^{\mathfrak{g}} \hookrightarrow \mathbb{C}[\Sigma_p]^*$$

compatible with $p,q \mapsto p+q$. Wait: Do this carefully with attention to the invariant-coinvariant problem

$$\begin{tikzcd}
\left[(\mathfrak{g}^*)^{\otimes p}\right]^{\mathfrak{g}} \otimes \left[(\mathfrak{g}^*)^{\otimes q}\right]^{\mathfrak{g}} \arrow[r] \arrow[d] & \mathbb{C}[\Sigma_p]^* \otimes \mathbb{C}[\Sigma_q]^* = \mathbb{C}[\Sigma_p \times \Sigma_q]^* \\
\left[(\mathfrak{g}^*)^{\otimes (p+q)}\right]^{\mathfrak{g}} \arrow[r] & \mathbb{C}[\Sigma_{p+q}]^*
\end{tikzcd}$$

I would like an algebra map. But there is no obvious way to map $\mathbb{C}[\Sigma_p \times \Sigma_q]^* \longrightarrow \mathbb{C}[\Sigma_{p+q}]^*$ except by an induction process. Does there exist a map when the [redacted] sign quotient by the symm. group is [redacted] done?

I asked Atiyah the following: Given a flat bundle over a manifold M one knows by Chern-Weil that its Chern classes are zero in real cohomology. However, suppose M is a fibre bundle over a base B and E is a vector bundle over M with a flat connection along the fibres. Do the Chern classes of E come from the base B?

<u>Counterexample</u>: Take $B =$ a torus, $M = B \times \overbrace{\text{dual torus}}^{B^*}$ and $E =$ the canonical family of flat line bundles on B^* parameterized by B. Then $c_1(E)$ doesn't come from B.

Here's how Gromo[redacted] + Lawson use families of flat bundles. Notation: M is a given manifold and we want to consider families of flat vector bundles over M. [redacted] Suppose M Riemannian + spin so that we can form the Dirac operator over M and tensor it with any vector bundle with connection. By Lichnerowicz if the scalar curvature is ≥ 0 and > 0 somewhere then M has no harmonic spinors. [redacted] Since curvature is local, the same is true for spinors tensored with a flat bundle. So now if one has a family of flat vector bundles over M parameterized by B, then $R>0$ [redacted] in M will imply that the index of this family is 0. However when the index is computed by the index thm., one gets [redacted] $\int_M \mathrm{char}(\tilde{E})\cdot\hat{A}(M) \in H^*(B)$, and so it [redacted] can happen that one can disprove $R>0$ in this way. (e.g. a torus where there are lots of flat line bundles whose forms cup to give something non-zero.)

Why Bott thinks of Connes homology as being related to the ~~fase~~ loop space on M. Denote [redacted] ΩM the free loop space and consider its equivariant cohomology for the S^1 action. If one had a localization thm., then the localized homology is that of the fixpts.

ΩM = free loop space

$$H^*_{S^1}(\cancel{M^{S^1}}\,\Omega M)[u^{-1}] \xrightarrow[\text{res.}]{\sim} H^*(M)[u,u^{-1}]$$

Better:

$$\cancel{H^*_{S^1}(\Omega M)[u^{-1}] \to}$$

$$H^p_{S^1}(\Omega M) \longrightarrow H^p_{S^1}(M) = \bigoplus_{k=0}^{[\frac{p}{2}]} H^{p-2k}(M)\,u^k$$

should become an isomorphism upon inverting u. Thus one should have

$$\lim\left\{ H^p_{S^1}(\Omega M) \xrightarrow{u} H^{p+2}_{S^1}(\Omega M) \to \cdots \right\} = \begin{cases} H^{ev}(M) & p \text{ even} \\ H^{odd}(M) & p \text{ odd} \end{cases}$$

(Now in fact I can use the Gysin sequence in this situation, so that I get an exact sequence

$$\to H^{p-2}_{S^1}(\Omega M) \xrightarrow{u} H^p_{S^1}(\Omega M) \to H^p(\Omega M) \longrightarrow H^{p-1}_{S^1}(M) \xrightarrow{u} H^{p+1}_{S^1}(M) \to \cdots$$

but this doesn't help. It shows if anything that $H_S(\Omega M)$ perhaps isn't the Connes homology (or cohomology?).)

Anyway a possible finite-dimensional approx. to ΩM is to use sequences $(m_1,\ldots,m_k)$ where m_i, m_{i+1} and m_k, m_1 are close. Then $\mathbb{Z}_k$ acts.

November 26, 1982

k field of char. 0, $\mathfrak{g}$ reductive Liegebra, A an algebra over k. We can form $\mathfrak{g} \otimes A$ which is a Lie algebra over k, for example

$$\mathfrak{gl}_n \otimes A = \mathfrak{gl}_n A .$$

We have the cochain complex

$$C.(\mathfrak{g} \otimes A) = \Lambda^{\cdot}(\mathfrak{g} \otimes A)$$

and have an action of $\mathfrak{g} \otimes A$ on it. Restrict the action to $\mathfrak{g}$ and consider the map

$$(*) \qquad C.(\mathfrak{g} \otimes A) \longrightarrow\!\!\!\!\rightarrow C.(\mathfrak{g} \otimes A)_{\mathfrak{g}}$$

of taking coinvariants with respect to the $\mathfrak{g}$-action.

<u>Assertion</u> 1: If $\mathfrak{g}$ is reductive, then $(*)$ is a quasi-isomorphism.

This follows from the facts

a) $C.(\mathfrak{g} \otimes A)$ is semi-simple as a $\mathfrak{g}$-module

b) $\mathfrak{g}$ acts trivially on the homology

Obvious generalization is to suppose that one has a homomorphism of Liegebras $\mathfrak{g} \longrightarrow \mathfrak{g}_1$ such that as a $\mathfrak{g}$-module $\mathfrak{g}_1$ is semi-simple. Then one sees that

$$C.(\mathfrak{g}_1) \longrightarrow\!\!\!\!\rightarrow C.(\mathfrak{g}_1)_{\mathfrak{g}}$$

is a quasi-isomorphism. Hence also

$$C.(\mathfrak{g}_1)^{\mathfrak{g}} \subset C.(\mathfrak{g}_1)_{\mathfrak{g}}$$

is a quasi-isom., since invariants $\simeq$ coinvariants.

I want to think of $C.(\mathfrak{g} \otimes A)$ as the chains on $\mathfrak{g} \otimes A$ and $C.(\mathfrak{g} \otimes A)_{\mathfrak{g}}$ as the "chains on the plus construction" of $\mathfrak{g} \otimes A$. How functorial? Clearly a morphism $\mathfrak{g} \to \mathfrak{g}_1$ of reductive Liealgebras induces a map

$$C_\cdot(\mathfrak{g}\otimes A)_{\mathfrak{g}} \longrightarrow C_\cdot(\mathfrak{g}_1\otimes A)_{\mathfrak{g}_1}.$$

If we have two reductive Liegebras $\mathfrak{g}_1, \mathfrak{g}_2$ then

$$C_\cdot\left[(\mathfrak{g}_1\oplus\mathfrak{g}_2)\otimes A\right]_{\mathfrak{g}_1\oplus\mathfrak{g}_2} \cong \left[C_\cdot(\mathfrak{g}_1\otimes A)\otimes C_\cdot(\mathfrak{g}_2\otimes A)\right]_{\mathfrak{g}_1\oplus\mathfrak{g}_2}$$

$$= C_\cdot(\mathfrak{g}_1\otimes A)_{\mathfrak{g}_1}\otimes C_\cdot(\mathfrak{g}_2\otimes A)_{\mathfrak{g}_2}$$

and from this one can conclude in a standard way that $C_\cdot(\mathfrak{g}\otimes A)_{\mathfrak{g}}$ is a cogebra, (anti-)comm. and assoc.

The next point to check is that two embeddings of $\mathfrak{gl}_m$ into $\mathfrak{gl}_n$ induce the <u>same</u> map on $C_\cdot(\mathfrak{g}\otimes A)_{\mathfrak{g}}$. The adjoint group of $\mathfrak{g}$ acts on $C_\cdot(\mathfrak{g}\otimes A)$, and it acts trivially on $C_\cdot(\mathfrak{g}\otimes A)_{\mathfrak{g}}$. Hence two ~~embeddings of~~ morphisms $\mathfrak{g}_1 \rightrightarrows \mathfrak{g}$ which are conjugate under the adjoint group of $\mathfrak{g}$ will induce the same map in $C_\cdot(\mathfrak{g}\otimes A)_{\mathfrak{g}}$. But then we know that two embeddings of $\mathfrak{gl}_m$ in $\mathfrak{gl}_n$ are conjugate under SL_n. (I want to consider embeddings induced by an injection $\{1,\dots,m\}\to\{1,\dots,n\}$. These are conjugate under the Weyl gp of SL_n which is Σ_n.

Check more closely: Let $p+q=n$ and consider the two embeddings $\mathfrak{g}_p\times\mathfrak{g}_q$ into $\mathfrak{g}_n$ obtained from

$$\begin{tikzcd}
{[p]\cup[q]} \arrow[d, "\text{flip}"] \arrow[dr, "\text{obv}"] & \\
{[q]\cup[p]} \arrow[r, "\text{obv}"'] & {[n]}
\end{tikzcd}$$

So then a permutation matrix effecting this is

$$(-1)^{pq}\quad \begin{array}{c c} q \\ p \end{array}\begin{pmatrix} & (-1)^p \\ 1 \ \ddots \ 1 & \end{pmatrix}$$

but then signs have to be put in as indicated. Seems OKAY.

This tells me now that I get a comm. assoc. Hopf alg.

$$\bigoplus_n C_\cdot(\mathfrak{gl}_n \otimes A)_{\mathfrak{gl}_n}$$

which is bigraded by n and the exterior degree p.

Math Institute, Oxford
November 26, 1982

Dear Loday,

I have found a proof of your conjecture that Connes homology = primitive part of the Lie homology of $\mathfrak{gl}$ in characteristic zero. The proof uses the main theorem for the invariants of GL_n in the following formulation: Let V be an n-dimensional vector space over a field k of char. 0, let $\mathfrak{g} = \mathfrak{gl}(V) = \mathrm{Hom}(V,V)$, and let Σ_p be the permutation group on p letters. One has a map

$$k[\Sigma_p] \longrightarrow \mathrm{Hom}(V^{\otimes p}, V^{\otimes p}) = \mathfrak{g}^{\otimes p}$$

which associates to a permutation σ the obvious effect on $V^{\otimes p}$. The main theorem states that the image is the subspace of invariants for the action of $\mathfrak{g}$ on $\mathfrak{g}^{\otimes p}$. Hence

$$k[\Sigma_p] \twoheadrightarrow (\mathfrak{g}^{\otimes p})^{\mathfrak{g}}$$

is surjective, and, as can easily be seen, injective for $n \geq p$. (For a concise proof see Atiyah, Bott, Patodi, Inv. Math 19 (1973) p.324 last paragraph, or the book of Hermann Weyl).

Here is a sketch of the proof of your conjecture. Let $C.(\mathfrak{g})$ denote the standard complex for computing the homology of a Lie algebra $\mathfrak{g}$. Let A be a k-algebra and form the Lie algebra $\mathfrak{g} \otimes A$, e.g.

$$\mathfrak{gl}_n A = \mathfrak{gl}_n \otimes A.$$

Let's consider the map

$$(*) \qquad C.(\mathfrak{g} \otimes A) \twoheadrightarrow C.(\mathfrak{g} \otimes A)_{\mathfrak{g}}$$

of taking coinvariants for the $\mathfrak{g}$-action, where $\mathfrak{g} \subset \mathfrak{g} \otimes A$ using the $1 \in A$.

<u>Assertion 1</u>: If $\mathfrak{g}$ is reductive (e.g. $\mathfrak{gl}_n$), then $(*)$ induces isomorphisms on homology.

This follows from the fact that $C.(\mathfrak{g} \otimes A)$ is semi-simple as a $\mathfrak{g}$-module, hence this complex can be written as a direct sum of complexes according to the simple $\mathfrak{g}$-modules, and because only the trivial representation of $\mathfrak{g}$ can occur in the homology.

I propose to think of $(*)$ as analogous to the map $X \to X^+$ for the following reason. Suppose one has two Lie algebra morphisms $\mathfrak{g}_1 \rightrightarrows \mathfrak{g}$ which are conjugate under the action of the adjoint group of $\mathfrak{g}$. Then the induced homomorphisms

$$C.(\mathfrak{g}_1 \otimes A)_{\mathfrak{g}_1} \rightrightarrows C.(\mathfrak{g} \otimes A)_{\mathfrak{g}}$$

will coincide since the adjoint group of $\mathfrak{g}$ acts trivially on the latter. In particular we can apply this to the two standard ways of embedding $\mathfrak{gl}_m$ in $\mathfrak{gl}_n$ and deduce by familiar arguments that

$$C.(\mathfrak{gl} \otimes A)_{\mathfrak{gl}} = \varinjlim_n C.(\mathfrak{gl}_n \otimes A)_{\mathfrak{gl}_n}$$

has an algebra product given by direct sum of matrices. Thus it is a Hopf algebra, and so it is the (anti-commutative) symmetric algebra on its primitive elements.

<u>Assertion 2</u>: $\mathrm{Prim}\{C.(\mathfrak{gl} \otimes A)_{\mathfrak{gl}}\}$ can be identified with the Connes complex $C.(A)$.

We can determine $C_p(\mathfrak{g} \otimes A)_{\mathfrak{g}}$ $\quad \mathfrak{g} = \mathfrak{gl}_n \quad n \geq p$ using the theorem on invariants

$$\begin{aligned}
\Lambda^p(\mathfrak{g} \otimes A)_{\mathfrak{g}} &= \left[(\mathfrak{g}^{\otimes p} \otimes A^{\otimes p}) \otimes_{\Sigma_p} (\mathrm{sign})\right]_{\mathfrak{g}} \\
&= \left[(\mathfrak{g}^{\otimes p})_{\mathfrak{g}} \otimes A^{\otimes p}\right] \otimes_{\Sigma_p} (\mathrm{sign}) \\
&= \left[k[\Sigma_p] \otimes A^{\otimes p}\right] \otimes_{\Sigma_p} (\mathrm{sign})
\end{aligned}$$

Here Σ_p acts on the group ring by conjugation, and so the last space will decompose into a sum over the conjugacy classes of Σ_p, which are described by partitions of p and giving cyclic permutations in each each block of a partition. The partition of p with only one block describes the cyclic permutations of degree p, hence

$$\Lambda^p(\mathfrak{g}\otimes A)_{\mathfrak{g}} = \left(k[\Sigma_p/(\mathbb{Z}/p\mathbb{Z})] \otimes A^{\otimes p}\right)_{\Sigma_p}(\text{sign})$$

$$\oplus \bigoplus_{\substack{p = q_1+\cdots+q_r \\ q_1 \geq q_2 \cdots \geq q_r > 0 \\ r \geq 2}} \cdots\cdots\cdots$$

The sum over the partitions with more than one block is decomposable with respect to the algebra structure, hence

$$\text{Prim}\{\Lambda^p(\mathfrak{g}\otimes A)_{\mathfrak{g}}\} = [k[\Sigma_p/\mathbb{Z}/p\mathbb{Z}] \otimes A^{\otimes p}] \otimes_{\Sigma_p} (\text{sign})$$

$$= A^{\otimes p} \otimes_{\mathbb{Z}/p\mathbb{Z}} (\text{sign})$$

is the part of degree p of Connes complex.

From Assertion 2 follows your conjecture

$$\text{Prim}\{H_\bullet(\mathfrak{gl}\otimes A)_{\mathfrak{gl}}\} = HC_\bullet(A)$$

by taking homology.

Best Regards,

Daniel G. Quillen

November 28, 1982

Tomorrow Atiyah talks on Witten's ideas for $\mathrm{Map}(S^1, M)$ and as preparation I think I should look at the group case. Let G be compact simply-connected, e.g. SU_n and consider ΩG as the space of connections in the trivial bdle $S^1 \times G \longrightarrow G$ with trivial monodromy. In this way one sees an action of S^1 on ΩG, as well as a symplectic structure, moment map, etc. Combine the translation S^1 with a maximal torus T in G to get the torus for the moment map. It seems reasonable ~~to~~ think of $S^1 \times T$ as being the Poincaré group translations since the corresponding moment map looks like energy and momentum.

The fixpts. of the torus $S^1 \times T$ are closed geodesics contained in T passing thru the identity, and are described by vertices in the diagram. Atiyah wants to apply, or to make sense out of an integral over ΩG

$$\int e^{-tE + ixp} \frac{\omega^n}{n!} \quad \text{"the" symplectic volume}$$

as a sum over the fixpoints. I think that the sort of sum one gets is of the form

$$\sum_{\text{Weyl gp orbits}} \frac{e^{-tE + ixp}}{(\text{Weyl denominator})}$$

and should coincide with the character of the fundamental representation of ΩG. This representation is a Hilbert space on which we have energy + momentum operators taking integral values.

The point is that there is a Hilbert space around in the group situation which we could relate to path integrals. In any case this Hilbert space has a character coinciding with the above sum over fixpoints. What I don't see is the difference of two characters, which is what one might expect.

Perhaps what can be explained this way is the fudge factor that occurs when one naively computes the heat kernel of the Laplacian via stationary phase. Hence it seems profitable to go over the heat kernel over a compact Lie gp.

Graeme's idea is to expand the heat kernel according to the characters of the group and then rewrite the result using the Poisson summation formula. Start with general comments. We work with functions on G and we know that $L^2(G) = \bigoplus_\lambda W_\lambda \otimes W_\lambda^*$ where λ indexes the irreducible repns. The Laplacean is essentially the Casimir operator.

$$e^{-t\Delta} = \sum_\lambda e^{-tc_\lambda} E_\lambda$$

$$E_\lambda = \text{projection on the } \lambda\text{th part.}$$

I think E_λ is essentially given [redacted] by convolution with the character χ_λ of [redacted] λ. [redacted] Here λ runs over dominant weights and the Weyl character formula will expand each λ to a Weyl orbit. Thus one should get a sum over the characters of the torus with a Weyl type denominator. The numerator is some kind of θ function on the torus (One calculates the heat kernel restricted to the torus), which can be written as a [redacted] sum over the lattice points by the Poisson formula.

