Problem. The key to determinants is the formula
\[
\log \det D = \text{Tr}_{\text{reg}} (D^T D)
\]
A regularization process for the trace defines a 1-form on the space of operators, and when this form is locally closed a determinant function is defined up to a constant factor.

So clearly an interesting question is to understand the possible regularization processes and when they give closed 1-forms. Let's look at this for a \( \overline{\delta} \)-operator where the variations in the operator occur in a small coordinate patch. I would like eventually to handle variations which are first order operators.

We begin with a \( \overline{\delta} \)-operator in \(|z| < 1\) and suppose boundary conditions are given which specify a Green's function for the \( \overline{\delta} \)-operator. Let \( D_0 \) be the \( \overline{\delta} \)-operator, \( G_0 \) the Green's function. Then for \( D = D_0 + B \) we have the Green's fn.

\[
G = G_0 - G_0 B G_0 + G_0 B G_0 B G_0 - \ldots .
\]

The one form of interest is \( \text{Tr}_{\text{reg}} (G S B) \).

It seems that the first thing to understand is the nature of the singularities of \( G S B \). From a local viewpoint a \( \overline{\delta} \)-operator \( D \) is of the form

\[
D = \xi \partial_{\xi} \varphi^{-1}, \quad \varphi \text{ smooth invertible}
\]

hence

\[
G(z, z') = \frac{\varphi(z) \varphi(z')^{-1}}{z-z'} + \text{smooth kernel}
\]

\[
= \frac{1}{z-z'} \left\{ \frac{\varphi(z')}{z'} + \partial_{z'} \varphi(z') + (z-z') \partial_{z} \varphi(z') + \ldots \right\} \varphi(z')^{-1}
\]

\[
= \frac{1}{z-z'} \sum_{n=0}^{\infty} \frac{(z-z')^{n}}{n!} \left( \partial_{z}^{n} \varphi \cdot \varphi^{-1} \right)_{z'} + \text{mod smooth kernels}
\]

Here I have dropped the \( \pi \) and will do integrals with \( d\xi \).

Next consider \( B \) of 0-th order. Then \( dB = dx(\xi) \)
any parametric has the form \( \frac{1}{z-z'} + \text{smooth} \).

Thus
\[
G_b = \frac{F_b(z, z')}{z-z'} + \text{smooth}
\]
and this will be a parametric \( \iff F_b - I \) is divisible by \( z-z' \).

Let's look at the case where \( n^2(z, z') = |z-z'|^2 \), \( z'=0 \) so that the geodesic from \( 0 \) to \( z \) is the straight line. Then \( F_b(z, 0) \) satisfies
\[
\frac{\partial}{\partial z} F_b = 0, \quad F_b = I \text{ at } z=0.
\]

If
\[
\nabla = (\partial_z + \beta) \partial z + \partial \bar{z} \partial \bar{z},
\]
this becomes
\[
\left[ z(\partial_z + \beta) + \bar{z} \partial \bar{z} \right] F_b = 0
\]
or
\[
\left[ z \partial_z + \bar{z} \partial \bar{z} \right] F_b = -z \beta F_b.
\]

Now you solve this by a power series
\[
F_b = I + F_1 + F_2 + \ldots
\]
where \( F_n \) is homogeneous of degree \( n \) in \( z, \bar{z} \). Then
\[
hF_n = (z \partial_z + \bar{z} \partial \bar{z}) F_n = -z (\beta F_b)_{n-1}
\]
which shows that \( F_n \) is divisible by \( z \).

General case: Put \( u = n^2(z, 0) \). Then we have
\[
|\partial_z u|^2 = \rho u, \quad u = |z|^2 e^w
\]
where \( w \) is smooth. Also
\[
\nabla \frac{\partial}{\partial z} F_b = \frac{2}{\rho} \left[ \partial_z u \cdot (\partial_z + \beta) + \partial_z u \cdot \partial \bar{z} \right]
\]
\[
\partial_z u = \bar{z}(1 + z \partial_z w) e^w
\]
so
\[
\nabla \frac{\partial}{\partial z} F_b = 0 \quad \text{becomes}
\]
\[
\left[ z (1 + \bar{z} \partial \bar{w}) (\partial_z + \beta) + \bar{z} (1 + \bar{z} \partial \bar{w}) \partial \bar{z} \right] F_b = 0
\]
\[(2 \partial_z + \overline{z} \partial_{\overline{z}}) F_n = -z \left[ \beta + \partial_z w \, \overline{z} \partial_{\overline{z}} + \partial_z \overline{w} \, \beta + \partial_z \overline{\beta} \right] F_n \]

does not lower degree.

Hence as before you get
\[n F_n = (2 \partial_z + \overline{z} \partial_{\overline{z}}) F_n = z \cdot \text{expression in } F_j \quad j < n\]
which shows that \( F_n \) is divisible by \( z \), and hence the flat Green's \( F_n \) is a parametrix.

It would be nice to understand the algebra behind parametrixes and determinants. Suppose we have an operator \( D \) and a quasi-inverse \( P \), better an inverse modulo compacts:

\[PD = I - K^0 \quad DP = I - K^1\]

Here \( K^0, K^1 \) are compact but not of trace class, so you can't compute the index as \( \text{tr}(K^0) - \text{tr}(K^1) \). However, one has an algebraic process for refining \( P \) to a

\[P_n^+ \text{ such that}\]

\[P_n D = I - (K^0)^{n+1} \quad DP_n = I - (K^1)^{n+1}\]

and then the index can be computed as

\[\text{tr} (K^0)^{n+1} - \text{tr} (K^1)^{n+1} .\]

The algebra is explained by thinking of \( P \) as a chain homotopy between \( \text{Id} \) and \( K = K^0, K^1 \) for the complex

\[E \xrightarrow{D} F.\]

Then \( P_n \) is a homotopy between \( I \) and \( K^{n+1} \).

\[I - K^2 = I - K + K - K^2\]
\[= PD + DP + K( PD + DP)\]
\[= (P + KP) D + D (P + KP)\]

so \( P_n^+ = P + KP \).
(One of the reasons this is interesting comes from the fact that with a suitable anti-symmetrization \( \text{tr}(K^n) = \text{tr}(K')^{n+1} \) will depend only on the symbol of \( D \) and only on first derivatives.)

If \( D = D_0 - B \) and \( P = D_0^{-1} \), then \( K_0 = I - D_0^{-1}(D_0B) \) and so
\[
P_1 = D_0^{-1} + D_0^{-1}BD_0^{-1}.
\]
Hence the refinement process is just giving us the geometric series:
\[
D^{-1} = D_0^{-1} + D_0^{-1}BD_0^{-1} + \cdots.
\]

Now you want to understand determinants. It would seem that the only general statement possible is that \( \text{tr}(D_0^{-1}B)^n \) is defined for large \( n \) and hence one can define locally
\[
\log \det_n (D_0B) = -\sum_{n \geq n_0} \frac{1}{n} \text{tr}(D_0^{-1}B)^n.
\]
Then
\[
\log \det_n (D_0 - B) = -\sum_{n > n_0} \text{tr}((D_0^{-1}B)^nD_0^{-1}BB) = -\text{tr}((D_0^{-1} - D_0^{-1}BD_0^{-1} + \cdots (D_0^{-1}B)^{n-1}D_0^{-1}))BB)
\]

Hence we see this kind of determinant corresponds to regularizing by subtracting off from \( D^{-1} \) the approximation
\[
P_n = D_0^{-1} + D_0^{-1}BD_0^{-1} + \cdots + (D_0^{-1}B)^{n-1}D_0^{-1}.
\]

What happens in the Riemann surface case? Here the regularization is done by subtracting a flat Green's fn. This is somehow much more subtle. Technically the same effect is achieved by removing \( D_0^{-1} \) and then using a principal value interpretation for the diagonal values of \( D_0^{-1}BD_0^{-1} \).
I should perhaps understand the analysis behind the heat kernel à la Seeley. One has

$$e^{-tA} = \oint \frac{1}{\lambda - A} e^{-\lambda t} \frac{d\lambda}{2\pi i}$$

over a contour circling the positive real axis. Here $A$ is a positive definite operator. An interesting point is that $\text{Tr}(e^{-tA})$, $t \succ 0$ is defined although $\text{Tr}(\frac{1}{\lambda - A})$ is not. We even $\langle x | e^{-tA} | x \rangle$ defined with an asymptotic expansion

(1) $\langle x | e^{-tA} | x \rangle \sim \sum a_n(x) t^n$ \quad $t \to 0$

although we have no clear idea of $\langle x | \frac{1}{\lambda - A} | x \rangle$.

One might try to define

$$\langle x | \frac{1}{\lambda - A} | x \rangle = \int_0^\infty \langle x | e^{-tA} | x \rangle e^{\lambda t} dt$$

however there are problems because of the terms with positive powers of $t$ in the asymptotic expansion (1).

However, it is clear from just these formulas that the singularities of $\langle x | \frac{1}{\lambda - A} | x \rangle$ as $x \to x$ should be intimately connected with the expressions in the asymptotic exp. (1).
October 31, 1982

\[ e^{-tA} = \int \frac{1}{\lambda - \Lambda} e^{-t\lambda} \frac{d\lambda}{2\pi i} \]

What's interesting is that the diagonal values \( \langle x | e^{-tA} | x \rangle \) are defined, but those for \( \langle x | \frac{1}{\lambda - \Lambda} | x \rangle \) are not.  