In general the convolution algebra (group ring) of G acts on representations of G; one assigns to $f(g)$ the operator $\int f(g)\rho(g)$. Composition corresponds to convolution, and central G functions commute with the G-action. The δ function on G corresponds to the identity operator, so

$$\int \delta(g)\rho(g) = id$$

so for a finite group we have

$$\delta(g) = \begin{cases} 0 & g \neq 1 \\ |G| & g = 1 \end{cases}$$

since we normalize Haar measure to $\int_G 1 = 1$.
Thus

$$\delta = \sum_\lambda d_\lambda \chi_\lambda$$

where λ runs over the irreducible characters, and

$$e^{-t\Delta} = \sum d_\lambda e^{-t\varepsilon_\lambda} \chi_\lambda$$

which means

$$\langle x | e^{-t\Delta} | y \rangle = \sum_\lambda d_\lambda e^{-t\varepsilon_\lambda} \chi_\lambda (xy^{-1}).$$

So now I want to take $y = 1$ in G and let x be in the maximal torus, so that I can use the Weyl character formula.

Now there are some obvious normalization problems connected with Δ. For example if we use the Killing form to define the Riemannian metric on G, then one can ask what the volume of G is. My immediate goal is to get some formulas for $SU(2)$.

Irreducible repns. of $SU(2)$ described by

$$\begin{array}{llllll} \text{spin} = l & = 0 & \frac{1}{2} & 1 & \frac{3}{2} & \cdots \\ \dim & = 1 & 2 & 3 & 4 & \cdots \quad = 2l+1 \end{array}$$

$$\text{character} = \frac{e^{i(l+\frac{1}{2})\theta} - e^{-i(l+\frac{1}{2})\theta}}{e^{i\frac{1}{2}\theta} - e^{-i\frac{1}{2}\theta}}$$

eigenvalue of Casimir is proportional to $(\lambda+\rho)^2 - \rho^2 = (l+\frac{1}{2})^2 - (\frac{1}{2})^2 = l(l+1)$

So look at

$$\sum_{l+\frac{1}{2} = \frac{1}{2}, 1, \frac{3}{2}, 2, \cdots} e^{-t\left[(l+\frac{1}{2})^2 - \frac{1}{2}^2\right]} \left[e^{i(l+\frac{1}{2})\theta} - e^{-i(l+\frac{1}{2})\theta}\right] \times (2l+1)$$

November 30, 1982

Let M be a Riemannian manifold, let E be a hermitian vector bundle over M, and let P be a 2nd order self-adjoint operator on sections of E whose symbol $\sigma(P,\xi)$ is multiplication by $|\xi|^2$ in the sense of the Riemann metric.

Question: Do the first order terms in P determine in an invariant way a connection on E?

The idea would be to use the normal coordinates at each point of M in a definite way. Consider

$$e^{-it\varphi}\,P\,e^{it\varphi}$$

which is a quadratic polynomial in t with leading term $t^2\sigma(D,d\varphi) = t^2|d\varphi|^2$. I want to remove this term and look at the linear term. At a point P this linear term will depend not just on $d\varphi$ at P but also the second derivatives of φ at P. For example

$$e^{-it\varphi}\left(\tfrac{1}{i}\partial_\mu\right)\left(\tfrac{1}{i}\partial_\nu\right)e^{it\varphi} = \left(\tfrac{1}{i}\partial_\mu + t\partial_\mu\varphi\right)\left(\tfrac{1}{i}\partial_\nu + t\partial_\nu\varphi\right)1$$

$$= \left[\cancel{-\partial_\mu\partial_\nu} + \tfrac{1}{i}\partial_\mu\, t\,\partial_\nu\varphi + \cancel{\tfrac{1}{i}t\,\partial_\mu\varphi\,\partial_\nu} + t^2\partial_\mu\varphi\,\partial_\nu\varphi\right]1$$

$$= t^2(\partial_\mu\varphi)(\partial_\nu\varphi) + t\left(\tfrac{1}{i}\partial_\mu\partial_\nu\varphi\right)$$

However if I use the Riemannian metric, then given a covector ξ at P, I can take the inverse of the expenential map at P and compose with ξ regarded as a linear map on the tangent space. This gives me a fn. φ defined near P such that $d\varphi|_P = \xi$. Next consider the linear term in $e^{-it\varphi}Pe^{it\varphi}$. ?

Let's work in coordinates. Choose normal coordinates at P on M, and choose an orthonormal frame in E. Then the operator P can be written

$$g^{\mu\nu}\partial_\mu\partial_\nu + a^\mu\partial_\mu + b$$

where a^μ, b are matrix functions. Because the coords. are normal we have $g^{\mu\nu} = \delta^{\mu\nu}$ + 2nd order terms.

Let's consider a gauge transformation in E.

$$\cancel{\phi^{-1} a^\mu \partial_\mu \phi + g^{\mu\nu} \partial}$$

$$\phi^{-1}\left(g^{\mu\nu}\partial_\mu\partial_\nu + a^\mu\partial_\mu + b\right)\phi$$

$$= g^{\mu\nu}\left(\partial_\mu + \phi^{-1}\partial_\mu\phi\right)\left(\partial_\nu + \phi^{-1}\partial_\nu\phi\right) + \phi^{-1}a^\mu\phi\left(\partial_\mu + \phi^{-1}\partial_\mu\phi\right) + \phi^{-1}b\phi$$

$$= g^{\mu\nu}\left(\partial_\mu\partial_\nu + (\phi^{-1}\partial_\mu\phi)\partial_\nu + (\phi^{-1}\partial_\nu\phi)\partial_\mu\right) + \phi^{-1}a^\mu\phi\,\partial_\mu$$
$$+ \text{ 0th order}$$

So now evaluate at P where $g^{\mu\nu} = \delta_{\mu\nu}$.

$$\partial_\mu^2 + \left(2\phi^{-1}\partial_\mu\phi + \phi^{-1}a^\mu\phi\right)\partial_\mu + \text{ 0th order}$$

One can certainly choose ϕ satisfying

$$\partial_\mu\phi + \tfrac{1}{2}a^\mu\phi = 0 \qquad \text{at } P$$

in which case we have a frame in E such that $a^\mu = 0$. This is unique to first order, hence it seems that we get a connection in E.

Let M be a Riemannian manifold, let E be a v.b. over M, let $\Box$ be a 2nd order DO on E whose symbol is multiplication by $|\xi|^2$. I claim there is a unique connection ∇ on E such that $\Box$ differs from the covariant Laplacean $\nabla^*\nabla$ from an operator of 0th order. This can't quite be correct, because I can't talk about ∇^* unless I am given a metric on E. However we can compose

$$E \xrightarrow{\nabla} E\otimes T^* \xrightarrow{\nabla} (E\otimes T^*)\otimes T^*$$

where the second operator is the connection on $E\otimes T^*$ obtained from those on E and T^*. Then we can use the Riemann metric $T^*\otimes T^* \longrightarrow \mathcal{O}$. This is the correct definition of the covariant Laplacean.

Why should this be [redacted] true? Consider the map $\nabla \longmapsto -\nabla^2$ going from connections to 2nd order operators with symbol $|\xi|^2$. ~~As I don't care about [redacted] scale factors on the right~~ Write this map in the form

$$\nabla_0 + B \longmapsto -(\nabla_0+B)^2 + \nabla_0^2 \qquad \text{mod 0th order}$$

$$-\left(\nabla_0 B + B\nabla_0\right) \qquad ''$$

The idea is that the possible B's are $\Gamma(\mathrm{End}(E)\otimes T^*)$ whereas the possible first order operators mod 0th order operators is $\Gamma(\mathrm{Hom}(E\otimes T^*, E)) = \Gamma(\mathrm{End}(E)\otimes T)$, and these are the same via the metric which goes into defining the <u>square</u>.

Now I want to place myself in the above situation of a $\Box : E \longrightarrow E$ with symbol $|\xi|^2$ and calculate the kernel $\langle x|e^{-t\Box}|y\rangle$ [redacted] as a path integral. We have the exact formula

$$\langle x|e^{-t\Box}|y\rangle = \int dx_{N-1}\cdots dx_1\ \langle x|e^{-\frac{t}{N}\Box}|x_{N-1}\rangle\cdots\langle x_1|e^{-\frac{t}{N}\Box}|y\rangle$$

and following Feynman we want to plug in what we know about the small-time asymptotics of the heat kernel. We have

$$\langle x|e^{-t\Box}|y\rangle \sim \frac{e^{-\frac{d(x,y)^2}{4t}}}{(4\pi t)^{n/2}}\left\{A_0(x,y) + A_1(x,y)t + \cdots\right\}$$

$$\frac{dx_{N-1}\cdots dx_1}{\left((4\pi \underbrace{t/N}_{\varepsilon})^{n/2}\right)^N}\left[e^{-\frac{1}{4\varepsilon}\sum d(x_i,x_{i-1})^2} \prod_{i=1}^{N} \left\{ A_0(x_i,x_{i-1}) + A_1(x_i,x_{i-1})\varepsilon + \cdots \right\}\right]$$

Then the idea is to take the $N\to\infty$ limit and to interpret the [redacted] factor in brackets as approaching some function of a path φ [redacted] with $\varphi(0)=y$, $\varphi(t)=x$. In fact one takes a smooth path and puts $x_i = \varphi(\underbrace{i\varepsilon}_{t_i})$ $\varepsilon=\frac{t}{N}$ forms the above expression and lets $N\longrightarrow\infty$.

The goal will be to calculate this fnl. of φ. One has

$$\frac{1}{\varepsilon}\sum d(x_i,x_{i-1})^2 \longrightarrow \int_0^t |\dot\varphi|^2\,dt$$

Next $\quad A_0(x_i,x_{i-1}) = I + dA_0\cdot\dot\varphi(t_{i-1})\,\varepsilon + O(\varepsilon^2)$.

I should be careful how I write this down, because the heat kernel has its values in $\mathrm{Hom}(E_y,E_x)$. So the correct thing to do is to use the basic connection we have in E ~~to~~ [redacted] do the parallel transport [redacted] along the curve and then to measure the first order deviation of $A_0(x_i,x_{i-1}) + A_1(x_i,x_{i-1})\varepsilon$ from the $\parallel$ transport.

To compute the ε-contribution I go thru the asymptotic calculation of the heat kernel around a point, and I will assume that I have trivialized the bundle E using the connection in the radial direction, and I also use normal coordinates.

Let's suppose we are on flat space. Then what does a connection on a trivial vector bundle look like when the connection is constant in the radial direction

$$\nabla = \nabla_\mu dx^\mu = (\partial_\mu + A_\mu)\,dx^\mu$$

and we have that $\quad \sum x^\mu A_\mu = 0$

so that constant vector fns. satisfy $\quad x^\mu \nabla_\mu f = 0$

But if one gives a family of smooth functions A_μ such that $x^\mu A_\mu = 0$, then one knows by exactness of the Koszul complex that $\exists\ F_{\mu\nu}$ smooth skew-symm. with

$$A_\mu = \frac{1}{2} F_{\mu\nu} x^\nu .$$

In this case $F_{\mu\nu} = \partial_\nu A_\mu - \partial_\mu A_\nu + [A_\mu, A_\nu]$ at $x = 0$.

Let $\Box = -\nabla_\mu \nabla_\mu$ the covariant Laplacean.

$$e^{\frac{u}{t}} \left(\partial_t + \Box\right) e^{-\frac{u}{t}} = \partial_t + \frac{u}{t^2} - \left(\nabla_\mu - \frac{1}{t}\partial_\mu u\right)\left(\nabla_\mu - \frac{1}{t}\partial_\mu u\right)$$

$$= \frac{1}{t^2}\left(u - \partial_\mu u\, \partial_\mu u\right) + \frac{1}{t}\left(2\partial_\mu u\, \nabla_\mu + \partial_\mu^2 u\right)$$

$$+ \partial_t + \Box$$

$$u = \frac{|x|^2}{4} \qquad \partial_\mu^2 u = \frac{n}{2}$$

$$\partial_\mu u = \frac{x^\mu}{2}$$

$$\frac{e^{+\frac{u}{t}}}{t^{n/2}} \left(\partial_t + \Box\right) \frac{e^{-\frac{u}{t}}}{t^{n/2}} = \frac{1}{2t} x^\mu \nabla_\mu \cdot + \partial_t + \Box$$

[redacted] $$\left(\frac{1}{2} \underbrace{x^\mu \nabla_\mu}_{x^\mu \partial_\mu} + n\right) A_n + \Box A_{n-1} = 0 .$$

So we have $A_0 = 1$. and

$$\left(\frac{1}{2} x^\mu \partial_\mu + 1\right) A_1 + \Box A_0 = 0$$

hence

$$A_1(0) = -\Box 1 = +\nabla_\mu \nabla_\mu 1$$

$$= \left(\partial_\mu + A_\mu\right)\left(\partial_\mu + A_\mu\right) 1 \quad \text{at } x = 0$$

$$= \partial_\mu A_\mu \big|_0 = \partial_\mu \left(\frac{1}{2} F_{\mu\nu} x^\nu\right)\Big|_0 = \frac{1}{2} F_{\mu\mu} \big|_0 = 0$$

December 1, 1982

$$\Box = -g^{\mu\nu}\nabla_\mu \nabla_\nu + B \qquad B \text{ zeroth order on } E$$

It's important to remember in using this notation that when ∇_μ is applied to $\nabla_\nu s$ with $s \in \Gamma(E)$, that one is taking the covariant derivative in $E \otimes T^*$. Thus

$$\left(\frac{e^{-\frac{u}{t}}}{t^{n/2}}\right)^{-1}(\partial_t + \Box)\frac{e^{-\frac{u}{t}}}{t^{n/2}} = \partial_t + \frac{u}{t^2} - \frac{n}{2t} - g^{\mu\nu}\left(\nabla_\mu - \tfrac{1}{t}\partial_\mu u\right)\left(\nabla_\nu - \tfrac{1}{t}\partial_\nu u\right) + B$$

$$= \frac{1}{t^2}\left[u - g^{\mu\nu}\partial_\mu u\, \partial_\nu u\right] + \frac{1}{t}\left[-\frac{n}{2} + g^{\mu\nu}(\partial_\mu u \cdot \nabla_\mu + \nabla_\mu \partial_\nu u)\right] + \partial_t + \Box .$$

Now look carefully at $\nabla_\mu \partial_\nu u$. Really this should be written in terms of

$$\begin{array}{ccccc} E & \xrightarrow{\nabla} & E\otimes T^* & \xrightarrow{\nabla} & (E\otimes T^*)\otimes T^* \\ s & \longmapsto & \nabla_\nu s\, dx^\nu & \longmapsto & \nabla_\mu(\nabla_\nu s\, dx^\nu)\, dx^\mu \end{array}$$

Thus we have to think of $\nabla_\mu[(\partial_\nu u)s]$ as the covariant derivative of $s\,du$ in $E\otimes T^*$. This means that

$$\nabla(s\,du) = (\nabla_\mu s \cdot du)\,dx^\mu + s\,\underbrace{\nabla_\mu du}_{\text{cov. derivative in } T^*}\,dx^\mu$$

and so

$$g^{\mu\nu}\nabla_\mu \circ \partial_\nu u = (g^{\mu\nu}\partial_\nu u)\nabla_\mu + g^{\mu\nu}\nabla_\mu\nabla_\nu u$$

$$= \operatorname{grad} u \cdot \nabla + \Delta u .$$

So what this calculation amounts to is the formula

$$e^{\lambda f}\Delta e^{-\lambda f} = \lambda^2|\operatorname{grad} f|^2 - \lambda[2\operatorname{grad} f\cdot\nabla + \Delta f] + \Delta$$

The final formula is therefore

$$\left(\frac{e^{-\frac{u}{t}}}{t^{n/2}}\right)^{-1}(\partial_t+\Box)\left(\frac{e^{-\frac{u}{t}}}{t^{n/2}}\right) = \frac{1}{t^2}\left[u-(\text{grad}\,u)^2\right]$$
$$+\frac{1}{t}\left[2\,\text{grad}\,u\cdot\nabla + \Delta u - \frac{n}{2}\right] + \left(\partial_t+\Box\right).$$

Then we choose $u=\frac{r^2}{4}$ so that $|\text{grad}\,u| = \frac{r}{2}$

and $2\,\text{grad}\,u\cdot\nabla = \nabla_{r\frac{d}{dr}}$ $\quad |\text{grad}\,u|^2 = \frac{r^2}{4} = u.$

So next consider the heat kernel

$$\langle x|e^{-t\Box}|0\rangle \approx \frac{e^{-\frac{r^2}{4t}}}{(4\pi t)^{n/2}}\left[A_0(x)+A_1(x)t+\cdots\right]$$

where the A_i satisfy the recursion relations

$$\left[\nabla_{r\frac{d}{dr}} + \left(\Delta\frac{r^2}{4}-\frac{n}{2}\right)+k\right]A_k + \Box A_{k-1} = 0.$$

Thus

$$\left[\nabla_{r\frac{d}{dr}} + \left(\Delta\frac{r^2}{4}-\frac{n}{2}\right)\right]A_0 = 0 \qquad A_0(0)=I.$$

defines A_0. Put $\Delta\frac{r^2}{4}-\frac{n}{2} = -g(x)$. It seems reasonable to expect that $g(x)=O(x^2)$ and I am sure of this for a Riemann surface. If ~~so~~, then

$$A_0 = I + \frac{1}{2}g(x) + O(x^3)$$

because $\nabla_{r\frac{d}{dr}}$ multiplies the homogeneous coefficients by their degree.

More carefully: Choose normal coords around O and also trivialize the v.b. by the radial parallel transport. Then certainly

$$\nabla_{r\frac{d}{dr}} = \sum x^\mu\partial_\mu$$

because if $\nabla_\mu = \partial_\mu + A_\mu$, to say radial $\parallel$ transport is constant means $x^\mu A_\mu = 0$.

Thus at least to the first order $A_0(x, y)$ is just parallel translation from y to x in the sense of the connection.

Next look at the relation for A_1:

$$\left[\nabla_{r\frac{d}{dr}} - g(x) + 1 \right] A_1 + \Box A_0 = 0.$$

In order to

December 8, 1982

Possible approach to a local index theorem for the Dirac operator $\not{D} = \left(\begin{array}{c|c} & D^* \\ \hline D & \end{array}\right)$. We first express

(*) $\operatorname{tr} \langle 0 | e^{-tD^*D} | 0 \rangle - \operatorname{tr} \langle 0 | e^{-tDD^*} | 0 \rangle$

where tr denotes the spinorial trace as a path integral over paths going from 0 to 0. Better we first express the kernel

$$\langle x | e^{-t\Delta} | y \rangle$$

where Δ is a Laplacean as a path integral over paths going from y to x. Then the particular expression (*) will be of the form

$$\int e^{-\frac{1}{t}E(\varphi)} F(\varphi)$$

where $F(\varphi)$ vanishes to high order at $\varphi =$ the constant path. Because of this asymptotic expansion as $t \to 0$ will be particularly easy to evaluate and will provide a local index theorem.

It ought to be true that if

$$\langle x | e^{-t\Delta} | y \rangle = \int e^{-\frac{1}{t}E(\varphi)} F(\varphi)\, \mathcal{D}\varphi$$

is true, then the $t \downarrow 0$ asymptotics of the path integral yield the standard asymptotic expansion of the heat kernel. So what is important is why the particular case (*) yields no negative powers of t, and this hopefully results because the integrand vanishes to high order at the critical pt. of the energy functional.

It will be necessary to understand a finite dimensional case first. Suppose then we have an integral

$$(*) \qquad \int e^{-\frac{1}{t}f(x)} F(x)\, d^n x$$

where $f(x)$ has a non-degenerate minimum at $x=0$ and no other critical point in $\mathrm{Supp}(F)$. To evaluate the asymptotics as $t \to 0$ we can change variables and assume that $f(x) = \sum_1 x_i^2$, and write $F(x)\, d^n x$ as a power series with terms $x^\alpha d^n x$. Then

$$\int e^{-\frac{1}{t}|x|^2}\, x^\alpha\, d^n x = \int e^{-|x|^2}\, t^{|\alpha|/2} x^\alpha\, t^{n/2} dx^n$$

$$= t^{n+(|\alpha|/2)} \cdot \text{constant}$$

and the constant $=0$ if some α_i is odd. So we will get a simple formula

$$\int e^{-\frac{1}{t}f(x)} F(x)\, d^n x \approx \text{const} + O(t)$$

provided $F(x)$ vanishes to order $n/2$ at $x=0$

So next let's go back to the representation of $\langle x | e^{-t\Delta} | y \rangle$ as a path integral and try to keep track of powers of t. We have

$$\langle x | e^{-t\Delta} | y \rangle = \int d^n x_{N-1} \cdots d^n x_1 \, \langle x | e^{-\varepsilon\Delta} | x_{N-1} \rangle \cdots \langle x_1 | e^{-\varepsilon\Delta} | y \rangle$$

and we are going to let $t \downarrow 0$ $\qquad \varepsilon = t/N$ and $N \to \infty$ in some way. Let's first keep N fixed and let's use the known asymptotic expansion for the kernel of $e^{-t\Delta}$. We have

$$\langle x | e^{-t\Delta} | y \rangle \sim \frac{e^{-\frac{d(x,y)^2}{t}}}{(4\pi t)^{n/2}} \left[A_0(x,y) + A_1(x,y)t + \cdots \right]$$

and hence we are going to get [redacted] a formula

$$\langle x | e^{-t\Delta} | y \rangle = \int d^n x_{N-1} \cdots d^n x_1 \; \frac{e^{-\frac{N}{t}\sum' d(x_i,x_{i-1})^2}}{(4\pi t)^{n/2} N} F(x, x_{N-1}, \ldots, x_1, y, \tfrac{t}{N})$$

where F has an asymptotic expansion in powers of t

$$F \sim F_0 + F_1 t + \cdots\cdots$$

Now the question becomes what sort of vanishing on F do we need? We take $x = y = 0$. Then we are integrating over a space of $(x_{N-1}, \ldots, x_1)$ of dim $(N-1)n$ and we have a factor of $t^{\frac{n}{2}N}$ in the denominator, so in order to get a result without negative powers of t we need to have $F_0(0, x_{N-1}, \ldots, x_1, 0)$ to vanish to order $n/2$, which is nicely independent of N. We also need to know that F_1 vanishes to order $\frac{n}{2}-1$, that F_2 vanishes to order $\frac{n}{2}-2$, etc. For example if I take $N = 1$, then there is no integration and one is requiring the asymptotic exp. [redacted] for

$$\mathrm{tr}_{sp} \langle 0 | e^{-tD^*D} | 0 \rangle - \mathrm{tr}_{sp} \langle 0 | e^{-tDD^*} | 0 \rangle$$

to begin with t^0 term.

December 10, 1982:

It's time to understand Kirwan's Morse theory approach to moduli problems. We have a complex group G_c acting on a compact complex manifold X and an equivalence holomorphic line bundle L. Inside G_c is a maximal compact subgroup G and G_c is the complexification of G. Put a metric on L invariant under G and assume the curvature is everywhere > 0, then X should be a Kähler manifold, in fact projective by Kodaira, the projective embedding coming from $L^{\otimes N}$. The compact group G acts as symplectic transformations on X, and because the lift of the G-action on X is given to L, a moment map is defined:

$$X \longrightarrow \mathrm{Lie}(\mathfrak{g})^*$$

Let's recall the moment map formulas and change X to M to avoid confusion. The line bundle L has a connection ∇ which is invariant under G. A connection we can think of as assigning to each vector field Y on M a differential operator ∇_Y on sections of L such that

$$\nabla_Y(fs) = f\nabla_Y s + (Yf)s$$

If $X \in \mathfrak{g} = \mathrm{Lie}(G)$, then X acts on sections of L such that

$$X * (fs) = \bar{X}f \cdot s + f(X * s).$$

where we denote by $\bar{X}$ the vector field on M induced by X. Thus

$$X * s = \left(\nabla_{\bar{X}} + \varphi_X\right) s$$

where φ_X is a function on X. Because both the G-action and connection preserve the metric

$$\bar{X}|s|^2 = \langle s | Xs \rangle + \langle Xs | s \rangle$$

$$\bar{X}|s|^2 = \langle s | \nabla_{\bar{X}} s \rangle + \langle \nabla_{\bar{X}} s | s \rangle$$

we have

$$\langle s | \varphi_X s \rangle + \langle \varphi_X s | s \rangle = (\varphi_X + \bar{\varphi}_X)|s|^2 = 0.$$

Thus φ_X has purely imaginary values, just like the connection and curvature forms.

As the G-action on L preserves the connection we have

$$[X, \nabla_Y] = \nabla_{[\bar{X}, Y]}$$

$$\parallel$$

$$[\nabla_{\bar{X}} + \varphi_X, \nabla_Y]$$

or

$$Y\varphi_X = [\nabla_{\bar{X}}, \nabla_Y] - \nabla_{[\bar{X}, Y]}$$

$$= \omega(\bar{X}, Y)$$

where ω is the curvature form. This says

$$d\varphi_X = i(\bar{X})\omega$$

or that φ_X is a Hamiltonian for $\bar{X}$.

The condition that we have an action of $\mathfrak{g}$ on L is

$$[\nabla_{\bar{X}} + \varphi_X, \nabla_{\bar{Y}} + \varphi_Y] = \nabla_{[\bar{X}, \bar{Y}]} + \varphi_{[X,Y]}$$

$$\omega(\bar{X}, \bar{Y}) + \underbrace{\bar{X}\varphi_Y - \bar{Y}\varphi_X}_{\omega(\bar{Y}, \bar{X}) - \omega(\bar{X}, \bar{Y})} = \varphi_{[X,Y]}$$

or

$$\varphi_{[X,Y]} = -\omega(\bar{X}, \bar{Y}).$$

Next I want to discuss some examples. First of all I really don't want to assume X is compact. Perhaps the most interesting case to begin with is where X is a complex vector space which is a representation of G. Ultimately we get this situation out of a projective variety anyway. Not too clear!

Let's begin with $G = U_n$ acting on $\mathbb{C}^n$ and take L to be the trivial line bundle with the Gaussian

metric $$\|s\|^2 = e^{-|z|^2},$$
where s is the canonical section of the trivial bundle. The moment map in this case must be a mapping of $V = \mathbb{C}^n$ into the real dual of $\mathrm{Lie}(U_n) =$ skew-symm. matrices which is invariant under the U_n-action. The only natural way I can see to associate to a vector v an operator is to use $|v\rangle\langle v|$.

Let's compute. Using the canonical section s we have the connection given by
$$\nabla s = \theta s \qquad \theta = \partial \log |s|^2 = -\bar{z}^\mu dz^\mu$$
Also the canonical section is invariant under the action, so
$$(\nabla_{\tilde{X}} + \varphi_X) s = 0. \qquad X \in \mathrm{Lie}(U_n).$$
Thus
$$\varphi_X(v) = -\theta(X)\big|_v = \langle v|X|v\rangle$$

To summarize, given a unitary representation of a compact Lie group in a Hilbert space, the moment map for the standard Kähler metric assigns to each v the linear function on the Lie algebra
$$X \longmapsto \frac{\pm 1}{i}\langle v|X|v\rangle$$
($\pm$ depends on sign conventions)

Next example: Consider the adjoint action of $gl_n\mathbb{C} = g_c$ on itself. Here the orbits are given by the Jordan normal forms for matrices. Semi-simple elements give closed orbits, which are described by the invariant polynomials.

General situation: G_c is a reductive group, say connected and the complexification of a connected compact group G. V is a complex repn. of G, hence an alg. rep. of G_c. One can form the ring $S(V^*)^G$ of inv.

polynomials and the problem is to describe the map

$$G_c \backslash V \longrightarrow \mathrm{Var}\{S(V^*)^G\}.$$

Clearly a closed orbit of G_c in V contains points which are closest to 0, and the set of these is acted on by G. In the case of the adjoint action a closed orbit is a conjugacy class of semi-simple elements, and the natural conjecture is that the points of minimum distance are those whose eigenspaces are $\perp$. These are all conjugate under the compact gp. G.

December 11, 1982

One knows that for Laplaceans Δ the heat kernel has a small asymptotic expansion

$$(*) \quad \langle x | e^{-t\Delta} | y \rangle = \frac{e^{-\frac{d(x,y)^2}{4t}}}{(\sqrt{4\pi t})^{n/2}} \left[A_0(x,y) + A_1(x,y)t + \cdots \right].$$

I am proposing to use this asymptotic expansion to set up a path integral expression for the heat kernel. The path integral is supposed to use only the first two ~~terms~~ A_0, A_1, which are functions of the first and zeroth order terms of Δ.

However it seems that Seeley's work shows that the above sort of asymp. expansion exists, where A is a ψDO, at least through the $A_{n/2}t^{n/2}$-term. So we reach a paradox: If I have a way to derive a path integral expression from the first two terms of the asymptotic expansion, then it should also work for Δ + smooth which has the same asymptotic expansion. (Actually, don't worry the first two terms)

Let me ~~state~~ this paradox more clearly. The asymptotic expansion of the heat kernel depends on

$$e^{\frac{u}{t}} \Delta e^{-\frac{u}{t}}$$

having an asymptotic expansion in t. This remains true when Δ is a ψDO, and the asymptotic exp. depends only on the symbol of Δ, so one gets exactly the same asymptotic expansion for Δ+ smooth-kernel-op. Therefore we can't [redacted] derive a path integral expression for the heat kernel solely using the asymptotic expansion. Some other ingredient must be added.

Let's check this with a constant coefficient [redacted] example.

Consider $\Delta = -\frac{1}{2}\nabla^2 + M$ where M is a convolution operator

$$(Mf)(x) = \int d^n y \; m(x-y)\, dy \qquad m \in C_0^\infty(\mathbb{R}^n)$$

Then M and $-\frac{1}{2}\nabla^2$ commute so we have

$$e^{-t\Delta} = e^{-t\left(-\frac{1}{2}\nabla^2\right)} \; e^{-tM}$$

$$\langle x | e^{-t\Delta} | 0 \rangle = \int d^n y \; \frac{e^{-\frac{|x-y|^2}{2t}}}{(2\pi t)^{n/2}} \underbrace{\langle y | e^{-tM} | 0 \rangle}_{\sum_{k=0}^{\infty} \frac{(-t)^k}{k!} \langle y | M^k | 0 \rangle}$$

But

$$\int d^n y \; \frac{e^{-\frac{|x-y|^2}{2t}}}{(2\pi t)^{n/2}} f(y) = \int d^n y \; \frac{e^{-\frac{|y|^2}{2t}}}{(2\pi t)^{n/2}} f(x+y)$$

$$= f(x) + t \cdot \text{second deriv of } f \text{ at } x + \cdots$$

$$= f(x) + \left[t\left(\tfrac{1}{2}\nabla^2\right) f\right](x) + \left[\frac{t^2}{2!}\left(\tfrac{1}{2}\nabla^2\right)^2\right] f(x) + \cdots$$

for f smooth such as $\langle y | M^k | 0 \rangle$ for $k \geq 1$. Hence we are getting an expansion

$$\langle x | e^{-t\Delta} | 0 \rangle = \frac{e^{-\frac{|x|^2}{2t}}}{(2\pi t)^{n/2}} + \sum_{k=1}^{\infty} \frac{(-t)^k}{k!} \; e^{t\left(\frac{1}{2}\nabla^2\right)} \langle x | M^k | 0 \rangle$$

where the second term is a power series in t without the exponential factor.

We have discovered in this example that when Δ is a ψDO, then the heat kernel needn't decay exponentially as $t \downarrow 0$ off the diagonal. Thus it appears that the paradox is resolved, namely the asymp. exp. required for setting up the path integral exists only for Δ which are differential operators.

The problem is the following with a convolution operator:

$$e^{+\frac{|x|^2}{2t}} M e^{-\frac{|x|^2}{2t}} f = \int d^n y \; e^{\frac{|x|^2}{2t}} m(x-y) e^{-\frac{|y|^2}{2t}} f(y)$$

$$= \int d^n y \, e^{\frac{|x|^2 - |y|^2}{2t}} m(x-y) f(y)$$

and we can't say anything as $t \downarrow 0$ because the integrand blows up when $|x| > |y|$. If we use imaginary time we get a contribution at the critical point $y = 0$, but then the $e^{|x|^2/2t}$ factor is wrong.

To summarize: Only for ~~[illegible]~~ differential operators can we expect and asymptotic expansion (*). The example of a convolution type ΨDO gives ΨDO's where the heat kernel does not decay exponentially for $x \neq y$ as $t \downarrow 0$.

I next want to go over the derivation of the Kolmogorov-Chapman type DE. The idea here is that we have a 1-parameter semi-group of operators e^{-tH}, for $t > 0$, and assuming the kernel $\langle x | e^{-tH} | y \rangle$ has an asymptotic expansion of the form (*), one shows that $e^{-tH} f$ satisfies a certain heat type equation.

Put $f_t(x) = (e^{-tH} f)(x) = \int dy \, \langle x | e^{-tH} | y \rangle f(y)$. Then

$$f_{t+\Delta t}(x) = \int dy \, \langle x | e^{-(\Delta t)H} | y \rangle f_t(y)$$

Presumably one could plug in the asymptotic expansion into this and do the Gaussian type integral as $\Delta t \downarrow 0$. But it is easier to use a test function

$$\int dx \, \varphi(x) f_{t+\Delta t}(x) = \int dy \left[\int dx \, \varphi(x) \langle x | e^{-\Delta t H} | y \rangle \right] f_t(y)$$

Because of the asymptotic expansion

$$\int dx\, \varphi(x) \langle x | e^{-\Delta t H} | y \rangle \sim \left(1 + \Delta t\, p_1 + (\Delta t)^2 p_2 + \cdots \right) \varphi(y)$$

where the p_n are differential operators. This results from the factor that because φ is smooth, the asymptotic expansion of a Gaussian style integral depends only on the derivatives of φ at the center. So now

$$\int dx\, \varphi(x)\, f_{t+\Delta t}(x) \sim \int dy \left[1 + \Delta t\, p_1 + \cdots \right] \varphi(y)\, f_t(y)$$

$$= \int dy\; \varphi \left(1 + \Delta t\, p_1^* + \cdots \right) f_t(y)$$

using the defn. of diff'l operators on distributions. Thus

$$\partial_t \int dx\, \varphi(x)\, f_t(x) = \int dx\, \varphi(x) \left(p_1^* f_t \right)$$

so in some weak sense

$$\partial_t f_t = p_1^* f_t .$$

Hence $p_1^* = H$, and so H is a differential operator.

Our goal is to prove a local index formula for the Dirac operator over a spin manifold of even dimension n, using path integral formulas for

$$\mathrm{tr}_{sp} \left(\langle x | e^{-t \not{D}^2} | x \rangle \gamma_5 \right),$$

and then use asymptotics on the path integral as $t \downarrow 0$. Let's try to understand what is happening in general terms. We have a Riemannian manifold M and the spinor bundle S over it. This is a vector bundle with metric which comes with a connection and a module structure over $C(T^*)$. The Dirac operator is

$$S \xrightarrow{\nabla} T^* \otimes S \xrightarrow{\text{Clifford mult.}} S.$$

Suppose now that we wish to represent the heat kernel $\langle x | e^{-tD^2} | y \rangle$ as an integral over the paths going from y to x in time t. Then we must associate to each path a maps from S_y to S_x. One candidate is parallel translation along the curve using the connection. Another possibility would be to take the element of $\Lambda^2 T$ along the curve given by $\dot\varphi \wedge \ddot\varphi$ and convert this to an element of the Clifford algebra either directly or via the curvature, and use this to modify the parallel translation along the curve. However according to Atiyah's talk, one only wants to worry about $\|$ translation, because of the identity

$$\sqrt{\det(1 - T_\varphi)} = tr(S_\varphi \gamma_5)$$

S_φ = monodromy of φ ~~on~~ spinors S

for a closed path.

Basic maps

$$\Omega(M, x) \xrightarrow{\|\text{ transport}} Spin(n) \xrightarrow{\sqrt{\det(1-T)}} \mathbb{R}$$

$$\varphi \longmapsto T_\varphi$$

The function $\sqrt{\det(1-T)}$ vanishes to order $n/2$ at $T = 1$, hence the fn. $tr(S_\varphi \gamma_5)$ will vanish to order $\geq n/2$ at $\varphi = 0$ = constant path with value x.

A nice problem is to describe the homog. fn. of degree $n/2$ on the tangent space to $\Omega(M,x)$ at $\varphi = 0$ which is the leading term of $\sqrt{\det(1-T_\varphi)}$. Notice that this tangent space consists of maps $S^1 \to T_x$ sending $t=0$ to 0.

This problem has nothing to do with the spinors being connected with the tangent bundle of M. One could start with a principal G-bundle P over M with a connection. Then // transport furnishes a map

$$\Omega(M, x) \longrightarrow G$$

the monodromy. Given $S^1 \longrightarrow M$ one pulls back P to S^1 with its connection. Then the monodromy is a map from P_x to P_x commuting with right mult. by G. So the correct thing to say is that we have

$$\Omega(M, x) \longrightarrow \mathrm{Aut}_G(P_x)$$

and so if we choose a basept. in P_x, then we [redacted] can identify $\mathrm{Aut}_G(P_x)$ ~~with~~ G.

Now take one of the sly index-type functions on G, I mean a character of a virtual representation and one can ask the same kind of question.

One can also ask [redacted] the same sort of question with free loops, in which case things might be easier [redacted] because of the Fourier series.

December 12, 1982

Path integral approach to a local index formula:

The local index at a point $O \in M$ is to be an integral over paths $S^1 \longrightarrow M$ with this basepoint: notation $\Omega(M, O)$. There is a Gaussian measure obtained from the energy and an integrand of the form $(\det 1-T)^{1/2}$ where T is the monodromy. Now $\det(1-T)$ = the determinant of the cov. deriv. ∇ along the path. The energy is somehow ∇^2, so heuristically the Gaussian, or boson, integral contributes a $(\det \nabla^2)^{-1/2} = (\det 1-T)^{-1}$, so the total boson + fermion contribution is $\det(1-T)^{-1/2}$, which puts the sines in the denominator to give the $\hat{A}$-genus.