Ideas:  
1) Up to now I have thought in terms of \( A \) being a 2nd order positive elliptic operator, whence \( e^{-tA} \) is a heat kernel.  However Hörmander considers \( A = (D^*D)^{1/2} \) a first order positive elliptic \( \Psi DO \), and this leads to interesting results concerning spectrum and bicharacteristics.

2) Seeley's analysis of the heat kernel assumes the symbol is positive, hence one could take \( A = D^*D \).

\[ \text{determinant formalism: It seems that the following heuristic formula} \]
\[ \log \det (A) = - \int_0^\infty \frac{\text{Tr} \left( e^{-tA} \right) dt}{t} + \text{const} \]

is good for the intuition, because it "explains" more precise formulas.  E.g.
\[ 8 \log \det (A) = - \int_0^\infty \frac{\text{Tr} \left( e^{-tA} (-tSA) \right) dt}{t} \]

\[ = \int_0^\infty \text{Tr} \left( e^{-t\Lambda} \right) dt = \text{Tr} \left( A^{-1} sA \right) \]

\[ s \]
\[ \frac{s}{\int s \gamma(s)} = \frac{1}{s \Gamma(s)} \int_0^\infty \text{Tr} \left( e^{-tA} \right) t^s \frac{dt}{t} = \frac{\int \{ 0 \} \}{s} \]

\[ \downarrow \]
\[ \int \{ 0 \} = \int_0^\infty \text{Tr} \left( e^{-tA} \right) \frac{dt}{t} \]

\[ \downarrow \]
\[ \text{const} \]
This seems to show that one should not think of $-i\phi'(0)$ as log det $(A)$ unless it is clear that $\phi(0)$ is constant for the variations of $A$ considered.

What I want to do next is to understand both the Seeley and Hormander representations. In both cases you have a kernel $K(x,t, x',t')$ which you represent as an oscillatory integral. Oscillatory integral means an integral of terms

$$e^{isiA}$$

rapidly varying asymptotic exp in homog. fns. phase as $t \to \infty$ of $\frac{i}{2}$, as $t \to \infty$.

Let's consider $A = -\frac{d^2}{dx^2} + g(x)$ on $\mathbb{R}$ and try to derive the representations for the kernels of $e^{-itA^{1/2}}$ and $e^{-itA}$.

Now $A^{1/2} = \sqrt{D^2 + g}$, $D = \frac{i}{\xi} \frac{d}{d_x}$, is a PDO with

$$\sqrt{D^2 + g} = |D| \left( 1 + \frac{1}{2} gD^{-2} + \cdots \right)$$

so $K(x,t) = \langle x | e^{-itA^{1/2}} | 0 \rangle$ is a solution of the pseudo-diff equation

$$(i \partial_t + A^{1/2}) K(x,t) = \delta(x,t).$$

How does one solve this? Suppose $g$ is a constant and let's use F.T.

$$\langle x | e^{-it\sqrt{D^2 + g}} | 0 \rangle = \int \frac{d^n\xi}{(2\pi)^n} e^{-ix \xi - it\sqrt{\xi^2 + g}}$$

Now use

$$\sqrt{\xi^2 + g} = |\xi| \left( 1 + \frac{1}{2} g|\xi|^{-2} \right)^{1/2} = |\xi| \left( 1 + \frac{1}{2} g |\xi|^{-2} + \cdots \right)$$
Thus in this case at least we have
\[
\langle x | e^{-itA^{1/2}} | 0 \rangle = \int \frac{d^n x}{(2\pi)^n} e^{i(\xi x - 1/2 t\xi^2)} e^{it[1/2 - \xi^2 + \xi^4]} \]
This has an asymptotic expansion in homogeneous form of \( \xi \) as \( |\xi| \to \infty \). So in the general case I want to do something similar. This means I want to construct asymptotic solutions of \((i\partial_t + A^{1/2})\psi = 0\)
of the form
\[
e^{i(\xi x - \omega(\xi) t)} (1 + t[\ldots])
\]
where \( \omega(\xi) \) is the symbol of \( A^{1/2} \). Then integrating these should give a parametrix for \( i\partial_t + A^{1/2} \).

Now this method is very close to the way one constructs the heat kernel, where one conjugates by an exponential factor so as to be able to recursively grind out an asymptotic expansion in \( t \).
Heat kernel $e^{-tA}$ where $A = \text{Laplacean} + \text{lower order terms}$. For example $A = -\partial_x^2 + q$. If $A$ has constant coefficients, then

$$\langle \alpha | e^{-tA} | \alpha \rangle = \int \frac{d^n x}{(2\pi)^n} e^{i\frac{x}{t}} e^{-t(\frac{x^2}{2} + q)}$$

The idea is to construct a parametrix for the operator $\partial_x + A$ as such an integral. This should be an oscillatory integral made up out of asymptotic solutions.

The logical idea is that one writes down an oscillatory integral

$$\sum \text{e}^{S_i F_i}$$

which will have singularities because of the growth of the $F_i$. Applying the operator

$$\sum \text{p}(\text{e}^{S_i F_i}) = \sum \text{e}^{S_i G_i}$$

will give a smooth function because the $G_i$ have rapid decrease.

So let's see how this idea might work for

$$\partial_t + A = \partial_x - \partial_x^2 + q$$

I need a supply of asymptotic solutions and I proceed as in WKB. This seems geared to the Schroedinger operator

$$\frac{i}{t} \partial_t - \partial_x^2 + q$$

and the resulting asymptotic solutions have phase
where $S$ satisfies the Hamilton-Jacobi equation
\[ e^{iS(x,t)} + \partial_t S + (\partial_x S)^2 = 0. \]

Up to now I have always taken the solution $S = \frac{x^2}{4t}$ and then derived the rest of the asymptotic expansion. This time I want to use other solutions such as
\[ S(x,t) = \frac{1}{4} x - t|\frac{x}{t}|^2 \]

There are some confusing aspects here. I want to construct solutions of
\[ \left[ \frac{i}{t} \partial_t - \partial_x^2 + g \right] \psi = 0 \]
which have the form
\[ \psi = e^{iS(x,t)} u \]
where $S$ is rapidly-varying and $u$ is slowly-varying. The function $u$ satisfies
\[ \left[ \frac{1}{4} \partial_t^2 + \partial_t \partial_x^2 + (\partial_x S)^2 + g \right] u = 0 \]
\[ \left[ (\partial_x S)^2 + \partial_t S + 2 \partial_x S \frac{1}{i} \partial_x + \frac{1}{i} \partial_x^2 S + \sqrt{i} \partial_t - \partial_x^2 + g \right] u = 0 \]

Because $S$ is rapidly-varying one wants
\[ \partial_t S + (\partial_x S)^2 = 0 \]
which is the HJ equation, and in the particular case of $S(x,t) = \frac{1}{4} x - |\frac{x}{t}|^2 t$ one has $\partial_x^2 S = 0$. So $u$ now satisfies
\[ \left[ \frac{2}{i} \partial_x + \left( \frac{1}{i} \partial_t - \partial_x^2 + g \right) \right] u = 0 \]

Now we are thinking of $u$ as slowly-varying, and $\psi$
as very large, so that our first approximation for 
\[ u \] 
\[ \partial_x u = 0 \]
i.e. \( u \) is a function of \( t \) but this is for one space dimension. In general one has
\[ i \partial_x u = 0 \]
This isn't quite accurate because the actual first approximation equation for \( u \) on the basis of the physics should be
\[ (2 \cdot \partial_x + \partial_t) u = 0 \]
The point is that the solution \( S \) of the H-J equation corresponds to a family of trajectories. These trajectories in the case of \( S = \frac{i}{\partial} x - |x|^2 t \) are solutions of \( \dot{x} = \frac{\partial H}{\partial p} = 2p \quad \dot{p} = -\frac{\partial H}{\partial x} = 0 \), since \( H(p) = p^2 \). Hence we get the family of trajectories
\[ \dot{x} = 2 \frac{i}{\partial} \]
which are the integral curves for the vector field \( 2 \cdot \partial_x + \partial_t \).

In some sense \( S \) is more rapidly varying in time than space so the \( \partial_t \) is of the same order as \( i \). So now it is clear that I can solve the transport equations
\[ \frac{1}{i} (2 \cdot \partial_x + \partial_t) u_n(x,t) = -(-\partial_x^2 + g(x)) u_{n-1}(x,t) \]
to obtain some kind of asymptotic series
\[ u(x,t) \sim \sum_{n=0}^{\infty} u_n(x,t) \]
Of course the \( u_n \) are not unique.
Let's compare the above ideas with the Feynman path integral approach. This says that
\[ \langle x | e^{-itA} | 0 \rangle \]
is an integral over paths starting at 0 and ending at x. We have
\[ (*) \quad \langle x | e^{-itA} | 0 \rangle = \int \frac{d^n \xi}{(2\pi)^n} e^{ix \xi} \langle \xi | e^{-itA} | 0 \rangle \]
and presumably \( \langle \xi | e^{-itA} | 0 \rangle \) is the integral over paths starting at 0 and ending with momentum \( \xi \). We are thinking of \( \xi \) as huge, hence \( g \) is negligible in the first approximation. Thus the path is a free motion with \( H(p) = p^2 \), so \( x = 2i \xi t' \) for \( 0 < t' < t \) is the trajectory and the action over it is
\[ \int L dt \quad L = \frac{\dot{p}}{2m} - H = p^2 - p^2 = \frac{1}{2} p^2 \]
or \[ S = \int L dt = \frac{1}{2} | \dot{\xi} |^2 t' . \] This would seem to give
\[ \langle \xi | e^{-itA} | 0 \rangle = e^{iS} = e^{i|\xi|^2 t'} \] which has the wrong sign.