It seems to me that an important question is how do we get differential forms as an answer. The analysis presents one with a density for the local index which depends on the orientation, and [redacted] is more naturally viewed as a differential form. (This is an essential ingredient of the ABP proof.) So, even if I can push this path integral approach to a proof of the local index fmla, I still have the awkward step of using the asymptotics to get a [redacted] number, which has to multiplied by the volume before [redacted] we get the good gadget.

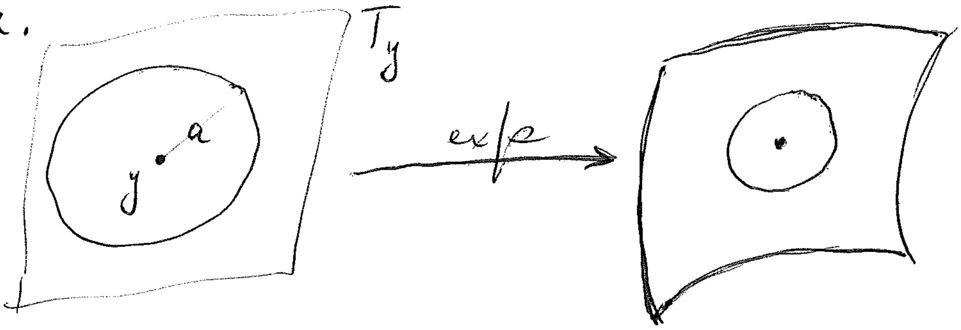
Recall yesterdays problem: Given a principal G bundle P over M with trivialization over $O \in M$, and a connection on P, we have the monodromy map

$$\Omega(M, O) \longrightarrow G.$$

Now take a virtual character $\chi \in R(G)$ [redacted] which vanishes to a high order m at the identity. Then the problem was to compute the leading term of the pull-back of χ to $\Omega(M,O)$ as a polynomial function on the tangent space to $\Omega(M,O)$.

December 13, 1982

Heat flow on a Riemannian manifold. Let us consider a random walk on a Riemannian manifold M of the following sort. We allow the particle to jump every Δt seconds and give the probability distribution $p_{\Delta t}(x,y)|dx|$ of finding the particle at x having started at y. A simple example is to make it jump a fixed distance a with random direction. Then $p_{\Delta t}(x,y)$ is the image under the exponential map at y of the spherically symmetric measure in T_y supported on the sphere of radius a.



Then if t is fixed and $\Delta t = \frac{t}{N}$, the distribution we get after N-steps is the convolution

$$p_{\Delta t} * p_{\Delta t} * \cdots * p_{\Delta t} \qquad N\text{-times}$$

and we take the limit as $N \to \infty$ to get the heat kernel.

We know from various examples that the specific nature of the probabilities $p_{\Delta t}$ is not important, only the first two moments matter. From one viewpoint we have a path P_t in the space of operators ~~starting~~ at the identity at $t=0$, and we are constructing the one-parameter semi-group

$$\lim_{N\to\infty} \left(P_{\frac{t}{N}}\right)^N$$

so all that matters is the infinitesimal generator. I also have a derivation (Chapman-Kolmogorov) of the heat equation which shows only the first two moments matter. Specifically I assume that for φ a smooth function

$$\int \varphi(x)\, p_{\Delta t}(x,y)\,|dx| \;=\; \varphi(y) - (H\varphi)(y)\,\Delta t \;+ O(\Delta t^2)$$

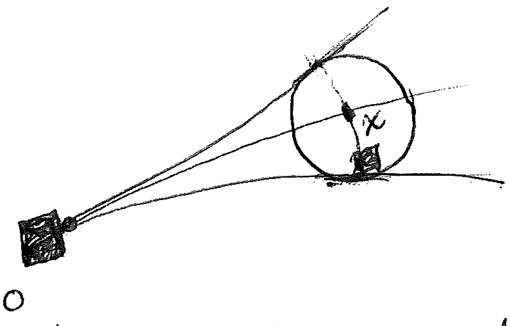
where H is a second order operator.

Problem: Suppose you have smooth measures $p_{\Delta t} \to \delta$ as $\Delta t \downarrow 0$, do the moments of $p_{\Delta t}$ have an intrinsic asymptotic meaning?

What does actual heat flow correspond to? It seems obvious that if $p_{\Delta t}(x,y)\,|dx|$ is the image under the exponential map at y of the Gaussian measure in T_y with variance $\sim \Delta t$, then we get the physical heat flow. So it is natural to ask what H this corresponds to. If we pull the above integral back to T_y, then it is clear that H has to be $-\Delta$ on T_y hence also on M.

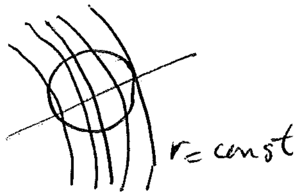
Next I would like to understand what $K_t(x,y)\,|dx|$ looks like, especially how it differs from the image of the Gaussian measure on T_y. Let's take a surface of constant curvature in which case we know by symmetry considerations that $K_t(x,y)$ is a function of $d(x,y)$. Fix an origin O on M and denote by r the distance from O. Start a particle at the origin at time 0, and look at the distance of the particle from O. This gives a random walk on the positive axis $r \geq 0$. We can try to understand what governs this random walk.

We suppose the particle is at a distance r_0 and then we want to calculate its distance, or rather the prob. dist. of its distance r, at a time Δt seconds later. By symmetry one can fix the position at x with $r(x) = r_0$. Then in time Δt the particle jumps from x to a sphere



of radius $\sim \sqrt{\Delta t}$ around x with random direction. Now we must compute the new distance and weight according to the volume on the sphere. The above picture shows a negatively curved surface. It's clear that the distance tends to increase faster in time than is the case for flat space.

Look in flat space $\mathbb{R}^n$ at $\langle |x|^2 \rangle$ = expectation value of r^2. Then $\langle |x|^2 \rangle = \sum_i \langle x_i^2 \rangle = n\Delta t$ so one gets a different behavior for $\langle r^2 \rangle$ for different n. This shows that curvature effect can be non-trivial, precisely that the r = constant curves



tend to favor larger r. I think the same thing should happen for a curved surface, but precise calculations are necessary.

It seems reasonable now to expect the heat kernel $K_t(x,0)$ for a curved space to differ from the image of the Gaussian measure under the exponential map. Perhaps we will get a different value for $\frac{\langle r^2 \rangle}{t}$ as $t \downarrow 0$.

December 14, 1982

Consider a flag manifold $U/T = G/B$. This has a Kahler metric invariant under U, hence the Laplacean on forms is invariant under the U-action.

On the other hand we have a homomorphism of the Lie algebra $\mathfrak{u}$ of U into the Lie alg. of vector fields on G/B which acts on the space of forms. This extends to an action of the universal enveloping algebra of

$$\mathfrak{u} \underset{\mathbb{R}}{\otimes} \mathbb{C} = \mathfrak{g}$$

on the sections of $\Lambda^p T^*$. Hence from the Casimir operator in $U(\mathfrak{u}\otimes\mathbb{C})$ we get a 2nd order operator on forms. What is the relation between this Casimir operator and the Laplacean?

Too vague.

Grassmannian: $U_{m+n}/U_m \times U_n$. Compute its homology via Lie algebra homology.

$$\mathfrak{gl}_{m+n}/\mathfrak{gl}_m \times \mathfrak{gl}_n = \left(\begin{array}{c|c} 0 & \operatorname{Hom}(W,V) \\ \hline \operatorname{Hom}(V,W) & 0 \end{array}\right) \qquad \begin{array}{l} W = \mathbb{C}^n \\ V = \mathbb{C}^m \end{array}$$

In general we know that

$$\left[\Lambda^\cdot (\mathfrak{g}/\mathfrak{h})^*\right]^{\mathfrak{h}}$$

is the algebra of invariant differential forms on G/H. In this case

$$\begin{aligned} \mathfrak{g}/\mathfrak{h} &= \operatorname{Hom}(V,W) \oplus \operatorname{Hom}(W,V) \\ &= V^*\otimes W \oplus W^* \otimes V \end{aligned}$$

and $\quad \mathfrak{h} = \operatorname{End} V \oplus \operatorname{End} W.$

$$\Lambda^k(\mathfrak{g}/\mathfrak{h}) = \bigoplus_{p+q=k} \Lambda^p(V^*\otimes W) \otimes \Lambda^q(W^*\otimes V)$$

and the p,q term is

$$\left[(V^*\otimes W)^{\otimes p} \otimes (W^*\otimes V)^{\otimes q}\right] \underset{\Sigma_p\times\Sigma_q}{\otimes} (\mathrm{sgn}\otimes\mathrm{sgn})$$

But now if I take coinvariants wrt $\mathrm{End}(V)$, we get something $\neq 0$ only if $p=q$. This is enough to show that the complex of forms is $\neq 0$ only in even dimensions, and that the invariant forms give the Lie algebra homology. This persists even when we put in the ring A, so this gives something only in even dimensions like the Weil algebra.

Next point: Let's finish the calculation using invariant theory, assuming $m,n \geq p$. We have from the above an embedding

$$\mathbb{C}\left[\Sigma_p \times \Sigma_p\right] \hookrightarrow (V^*\otimes W)^{\otimes p} \otimes (W^*\otimes V)^{\otimes p}$$

where I think of the latter as $\mathrm{End}(V^{\otimes p}) \otimes \mathrm{End}(W^{\otimes p}) = \mathrm{End}(V^{\otimes p}\otimes W^{\otimes p})$.

Look at $V^{\otimes p}\otimes W^{\otimes p}$. We have a way to make $\Sigma_p\times\Sigma_q$ act on this by permuting the factors, so this embeds $\mathbb{C}[\Sigma_p\times\Sigma_p] \subset \mathrm{End}(V^{\otimes p}\otimes W^{\otimes p})$. On the other hand given $\sigma\in\Sigma_p$ and an endo φ: of $V^{\otimes p}\otimes W^{\otimes p}$ we can act on φ by

$$(\sigma\otimes\mathrm{id})\,\varphi\,(\mathrm{id}\otimes\sigma^{-1}).$$

(This is the Σ_p action on $W^*\otimes V$.)

So if $\varphi = (\sigma_1\otimes\sigma_2)$, this is $\sigma\sigma_1\otimes\sigma_2\sigma^{-1}$. Thus I want the orbits of $\Sigma_p\times\Sigma_p$ acting on itself by

$$(\sigma,\tau)*(\sigma_1,\sigma_2) = (\sigma\sigma_1\tau^{-1}, \tau\sigma_2\sigma^{-1})$$

and the obvious invariant is the conjugacy class of $\sigma_1\sigma_2$. We get a cross-section for the orbits by taking $\sigma_2 = 1$

whence [redacted] $\sigma = \tau$ and we have $\mathbb{C}[\Sigma_p]$ (spanned by σ_1) acted on by conjugation. We have to tensor with $\sigma, \tau \longmapsto \operatorname{sgn}\sigma \cdot \operatorname{sgn}\tau$ which is the trivial character [redacted] on the subgp. $\sigma = \tau$. Thus [redacted] the primitive elements will correspond to cyclic permutations, and there will be one of each even degree. When a ring A is put in, the same calculation gives the symmetric Connes group

$$(A^{\otimes p}) \underset{\mathbb{Z}/p}{\otimes} (\mathbb{1})$$

for the primitive part of degree $2p$.

Incredibly interesting point maybe: The homology of $\mathbb{Z} \times BU$ is the same as the Fock space of $L^2(S^1)$.

December 17, 1982

On Witten's talk yesterday.

Low energy description of QCD: Basic symmetry is $SU(3)_L \times SU(3)_R$ which is broken spontaneously. The different vacuum states are permuted transitively and the stabilizer of one of them is the diagonal subgroup. One can thus identify the set of vacuum states with

$$G \times G / \Delta G = G$$

where $G \times G$ acts by left and right multiplication.

Now a low energy state should at each point be close to a vacuum state. Hence we can try to construct a low energy model by taking smooth functions $U(\vec{x})$ $\vec{x} \in \mathbb{R}^3$ with values in G. The dynamics is to be given by an effective Lagrangian, which is to be made up of as few derivatives of U as possible, in keeping with the idea that low energy means that $U(\vec{x})$ is slowly varying in x.

First candidate is

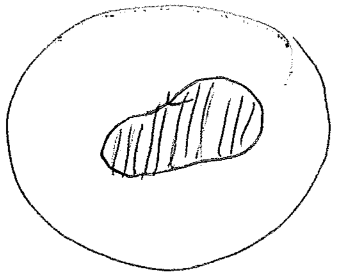
$$\int d^4x \ tr\left(U^{-1}\partial_\mu U \ U^{-1}\partial_\mu U\right)$$

This Lagrangian gives rise to the σ-model. It is very successful at describing low energy QCD, but it has too many symmetries. In particular boson number mod 2 is conserved, which forbids certain reactions which occur. So one wants to add a term to the Lagrangian, something done by Wess-Zumino.

Witten derived the equation of motion for σ-model Lagrangian and added a suitable quartic term which looked good. The problem is to find a Lagrangian producing this quartic term. To explain how one produces the

Lagrangian he [redacted] gave a simpler examples.

Consider particle motion on S^2. This is decribed by giving the kinetic energy for Lagrangian. Now put a magnetic monopole at the origin and assume the particle is charged. Then we get a Lorentz force added to the equation of motion. How do we get this from a Lagrangian? We choose a vector potential A for the magnetic field B, and add the term $A_\mu dx^\mu$ to the Lagrangian. But in this monopole example A has a singularity (the Dirac string), so you want to eliminate A in favor of B. To do this, assume you want periodic motions. Then a [redacted] particle path will be a loop on S^2



and
$$\oint_\gamma A_\mu dx^\mu = \int_{\text{surface bounding } \gamma} B_{\mu\nu}\, dx^\mu dx^\nu$$

However the surface bounding γ won't be unique, one can also use the other side of γ. Actually one is [redacted] using a contracting homotopy of γ, and the difference of 2 contractions is a map of S^2 into S^2, hence different choices change $\int_{\text{surface bdg } \gamma} B_{\mu\nu}\, dx^\mu dx^\nu$ by an integral multiple of

$$\beta = \int_{S^2} B_{\mu\nu}\, dx^\mu dx^\nu$$

[redacted] So if we add to the $\int$ Kinetic energy the term

$$\frac{2\pi n}{\beta} \int_{\text{surface bdgs. } \gamma} B_{\mu\nu}\, dx^\mu dx^\nu \qquad n \in \mathbb{Z}$$

then the ambiguity disappears in the path integral

$$\int e^{iL}$$

The requirement that $n \in \mathbb{Z}$ is the same condition derived by Dirac for monopoles (Existence of a monopole + QM. $\Longrightarrow$ all charges are multiples of election charge e)

Now return to Wess-Zumino. This time instead of maps $S^1 \to S^2$ we have maps $S^4 \to SU(3)$ which are all null-homotopic. The role of B is played by a biinvariant 5-form ω on $SU(3)$, that is, a closed diff'l form representing an integral coh. class. Then for any map $S^4 \xrightarrow{f} SU(3)$ we can extend to the disk $D^5 \longrightarrow SU(3)$ and integrate the pull-back of ω. Two different extensions of f to the disk give a map $S^5 \longrightarrow SU(3)$ whose ω integral is integral. Thus

$$e^{2\pi i \int_{\text{extension of } f \text{ to } D^5} \omega}$$

is a well-defined function on maps $S^4 \longrightarrow SU(3)$.

Next put in an integer n to get the Wess-Zumino style Lagrangian. The question then is which n corresponds to QCD? He couples the above theory to the EM field, and calculates the process

$$\pi^0 \longrightarrow \gamma\gamma$$

which one knows about and finds $n = 3$.

The exact nature of coupling is unclear to me, except that derivatives ∂_μ become covariant derivatives. Atiyah translated this into a precise mathematical problem, which I should now try to understand.

First example: Canonical maps $\Omega^{2n} U \longrightarrow S^1$ defined by differential forms. More generally for maps $f: S^{2n} \longrightarrow U$. Take the biinvariant form ω on U describing the primitive generator of $H^{2n+1}(U, \mathbb{Z})$. Because f is null-homotopic it extends to D^{2n+1} and one can take

$$e^{2\pi i \int_{\text{ext. of } f \text{ to } D^{2n+1}} \omega}$$

Special case $\Omega^2 U_n \longrightarrow S^1$. This depends on the biinvariant 3-form on U_n given up to a constant by

$$\mathrm{Tr}\,(g^{-1}dg)^3$$

which we know in the fibration

$$U_n \longrightarrow * \longrightarrow BU_n$$

transgresses to c_2. On the other hand in the fibration

$$\Omega U_n \longrightarrow * \longrightarrow U_n$$

this class in $H^3(U_n)$ comes from a class in $H^2(\Omega U_n)$ which one can conjectures to be the first Chern class of the determinant line bundle. But we know the determinant line bundle has a canonical connection, hence taking a map $S^2 \longrightarrow U_n$ and regarding it as a loop in ΩU, we can parallel translate and get an element of S^1. Close to Hilbert space.

Another possibility is to regard maps $S^2 \to U_n$ as being gauge transformations. Make them act on the determinant line and use the anomaly somehow.

Maybe the map $\Omega^2 U_n \longrightarrow S^1$ viewed as a map $S^1 \longrightarrow \Omega^1 U_n$ can be interpreted as giving the phase for the vacuum-vacuum amplitude?

December 18, 1982

Path integrals again. Let's start with the idea of random walk on a Riemannian manifold M, say compact. Random walk gives a flow on probability measures, namely, given the probability measure for finding the particle at time zero, then after walking for time t you end up with a new probability measure.

Let the initial prob. distribution be given by

$$f(y)\sqrt{g(y)}\,d^ny \qquad\qquad g = \det g_{\mu\nu}$$

then the prob. dist. after time t walking is

$$\left(\int K_t(x,y)\,f(y)\sqrt{g(y)}\,d^ny\right)\sqrt{g(x)}\,d^nx$$

for some kernel $K_t(x,y)$. ~~The Dirac measure located at the point y is given by~~

$$\cancel{f(y)\,g(y)^{1/2}d^ny = \delta(y)\,d^ny}$$

$$\cancel{\text{or } f = g}$$

Let's take the initial prob. dist. to be the Dirac measure at the point z:

$$f(y)\sqrt{g(y)}\,d^ny = \delta(y-z)\,d^ny$$

and so

$$* \quad \boxed{K_t(x,z)\,g(x)^{1/2}d^nx = \text{prob. dist. of finding the particle after walking for time } t \text{ starting at } z}$$

Next let's describe the actual random walk process. Subdivide t into N steps of time $t/N = \Delta t$

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* Note: $\quad K_t(x,z)\,g(x)^{1/2}d^nx \longrightarrow \delta(x-z)\,d^nx \quad$ as $t\to 0$

so $\quad K_t(x,z) \longrightarrow \delta(x-z)\,g(x)^{-1/2}$ (or z) $\quad$ as $t \to 0$

I give a rule for how the particle moves in a small time Δt. This is a probability distribution

$$L_{\Delta t}(x,y)\,(g(x))^{1/2} d^n x$$

for each point y which is peaked around the point y in a sense to be made precise. I will proceed to collect the properties of $L_{\Delta t}(x,y)$ that are needed as we go along.

Now we put

$$K_t = \lim_{N\to\infty}\left(L_{t/N} * \cdots\cdots * L_{t/N} \quad N\text{-times}\right)$$

and assume the limit exists. Then K_t should give a flow on probability measures. This should be a formal consequence of the fact that if one has a path $\gamma(t)$ of operators starting at the identity, then

$$\lim_{N\to\infty} \gamma(t/N)^N = \rho(t)$$

will be a 1-parameter semi-group. I need that $L_{\Delta t} \to I$ as $\Delta t \to 0$ i.e.

$$L_{\Delta t}(x,y) \longrightarrow \delta(x-y)\,g(y)^{-1/2}$$

I probably also need to know that $\gamma(t)$ has a tangent vector at $t=0$:

$$\gamma(t) = I + \gamma'(0)\Delta t + O(\Delta t^2)$$

in order to guarantee the existence of the limit $N\to\infty$. What does this mean for $L_{\Delta t}(x,y)$? Look "weakly" i.e. choose a smooth function $\varphi(x)$ on M and form

$$\int \varphi(x)\, L_{\Delta t}(x,y)\, g(x)^{1/2} d^n x$$

and look for an asymptotic expansion in Δt.

Fix y, call it O, and use normal coordinates on M around y. The sort of L_t I am interested in has the form

$$L_t(x,0)\, g(x)^{1/2} d^n x = \frac{e^{-\frac{|x|^2}{2t}}}{(\sqrt{2\pi t})^n}\left(\ell_0(x) + \ell_1(x)t + \cdots\right) d^n x$$

or better, has this asymptotic expansion. Then

$$\int \varphi(x) L_t(x,0)\, g(x)^{1/2} d^n x = \int \frac{e^{-\frac{|x|^2}{2t}}}{(2\pi t)^{n/2}} d^n x \left[\varphi \ell_0 + \varphi \ell_1 t + \cdots\right]$$

$$= \varphi(0)\ell_0(0) + \left[\frac{1}{2}\partial_\mu\partial_\mu(\varphi\ell_0)(0) + (\varphi\ell_1)(0)\right]t + \cdots$$

Here I am using that the Gaussian moments are:

$$\int \frac{e^{-\frac{|x|^2}{2t} + i\xi x}}{(2\pi t)^{n/2}} d^n x = e^{-\frac{t}{2}|\xi|^2}$$

$$\therefore \int \frac{e^{-\frac{|x|^2}{2t}}}{(2\pi t)^{n/2}} d^n x\, x^\mu x^\nu = -\frac{\partial^2}{\partial\xi^\mu\partial\xi^\nu} e^{-\frac{t}{2}|\xi|^2} \text{ at } \xi = 0$$

$$= \frac{t}{2}\partial_{\xi^\mu}\underbrace{\partial_{\xi^\nu}\xi^\alpha\xi^\alpha}_{2\xi^\alpha\delta^\alpha_\nu} = t\,\delta_{\mu\nu}$$

I want $L_t(x,y)\, g(x)^{1/2} d^n x$ to be a probability measure, hence taking $\varphi \equiv 1$ we want

$$1 = \ell(0) + \left[\frac{1}{2}\left(\partial_\mu^2 \ell_0\right)(0) + \ell_1(0)\right]t + \cdots$$

which gives the conditions

$$\ell(0) = 1$$

$$\left[\frac{1}{2}\partial_\mu^2 \ell_0 + \ell_1\right](0) = 0$$

Another feature of my random walk is that it shouldn't have any "drift", that is, as I move away from y I should favor all directions evenly. Hence we want $(\partial_\mu l_0)(0) = 0$. Then we will have

$$\int \varphi(x)\, L_t(x,0)\, g(x)^{1/2} d^n x \sim \varphi(0) + \tfrac{1}{2}(\Delta\varphi)(0)\, t + O(t^2)$$

By basic invariance it follows that

$$\int \varphi(x)\, L_t(x,y)\, g(x)^{1/2} d^n x \sim \varphi(y) + \tfrac{1}{2}(\Delta\varphi)(y)\, t + O(t^2)$$

hence

$$L_t \sim I + \tfrac{1}{2}\Delta t + O(t^2).$$

Actually this last step uses the self-adjointness of Δ: We have

$$\langle \varphi | L_t \psi \rangle = \int g(x)^{1/2} d^n x\, \varphi(x) \int g(y)^{1/2} d^n y\, L_t(x,y)\psi(y)$$

$$= \int g(y)^{1/2} d^n y \Big[\underbrace{\int g(x)^{1/2} d^n x\, \varphi(x)\, L_t(x,y)}_{\sim \varphi(y) + \frac{1}{2}(\Delta\varphi)(y) t + \cdots\cdots} \Big] \psi(y)$$

$$\sim \langle \varphi + \tfrac{1}{2}\Delta\varphi t | \psi \rangle +$$

$$= \langle \varphi | \psi \rangle + \tfrac{t}{2} \langle \varphi | \Delta\psi \rangle + \cdots\cdots$$

Therefore I see from the above that the coefficients l_0 and l_1 are not fixed. But when we grind out the heat kernel we do get specific coefficients. Hence among the choices of L_t is a good one that may have many virtues when it comes to the analysis.

I can attempt to describe L_t more invariantly. Thus we have

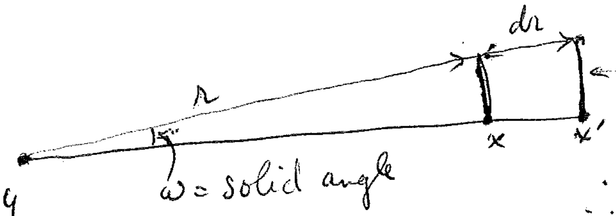
$$L_t(x,y)\, g(x)^{1/2} d^n x = \frac{e^{-\frac{d(x,y)^2}{2t}}}{(2\pi t)^{n/2}} \left[l_0(x,y) + l_1(x,y)t + \dots \right] (\exp_y)_* \begin{pmatrix} \text{vol. on} \\ T_y \end{pmatrix}$$

and hence

$$L_t(x,y) = \frac{e^{-\frac{d(x,y)^2}{2t}}}{(2\pi t)^{n/2}} \left[l_0(x,y) + l_1(x,y)t + \dots \right] J(x,y)$$

where $J(x,y) = \dfrac{(\exp_y)_* (\text{vol. on } T_y)}{g(x)^{1/2} d^n x}$. I have seen that the simplest solution is to take $l_0(x,y) = 1$ $l_1(x,y) = 0$ in which case $L_t(x,y)\, g(x)^{1/2} d^n x$ is the Gaussian measure on T_y pushed forward under the exponential map.

The next project will be to calculate the exact heat kernel asymptotics. I will ultimately need to know $\Delta\left(\frac{r^2}{2}\right)$, where $r(x) = d(x,y)$.



← has area $(r+dr)^{n-1}\omega$ in the metric on T_y

$\therefore$ has area $(r+dr)^{n-1}\omega\, J(x',y)^{-1}$ in M

By divergence thm.

$$\nabla\left(\frac{r^2}{2}\right) = r \cdot \hat{u}_r$$

$$\int_{\square} \Delta\left(\frac{r^2}{2}\right) \cdot \text{vol} = \int_{\text{boundary}} \nabla\left(\frac{r^2}{2}\right) \cdot \hat{n} \cdot \text{area}$$

$$= (r+dr) \text{ area of } \Big|_{x'} - (r) \text{ area of } \Big|_{x}$$

$$= (r+dr)^n \omega\, J(x',y)^{-1} - r^n \omega\, J(x,y)^{-1}$$

But $\quad \text{vol } \square = r^{n-1} \omega\, J(x,y)^{-1}\, dr$

So
$$\Delta\left(\frac{r^2}{2}\right) = r\frac{d}{dr}\log\left(r^n J^{-1}\right)$$

In normal coords around y we have
$$J(x,y) = \frac{(\exp_y)_*(\text{vol on } T_y)}{g(x)^{1/2}\, d^n x} = g(x)^{-1/2}$$

Now let's use the formula in normal coordinates
$$g_{\mu\nu}(x) = \delta_{\mu\nu} - \frac{1}{3} R_{\mu\alpha\nu\beta}\, x^\alpha x^\beta + O(x^3)$$

whence
$$\frac{\exp^*\left(g(x)^{1/2} d^n x\right)}{\text{vol. on } T_y} = g(x)^{1/2} = 1 - \frac{1}{6} R_{\mu\alpha\mu\beta}\, x^\alpha x^\beta + O(x^3)$$
$$\log g(x)^{1/2} = -\frac{1}{6} R_{\mu\alpha\mu\beta}\, x^\alpha x^\beta$$

$$\boxed{\Delta\left(\frac{r^2}{2}\right) = n - \frac{1}{3} R_{\mu\alpha\mu\beta}\, x^\alpha x^\beta + O(x^3)}$$

Next grind out the asymptotic expansion for the heat kernel.
$$\left(\frac{e^{-\frac{u}{t}}}{t^{n/2}}\right)^{-1}\left(\partial_t - \frac{1}{2}\Delta\right)\frac{e^{-\frac{u}{t}}}{t^{n/2}} = \partial_t + \frac{u}{t^2} - \frac{n}{2t} - \frac{1}{2}\left(\nabla - \frac{1}{t}\nabla u\right)\cdot\left(\nabla - \frac{1}{t}\nabla u\right)$$
$$= \frac{1}{t^2}\cancel{\left(u - \frac{1}{2}\nabla u\cdot\nabla u\right)} + \frac{1}{2t}\left[2\nabla u\cdot\nabla + \Delta u - n\right] + \partial_t - \frac{1}{2}\Delta$$
$u = \frac{r^2}{2}$
$$= \frac{1}{t}\Big[\nabla_{r\frac{d}{dr}} + \underbrace{\frac{1}{2}\left(\Delta\frac{r^2}{2} - n\right)}_{q = q_{\alpha\beta}x^\alpha x^\beta + O(x^3)}\Big] + \partial_t - \frac{1}{2}\Delta \qquad q_{\alpha\beta} = \frac{1}{6}R_{\mu\alpha\mu\beta}$$

$$\left[\nabla_{r\frac{d}{dr}} + q\right]A_0 = 0 \qquad A_0 = 1 - \frac{1}{2} q_{\alpha\beta}\, x^\alpha x^\beta + \cdots$$

$$\left[\nabla_{r\frac{d}{dr}} + q + 1\right]A_1 - \frac{1}{2}\Delta A_0 = 0 \qquad A_1 = \frac{1}{2}\Delta A_0 = -\frac{1}{2} q_{\alpha\alpha}$$

Hence the actual asymptotic expansion for the heat kernel is

$$\boxed{K_t(x,0) = \frac{e^{-\frac{|x|^2}{2t}}}{(2\pi t)^{n/2}} \left[\left(1 + \frac{1}{12} R_{\mu\alpha\mu\beta} x^\alpha x^\beta + \dots\right) + \left(+\frac{1}{12} R_{\mu\alpha\mu\alpha}\right) t + \dots \right].}$$

The obvious candidate is

$$\frac{e^{-\frac{|x|^2}{2t}}}{(2\pi t)^{n/2}}\, g(x)^{-1/2} = \frac{e^{-\frac{|x|^2}{2t}}}{(2\pi t)^{n/2}} \left[1 + \frac{1}{6} R_{\mu\alpha\mu\beta} x^\alpha x^\beta + \dots \right]$$

What are the l_0, l_1 corresponding to the true asymptotic expansion?

$$K_t(x,0)\, g(x)^{1/2} d^n x = \frac{e^{-\frac{|x|^2}{2t}}}{(2\pi t)^{n/2}} \left[l_0(x) + l_1(x) t + \dots \right]$$

so

$$l_0(x) + l_1(x) t + \dots = \left(1 - \frac{1}{6} R_{\mu\alpha\mu\beta} x^\alpha x^\beta + \dots\right)\left(\left(1 + \frac{1}{12} R x^\alpha x^\beta\right) + \left(+\frac{1}{12} R\right) t + \dots\right)$$

$$= \underbrace{\left(1 - \frac{1}{12} R_{\mu\alpha\mu\beta} x^\alpha x^\beta + \dots\right)}_{l_0} + \underbrace{\left(+\frac{1}{12} R_{\mu\alpha\mu\alpha} + \dots\right)}_{l_1} t + \dots$$

and this does satisfy $\left(\frac{1}{2}\Delta l_0 + l_1\right)_0 = 0$

Review: The obvious candidate for the heat kernel is

$$L_t(x,y) = \frac{e^{-\frac{d(x,y)^2}{2t}}}{(2\pi t)^{n/2}} \; \frac{(\exp_y)_* (\text{vol on } T_y)}{g(x)^{1/2} d^n x}$$

and this has to be the only possibility where there is no linear A term. This is probably not symmetric in x, y.

Calculate: Put $S = + \frac{d(x,y)^2}{2}$. Then we know that

$$p_\mu = -\frac{\partial}{\partial y^\mu} S(x,y)$$

is the initial momenta for the trajectory going from y to x.

Hence the tangent vector to this trajectory is

$$v^\mu(x) = g^{\mu\nu}(y)\, p_\nu = -\, g^{\mu\nu}(y) \frac{\partial S(x,y)}{\partial y^\nu}$$

Thus the Jacobian matrix of $x \mapsto v(x)$ is

$$\frac{\partial v^\mu}{\partial x^\lambda} = -\, g^{\mu\nu}(y) \frac{\partial^2 S}{\partial x^\lambda \partial y^\nu}(x,y)$$

and so

$$\frac{d^n v}{d^n x} = g(y)^{-1} \det\left(-\frac{\partial^2 S}{\partial x \partial y}\right)$$

or

$$J(x,y) = \frac{(\exp_y)_*(d^n v)}{g(x)^{1/2} d^n x} = g(x)^{-1/2} g(y)^{-1} \det\left(-\frac{\partial^2 S}{\partial x \partial y}\right)$$

Check: Assume the coords are normal at y. Then

$$J(x,y) = g(x)^{-1/2} \qquad \text{which checks.}$$

~~Assume the coords are normal at x~~

Thus $J(x,y)$ is <u>not</u> symmetric in x, y.
What is symmetric is

$$J(x,y)\, g(y)^{1/2} = g(x)^{-1/2} g(y)^{-1/2} \det\left(-\frac{\partial^2 S}{\partial x \partial y}\right)$$

which in normal coords at y gives

$$g(x)^{-1/2}.$$

According to the van Vleck business the good heat candidate should start with

$$\frac{e^{-\frac{d^2(x,y)}{2t}}}{(2\pi t)^{n/2}} g(x)^{-1/4} g(y)^{-1/4} \det\left(-\frac{\partial^2 S}{\partial x \partial y}\right)$$

$$= \frac{e^{-\frac{|x|^2}{2t}}}{(2\pi t)^{n/2}} \underbrace{g(x)^{-1/4}}_{\left(1-\frac{1}{6}Rxx + \cdots\right)^{-1/2}} = \frac{e^{-\frac{|x|^2}{2t}}}{(2\pi t)^{n/2}} \left(1 + \frac{1}{12} Rxx + \cdots\right)$$

<u>Problems</u>: 1) One now has a feeling for the Wiener or Brownian metric $d\mu_t$ on paths $[0,1] \to M$ starting from y. The next thing to do would be to put in a vector bundle with connection. Is parallel translation along a curve going to be defined almost everywhere for the Wiener measure? This question makes sense already in Euclidean space.

2) Can one obtain the asymptotic expansion of the heat kernel to all orders by steepest descent expansion about the critical path. There should be a set of <u>formal</u> perturbation <u>theory</u> rules for deriving this expansion, and these rules are all you will need to prove the index theorem.

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3) What is the path integral in the momentum space formulation? How do you bring in the parallel translation along the curve? Can you work in the actual generator for $K(T^*_X)$ using the supersymmetry?

$-\nabla^2 + \frac{R}{6}$ has <u>no</u> potential from the path integral viewpoint.

December 20, 1982

Supersymmetry gives meat to the following analogy

bosons	fermions
real symplectic vector space	real orthogonal vector space
Weyl algebra	Clifford algebra
metaplectic repn.	Clifford module
metaplectic gp.	spin group

Let us now develop the path integral concepts on both sides, and let us stick to quadratic Hamiltonians. Let's start on the boson side with

$$H = \text{quad. fn. of } p, q.$$

This is then a path in the Lie algebra of the symplectic group with scalars added and lifts ~~into~~ the metaplectic group $* S^1$. One can integrate this to obtain a metaplectic monodromy transformation.

So suppose I fix a time interval $[0,1]$ and someone gives me H_t, a quadratic fn. of p,q, for each $t \in [0,1]$. Then I know that I have a path integral formula of Feynman type for the unitary time evolution operator

$$\int \mathcal{D}q\,\mathcal{D}p\; e^{\frac{i}{\hbar}\int p\,dq - H\,dt}.$$

The exponent being quadratic in q,p this can be evaluated exactly.

What about positivity of the energy and imaginary time? It doesn't appear in what has happened so far. Perhaps (à la Witten) it will follow from a supersymmetry requirement.

Notice that there is a positive cone in the Liegebra of the symplectic group obtained by taking squares of real linear functions in the p, q's. How does one work in the scalars?

This raises the whole question of positivity in the symplectic group. It seems clear that if ~~H is positive ≥ 0 for each t, then the unitary~~ H is a positive form in the p, q's, then the unitary group e^{-itH} extends to the lower half t-plane as a contraction operator.

The Liegebra of the symplectic group can be identified with all homogeneous quadratic functions in q, p. The positive cone consists of positive definite quadratic forms. One knows that $H =$ const is preserved by the Hamiltonian flow belonging to H, so the one-parameter subgp associated to a positive H lies in a compact subgrp. ($\simeq$ unitary gp.) of the symplectic gp. This picture shows that although there is an intrinsic notion of positivity in the Liegebra it does not extend to the group elements, as a positive H can generate a periodic flow, e.g. the harmonic oscillator.