Another problem with (*) is that \( \langle \xi | e^{-itA} | 0 \rangle \) is a function of \( t' \) but not \( x \), so I can't see how to get it from the transport equations.
November 2, 1982

Possible idea: One can obtain the $I$ fr. of $A$ from either $\text{Tr}(e^{-tA})$ or $\text{Tr}(e^{-tA''})$, hence it might be possible to use the known relations between $\text{Tr}(e^{-itA''})$ and geodesics in the theory of determinants.

Today I want to understand Fourier integral operators. I think the basic idea is the following. Take a Schrödinger operator $e^{-itH}$, where $H$ is a pseudo-differential operator. Then fix $t$ and try to understand the operator $e^{-itH}$. Look at the Schwartz kernel $\langle x | e^{-itH} | x' \rangle$ and look at its singularities. Now we know what the singularities look like from the theory of hyperbolic equations. If one starts with a $S$-function at $t = 0$, then there is a light cone of singularities. Singularities propagate along bicharacteristics.

I want to assume $P$ is of order 1 so that $\frac{i}{t} \partial_x + P$ is hyperbolic. Then the symbol of $P$ is a real function $H(x, \xi)$ homogeneous of degree 1 in $\xi$. Then you get a Hamiltonian flow on $(x, \xi)$ space in particular a symplectic diffeomorphism $x, \xi' \mapsto x, \xi$ by letting the flow work for the time $t$. This symplectic diffeomorphism controls the singularities of $e^{-itP}$. There is a Huyghens' principle operating on the level of singularities.

**Example:** Suppose $H(x, \xi) = H(\xi)$ is homogeneous of degree 1. The Hamilton flow is

$$\dot{x} = \partial_\xi H \quad \text{and} \quad \dot{\xi} = -\partial_x H = 0.$$ 

For example, take $H(\xi) = |\xi|$, whence the symplectic
\[ \dot{x} = \frac{\xi}{|\xi|} \]
\[ \dot{\xi} = \xi' \]

Notice that one has \(|x-x'| = t\) and hence the graph of this symplectic diffeomorphism does not project nicely onto \(x,x'\) space, and hence it can't be described in terms of an action \(S(x,x')\).

This behavior will be typical of any of the flows belonging to \(H(x,\xi)\) which is homogeneous of degree 1 in \(\xi\). The equations
\[ \dot{x} = \xi H \]
\[ \dot{\xi} = -\xi x H \]

will be preserved under homotheties in \(\xi\), so the graph of the map \(x'\xi' \mapsto x\xi\) will contain directions projecting trivially in \(x,x'\) space. The graph should project into a hypersurface in \(x,x'\) space with 1-dimensional fibres which are orbits under homothety.

Pseudo-differential operators correspond to the identity \(x'\xi' \mapsto (x\xi) = (x'\xi')\) which has image the diagonal in \((x,x')\) space and \(n\)-dimensional fibres. So things are slightly more degenerate than in the case of \(e^{-itH}\).

Let's take the case of \(e^{-itH}\) where \(H = H(\xi)\) has constant coefficients. Then
\[ \langle x | e^{-itH} | x' \rangle = \int \frac{d^{\frac{n}{2}} \xi}{(2\pi)} e^{i\frac{1}{2} (x-x') \cdot \xi - itH(\xi)} \]

do we actually are getting the Fourier transform of the kernel. Where are its singularities? Let's use
the method of stationary phase, i.e. look when
the exponent is stationary in $\xi$:
\[
0 = \nabla_\xi \left[ \frac{1}{2} \cdot (x-x') - \frac{1}{2} H(\xi) t \right] = (x-x') - \nabla_\xi H(\xi) t.
\]
This happens when $x, \xi$ is the image of $x', \xi$ under the symplectic transformation.

What is it that I want to understand? Somehow I have to see the singularities exactly in this constant coefficient case before I can generalize. This is a simple question, namely, the singularities of a sum of exponentials.

It might be easier to first look at what happens over $(x,t)$ space and then restrict to $t = constant$, since restriction is a delicate process for distributions. So what we have is a hypersurface $
abla \mapsto (\xi, -H(\xi))$ in the dual of $(x,t)$ space, in fact a cone because it's the graph of the homogeneous fn.

\[H(\xi).\]

Review: I start with a positive PDO operator $P$ of order 1, say $P = A^{1/2}$ where $A = -\Delta + g$, and want to understand the unitary operator $e^{-itP}$ with $t$ fixed. The symbol of $P$, call it $H$, gives a Hamiltonian flow on $(x,\xi)$ space, hence taking the flow at time $t$ we get a symplectic diffeomorphism of $(x,\xi)$ space which is the classical counterpart of $e^{-itP}$. Now the hope would be that via the singularities the classical + quantum pictures can be related. For example I can look at the singular support of the Schwartz kernel of $e^{-itP}$ which is a closed subset of $(x,\xi)$ spaces. This should be the image of the graph of symplectic diffeomorphism.
In some generic way, the graph of the diﬀerent.
sits over a hypersurface in \((x, x')\) space. Where
the hypersurface is smooth one gets a line in the
cotangent space which should be in the graphs.
This is a classical statement which I should be
able to prove.

One wants to understand Lagrangian submanifolds
of the cotangent bundle. Those projecting
non-singularity on the base I believe are locally sections
of the form \(df\). Then perhaps you can classify ones
which project non-degenerately onto a smooth hypersurface.

Consider the example \(H(\delta) = 1/\delta\). Then the diﬀerentia
is
\[ x = x' + \frac{x}{|\delta|} \]
\[ y = y' \]
and the hypersurface is \(|x - x'| = \delta\).

The tangent space to this hypersurface at \((x, x')\) is
\[ \{(dx, dx') \mid (x - x')(dx - dx') = 0 \} \]
and a linear function describing this hyperplane is
\[ \frac{1}{\delta} dx - \frac{1}{\delta'} dx' \] where
\[ x = y = x - x'. \]

All I am doing so far is classical geometry
which depends only on the symbol of the operator \( \mathcal{P} \).
It's clear that I will get at least for \( \delta \) small, a
nice hypersurface in \((x, x')\) space and if I use the
known results about propagation of singularities for
hyperbolic equations, then the singularities of the kernel
of \( e^{-i \mathcal{P} \delta} \) are contained in this hypersurface.
My goal is to understand what is known, mainly thru Hormander's work, about \( \text{Tr} (e^{-itA^{1/2}}) \) where \( A \) is a positive 2nd order elliptic operator. The idea is that \( e^{-itA^{1/2}} \) solves the Schroedinger equation \( (i \partial_t + A^{1/2}) \psi = 0 \) with the Hamiltonian operator \( A^{1/2} \). This is a quantum mechanical situation whose corresponding classical situation is the Hamiltonian flow

\[
\dot{x} = \partial_x H, \quad \dot{t} = -\partial_t H
\]

associated to the symbol \( H(x, \partial_x) \) of \( A^{1/2} \). Hence on one hand we have a unitary operator and on the other a symplectic diffeomorphism. Because there is no Planck's constant here, the QM-classical correspondence here is given by singularities, namely the singularities of the kernel \( \langle x | e^{-itA^{1/2}} | x' \rangle \) should be described in classical terms.

The precise relationship is what I want to work out. The singularities of \( K(x, t; x') = \langle x | e^{-itA^{1/2}} (x') \rangle \) lie (in general) on a hypersurface in \( x, t, x' \) space. A point lies on this hypersurface iff there is a trajectory of the Hamilton flow starting at \( x' \) and ending at \( x' \) after the time \( t \). Light cone picture

\[
\text{Fix } t \text{ and } x' \text{ and look at } K(x, t; x'). \text{ Then it is singular on some hypersurface in } x \text{-space.}
\]

Hence I want to understand what distributions with singularities along a hypersurface might look
like. I want the singularities to be in the transversal direction, to involve things like $H^K$, where $f = 0$ is the hypersurface, but not a $g$ function supported at a point of the hypersurface.

Look first at dim 1, i.e. $R$, and suppose the hypersurface is $x = 0$. Then the singularities I am interested in are the sort of things encountered with $400$'s:

$$K(x) = \int_{-\infty}^{\infty} e^{i\xi x} f(\xi) \frac{d\xi}{2\pi}$$

where $f$ has an asymptotic expansion in homogeneous functions of $\xi$ as $|\xi| \to \infty$. (Note: homogeneous functions in $R^*$ are very simple, of two types for each degree supported on either side of 0. So the possible singularities at $x = 0$ are like

$$\int_{0}^{\infty} e^{i\xi x} \xi^n d\xi = \frac{\Gamma(n+1)}{(-ix)^{n+1}}$$

Anyway, I might hope to show that the singularities of $K(x, t; x')$ for fixed $t, x'$ could be described by an integral

$$K(x) = \int_{-\infty}^{\infty} e^{i\xi} q(\xi) F(y, \xi) d\xi + \text{smooth}$$

where $q = 0$ is the hypersurface and $y = \xi y'$ are coordinates along the hypersurface. $F$ should have a nice expansion as $|\xi| \to \infty$.

So let's try this out in a constant coefficient case, say $A''z = P(D)$, e.g. $\sqrt{D^2 + g}$, $g \in R^{2\infty}$. Then

$$\langle x | e^{-itP(D)} | x' \rangle = \int d^n z e^{i\frac{1}{2} (z \cdot (x-x') - tP(z))}$$

Let's take $x' = 0$, $P(z) = |z|^2$. Then we have to
Understand the singularities of
\[ K(x, t) = \int \frac{d^n q}{(2\pi)^n} e^{i(q \cdot x - |q|^2 t)} \cdot \left( \frac{x_1 - t}{|x|^2} \right). \]

This time let's allow \( t \) to vary also. The exponent is stationary when
\[ 0 = \partial_q (q \cdot x - |q|^2 t) = x_1 - \frac{t}{|x|^2}, \]
and so no such \( q \) will exist unless \( |x| = t \).

Hence \( \varphi = t - |x| \) describes what should be the singularity locus of \( K(x, t) \).

Let's try to understand the nature of the singularities of \( K(x, t) \). Let's do the radial integral first:
\[ K(x, t) = c \int d^n \omega \int_0^\infty r^{n-1} dr e^{ir(\omega \cdot x - t)} \frac{\Gamma(n)}{[i(t - \omega \cdot x)]^n}. \]

I see from this that for all \( \omega \in S^{n-1} \) one has \( t - \omega \cdot x \neq 0 \), this is equivalent to \( t > |x| \), then \( K(x, t) \) is smooth. But this doesn't show that \( K \) is smooth when \( |x| > t \), (which one would think would be easier by causality).

Let's instead do the radial integral last:
\[ K(x) = \int d^n \omega e^{-irt} \int_0^\infty r^{n-1} dr e^{ir(\omega \cdot x)} \int_0^{2\pi} d\phi \int_0^\infty r dr e^{ir(\omega \cdot x)} \]
critical points $\omega = \pm \frac{x}{|x|}$ hence one has

\[
\int_{\mathbb{R}^{n-1}} c \cdot d^{n-1}e^{i\omega \cdot x} \sim \frac{e^{-i|x|}}{\lambda^{(n-1)/2}} \left[ a_0 + \frac{a_1}{\lambda} + \frac{a_2}{\lambda^2} + \ldots \right] + \frac{e^{-i|x|}}{\lambda^{(n-1)/2}} \left[ \tilde{a}_0 + \frac{\tilde{a}_1}{\lambda} + \frac{\tilde{a}_2}{\lambda^2} + \ldots \right]
\]

So it's clear that we get

\[
K(x,t) = \int_0^\infty e^{-i(t+1|x|)} F_1(r) \, dr + \int_0^\infty e^{-i(t-1|x|)} F_2(r) \, dr.
\]

(Use a partition of unity over $\mathbb{R}^d$ to get sum over critical points), where $F_1, F_2$ have asymptotic expansions in powers of $r$ as $r \to \infty$. For $t > 0$, the first integral is smooth.

This example convinces me of the essential correctness of the idea that $\langle x | e^{-itA^{1/2}} | x' \rangle$ near points where the hypersurface is smooth will be a 1-diml oscillatory integral modulo smooth functions. However, for the applications to $\text{Tr}(e^{-itA^{1/2}})$ I want the behavior for $t \to 0$, and I think this involves one with a conical singularity.

Let's consider a more complicated case with $A^{1/2} = P(D)$, $P(\xi)$ homogeneous of degree 1 and $\theta > 0$. Then

\[
\langle x | e^{-itP(\xi)} | 0 \rangle = \int \frac{d^n \xi}{(2\pi)^n} e^{i(\xi x - P(\xi)t)}
\]

Now $P : \mathbb{R}^n \to \mathbb{R}_{>0}$ has no critical points by homogeneity, hence $\{ P(\xi) = 1 \}$ is a smooth star-like surface surrounding the origin.
and we can write
\[ \langle x | e^{-i t P(\omega)} | 0 \rangle = \int n^{-1} d\omega \int e^{-i t \lambda} \int e^{i n(\omega \cdot x)} d\mu \]
where $d\mu$ is a suitable "solid angle" type measure on the surface $P(\omega) = 1$. (i.e. $\frac{d^n f}{(2\pi)^n} = n^{-1} d\omega \cdot d\mu$).

Now apply stationary phase to the last integral so as to get an asymptotic expansion as $n \to +\infty$. First we look for critical points. Now use Lagrange multipliers to understand critical pts. for $\xi \cdot x$ subject to $P(\xi) = 1$. Consider
\[ \xi \cdot x - \lambda (P(\xi) - 1) \]
and then we get the conditions
\[ x = \lambda \partial_\xi P(\xi) , \quad P(\xi) = 1. \]

In other words $x$ is a critical point if grad $P$ points in the same direction as $x$.

Actually $\omega \mapsto \omega \cdot x$ is "the height above the table" type of function, so we can easily visualize the critical points.

Consider the case where $P(\xi) = 1$ is strictly convex. In this case there are 2 critical points for each $x$, a maximum and a minimum. Call $m_+(x)$ the maximum value of $\omega \cdot x$ on $P(\omega) = 1$ and $m_-(x)$
the minimum value. Then I have the asymptotic expansion
\[ \int e^{i\pi(x,x)} \, dp \sim \frac{e^{-i\pi m_+}}{\pi^{(n-1)/2}} \left[ a_0(x) + \frac{a_1(x)}{\pi} + \ldots \right] \]
\[ + \text{ similar thing with } e^{-i\pi m_-(x)} \]
So \[ \langle x | e^{-it\pi(p)} | \xi \rangle = \int_0^\infty e^{-i(t-m_+ x) r} \, dr \]
+ smooth
assuming \( t > 0 \).

Actually I should make this more precise, namely I can write the kernel as 1-diml oscillatory integral
\[ \langle x | e^{-it\pi(p)} | \xi \rangle = \int_0^\infty e^{-i\pi(t-m_+ x) r} \, dr \]
+ smooth
where \( F(x, r) \sim \frac{1}{\pi^{(n-1)/2}} \left[ a_0(x) + \frac{a_1(x)}{\pi} + \ldots \right] \) as \( r \to \infty \).

This is exactly the form I wanted, namely the singular hypersurface is \( t-m_+ = 0 \), and the coefficients of the asymptotic expansion are expressed in the coordinates \( x \) which are complementary to \( y = t-m_+(x) \).

Question: I know that
\[ e^{-it\pi} = \int \frac{1}{\lambda - \pi} \, e^{-it\lambda} \, d\lambda \]
\[ = \int_0^\infty e^{-it\lambda} \frac{dE_\lambda}{d\lambda} \, d\lambda \]
where \( \frac{dE_\lambda}{d\lambda} \) is the jump in \( \frac{1}{\lambda - \pi} \) across the real axis.
Do I get any useful information about the resolvent in this situation?

\[
\langle x | e^{-itP} | 0 \rangle = \int_0^\infty e^{-it\lambda} \langle x | \frac{dE}{d\lambda} | 0 \rangle \, d\lambda
\]

\[
= \int_0^\infty e^{-it\lambda} \left[ \int \frac{e^{i\lambda(\omega,x)}}{P(\omega)} \, d\omega \right] r^{n-1} \, d\lambda
\]

Hence for \( x \neq 0 \) I would get an asymptotic expansion for \( \langle x | \frac{dE}{d\lambda} | 0 \rangle \) as \( \lambda \to \infty \).

The above formulas in the simple case \( P(0) \) homogeneous of degree 1 are exact for \( x = 0 \). So we get

\[
\langle 0 | e^{-itP} | 0 \rangle = \int_0^\infty e^{-it\lambda} \left[ \text{vol}[P = 0] \right] r^{n-1} \, d\lambda
\]

\[
= \frac{\Gamma(n) \left[ \text{vol} \right]}{(it)^n}
\]

Also

\[
\langle 0 | \frac{1}{\lambda-P} | 0 \rangle = \int_0^\infty \frac{1}{\lambda-P} \left[ \text{vol} \right] r^{n-1} \, d\lambda
\]

is undefined.

My idea is that as I pass to the general case I will perhaps be able to get an asymptotic expansion for \( \langle x | e^{-itP} | x \rangle \) as \( t \downarrow 0 \).

How do I handle the general case? I should consider things like \( P = \sqrt{D^2 + \delta} \), or \( \sqrt{D^*D} \) where \( D \) is a \( \bar{\partial} \)-operator on a torus. The idea is that the symbol is constant coefficients, and so you will have to understand the changing lower order terms.
What might be the analogue of the action in the Hörmander theory? In QM one has
\[
\langle x | e^{-\frac{i}{\hbar}H} | x' \rangle \sim e^{i S(x,t,x')/\hbar}
\]
so the action appears using an asymptotic expansion w.r.t. \(\hbar\). The action should be a classical gadget.

It might be useful to forget about \(x'\) and to think of solutions of the Schrödinger equation as having the form
\[
\psi(x,t) \sim e^{-i S(x,t)/\hbar}
\]
where \(S\) is a solution of the HJ equation.