December 21, 1982

Consider the family of $\bar\partial$-operators on a vector bundle E over a Riemann surface M, and the determinant line bundle $\mathcal{L}$ over $\mathcal{A}$ with its action of $\mathcal{G}_c =$ cx. gauge transf. gp. Fix the metrics on M, E. Then on $\mathcal{L}$ we have a connection defined by ζ fn. methods, which is invariant under $\mathcal{G} =$ gauge transf. gp. But the orbits of $\mathcal{G}$ in $\mathcal{L}$ are not horizontal for this connection, the deviation being given by the anomaly formula.

I should perhaps think of $\mathcal{A}$ as a principal $\mathcal{G}$-bundle over $\mathcal{A}/\mathcal{G}$. The problem is then to compute the Chern class of the induced determinant line bundle $\tilde{\mathcal{L}}$ over $\mathcal{A}/\mathcal{G}$. In any case I have a definite type of problem with differential forms on $\mathcal{A}$.

So what is the general sort of result to be expected? Put $P = \mathcal{A} \longrightarrow \mathcal{A}/\mathcal{G} = \mathcal{B}$. Then we have an equivariant line bundle $\mathcal{L}$ over P equipped with an equivariant connection.

From this process I can derive a 2-form ω on $\mathcal{A}$ invariant under $\mathcal{G}$, and a moment map from $\mathcal{A}$ to $(\mathrm{Lie}\,\mathcal{G})^*$. This is reminiscent of Duistermaat-Heckman where one has the circle S acting on a symplectic (M,ω) with moment map given by f. One wants to define an equivariant 2-form out of ω and f. I saw how to do this in the equivariant cohomology setting, namely over $M \times P_S$ one takes the form

$$\omega + d(f\eta)$$

(Check: $i(X)\omega = df$ by defn. of f. $i(X)\eta = 1$ by definition of the form η on P_S. Clearly $\theta(X)[\omega + d(f\eta)] = 0$

$$i(X)[\omega + d(f\eta)] = df + i(X)[(df)\eta + f\,d\eta]$$
$$= df - df\, i(X)\eta = 0.$$

Conversation with Donaldson. He describes a map

$$H_2(M) \longrightarrow H^2(\mathcal{A}/\mathcal{G})$$

as follows geometrically. Realize a 2 cycle by a Riemann surface Y mapping in, then pull back the v. bundle E + connection ∇_A to Y, use the analytic structure to specify a divisor in $\mathcal{A}/\mathcal{G}$, namely those ∇_A which have a singular $\bar\partial$ operator over Y. Then one gets an element of $H^2(\mathcal{A}/\mathcal{G})$ belonging to this divisor.

In other words, $\mathcal{A}/\mathcal{G}$ parametrizes a family of bundles + connection over M. Pulling back to Y we get a family of holomorphic vector bundles over Y, hence a determinant line bundle over $\mathcal{A}/\mathcal{G}$. Then you take the chern class of this line bundle.

I thought one might consider the canonical bundle + connection over $M \times \mathcal{A}/\mathcal{G}$, take its second Chern class

$$c_2(\tilde{E}) \in H^4(M \times \mathcal{A}/\mathcal{G})$$

and apply Künneth. However I might also want to put in $c_1(\tilde{E})^2$.

What's nice about Donaldson's description is that one sees the cycles in M coming in naturally. Connections, unlike holomorphic structures can be pulled back and integrated using elliptic operators.

This suggests a very general idea. Recall that E is given over M, and that $\mathcal{G}$ is the auto. gp. of E, and that $\mathcal{A}$ is acted on freely by $\mathcal{G}$. Then there is a universal bundle $\tilde{E}$ over $M \times \mathcal{A}/\mathcal{G}$, i.e. one can think topologically of any family of bundles over M isom. to E as being induced from $\tilde{E}$.

Anyway $\tilde{E}$ defines a class in $K^0(M \times \mathcal{A}/\mathcal{G})$. Given a class $\alpha \in K_0(M)$ one can take

$$\alpha \cap [\tilde{E}] \in K^0(\mathcal{A}/\mathcal{G}).$$

In practical terms α is given by a pair (Y, D) and a map $Y \to M$, where D is an elliptic operator over the compact manifold Y. Then $\alpha \cap [\tilde{E}]$ is the index of the family of operators on Y parametrized by $\mathcal{A}/\mathcal{G}$, obtained by tensoring $D \otimes \mathrm{id}$ on $Y \times \mathcal{A}/\mathcal{G}$ with the pull-back of $\tilde{E}$.

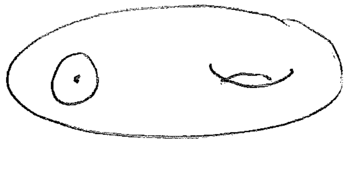
December 22, 1982

Problem: Calculate the equivariant cohomology of the gauge group $\mathcal{G}$ acting on the space of connections $\mathcal{A}$ using differential forms.

For example, let's start with the symplectic form ω on $\mathcal{A}$ which we know is $\mathcal{G}$-invariant. Does it come from an equivariant [redacted] cohomology class? Yes, because we know that the determinant line bundle is a $\mathcal{G}$-line bundle over $\mathcal{A}$, hence its first Chern class is an equivariant cohomology class.

So the problem becomes how are we going to describe equivariant cohomology classes by differential forms. Consider then a [redacted] connected compact Lie group G acting on a manifold M. The equivariant coh. $H_G^\bullet(M)$ is the cohomology of $P \times^G M$, where P is a universal G-bundle. I can approximate P by a free G-manifold, and I know already that one can effectively replace the differential forms on P by the Weil algebra. So the [redacted] thing to be checked is whether the equivariant cohomology is obtained by tensoring the forms $\Omega^\bullet(M)$ on M with $W(\mathfrak{g})$ and taking the "basic" forms, i.e. those killed by $\theta(X), i(X)$.

G. Wilson conversation, preparation for.



Consider a holom. v.b. E over a RS M and a small disk. Let $\mathcal{G} = \Gamma(\mathrm{Aut}\,E \mid S^1)$.

By the clutching construction we get a family of holom. vector bundles on M parametrized by $\mathcal{G}$. The complex gauge group in this situation is $\mathcal{G}^+ \times \mathcal{G}^-$ where $\mathcal{G}^+$ is the group of holom. autos. of $\mathcal{G}$ inside, resp. $\mathcal{G}^-$ outside. [redacted] The determinant line bundle $\mathcal{L}$ of the family is equivariant for the $\mathcal{G}^+ \times \mathcal{G}^-$ action.

Now I am basically going to be interested in constructing determinants for this family and their transformation properties under the gauge group. I think it is a good idea to replace the outside of the disk by the half-space W in $\Gamma(S^1, E)$ consisting of the boundary values of holomorphic sections outside the disk.

I should be clearer: I have over $\mathcal{G}$ a determinant line bundle $\mathcal{L}$ equivariant for an action of $\mathcal{G}^+ \times \mathcal{G}^-$. Using the Cech calculation of cohomology I know that the fibre of $\mathcal{L}$ at $\varphi \in \mathcal{G}$ is the determinant line of the Fredholm operator

$$H^+ \oplus W \xrightarrow{\;in - \varphi\;} V. \qquad V = \Gamma(S^1, E)$$

Hence the situation depends only on the two subspaces H^+, W (and possibly $\mathcal{G}^-$ although I think $\mathcal{G}^-$ should be the stabilizer in $\mathcal{G}$ of W.)

So let's enlarge the problem slightly, and take the

December 23, 1982:

Equivariant cohomology via differential forms.

G connected compact gp. acting on M. I want a differential form procedure for calculating the equivariant cohomology of M. For example if the action is free, then we know the forms on $G\backslash M$ can be identified with the "basic" subcomplex of $\Omega^\bullet(M)$, i.e. forms which are killed by $i(X)$ and $\theta(X)$ for each $X \in \mathfrak{g}$.

A basic criterion is that we can define characteristic classes in the equivariant cohomology for equivariant vector bundles. [redacted] Given such a bundle E, we can consider the connections on E, and by averaging over the compact group G, we can construct an invariant connection. From this we should have equivariant characteristic classes.

Consider an equivariant line bundle L, choose an invariant connection ∇. Then [redacted] the curvature is an invariant [redacted] closed 2-form ω on M:

$$[\nabla_Y, \nabla_{Y'}] - \nabla_{[Y,Y']} = i(Y')\,i(Y)\,\omega$$

The G-action on L gives a diff'l operator on sections of L with the same symbol as ∇.

$$\theta(X)\,s = \left(\nabla_{\bar{X}} + \varphi_X\right) s \qquad X \in \mathfrak{g}$$

The invariance of the connection means that

$$[\underset{\nabla_{\bar{X}} + \varphi_X}{\underset{\parallel}{\theta(X)}}, \nabla_Y] = \nabla_{[\bar{X}, Y]}$$

or
$$[\nabla_{\bar{X}}, \nabla_Y] - \nabla_{[\bar{X}, Y]} = [\nabla_Y, \varphi_X]$$

or
$$[\nabla_Y, \varphi_X] = \omega(\bar{X}, Y)$$

Also the fact we have an action of $\mathfrak{g}$ on L means

$$\nabla_{\overline{[X,X']}} + \varphi_{[X,X']} = [\nabla_{\bar X} + \varphi_X, \nabla_{\bar X'} + \varphi_{X'}]$$

$$\varphi_{[X,X']} = \cancel{\omega(\bar X, \bar X')} + [\nabla_{\bar X}, \varphi_{X'}] - \cancel{[\nabla_{\bar X'}, \varphi_X]} + [\varphi_X, \varphi_{X'}]$$

or more simply $\quad [\theta(X), \varphi_{X'}] = \varphi_{[X,X']}$.

Suppose I have a line bundle. Then

$$\omega \in \Omega^2(M)^G \quad \text{and} \quad d\omega = 0.$$

$$\varphi \in [\Omega^0(M) \otimes \mathfrak{g}^*]^G.$$

The reason φ is invariant is that

$$\underbrace{[\underset{\nabla_{\bar X} + \varphi_X}{\underset{\shortparallel}{\theta(X)}}, \varphi_{X'}]}_{\bar X \varphi_{X'}} = \varphi_{[X,X']}$$

and hence $\varphi : \mathfrak{g} \longrightarrow \Omega^0(M)$ commutes with the $\mathfrak{g}$ action.

For equivariant vector bundles E, we will get the data

$$F \in \Omega^2(M, \operatorname{End} E) = \Gamma(\operatorname{End} E \otimes \Lambda^2 T^*)$$

$$\varphi \in \Omega^0(M, \operatorname{End} E) \otimes \mathfrak{g}^* = \Gamma(\operatorname{End} E \otimes \mathfrak{g}^*)$$

subject to various conditions. The game is now to apply an invariant polynomial, like $\operatorname{tr}(e^A)$, to the data F, φ. Then one will get various forms which should constitute an equivariant class on M. It seems that by taking monomials in F, φ one gets forms in $\Omega^{2p}(M) \otimes (\mathfrak{g}^*)^{\otimes q}$, but it is not clear yet whether I want $\Lambda^q(\mathfrak{g}^*)$ or $S^q(\mathfrak{g}^*)$.

Extreme case: Take $M = pt$ and let E be given by a repn. of G. Then in the above

$$\theta(X) = \varphi_X \in \mathrm{End}(E)$$

so $\varphi : \mathfrak{g} \longrightarrow \mathrm{End}(E)$ is just the representation of the Lie algebra. Then an invariant polynomial on $\mathrm{End}(E)$ pulls back to an element of $S(\mathfrak{g}^*)^G$.

The conjecture so far is that the equivariant cohomology can be computed by a complex of the form

$$\left[\Omega(M) \otimes S(\mathfrak{g}^*[2])\right]^G.$$

This will be the same as

$$\left[\Omega(M) \otimes W(\mathfrak{g})\right]^{\mathfrak{h}}$$

provided we can see that

$$\Omega(M) \simeq \{\alpha \in \Omega(M) \otimes \Lambda\mathfrak{g}^* \mid i(X)\alpha = 0 \quad \forall X \in \mathfrak{g}\}.$$

Notice that the $i(X)$ operators define a $\Lambda\mathfrak{g}$ module structure on $\Omega(M)$ and that

$$\Lambda\mathfrak{g}^* = \mathrm{Hom}(\Lambda\mathfrak{g}, k)$$

as $\Lambda\mathfrak{g}$ modules. So we should have isomorphisms

$$\mathrm{Hom}_{\Lambda\mathfrak{g}}(k, \Omega(M) \otimes \Lambda\mathfrak{g}^*) = \mathrm{Hom}_{\Lambda(\mathfrak{g})}(k, \mathrm{Hom}(\Lambda\mathfrak{g}, \Omega(M)))$$

$$= \mathrm{Hom}_k(k \otimes_{\Lambda(\mathfrak{g})} \Lambda(\mathfrak{g}), \Omega(M)) = \Omega(M).$$

Special case: Let $\mathfrak{g}$ be 1-dimensional with gen. X and dual base $\eta \in \mathfrak{g}^*$. Then we have

$$\Omega(M) \longrightarrow \Lambda\mathfrak{g}^* \otimes \Omega(M)$$

$$\alpha \longmapsto \alpha - \eta \otimes i(X)\alpha$$

and clearly one has

$$i(X)\left[\alpha - \eta\, i(X)\alpha\right] = i(X)\alpha - i(X)\alpha + \cancel{\eta\, i(X)^2\alpha} = 0.$$

In general you can take a basis X_j for $\mathfrak{g}$ and a dual basis η_j for $\mathfrak{g}^*$. Then consider the operators

$$1 - e(\eta_j)\, i(X_j) \quad \text{on} \quad \Lambda\mathfrak{g}^* \otimes \Omega(M).$$

$$\left[1 - e(\eta_j)\, i(X_j)\right]\left[1 - e(\eta_k)\, i(X_k)\right]$$

$$= 1 - e(\eta_j)\, i(X_j) - e(\eta_k)\, i(X_k) + e(\eta_j)\left[\delta_{jk} - e(\eta_k)\, i(X_j)\right] \times i(X_k)$$

So these operators commute, and are projection operators. Since they commute their product makes sense and since

$$i(X_j)\left[1 - e(\eta_j)\, i(X_j)\right] = i(X_j) - i(X_j) = 0$$

it follows that the product is killed by the $i(X_j)$. Thus we have

$$\Omega(M) \xrightarrow{\ \sim\ } \left\{\alpha \in \Lambda\mathfrak{g}^* \otimes \Omega(M) \;\middle|\; i(X)\alpha = 0 \quad \forall X \in \mathfrak{g}\right\}$$

$$\prod \left(1 - e(\eta_j)\, i(X_j)\right).$$

Now the pieces fall together. The equivariant cohomology can be calculated by the complex

$$\left[\Omega(M) \otimes W(\mathfrak{g})\right]^{\natural}.$$

Using $W(\mathfrak{g}) = \Lambda(\mathfrak{g}^*) \otimes S(\mathfrak{g}^*)$ and the above isom, it follows that on taking the $i(X)$ kernels we get a complex of the form

$$\Omega(M) \otimes S(\mathfrak{g}^*)$$

not quite as d is not defd.

and so we have

$$\left[\Omega(M) \otimes W(\mathfrak{g})\right]^{\natural} \simeq \left[\Omega(M) \otimes S(\mathfrak{g}^*)\right]^{G}$$

One thing to be checked is that this is compatible with quasi-isomorphisms, i.e. I want to know that an $M \to M'$ inducing isom. on cohomology induces isoms on the functor

$$\left[\Omega(?) \otimes S(\mathfrak{g}^*)\right]^G .$$

It would be simpler to set up the spectral sequence for the fibration over BG. This should result by the obvious filtration generated by the augmentation ideal in $S(\mathfrak{g}^*)$. The quotients are clearly

$$\left[\Omega(M) \otimes S^p(\mathfrak{g}^*)\right]^G$$

with differential from the first factor. But now decompose $S^p(\mathfrak{g}^*)$ according to the irred. repns. of G and use that the isotypical part of $\Omega(M)$ corresponding to a non-trivial irreducible repn. is acyclic. Thus

$$\left[\Omega(M) \otimes S^p(\mathfrak{g}^*)\right]^G \supset \Omega(M)^G \otimes S^p(\mathfrak{g}^*)^G$$

is a qis.

Clearly, one point in the Atiyah-Bott paper must be that the moment map can be viewed as a way to convert the symplectic form into an equivariant cohomology class.

Hence the connection between the curvature of the determinant line bundle and the anomaly formula (or moment map). Both are somehow coming from an equivariant coh. class, coming from an invariant connection on the determinant line bundle.

Problem: Define an equivariant connection on the index virtual bundle.

December 25, 1982:

Problem: To describe the equivariant cohomology of the gauge tr. gp. $\mathcal{G}$ acting on the space $\mathcal{A}$ of connections using differential forms. We know $\mathcal{A}$ is contractible, hence this equivariant cohomology is just the cohomology of the classifying space $B\mathcal{G}$.

(We know that the Lie algebra cohomology and cohomology of $\mathcal{G}$ are different, hence there is some possible ambiguity in the concept of the cohomology of $B\mathcal{G}$. On one hand one has the actual cohomology of this classifying space which treats the coefficients as discrete. On the other hand one has continuous cochain cohomology. For example one has van Est:

$$H^p_{cont}(B\mathcal{G}, H^q(\mathcal{G})) \Longrightarrow H^{p+q}(\mathrm{Lie}\,\mathcal{G})$$

In geometric terms, there a "fibration"

$$\mathcal{G} \longrightarrow \mathrm{Lie}\,\mathcal{G} \longrightarrow B_{cont}\mathcal{G}.$$

This "fibration" is supposed to result by forming over $B_{cont}\mathcal{G}$ the associated fibre space with fibre the top. space $\mathcal{G}$. I don't know how to do this precisely, except I know that there is a map of fibrations:

$$\begin{tikzcd}
\mathcal{G} \arrow[r] \arrow[d, equal] & B(\mathcal{G}^{dis} \text{ on } \mathcal{G}) \arrow[r] \arrow[d] & B\mathcal{G}^{dis} \arrow[d] \\
\mathcal{G} \arrow[r] \arrow[d, equal] & \mathrm{Lie}\,\mathcal{G} \arrow[r] \arrow[d] & B_{cont}\mathcal{G} \arrow[d] \\
\mathcal{G} \arrow[r] & P\mathcal{G} \arrow[r] & B\mathcal{G}
\end{tikzcd}$$

$$P\mathcal{G} \wr\, pt$$

One can perhaps get some idea of these gadgets from the case of Gl_n defined over a finite field.)

The main idea in the above discussion is that because there is already a difference between the Liegebra coh + coh. for $\mathcal{G}$, we can expect there to exist different kinds of equivariant cohomology.

Such a difference might arise when we try to form equivariant Chern classes for the tautological bundle. Recall that $\mathcal{G}$ is the group of autos. of a vector bundle E over M and hence gives an equivariant vector bundle $\tilde{E}$ over $M \times \mathcal{A}$, the tautological bundle.