In the Hörmander situation solutions of the hyperbolic \(\text{DE} \) have \(\text{singularities}\), not an asymptotic expansion. So what might the action be? A key idea I think is that one represents the solution of the hyperbolic \(\text{DE} \) as an oscillatory integral where the integration takes place over a cone, i.e. has this homogeneity. Thus
\[
\psi(x,t) = \int e^{-i \varphi(x,t,\xi)} \{ \text{---} \}
\]
where \(\varphi(x,t,\xi)\) is homogeneous of degree 1 in \(\xi\).

Thus when one applies stationary phase one will get a ray of critical points
\[
\nabla_\xi \varphi(x,t,\xi) = 0 \quad \text{homogeneous of deg } 0
\]
\[
\Rightarrow 0 = \xi \cdot \nabla_\xi \varphi(x,t,\xi) = \varphi(x,t,\xi) \quad \text{by Euler}.
\]

I have the feeling that the action is going to be zero on the singular surface and that if we have a smooth singular hypersurface, then it is
given by \( S = 0 \) where \( S \) is the action. My idea now is that action is an equation for the singular hypersurface in some sense.

I should be thinking of \( S \) as a solution of the HJ equation. But then \( S + \text{constant} \) will also be a solution.

This tells me that \( S \) is more than the singularities of a particular solution.Usually an \( S \) is associated to a family of trajectories for the Hamilton flow which fill up \( x \)-space at any fixed time, assuming we are not at a bad point.

Picture of the singularities of a solution of the wave equation

![Picture of singularities]

Picture of a family of solutions of Hamilton's equations would be the union of these cones at different times.

At time \( t = 0 \) you could start with \( S = \frac{m}{\hbar} x \), i.e. all the particles have momentum \( \frac{m}{\hbar} \). Then you solve the HJ equation, which amounts to determining all trajectories with initial momentum \( \frac{m}{\hbar} \). What is then \( S(x,t) \) assuming \( H(q) = 1/2 \)? Clearly

\[
S(x,t) = \frac{m}{\hbar} x - \frac{m}{2\hbar} t.
\]
I want next to work out the Hormander theory for $e^{-itA^{1/2}}$ where $A$ is a simple positive second order operator such as $-\partial_x^2 + g(x)$. The first step is to construct asymptotic solutions of

$$(\frac{i}{t} \partial_t + A^{1/2}) u = 0$$

of the form

$$u = e^{irS(x,t)} \left[ a_0 + \frac{a_1}{n} + \frac{a_2}{n^2} + \ldots \right].$$

Because $A$ doesn't involve $t$, it follows that any solution of the Schrodinger equation is also a solution of the wave eqn.

$$(-\frac{i}{2} \partial_t + A^{1/2})(\frac{i}{2} \partial_t + A^{1/2}) u = (\partial_t^2 + A) u = 0,$$

and hence I think I can avoid having to explicitly work with the symbol of $A^{1/2}$. So if we put

$$u = e^{-irS(x,t)} \psi,$$

then the wave equation for $u$ becomes

$$\left[ -\left(\frac{i}{t} \partial_t + n \partial_x S \right)^2 + \left(\frac{i}{t} \partial_x + n \partial_x S \right)^2 + g(x) \right] \psi = 0$$

$$-n^2(\partial_t S)^2 - 2n \partial_t S \cdot D_t - n^2(\partial_x S)^2 + 2n \partial_x S \cdot D_x$$

The HT equation is then

$$- \left(\partial_t S \right)^2 + \left(\partial_x S \right)^2 = 0$$

and I am interested in the branch

$$\partial_t S + |\partial_x S| = 0$$

which is the HT equation for $\frac{i}{t} \partial_t + A^{1/2}$. The simplest solution is

$$S(x,t) = \xi x - \frac{1}{2} |\xi| t$$

and I think in Hormander's method we solve this
asymptotically for each \( \xi \). Take \( \xi > 0 \) and call it \( \xi_j \); then we are after an asymptotic solution of the wave equation
\[
\epsilon^{-i(\xi_j x - \xi_j t)} \left[ a_0 + \frac{a_1}{\xi_j} + \ldots \right].
\]
The \( a_n \) which in general are functions of \( x,t \) are subject to the following equations (in general)
\[
\left[-2 \frac{d}{dt} D_t + 2 \frac{d}{dx} D_x - \frac{i}{\xi_j} \frac{d}{dx} S + \frac{i}{\xi_j} \frac{d}{dx} S \right] a_n
= -\left[ -D_t^2 + D_x^2 + g(x) \right] a_{n-1},
\]
In the present case where \( S = x - t \) we get simply
\[
2(D_t + D_x) a_n = -\left[ -D_t^2 + D_x^2 + g(x) \right] a_{n-1},
\]
which determines \( a_n \) up to a function of \( x-t \).

At this point I have some understanding of the possible asymptotic solutions of the wave equation, and I have to figure out how these are combined to get \( \langle x | e^{-iAt/\epsilon} | x' \rangle \).

**Example:** Suppose that \( g \) is a constant (i.e. \( m^2 \)
where \( m \) is a "mass") Then the exact solution is
\[
\langle x | e^{-iAt/\epsilon} | x' \rangle = \int \frac{d\xi}{2\pi} e^{i\xi(x-x')-1/2+\xi^2 + g^2 \tau} \]
\[
= \int \frac{d\xi}{2\pi} e^{i\left|\xi(x-x')-1/2\right|} e^{i(\xi^2 + g^2)\tau} \]

Because
\[
\sqrt{x^2 + g^2} - 1/2 = \left|\xi\right|\left((1 + \frac{\xi^2}{1/2})^{1/2} - 1\right) = \frac{\xi}{\left|\xi\right|} + \frac{1/2}{1/2} + \ldots
\]
the last factor has an asymptotic expansion
\[
e^{i(\left|\xi\right| - \sqrt{x^2 + g^2})\tau} = (1 - \frac{\xi \tau}{\left|\xi\right|} + \ldots)
\]
in which the coefficients vanish at \( \tau = 0 \), except for the first
If we were to solve
\[ 2(D_t + D_x) a_n = \left[-D_t^2 + D_x^2 + \delta \right] a_{n-1} \]
requiring \( a_n = 0 \) for \( t=0 \) \( n \geq 1 \), and \( a_0 = 1 \)
then clearly \( a_n \) is a polynomial in \( t \). Thus
\[ \text{this is the appropriate boundary condition in general.} \]

Let's now summarize what ought to be true for \( \langle x | e^{-itA_{1/2}} | x' \rangle \)
where \( A = -D_x^2 + \delta \).

For each \( \delta \neq 0 \) I can construct an asymptotic solution of the wave equation
\( (D_t^2 - D_x^2 + \delta) u = 0 \)
of the form
\[ e^{i \frac{a_1}{h} (x-1 \delta t)} \left( 1 + \frac{a_1}{h} + \frac{a_2}{h^2} + \ldots \right) \]
where the \( a_n \) are determined recursively by solving
\[ 2 \left( i \delta | D_t + \delta D_x \right) a_n = -(-D_t^2 + D_x^2 + \delta (x)) a_{n-1} \]
subject to the boundary condition \( a_n = 0 \) when \( t=0 \).

(Note that \( a_n \) is homogeneous of degree \(-n\) in \( \delta \)).

Next select a \( F(x,t,\delta) \) which is smooth, vanishes near \( \delta = 0 \), and which has the above asymptotic expansion \( (x \to \infty) \)

The claim is then that \( \langle x | e^{-itA_{1/2}} | x' \rangle \equiv \int \frac{d\delta}{2\pi} e^{i (\frac{\delta}{h} - 1 \delta t)} F(x,t,\delta) \)

modulo smooth functions.

One thing that is clear is that if the wave operator \( D_x^2 - D_x^2 + \delta (x) \) is applied to the integral, then one gets a smooth function of \( xt \). I believe the same is true for the operator \( i \frac{D_t + A_{1/2}}{h} \). Next if I set \( t = 0 \) I get \( \delta (x-x') \) mod smooth functions.
What is mainly of interest to me is the diagonal behavior

\[
\langle x | e^{-i A \frac{t}{2} } | x \rangle \equiv \int \frac{d^3 \xi}{2\pi} e^{-i \frac{t}{2} \xi} F(x,t,\xi) \sim \left[ 1 + a_1(x,t,\xi) + a_2(x,t,\xi) + \ldots \right]
\]

In one dimension what actually happens?

\[
\int_0^\infty \frac{d\xi}{2\pi} e^{-i\xi t} = \frac{1}{2\pi i} \frac{1}{t}
\]

\[
\int_0^\infty \frac{d\xi}{2\pi} \frac{e^{-i\xi t}}{\xi} = -\frac{1}{2\pi} \log(t) + \text{constant}
\]

Let's determine \( a_1(x,t,\xi) \). It satisfies

\[
2(\partial_t + \partial_x) a_1 = -\xi \phi(x)
\]

First take \( \xi > 0 \). \( a_1 \) satisfies

\[
(\partial_t + \partial_x) a_1 = -\frac{i}{2\xi} \phi(x) \quad \text{and} \quad a_1 = 0 \text{ when } t = 0
\]

\[
a_1(x,t) = -\int_0^t \frac{d\xi}{2\xi} \phi(x-t+t',t') dt'
\]

Next take \( \xi < 0 \)

\[
(\partial_t - \partial_x) a_1 = \frac{-i}{2|\xi|} \phi(x)
\]

\[
a_1(x,t) = -\int_0^t \frac{d\xi}{2|\xi|} \phi(x+t-t',t') dt'
\]

General formula

\[
a_1(x,t) = -\int_0^t \frac{d\xi}{2|\xi|} \phi(x-\frac{t}{111}(t-t'),t') dt'
\]
But $a_1, a_2,$ etc. vanish at $t = 0,$ hence the worst singularity will be like $t \log(t)$ as $t$ goes to zero. In fact we have

$$a_1(x,t) \sim \frac{-i}{2|s|} g(x) t$$

as $t \to 0$

so that

$$\int \frac{d^3 \xi}{2 \pi} e^{-i|\xi| t} a_1(x,t,\xi) \sim \frac{1}{2\pi} \log t \frac{i}{2\pi} g(x) t$$

$$= \frac{1}{2\pi i} (t \log t) g(x).$$

Actually for later reference, it might be worthwhile to understand more precisely the function

$$g(t) = \int \frac{d^3 \xi}{2 \pi} e^{-i(\sqrt{s^2 + \xi^2})t}$$
$J_A(s) = \text{Tr}(A^{-S}) = \text{Tr}((A^{1/2})^{2S})$

$$= \frac{1}{\Gamma(s)} \int_0^\infty \text{Tr}(e^{-tA}) t^{s-1} \frac{dt}{t} = \frac{1}{\Gamma(2s)} \int_0^\infty \text{Tr}(e^{-tA^{1/2}}) t^{2S} \frac{dt}{t}$$

I should be able to use what I know about the asymptotic expansion of $\text{Tr}(e^{-tA})$ to say something similar about $\text{Tr}(e^{-tA^{1/2}})$. Let's assume that $\exists$ asymptotic exp.

$\text{Tr}(e^{-tA^{1/2}}) \sim \frac{a_{-n}}{t^n} + \ldots + a_o + a_1 t + \ldots$

Then

$$\int_0^\infty \text{Tr}(e^{-tA^{1/2}}) t^{2S} \frac{dt}{t} \sim \sum a_k \int_0^1 t^{k+2S} \frac{dt}{t}$$

$$\sim \sum_{k\geq -n} \frac{a_k}{2S+k}$$

has simple poles at $\frac{n}{2}, \frac{n-1}{2}, \ldots$ provided the coefficients are $\neq 0$. $\frac{1}{\Gamma(2S)}$ is 0 for $s=0, -\frac{1}{2}, -1, \ldots$ so consequently assuming $\exists$ we see that $f(s)$ will be regular at these points $0, -\frac{1}{2}, -1, \ldots$ and so the singularities will be at most simple poles at $s=\frac{n}{2}, \frac{n-1}{2}, \ldots, -\frac{1}{2}$.

On the other hand I know $\exists$ an asymptotic expression

$$\text{Tr}(e^{-tA}) = \sum b_n t^{n/2} + b_{n/2-1} t^{-3/2} + \ldots + b_{\frac{n}{2}+k} t^{-\frac{n+2k}{2}}$$

hence

$$\int_0^\infty \text{Tr}(e^{-tA}) t^{s} \frac{dt}{t} \sim \frac{b_{\frac{n}{2}}}{s^{\frac{n}{2}}} + \frac{b_{\frac{n}{2}-1}}{s^{\frac{n}{2}-1}} + \ldots$$

has at most simple poles at the points $\frac{n}{2}, \frac{n-1}{2}, \ldots$ $f(s)$ vanishes at $s=0, -1, -2, \ldots$ hence for $n$ odd $f(s)$ might have poles at $-\frac{1}{2}, -\frac{3}{2}, \ldots$. Hence it might
be the case that the asymptotic expansion for $\text{Tr}(e^{-itP})$ exists only through the $a_0$ term. Probably we encounter a $t^{\log_2 q}$ term, and thus I should see it already when $A = D^2 + \frac{q}{2} > \frac{q}{2}$ is constant.

Next I want to understand $\text{Tr}(e^{-itP})$ where $P(D)$ is a constant coefficient operator over a torus $\mathbb{R}^n/\mathbb{Z}^n$, e.g. $P(\xi) = \sqrt{\frac{q}{2} + \frac{1}{8}}$, $q > 0$. I can get at the kernel $\langle x | e^{-itP} | x' \rangle$ in two ways, either by eigenfunctions

$$\langle x | e^{-itP} | x' \rangle = \frac{1}{L^n} \sum_{\xi \in \frac{2\pi}{L} \mathbb{Z}^n} e^{i\frac{1}{2}(x-x') \cdot \xi} e^{-itP(\xi)}$$

as by taking the kernel in $\mathbb{R}^n$ and making it periodic

$$\sum_{\mu \in \mathbb{L}^n} \int_{\mathbb{R}^n} \frac{d^n \xi}{(2\pi)^n} e^{i\xi \cdot (x-x' + \mu)} e^{-itP(\xi)}$$

The equality of these is the Poisson summation formula. We have

$$\text{Tr}(e^{-itP}) = \sum_{\xi \in \frac{2\pi}{L} \mathbb{Z}^n} e^{-itP(\xi)}$$

$$= L^n \sum_{\mu \in \mathbb{L}^n} \int_{\mathbb{R}^n} \frac{d^n \xi}{(2\pi)^n} e^{i\frac{1}{2}\xi \cdot \mu} e^{-itP(\xi)}$$

Let's put $Q(x,t) = \int_{\mathbb{R}^n} \frac{d^n \xi}{(2\pi)^n} e^{i\xi \cdot x} e^{-itP(\xi)}$. This is the fundamental solution in $\mathbb{R}^n$ of \((\frac{1}{i} \partial_t + P) u = 0\), and has its singularities on the light cone.
We have
\[ \langle x | e^{-itP} | x \rangle = \sum_{\mu \in \mathbb{Z}^n} Q(- \mu, t) \]
and the point of Hörmander's analysis is that the essential term is \( Q(0, t) \), provided we work in \(|t| < \varepsilon\).

Each term \( Q(\mu, t) \) for \( \mu \neq 0 \) is a smooth around \( t = 0 \). We have
\[ (\partial_t^n) Q(\mu, t) = \int \frac{d^n \xi}{(2\pi)^n} \ e^{i\mu \cdot \xi} \ P(\xi)^n \ e^{-i\xi \cdot x} \]
so when \( t = 0 \) we get the F.T. of \( P(\xi)^n \). If \( P(\xi) = \sqrt{\xi^2 + \varepsilon^2} \), then for \( n \) even, \( P(\xi)^n \) is a polynomial in \( \xi \), hence its F.T. is supported at 0. This means that for \( x \neq 0 \), \( Q(x, t) \) is an odd smooth function of \( \xi \). If \( \varepsilon \to 0 \), \( Q(x, t) \) is analytic; in any case we have
\[ Q(x, t) \approx c_1 t + c_2 t^3 + \ldots \quad \text{as} \quad t \to 0. \]

Modulo the result about \( Q(x, t) \to 0 \) fast enough as \(|x| \to \infty \), this will guarantee uniform convergence of the \( \mu \)-sum, we see that the asymptotic expansion for \( Q(0, t) \) is essentially the same as for
\[ Q(0t) = \left( \frac{d^n \xi}{(2\pi)^n} \right) e^{-itP(\xi)} \]
So let's understand this for \( P(\xi) = \sqrt{\xi^2 + \varepsilon^2} \)
\[ \sqrt{\xi^2 + \varepsilon^2} = |\xi| (1 + \frac{\varepsilon}{|\xi|})^{1/2} \]
\[ = |\xi| + \frac{1}{2} \varepsilon - \frac{1}{8} \varepsilon^2 |\xi|^{-1} + \ldots \]
\[ Q(0t) = \int \frac{d^n \xi}{(2\pi)^n} e^{-it|\xi|} e^{-it(\sqrt{\xi^2 + \varepsilon^2} - |\xi|)} \]
\[
\int \frac{d^n \xi}{(2\pi)^n} \, e^{-it||\xi||^2} \left[ 1 - it \frac{\delta}{2!} \frac{1}{||\xi||^2} + \frac{1}{2!} \left( \frac{it^2}{2} \frac{1}{||\xi||^2} \right)^2 \right] \\
+ it \frac{\delta^2}{3!} \frac{1}{||\xi||^3} + \cdots 
\]

So we need
\[
\int \frac{d^n \xi}{(2\pi)^n} \, e^{-it||\xi||^2} \frac{1}{||\xi||^k} = c \int_0^\infty r^{n-k} dr \, e^{-itr} \frac{1}{r^k} 
\]
\[
= c \frac{\Gamma'(n-k)}{(it)^{n-k}}
\]

This will hold for \( k = 0, 1, \ldots, n-1 \). There won't be any possibility of log \( t \) terms until \( k = n \), and in this case they come multiplied by \( t \) from the numerator.

Let's return to the general situation. I have this 2nd order pseudoelliptic operator \( A \) and I am going to use Hormander's way to get at the kernel \( \langle x | e^{-itP} | x' \rangle \) where \( P = A^{1/2} \). For each \( \xi \neq 0 \) one lets \( S(x,t,\xi) \) be the solution of the HJ equation such that \( S(x,t,\xi) = \xi \cdot x \) when \( t = 0 \). Then we find a formal solution
\[
e^{-iS(x,t,\xi)} (1 + a_1(x,t,\xi) + a_2(x,t,\xi) + \cdots)
\]

of the hyperbolic equation \((\delta^2_x + A)u = 0\) where \( a_n(x,t,\xi) \) is homogeneous of degree \(-n\) in \( \xi \), and \( a_n(x,t,\xi) = 0 \) when \( t = 0 \).