Let's begin again. We are starting with a smooth vector b. with inner product E over M, and $\mathcal{G}$ is the group of its automorphisms. Hence E is an equivariant bundle over M with trivial $\mathcal{G}$-action, and so gives us a vector bundle $\tilde{E}$ over $M \times B\mathcal{G}$. Hence we get various characteristic classes

$$c_\alpha(\tilde{E}) \in H^{ev}(M \times B\mathcal{G})$$

which can be capped with homology classes in M to obtain cohomology classes for $B\mathcal{G}$. Notice that because we are considering <u>topological</u> questions, the space $\mathcal{A}$ plays no role because it is contractible.

In the same way (i.e. tautologically) we obtain over $M \times \mathcal{G}$ a pair (E,g) consisting of a vector bundle + automorphism, hence we get characteristic classes

$$c_\alpha(E,g) \in H(M \times \mathcal{G})$$

which can be capped with homology classes in M to obtain cohomology classes in $\mathcal{G}$.

The first thing we would like to understand is the nature of all these topologically-defined classes, at least in the stable range.

Problem: It seems that the technique of rational homotopy theory might be powerful enough to calculate the rational homotopy of $\mathcal{G}$ and $B\mathcal{G}$. Then arises the question of whether one could see the Lie algebra cohomology inside the cohomology of $\mathcal{G}$, i.e. state a precise [redacted] result calculating the Lie algebra cohomology [redacted] of the gauge group in general.

Consider $E = M \times U_n$, whence $\mathcal{G} = \mathrm{Map}(M, U_n)$. [redacted] Rational homotopy theory says that any connected H-space is a generalized EM space, i.e. [redacted] a product of EM spaces. Thus the cohomology of $\mathcal{G}$ is free, and the indecomposable space is dual to $\pi_* \mathcal{G}$. Now rationally

$$U_n \xrightarrow{\sim} \prod_{j=1}^{n} K(2j-1)$$

$$\mathcal{G} = \mathrm{Map}(M, U_n) \xrightarrow{\sim} \prod \mathrm{Map}(M, K(2j-1))$$

and

$$\pi_q \mathrm{Map}(M, K(2j-1)) = [M, \Omega^q K(2j-1)] = H^{2j-1-q}(M).$$

Hence the cohomology of $\mathcal{G}$ is free with

$$\mathrm{Ind}\left(H^q(\mathcal{G})\right) \simeq \bigoplus_{j=1}^{n} H_{2j-1-q}(M).$$

It's more or less clear that this isomorphism is given as follows. The evaluation map

$$M \times \mathcal{G} \longrightarrow U_n$$

~~which is a map~~ can be used to pull-back the primitive element $e_j \in H^{2j-1}(U_n)$ to a class [redacted]

$$\tilde{e}_j \in H^{2j-1}(M \times \mathcal{G}) = \bigoplus_q H^{2j-1-q}(M) \otimes H^q(\mathcal{G})$$

so capping with $\tilde{e}_j$ gives the desired map

$$\tilde{e}_j^{\#} : H_{2j-1-q}(M) \longrightarrow H^q(\mathcal{G}). \qquad 1 \leq q \leq 2j-1$$

In the above it is necessary to cut $\mathcal{G}$ down to its identity component $\mathcal{G}_{(0)}$[redacted].

The first interesting case is where $q = 1$ where we have

$$H^1(\mathfrak{g}) = A^* = \text{currents of degree } 0$$

$$H^1(\mathcal{G}_{(0)}) = H_{ev}(M)$$

The map

$$H^1(\mathfrak{g}) \longrightarrow H^1(\mathcal{G}_{(0)})$$

associates to a current the corresponding element of H_0[redacted](M). In other words $\tilde{e}_1^{\#}\{H_0(M)\}$ comes from the Liegebra but none of the higher ones do.

Analogous things work for $B\mathcal{G}$ which is the identity component of $\mathrm{Map}(X, BU_n)$. Rationally

$$BU_n \simeq \prod_{j=1}^{n} K(2j)$$

and instead of the e_j we use the components of the Chern character. [redacted] We see that $H^*(B\mathcal{G})$ is free with

$$[redacted]\ \mathrm{Ind}\left(H^q(B\mathcal{G})\right) = \bigoplus_{j=1}^{n} H_{2j-q}(M)$$

where the map comes from capping with

$$\tilde{ch}_j \in H^{2j}(M \times B\mathcal{G}) = \bigoplus_{q=0}^{2j} H^{2j-q}(M) \otimes H^q(B\mathcal{G}).$$

Let's recall that I am after a Lie analogue of $H^*(B\mathcal{G})$. What might the H^1_{Lie} be? We have

$$H^1(B\mathcal{G}) = [redacted]\ H_{odd}(M) = \bigoplus_{j=1}^{n} \tilde{ch}_j^{\#} \cdot H_{2j-1}(M)$$

By analogy, I can expect that only the $j=1$ part will be "Lie-theoretic". So the conjecture will be that

$$H^1_{Lie}(B\mathcal{G}) \subseteq \begin{array}{l}\text{closed currents}\\ \text{of dim } 1\end{array}$$

Problem: To find a Lie analogue of $H^*_{\mathcal{G}}(\mathcal{A}) = H^*(B\mathcal{G})$.

Yesterday's description of $H^*(\mathcal{G})$ where $\mathcal{G} = \mathrm{Map}(M, U_n)$.
Pull back primitive classes $e_j \in H^{2j-1}(U_n)$ under eval. map $M \times \mathcal{G} \to U_n$ to get

$$\tilde{e}_j \in H^{2j-1}(M \times \mathcal{G}) = \bigoplus_q H^{2j-1-q}(M) \otimes H^q(\mathcal{G})$$

then cap to obtain

$$\tilde{e}_j^{\#} : H_{2j-1-q}(M) \longrightarrow H^q(\mathcal{G}).$$

Then

$$\bigoplus_{j \le n} \tilde{e}_j^{\#} H_{2j-1-q}(M) \xrightarrow{\sim} \mathrm{Prim}\, H^q(\mathcal{G}_{(0)})$$

for $q \geq 1$.

For $B\mathcal{G}$ take the tautological bundle over $M \times B\mathcal{G}$ and its char. classes

$$\tilde{ch}_j \in H^{2j}(M \times B\mathcal{G}) = \bigoplus H^{2j-q}(M) \otimes H^q(B\mathcal{G})$$

then cap to obtain

$$\bigoplus_{j \le n} \tilde{ch}_j^{\#} H_{2j-q}(M) \xrightarrow{\sim} \mathrm{Prim}\, H^q(B\mathcal{G})$$

for $q \geq 1$.

For $q = 1$ we get (say n large) $\qquad$ deg 3 $\qquad 4-1-2=1$ $\qquad$ deg 5

$$H^1(\mathcal{G}) = H_{ev}(M) = \tilde{e}_1^{\#} H_0(M) \oplus \tilde{e}_2^{\#} H_2(M) \oplus \tilde{e}_3^{\#} H_4(M)$$

$$H^1(B\mathcal{G}) = H_{odd}(M) = \tilde{ch}_1^{\#} H_1(M) \oplus \tilde{ch}_2^{\#} H_3(M) \oplus \cdots$$

I know that the summand $\tilde{e}_1^{\#} H_0(M)$ comes from invariant differential forms on $\mathcal{G}$, i.e. from $H^1(\mathrm{Lie}\,\mathcal{G})$. In fact $H^1(\mathrm{Lie}\,\mathcal{G}) = \Omega^0(M)^* =$ distributions on M.

There two things of importance with the map

$$H^1(\mathrm{Lie}\,\mathcal{G}) \longrightarrow H^1(\mathcal{G})$$

On one hand it's image is $\tilde{e}_1^{\#} H_0(M)$ so that [redacted] the classes obtained from e_3 e_5 etc. are not realized by invariant differential forms on $\mathcal{G}$. On the other hand the map is not injective: [redacted] different distributions give distinct [redacted] invariant classes which are cohomologous in all differential forms. It seems worthwhile to understand the formulas better.

Let us then start with the formula for $e_j \in H^{2j-1}(U_n)$. This is represented by the differential form

$$\operatorname{Tr}\,(g^{-1}dg)^{2j-1}.$$

Explain: Think of U_n as a subset of $n \times n$ matrices and the tangent space to U_n at a point is a subsp. of matrices. A tangent vector at $g \in U_n$ is an [redacted] infinitesimal displacement $g + \delta g$. Then $g^{-1}(g+\delta g) = 1 + g^{-1}\delta g$ is the corresponding tangent vector at the identity, corresponding under left-translation. So we get a canonical matrix-valued form $g^{-1}dg$ on U_n. Then

$$\operatorname{Tr}\,(g^{-1}dg)^k$$

is a k-form on U_n. By cyclic symmetry of the trace and the skew-symmetry of forms, this is zero for k even. It is closed because for k odd

$$d\,\operatorname{Tr}\,(g^{-1}dg)^k = \cancel{k\operatorname{Tr}\left[(g^{-1}dg)^{k-1}\,dg^{-1}dg\right]} \quad \cancel{k\operatorname{Tr}\,(g^{-1}dg)^{k+1} = 0}$$

$$= \sum_{j=1}^{k} [redacted]\,(-1)^{j-1}\operatorname{Tr}\left[(g^{-1}dg)^{j-1}\underbrace{d(g^{-1}dg)}_{-(g^{-1}dg)^2}(g^{-1}dg)^{k-j}\right]$$

$$= \text{const}\quad \operatorname{Tr}\,(g^{-1}dg)^{k+1} = 0.$$

It should be obvious from construction that this differential form is left-invariant. Let h be a fixed element of U_n. The pull-back of the canonical form $g^{-1}dg$ under the map $g \mapsto hg$ is

$$(hg)^{-1}d(hg) = g^{-1}h^{-1}(\underset{0}{\underset{\shortparallel}{dh}}\cdot g + h\cdot dg) = g^{-1}dg .$$

Thus $g^{-1}dg$ and $\mathrm{Tr}\,(g^{-1}dg)^k$ are left invariant. But also

$$(gh)^{-1}d(gh) = h^{-1}(g^{-1}dg)\,h \qquad \text{and so}$$

$$\mathrm{Tr}\,[(gh)^{-1}d(gh)]^k = \mathrm{Tr}\,\big(h^{-1}(g^{-1}dg)^k h\big) = \mathrm{Tr}\,(g^{-1}dg)^k .$$

Thus the diff'l form is bi-invariant.

The next project is to pull the form e_j back under the evaluation map: $M \times \mathrm{Map}(M, U_n) \longrightarrow U_n$. More generally suppose we have a map $M \times Y \xrightarrow{A} U_n$. Then the pull-back of the form e_j is the form

$$\mathrm{Tr}\,(A^{-1}dA)^k \qquad k = 2j-1.$$

and this will be closed $\neq 0$ for k odd. If γ is a q-cycle on M, then

$$\int_\gamma \mathrm{Tr}\,(A^{-1}dA)^k$$

is a differential $k-q$ form on Y. This form is closed because integration over the fibre commutes with d. (First restrict to $\gamma \times Y$, then integrate over the fibre of $\gamma \times Y \longrightarrow Y$.)

We have just described a way to produce diff'l forms on $\mathcal{G} = \mathrm{Map}(M, U_n)$, namely we pull-back the form e_j on U_n via the evaluation map $M \times \mathcal{G} \longrightarrow U_n$, restrict to a cycle $\gamma \times \mathcal{G}$ and integrate over γ:

$$\begin{tikzcd} \gamma \times \mathcal{G} \arrow[r, phantom, "\subset"] \arrow[d] & M \times \mathcal{G} \arrow[r] & U_n \\ \mathcal{G} & & \end{tikzcd}$$

A natural question is when the resulting differential forms on $\mathcal{G}$ are (left or right) invariant, hence come from cocycles on $\mathrm{Lie}(\mathcal{G})$. Let's compute the possible dimensions using Connes calculation.

$$\begin{array}{ccccccccc}
 & & \mathrm{Prim}\, H^{g}(\mathfrak{g}) & = & (\Omega^{g-1}/d)^* & \oplus & H_{g-3} & \oplus & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow S & & \downarrow S \\
H_{2j-1-g}(M) & \xrightarrow{\tilde{e}_j^{\#}} & \mathrm{Prim}\, H^{g}(\mathcal{G}) & = & \cdots \oplus H_{g-1} & \oplus & H_{g-3} & \oplus & H_{g-5} \oplus \cdots
\end{array}$$

The critical dimension is when $2j-1-g = g-1$ or $j=g$.

Thus we are led to the question of whether given a $(j-1)$-cycle γ_{j-1}, can we see that the diff'l form on $\mathcal{G}$:

$$\int_{\gamma_{j-1}} \mathrm{Tr}(A^{-1}dA)^{2j-1}$$

$A : M \times \mathcal{G} \longrightarrow U_n$ is the eval. map

comes from a j-cocycle on the Liegebra. Now the idea I had is that γ_{j-1} defines a j-cocycle by

$$X_0, \dots, X_{j-1} \longmapsto \int_{\gamma_{j-1}} \mathrm{tr}\,(X_0\, dX_1 \cdots dX_{j-1})$$

So there should be a way to relate these formulas.

Already $j=2$ is interesting as it relates e_2 to the 2-cocycle on the Liegebra related to the dilog.

~~Think as follows~~

Think as follows. Given j, then for $p \le j-1$ a p-cycle γ in M should give a Lie cocycle by some process using the fact that p is smaller than half: $2p+1 \le 2j-1$. Somehow I also want to think of X as $A^{-1}dA$, then $dX = -(A^{-1}dA)^2$. This means we can absorb $(A^{-1}dA)^2$ into a single Lie variable.

Take $p = 1$.

Consider a vector bundle E over M but don't suppose it is trivial, and let $\mathcal{G}$ be the gauge group. Then we can still pose the problem of calculating the cohomology of $\mathcal{G}$ and $\mathrm{Lie}(\mathcal{G})$. By choosing E' such that $E \oplus E'$ is free we obtain an evaluation map

$$M \times \mathcal{G} \longrightarrow U_n \qquad n = \mathrm{rank}(E \oplus E')$$

via which we can pull back the forms e_j and cap with cycles in M so as to get forms in $\mathcal{G}$. We can ask when we get invariant forms on $\mathcal{G}$, hence Lie cocycles.

To fix the ideas suppose $M = S^1$. Then E is trivial, but let us not fix a trivialization. By capping $\tilde{e}_2$ with the fundamental class of M we obtain a 2-diml class on $\mathcal{G}$, which presumably comes from a Lie cocycle.

The Liegebra $\mathrm{Lie}(\mathcal{G})$ can be identified with $\Gamma(\mathrm{End}\,E)$. In the trivial case we get a cocycle on matrix valued functions on S^1 given by

$$\varphi(X_0, X_1) = \int \mathrm{tr}(X_0\, dX_1).$$

However d has no intrinsic meaning. Let us give it meaning by choosing a connection A and take $d = d_A$.

$$\varphi_A(X_0, X_1) = \int \mathrm{tr}(X_0\, d_A X_1)$$

Since $\quad d_A(X_0 X_1) = d_A X_0 \cdot X_1 + X_0\, d_A X_1$

$$\underset{\displaystyle \int d\, \mathrm{tr}(X_0 X_1) = 0,}{\int \mathrm{tr}\, d_A(X_0 X_1) \,\Vert} = \int \mathrm{tr}(X_1 d_A X_0) + \int \mathrm{tr}(X_0 d_A X_1)$$

φ_A is skew-symmetric, hence a 2-cochain on $\mathfrak{g}$.

The difference of two connections is an element of $\Gamma(\mathrm{End}(E)\otimes T^*)$. In the $M=S^1$ case write

$$d_A = d + A\,dt \qquad A\in\Gamma(\mathrm{End}\,E) = \mathrm{Lie}\,\mathcal{G}$$

and see how φ changes:

$$(\varphi_{A_1}-\varphi_{A_0})(X_0,X_1) = \int tr(X_0[B,X_1])\,dt \qquad A_1-A_0=B$$

$$= \int tr([X_1,X_0]B)\,dt$$

$$= -\int tr(B[X_0,X_1])\,dt.$$

Thus we see that $\varphi_{A_1}-\varphi_A$ is a coboundary.

Finally I should check that φ_A is a 2-cocycle. One can check this brutally, but a more intelligent argument goes as follows.

Recall that in the $M=S^1$ case, the gauge group orbits in the space of connections are homogeneous symplectic manifolds. Precisely, write a connection in the form $d_A = d + A\,dt$, then the inf. gauge action is

$$[X,d_A] = -dX + [X,A]\,dt = -d_A X/dt$$

or

$$X*A = -X' + [X,A].$$ Thus if we define for each $X\in\mathfrak{g}$ a fn. on $\mathcal{A}$ by

$$f_X(A) = \int tr(XA)\,dt$$

we have

$$(i(Y)\,df_X)(A) = -\int tr(X\cdot\overbrace{Y*A}^{-d_A Y})\,dt \qquad (\text{sign?})$$

$$= \int tr(X\cdot d_A Y).$$

Hence $\varphi_A(X,Y) = \int tr(X\,d_A Y)$ is the symplectic 2-form on the orbit of $\mathcal{A}$ pulled back to $\mathfrak{g}$.

Here's the point. The symplectic 2-form on the orbit is both invariant and closed. Hence pulling back via the map

$$\mathcal{G} \xrightarrow{\cdot A} \mathcal{G}A$$

we get a left-invariant closed 2-form on $\mathcal{G}$, whose restriction to the tangent space $\mathfrak{g}$ at the identity is φ_A. Thus φ_A is a 2-cocycle on $\mathfrak{g}$ by definition of Lie alg. cohomology.

Let's summarize the progress so far. I am still trying to understand the relation between cohomology and Lie algebra cohomology for the gauge group. I have decided to concentrate on the case of 2-diml. classes obtained by capping c_2 with 1-cycles. There should be a way to manipulate

$$\int_\gamma tr\,(g^{-1}dg)^3 \qquad \text{to} \qquad \int_\gamma tr\,(X_0\, dX_1)$$

I looked at the case of a non-trivial v.b. E and then one needs a connection to make sense out of the second expression, however the resulting Lie coh. class is independent of the choice of the connection.

Moment map theory is connected with 2-cocycles on $\mathfrak{g}$. Given a symp. G-manifold, one can map it into 2-cocycles as follows: A point m of M gives $G \xrightarrow{\cdot m} M$, and pulling back the symplectic form gives a closed left-inv. 2-form on G, which is a 2-cocycle. If the 2-cocycle is a coboundary it comes from an elt. of $\mathfrak{g}^*$, so M maps to $\mathfrak{g}^*$.