Then
\[
\langle x | e^{-itP} | x' \rangle \equiv \int \frac{d^n \xi}{(2\pi)^n} \, e^{is(x,t,\xi) - is(x',t,\xi)} (1 + a_1(x,t,\xi) + \cdots)
\]

To fix the ideas let \( A \) have symbol \( |\xi|^2 \). Then
\[
S(x,t,\xi) = \xi \cdot x - |\xi|^2 t
\]
and finally

$$\langle x | e^{-itP(x)} | x' \rangle = e^{-\frac{1}{i\hbar} \int (P(x') - P(x)) dx}$$

so now I want to do something similar in general for variable coefficients. Work with $\exp(-itH(x))$ and then $\exp(-it\phi(x,x'))$.

Also, we end up with

$$\frac{dx}{dt} = \mathbf{A}(x)$$

as time goes on. This means that we are solving $x(t) = x(t=0) + \frac{dx}{dt} \cdot t$.

The idea is to describe the solution to this equation, which is

$$\langle x | e^{-itA(x)} | x' \rangle$$

where the spectrum is discrete. For $P \neq 0$, we need the fermionic matrix $\mathbf{A}(x)$.

Now, the equation

$$(\frac{d}{dx} + \frac{1}{2}) \psi = 0$$

where $x$ is a discrete, self-adjoint operator on $\mathbb{R}$.
where $E$ is a constant. What does this mean in the Riemannian case where $H(x, \xi) = |\xi|$ at $x$.

I am after $S(xt, \xi) = W(x, \xi) - E(\xi)t$ which is homogeneous of degree 1 in $\xi$ and such that $\xi = (\partial_x W)|_{x'}$. Here $E(\xi) = |\xi|$. Thus $W(x, \xi)$ is a fn. whose gradient has size $|\xi|$ and also coincides with $\xi$ at $x'$. Picture:

![Diagram](image)

$W(x, \xi) = 0$

I guess it is clear that you get a $W(x, \xi)$ by choosing nicely a surface thru $x'$ with normal vector $\xi$, and then $W$ will be the distance of $x$ from that surface suitably scaled by $|\xi|$.

The problem with using this $W$ is that we will not have

$$\int \frac{d^n\xi}{(2\pi)^n} e^{iW(x, \xi)} = 8(x-x')$$

even modulo smooths? (Maybe this is okay. I should pick a $g(x) = 1$ near $x'$ of compact support and consider)

At least this isn't clear to me.

Now why would I like an $S(xt, \xi)$ of the form $W(x, \xi) = |\xi|t$? Because then when I compute
the trace I put $x = x'$ and I get a very simple time dependence. What might be the simplest type of $S(x,t,\xi)$ from a geometric viewpoint? This would have to be a family of trajectories all having the initial momentum $\xi$. The only way I can see to do this is to use the exponential map to extend $\xi$ to vectors in a nbhd of $x'$. You can also parallel translate along radial lines.

Formula for $W$ in the Riemannian case: Think of $\xi$ as being a tangent vector in the tangent space at $x'$. Let $\xi^\perp$ be the perpendicular hyperplane and also use the same notation for its image under the exponential map.

\[ d(x, \xi^\perp) = \text{distance calculated in the manifold} \]

Then
\[ W(x, \xi) = d(x, \xi^\perp) |\xi| \]

Evidently Hormander actually uses $S(x,t,\xi)$ of the form $W(x,\xi) - E(\xi) t$. Therefore he must have a way to understand
\[ \left[ \int \frac{d^n \xi}{(2\pi)^n} e^{i W(x,\xi)} \right] - S(x-x'). \]

At this point I get less enthusiastic about being able to apply the wave equation methods to understanding $Tr(e^{-tA})$ where $A = D^\ast D$, where the symbol of $A$ is given by the Riemannian metric. I don't see a nice choice for the $W(x,x',\xi)$ function.
November 6, 1982

Essential problem for me is to understand the asymptotic expansion of

\[ \text{Tr}(e^{-tAB}) \]

as \( t \to 0 \). The reason is because I need to know about regularizing the trace of the operator \( B \). Here \( B \) is a PDO, hence it has singularities along the diagonal. One can't pull \( B \) back to the diagonal and integrate, because this is not a transversal situation. Hence one smooths out the identity to \( e^{-tA} \) and then tries to extract something as \( t \to 0 \).

The extraction proceeds via taking the constant term in the asymptotic expansion. This is reasonable because

\[ \text{Tr}(A^{-s}B) = \frac{1}{\Gamma(s)} \int_0^\infty \text{Tr}(e^{-tA}B) t^{-s} \frac{dt}{t} \]

and so provided \( \text{Tr}(A^{-s}B) \) has an asymptotic expansion one gets

\[ \text{Tr}(A^{-s}B) \bigg|_{s=0} = \text{constant term}. \]

Let's go over the Hörmander theory for future understanding, even though it doesn't seem relevant to the problem of determinants. One has a first order positive elliptic \( P \) and one wants to understand its eigenvalue distribution: \( N(\lambda) = \text{no. of eigenvalues } \leq \lambda \). In the case of \( P(0) \) on a torus, say where \( P(0) \) is homog. of degree 1, one has

\[ N(\lambda) = \text{no. of lattice points } \xi \text{ with } P(\xi) \leq \lambda. \]

Let the torus be \( \mathbb{R}^n / L \mathbb{Z}^n \), then \( \xi \in \frac{2\pi}{L} \mathbb{Z}^n \) and so

\[ N(\lambda) \sim (\frac{2\pi}{L})^n \sim \text{vol} \{ P(\xi) \leq \lambda \} = 2^n \text{vol} \{ P(0) \leq 1 \} \]

\[ \Rightarrow N(\lambda) \sim \frac{\lambda^n}{(2\pi)^n} \cdot 2^n \text{vol}(P(0) \leq 1) = \frac{\lambda^n}{(2\pi)^n} \text{vol}(\{ \xi, \xi \mid P(\xi) \leq 1 \}) \]
Hörmander's general claim is that the error term is $O(\lambda^{-1})$. The method of proof uses the quantity

$$\text{Tr}(e^{-itP}) = \sum_{\lambda \text{ eigenvalue}} e^{-it\lambda} \frac{d\lambda}{\lambda} = \int e^{-it\lambda} \frac{dw}{dl} dl$$

which for $|t| < \varepsilon$ can be shown to be of the form

$$ln \int e^{-itP(x)} \frac{d^n x}{(2\pi)^n} + \text{ smooth } r(t)$$

(I am writing out the formula in the constant coeff. case)

WAIT. It seems that Hörmander's asymptotic solutions are of the form

$$e^{i(W(x,\xi) - tE(\xi))} [1 + a_1(x,\xi) + a_2(x,\xi) + \ldots]$$

where the second term is independent of $t$.

If I consider this in the case $P(x,D) = \sqrt{D^2 + q}$ on $\mathbb{R}$ then I am looking at an asymptotic solution of

$$(D^2 + q)u = \frac{\partial^2}{\partial x^2} u$$

of the form

$$u = e^{i\frac{\xi x}{2}} \left(1 + \frac{a_1(x)}{\frac{\xi}{2}} + \frac{a_2(x)}{\frac{\xi}{2}} + \ldots\right),$$

and I know such an asymptotic solution is unique up to multiplying by a series in $\frac{\xi}{2}$. So now the problem is to reconstruct $\delta(x)$ out of these modulo smooth functions. Thus I want to see that it is possible to find a formal series $f(i)$ such that

$$\int \frac{dx}{2\pi} e^{ix} \left(1 + \frac{a_1(x)}{\frac{\xi}{2}} + \frac{a_2(x)}{\frac{\xi}{2}} + \ldots\right) \equiv \delta(x) \mod \text{smooths}.$$ 

Now in fact the problem will be that $a_1(x) = cx$ so

$$\int \frac{dx}{2\pi} e^{ix} \frac{x}{\frac{\xi}{2}} \sim x \log x$$

has to be cancelled by the constant term in $a_2$. 
November 7, 1982:

There is an aspect of the Hörmander theory that I completely missed by looking at constant coefficient operators. Consider for example $P(D) = (D^2 + 8)^2$ is constant. Then

$$\langle x | e^{-itP(0)} | x' \rangle = \int \frac{d^n x}{(2\pi)^n} e^{i[\frac{1}{2} (x-x')^2 + tP(x)]}$$

is an exact formula which can be written in the form of an oscillatory integral

$$\int \frac{d^n x}{(2\pi)^n} e^{i[\frac{1}{2} (x-x')^2 - 18|t|]} \frac{e^{-it\sqrt{t^2 + 8} - 18|t|}}{\frac{1}{2} t^2 \frac{1}{18|t|} - \frac{t^2}{8} \frac{1}{18|t|^2} + \ldots}$$

However, Hörmander's representation is in the form

$$\int \frac{d^n x}{(2\pi)^n} e^{i[\frac{1}{2} (x-x')^2 - 18|t|]} \left[1 + a_1(x,x', \frac{t}{18} \right] + a_2(x,x', \frac{t^2}{18}, \ldots)$$

which means he is using asymptotic eigenfunctions for $P(D)$.