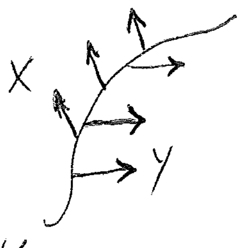
December 27, 1982

Basic problem: Relate $e_2 \in H^3(U_n)$ to the central extension of ΩU_n, more generally for any connected compact group.

Witten: $Map(S^2, G) \longrightarrow S^1$ defined as follows: Given a map $S^2 \to G$ it extends to $D^3 \to G$ and one can integrate the 3-form e_2 to get a real no. defined modulo $\mathbb{Z}$.

Another possibility: View a map $S^2 \to G$ as a loop in ΩG. Over ΩG one has a canonical line bundle with connection. Then take $\parallel$ transport along this loop which is a well-defined element of S^1.

Problem: Relate the above two definitions. Idea might be that a loop in ΩG is contractible, hence choosing an extension to the 2-disk, one can calculate the $\parallel$ transport using the curvature of the line bundle. This is the Kähler form on ΩG, hence all one must do is to relate a specific 2-form on ΩG with the 3-form e_2 on G. How might this go?



loop in G
with two tangent fields

This gives, together with the tangent field to the curve, three tangent fields, which can be capped against e_2 to obtain a function, which can be integrated along the curve. [redacted] This should give the symplectic 2-form on ΩG from e_2.

~~The above isn't right because [illegible] doesn't have two tangent fields along a loop in G.~~

Consider a vector bundle E over S^1 and let $\mathcal{G}$ be its gauge group. Yesterday I saw that any connection on E gave rise to a closed left-invariant 2-form on $\mathcal{G}$, and the resulting cohomology class is independent of the connection. Given a connection A one has a $\mathcal{G}$-map

$$\mathcal{G} \xrightarrow{*A} \mathcal{G}*A$$

and one knows $\mathcal{G}*A$ has an invariant symplectic 2-form. Pulling back gives the 2-form on $\mathcal{G}$.

We know that over $\mathcal{G}$ is a line bundle obtained [redacted] by using the clutching process to obtain a vector bundle over the Riemann sphere and taking the determinant line bundle. (Why not any Riemann surface?) [redacted] (We get a whole supply of line bundles in this way, quite possibly all isomorphic). It would be nice if the different 2-forms on $\mathcal{G}$ arose from different invariant connections on the line bundle.

In any case we have a closed invariant 2-form on $\mathcal{G}$ associated to a connection. The problem now is to see if this 2-form can be obtained from the e_2-class on U_n. Possibility: Given $g \in \mathcal{G}$ and two tangent vectors $\delta_1 g$, $\delta_2 g$ at g one has $X_i = g^{-1}\delta_i g \in \Gamma(\operatorname{End} E)$. Because of the connection we also have $g^{-1}[d_A, g]/dt \in \Gamma(\operatorname{End} E)$. So we have 3 endos. of E and can apply e_2 and integrate over the circle. It seems we get the number (up to a const)

$$\int \operatorname{tr}\left(g^{-1}[d_A, g]\,[X_1, X_2]\right)$$

because we must anti-symmetrize the three endos.

Unfortunately this vanishes if $g = 1$, so it can't be the correct expression. So let's go back to the idea that we have a map

$$M \times \mathcal{G} \longrightarrow \mathcal{U}$$

(at least on the homotopy level) because over $M \times \mathcal{G}$ is a vector bundle with automorphism.

<u>Question</u>: Given a vector bundle + automorphism how can we obtain its differential form characteristic classes.

December 28, 1982

$M, E, \mathcal{G}, \mathcal{A}$ as usual. $\tilde{E}$ = tautological bundle over $M \times B\mathcal{G}$. Then

1) $$ch(\tilde{E}) \in H^{ev}(M \times B\mathcal{G}) = \mathrm{Hom}^{(ev)}(H_*(M), H^*(B\mathcal{G}))$$

and from this one obtains generators for $H^*(B\mathcal{G})$. Similarly over $M \times \mathcal{G}$ we have a tautological automorphism $\tilde{g}$ of $pr_1^* E$, hence classes

2) $$e(\tilde{g}) \in H^{odd}(M \times \mathcal{G}) = \mathrm{Hom}^{(ev)}(H_*(M), H^*(\mathcal{G}))$$

and from this one obtains generators for $H^*(\mathcal{G})$. Presumeably the classes 2) can be obtained from 1) using the canonical map $S^1\mathcal{G} \longrightarrow B\mathcal{G}$.

The above is purely topological. The next step is to realize these classes in $H^*(B\mathcal{G})$ or $H^*(\mathcal{G})$ via differential forms.

Let us consider then the situation where a group G acts as automorphisms of E over M. If G were compact, then we could choose an invariant connection in E and then the characteristic classes would lie in $\Omega(M) \otimes S(\mathfrak{g}^*)^G$. But if G isn't compact, then there isn't an invariant connection. For example if $G = \mathcal{G}$, then there is no invariant connection because the stabilizers are compact.

However if we lift E up to $M \times \mathcal{A}$, then there should be a tautological connection. We have a natural operator

$$pr_1^* E \xrightarrow{\nabla''} pr_1^* E \otimes pr_2^* T^*_{\mathcal{A}}$$

because $pr_1^* E$ is constant in the $\mathcal{A}$ direction. Also we have a tautological op.

$$pr_1^* E \xrightarrow{\nabla'} pr_1^*(E \otimes T^*_M)$$

because each point of $\mathcal{A}$ gives us an op. $\nabla_A : E \to E \otimes T^*_M$.

Combining these we get

$$pr_1^*E \xrightarrow{\nabla' + \nabla''} pr_1^*E \otimes \underbrace{[pr_1^* T^*_M \oplus pr_2^* T^*_{\mathcal{A}}]}_{T^*_{M\times\mathcal{A}}}$$

I don't have a good notation for this, however given a family of connections $Y \to \mathcal{A}$ $\quad y \mapsto A_\mu(y)$ then

$$\nabla s = (\nabla' + \nabla'')s$$
$$= [\partial_\mu + A_\mu(y)]s \cdot dx^\mu + \frac{\partial s}{\partial y^\nu} dy^\nu .$$

This tautological connection on pr_1^*E over $M\times\mathcal{A}$ is invariant under $\mathcal{G}$. An element $g(x^\mu)$ is constant in the y^ν and hence $g\nabla g^{-1} = g\nabla' g^{-1} + \nabla''$.

So now we have an invariant connection on the tautological bundle and so we should get specific differential forms in

$$[\Omega(M\times\mathcal{A}) \otimes S(\mathfrak{g}^*)]^{\mathcal{G}}$$

for the character.

Recall that the equivariant curvature for an equivariant connection consists of the curvature which is a [redacted] End(E)-valued 2-form and the "moment" function $\phi : \tilde{\mathfrak{g}} \to \Gamma(\mathrm{End}(E))$. Let's now describe ϕ.

In general ϕ is given by the deviation [redacted] of the Lie algebra action from being horizontal wrt the connection. We start with E over M and pull back to get pr_1^*E over $M\times\mathcal{A}$. [redacted] The tautological connection combines the obvious taut. connection over $M\times\{A\}$ for each A with the flat connection over each $\{m\}\times\mathcal{A}$. The gauge orbits are contained in the pieces $\{m\}\times\mathcal{A}$. The action of a gauge transformation on pr_1^*E [redacted] combines the action in E with the action in $\mathcal{A}$. In fact we have

$$pr_1^* E = E \times \mathcal{A}.$$

Now it is clear what the "moment" fn. ϕ is. An infinitesimal gauge transformation moves both E and $\mathcal{A}$ and the connection lifts a motion in $\mathcal{A}$ to be constant in E. Thus the difference is just the motion in E. In other words

$$\phi : \tilde{\mathfrak{g}} \longrightarrow \Gamma(\operatorname{End} E)$$

is just the identity.

We can describe the equivariant curvature as follows: It belongs to

$$\Omega^{\cdot}(M, \operatorname{End} E) \otimes \Omega^{\cdot}(\mathcal{A}) \otimes S(\tilde{\mathfrak{g}}^*)$$

and consists of three pieces:

2,0,0 piece which assigns to A the curvature $F_A \in \Omega^2(M, \operatorname{End} E)$

1,1,0 piece: since $\mathcal{A}$ is affine over $\Omega^1(M, \operatorname{End} E)$ we have an embedding

$$\Omega^1(M, \operatorname{End} E)^* \subset \Omega^1(\mathcal{A})$$

as the translation-invariant 1-forms. Thus we get an element

$$id \in \Omega^1(M, \operatorname{End} E) \otimes \Omega^1(M, \operatorname{End} E)^* \subset \Omega^1(M, \operatorname{End} E) \otimes \Omega^1(\mathcal{A}).$$

0,0,2 piece is the image of

$$id \in \Omega^0(M, \operatorname{End} E) \otimes \underbrace{\Omega^0(M, \operatorname{End} E)^*}_{\tilde{\mathfrak{g}}^*} \subset \Omega^0(M, \operatorname{End} E) \otimes \tilde{\mathfrak{g}}^*.$$

Viewpoint: 1) One obtains generators for $H^*(B\mathcal{G})$ by taking

$$ch(\tilde{E}) \in H^{ev}(M \times B\mathcal{G}) = \mathrm{Hom}^{(ev)}(H_*(M), H^*(B\mathcal{G}))$$

and capping with homology classes in M.

2) Given an operator on M one can tensor it with the tautological bundle $\tilde{E}$ over $M \times B\mathcal{G}$, and obtain a family of elliptic ops. on M parametrized by $B\mathcal{G}$. The character of the index of this family lies in $H^*(B\mathcal{G})$, and the index thm. for families describes this class in terms of the classes 1) above. Namely the operator gives a K-homology class in M whose character lies in $H_{ev}(M)$; this character is capped with $ch(\tilde{E})$ to get the character of the index of the family. [redacted]

Note that this gives only even cohomology in $B\mathcal{G}$ so there should be also a $K_1(M)$-version.

So far we have emphasized the topology. Now [redacted] we bring in connections and differential forms.

3) If E is lifted to $M \times \mathcal{a}$, then it has a tautological G-invariant connection, hence the classes $ch(\tilde{E})$ can be realized by equivariant differential forms [redacted] on $M \times \mathcal{a}$. So by integrating over cycles in M one obtains specific equivariant differential forms on $\mathcal{a}$ [redacted] representing the generating classes for $H^*(B\mathcal{G})$.

4) If we take an operator on M such as the Dirac operator, then upon tensoring with the bundle E pulled up to $M \times \mathcal{a}$ we get a definite family of operators on M param. by $\mathcal{a}$ which is equivariant for the G-action. [redacted] I believe that it should be possible to put a G-invariant connection on the

index of the family. The idea is that a really good analytical proof of the index theorem should actually compute the character of the index virtual bundle.

December 29, 1982

Let E, F be two vector bundles with connection over a manifold M and let $f: F \to E$ be an isomorphism of the underlying vector bundles. Then there should be an odd form $ch(f)$ such that

$$ch(F) - ch(E) = d\,ch(f).$$

The reason is that $ch F$ can be viewed as the character of E computed using the connection obtained by pulling back the connection on E, and two different connections lead to the same char. coh. class. Special case: suppose $F=E$ with connections $=$. Then $d\,ch(f) = 0$ and $ch(f)$ is the odd class attached to a bundle automorphism, which I expect to involve the derivatives of f with respect to the connection $f^{-1}Df - D = f^{-1}[D,f]$ as well as the curvature of E. So I need to work out the formula for $ch(f)$.

The idea is that we will put together the two connections $f^{-1}Df$ and D at the ends of $M \times I$ and use the linear interpolation in between. The connection will be

$$\nabla = \underbrace{D + tB}_{D_t} + \partial_t\, dt \qquad\qquad B = f^{-1}Df - D$$

$$\nabla^2 = D_t^2 + \{\partial_t dt, D_t\} = D_t^2 - B\,dt$$

$$tr(e^{\nabla^2}) = tr(e^{D_t^2}) - tr(e^{D_t^2}B)\,dt$$

using $(dt)^2 = 0$ and cyclic symmetry of the trace.

In general given $\alpha + dt\,\beta$ on $I \times M$ which is closed we can integrate over the fibre

$$\int_0^1 dt\ i(\partial_t)[\alpha + dt\,\beta] = \int_0^1 dt\,\beta$$

and we have ~~d commutes with integration over the fibre~~

$$d_M \int_0^1 dt\,\beta = \cancel{\int_0^1 dt\ i(\partial_t)\, d[\alpha + dt\,\beta] = 0}$$

$$= \int_0^1 dt\, d_M\beta = \int_0^1 dt\,\partial_t\alpha = \alpha_1 - \alpha_0$$

because $0 = d\alpha - dt\, d\beta \implies \partial_t\alpha = d_M\beta$. Thus in this case

$$\boxed{\operatorname{tr}(e^{D_1^2}) - \operatorname{tr}(e^{D_0^2}) = d\int_0^1 dt\ \operatorname{tr}(e^{D_t^2} B)}$$

Check: Look at degree 2.

$$\operatorname{tr}(D_1^2) - \operatorname{tr}(D_0^2) = \operatorname{tr}\left((D+B)^2 - D^2\right)$$

$$= \operatorname{tr}(DB + BD) + \underset{\overset{\parallel}{0}}{\operatorname{tr}(B^2)}$$

On the other side $\int_0^1 dt\ \operatorname{tr}(B) = \operatorname{tr}(B)$

$$d\ \operatorname{tr}(B) \overset{?}{=} \operatorname{tr}\{D, B\} = \operatorname{tr}(DB + BD).$$

December 30, 1982

Yesterday I found a couple of good ideas by reviewing Connes' approach to the index thm. Let's recall his program. He starts with an elliptic operator D on M and uses it to construct a map $K^0(M) \longrightarrow \mathbb{Z}$. Represent a vector bundle E over M by an idempotent matrix e of functions on M. Then eDe is an elliptic operator and one takes the index. In more detail suppose $D: S^+ \longrightarrow S^-$ and let $E = \operatorname{Im} e$ in the trivial bundle of rank N. Then we have

$$E \otimes S^+ \xrightarrow{\ e\ } (S^+)^N \xrightarrow{\ D = D^N\ } (S^-)^N \xrightarrow{\ e\ } E \otimes S^- .$$

Thus the operator eDe is one way of tensoring D with the vector bundle E. Now suppose D is a first order operator, say $D = \alpha^\mu \partial_\mu$ locally. Then

$$eDe = \alpha^\mu(\partial_\mu + e\partial_\mu e)$$

which shows essentially that what is important about the representation of E by an idempotent is the connection induced on E. So it would seem that I can replace Connes' use of idempotent matrices by vector bundles with connection.

So D defines a map

$$\begin{aligned} K^0(M) &\longrightarrow \mathbb{Z} \\ e &\longmapsto \operatorname{Ind}(eDe). \end{aligned}$$

We want an analytical formula for this index. Suppose D invertible. Then $eD^{-1}e$ is an inverse for D modulo compacts.

$$\begin{aligned} eDe\, eD^{-1}e - e &= eDeD^{-1}e - eeDD^{-1}e \\ &= e[D,e]D^{-1}e \end{aligned}$$

$$eD^{-1}e\, eDe - e = -eD^{-1}[D,e]e$$

and the operator $[D,e]D^{-1}$ is compact. (Think of D as a ψDO of degree 0 so that $[D,f]$ is of degree -1 hence compact.) Now if we put

$$e - K_0 = eD^{-1}e\,eDe \quad \Rightarrow \quad K_0 = eD^{-1}[D,e]e$$

$$e - K_1 = eDe\,eD^{-1}e \quad \Rightarrow \quad K_1 = -e[D,e]D^{-1}e$$

then $K = (K_0, K_1)$ is homotopic to the identity map for the complex $E\otimes S^+ \xrightarrow{eDe} E\otimes S^-$. Hence

$$\mathrm{Ind}(eDe) = \mathrm{Tr}(K_0^N) - \mathrm{Tr}(K_1^N)$$

for any N such that these traces make sense.

What Connes does is to take

$$\mathrm{Tr}(K_0^N) = \mathrm{Tr}\left(D^{-1}[D,e]e\right)^N \qquad \text{also for } K_1$$

and consider more generally the multilinear expression

$$\varphi(a_1,\dots,a_{2N}) = \mathrm{Tr}\left(D^{-1}[D,a_1]a_2 \cdot D^{-1}[D,a_3]a_4 \cdots\cdots a_{2N}\right)$$

$$-\text{ similar thing for } K_1.$$

where the a_j are functions on M. Then he shows this is a stable cocycle in his sense. He computes the cohomology in his sense and stably gets $H_{ev}(M)$ by his theorems. Thus the operator D defines an even homology class. On the other hand D defines a K-homology class $\in K_0(M)$, whose character is an even homology class, the fund. cycle $[M]$ capped with the $\hat{A}$ genus for the operator. So presumably these classes are the same.

Interesting idea: The Connes cocycle φ is a distribution on M^{2N} with a certain cyclic skew-symmetry. It would be nice if this distribution had a simple structure, for example supported on the diagonal and equivalent in some sense to a specific collection of currents, or forms like the $\hat{A}$ genus. Maybe one can think of his cocycle as being the local index for the operator. Somehow the use of functions a_j on the manifold is the same as using functions to give infinitesimal gauge transfs.

The next idea was suggested by the similarity of the above-mentioned index formula with residues à la Tate. Let Q be the projection operator in $L^2(S^1)$ which projects onto the subspace H^+ of fns. extending holomorphically inside. Given an invertible matrix fn. g on S^1 we then have the Wiener-Hopf operator $Qg : H^+ \longrightarrow H^+$, or Toeplitz operator.

$$\mathrm{Ind}(Qg) = \mathrm{Tr}\,[Qg, Qg^{-1}]. \qquad \text{up to sign}$$

But more basic is the expression

$$\varphi(a,b) = \mathrm{Tr}\ [Qa, Qb]$$

where a, b are functions on S^1. I think this is a Connes 1-cocycle over S^1. The first Connes cohomology group $=$ closed 1 diml currents $= H_1(S^1) = \mathbb{C}$. So it should be that

$$\varphi(a,b) = \int_{S^1} a\,db$$

What is interesting here is that we are dealing with an odd-diml manifold M and are using invertible matrix functions on M. Thus instead of

$$\underset{\text{operators}}{K_0(M)} \times \underset{\text{vector bundles}}{K^0(M)} \longrightarrow \mathbb{Z}$$

- have

$$K_1(M) \otimes K^{-1}(M) \longrightarrow \mathbb{Z}$$

BDF extensions   bundle autos.

Question: Is there any reason that something like

$$\varphi(a,b) = \mathrm{Tr}\,[Qa, Qb]$$

should be related to a ~~[illegible]~~ Lie algebra cocycle?

Can you see directly that the map $g \mapsto \mathrm{Ind}(Qg)$ comes from a primitive Lie algebra cocycle?