In general, given $P(x,D)$ of order 1 one can look at the eigenvalue equation

$$P(x,D)u = Eu$$

and construct asymptotic solutions as $E \rightarrow \infty$. Look for

$$u = e^{iW(x)} \left[1 + a_1(x) + \ldots\right]$$

satisfying

$$P(x,D)u = \rho E u.$$ 

So if $u = e^{iW(x)} v$, then

$$e^{-iW(x)P(x,D)e^{iW(x)} v = \rho E v$$

$$r \rho(x, dW) + \ldots$$

and so we get the HJ equation

$$\rho(x, dW) = E$$
I have already seen that I can do this in the case of the ordinary DE

\[(D^2 + \gamma)u = k^2 u\]

whence we get asymptotic solutions

\[u = e^{ik\xi} \left(1 + \frac{a(\xi)}{k} + \ldots\right).

Here's how this works for \(P(\xi) = \sqrt{\xi^2 + \gamma}\). Write \(\xi = n\omega\) with \(n > 0\), \(|\omega| = 1\), and then change variables \(\xi \longrightarrow \eta = \frac{n'}{n}\omega\), \(n' = \sqrt{n^2 + \gamma}\).

(In general you want the change of variables

\[\xi \longrightarrow \eta = \frac{P(\xi)}{|\xi|} - 1.)\]

The point is that this is a deformation for \(|\xi|\) large. In the specific example

\[
\int d^m\xi \quad e^{i \left[ \frac{m}{2} \Delta x - t \sqrt{\xi^2 + \gamma} \right]} = \int d^{m-1}n' \int_{|\omega| = 1} e^{i \left[ n\omega \cdot \Delta x - t \sqrt{n^2 + \gamma} \right]} d\mu_1(\omega)
\]

\[
= \int \left( \frac{\sqrt{n^2 + \gamma}}{\sqrt{n^2 - \gamma}} \right)^{m-1} \frac{n'}{n^2} d\nu_1 \int_{|\omega| = 1} e^{i \left[ \frac{\sqrt{n^2 + \gamma}}{\sqrt{n^2 - \gamma}} \omega \cdot \Delta x - t n' \right]} d\mu_1(\omega)
\]

\[
= \int \left( \frac{\sqrt{n^2 + \gamma}}{\sqrt{n^2 - \gamma}} \right)^{m-2} \frac{n'}{n^2} d\nu_1 \int_{|\omega| = 1} e^{i \left[ \frac{\sqrt{n^2 + \gamma}}{\sqrt{n^2 - \gamma}} \omega \cdot \Delta x - t n' \right]} d\mu_1(\omega)
\]

\[
= \int \left( \frac{\sqrt{n^2 + \gamma}}{|\eta|} \right)^{m-2} \frac{d^m \eta}{|\eta|} e^{i \left[ \frac{\sqrt{n^2 + \gamma}}{|\eta|} \eta \cdot \Delta x - t |\eta| \right]}.
\]

Now you see that it is possible to write the exponent \(i \left[ \eta \cdot \Delta x - t |\eta| \right]\), and the rest an asymptotic expansion in \(\eta\) whose coefficients are fun. of \(x-x'\).
Next I would like to see if these asymptotic eigenfunctions can be related to the resolvent. I want to use the formula

$$e^{-it\mathcal{P}} = \mathcal{F} \left\{ \frac{1}{\lambda - \mathcal{P}} \right\} e^{-it} \frac{d\lambda}{2\pi i},$$

and to compute \( \langle x|\frac{1}{\lambda - \mathcal{P}}|x'\rangle \) by \( \Psi \)00 methods, treating \( \lambda \) as having a certain homogeneity. In the case \( P(0) = 10^2 + 8 \), the exact formula is

$$\langle x|\frac{1}{\lambda - \mathcal{P}}|x'\rangle = \int \frac{d^nx}{(2\pi)^n} e^{i\mathbf{x} \cdot \mathbf{\Delta}} \frac{1}{\lambda - P(\mathcal{P})}$$

i.e. the full symbol of \( \frac{1}{\lambda - \mathcal{P}} \) is \( \frac{1}{\lambda - \sqrt{10^2 + 8}}. \) If you think of the \( \lambda \) as coming from \( \int d\tau \) for the elliptic operator \( \Delta + P, \) then the obvious expansion for the symbol \( \frac{1}{\lambda - \sqrt{10^2 + 8}} \) involves powers of \( \lambda - |\lambda| \) in the denominator. But

$$\int \frac{1}{(\lambda - |\lambda|)^m} e^{-it} \frac{d\lambda}{2\pi i} = \int e^{-i(\lambda + |\lambda|)t} \frac{d\lambda}{\lambda^n 2\pi i}$$

$$= e^{-i|\lambda|t} \frac{(-it)^{m-1}}{(m-1)!}$$

Introducing these factors of \( t \) that the Hormander method gets rid of.

I have since looked at various papers including the Gulliver-Dinntrott one. It seems that Hormander originally used

$$\langle x|e^{-it\mathcal{P}}|x'\rangle = \int \frac{d^nx}{(2\pi)^n} e^{i(W(x,x',3) - P(x',3)t)} a(x,y,x',\bar{y})$$

so that \( t \) does appear in the amplitude. None of the papers seemed to say that \( a \) could be taken independent of \( t, \) so perhaps there exists an obstruction.
Summary of the wave equation approach:

Let $A$ be a positive 2nd order elliptic operator e.g. a Laplacian $-\Delta$ or $-\partial_x^2 + q$. Then

$$\tilde{S}_A(s) = \frac{1}{\Gamma(s)} \int_0^\infty \frac{\text{Tr}(e^{-tA})}{t} t^{-s} \frac{dt}{t^k}$$

asymptotic exp. $t^{-\frac{n}{2}} [a_0 + a_1 t + ...]$ 

$$\sim \frac{1}{\Gamma(s)} \left[ \frac{a_0}{s - \frac{n}{2}} + \frac{a_1}{s - \frac{n}{2} + 1} + ... \right]$$

so $\tilde{S}(s)$ has at most simple poles at $\frac{n}{2}, \frac{n}{2} - 1, ...$ which for $n$ even get cancelled by those of the $\Gamma$-fn.

This is because the heat kernel has an asymptotic exp. in powers of $t$.

On the other hand

$$\tilde{S}_A(s) = \text{Tr}((A^2)^{-2s}) = \frac{1}{\Gamma(2s)} \int_0^\infty \frac{\text{Tr}(e^{-tA^2})}{t} t^{-2s} \frac{dt}{t}$$

so for $n$ odd when $A$ has a pole at say $-\frac{1}{2}, -\frac{3}{2}, ...$ then there must be a $t \log t, t^2 \log t$, etc. term in the asymptotic expansion for $\text{Tr}(e^{-tA^2})$.

\[ \text{How to understand the kernel } \langle x | e^{-itP}|x' \rangle, \]

where $P = A^{1/2}$. One constructs approximate, or asymptotic eigenfunctions for the operator $P(x, \partial)$ of the form

$$e^{i(W(x,x',\partial) - p(x',\partial) t)} \left[ 1 + q_1(x,x',\partial) t + ... \right].$$

Here $p(x,\partial)$ is the symbol of $P$ and $W$ is Hamilton's $W$-function, i.e. a solution of

$$p(x, \nabla W) = p(x', \partial)$$

such that $W = 0$ at $x'$ and $\nabla W = \frac{1}{\partial} \partial x'$. In other words I take the momentum $\frac{1}{\partial}$ at $x'$ and construct a family of trajectories near $x'$ of the same
energy, and use this to define the idea of momentum \( \xi \) near \( x' \). The only point I don't understand well is whether I can get enough approximate eigenfunctions to build up the \( \delta \) fun. \( \delta(x-x') \) at least mod smooth functions.

However, Hörmander proves that there is a change of variables \( \xi = \xi(x, \eta) \) such that (take \( x'=0 \))

\[
W(x, \xi(x, \eta)) = \eta \cdot x
\]

Then

\[
\delta(x) = \int \frac{d^n \eta}{(2\pi)^n} e^{i \eta \cdot x} = \int \frac{d^n \eta}{(2\pi)^n} e^{i W(x, \xi(x, \eta))} = \int \frac{d^n \xi}{(2\pi)^n} \frac{1}{\text{det}(\frac{\partial \xi}{\partial \eta})} e^{i W(x, \xi)}
\]

so that if one takes \( \alpha(x, \xi) = \frac{1}{\text{det}(\frac{\partial \xi}{\partial \eta})} \), then one has

\[
\delta(x) = \int \frac{d^n \xi}{(2\pi)^n} e^{i W(x, \xi)} \alpha(x, \xi)
\]

From this viewpoint the amplitude is fixed at \( t=0 \) and so to integrate the transport equations one needs an amplitude \( \alpha(x, t, \xi) \). However, I still would like to understand these approximate eigenfunctions better in connection with \(-\Delta^2 + g(x)\), and in particular whether the above difficulty is the same as encountered with KdV versus MKdV.

Continue with the summary. From Hörmander's repn.

\[
\langle x | e^{-itP} | x' \rangle = \int \frac{d^n \xi}{(2\pi)^n} e^{i W(x, \xi(x', \eta)) - \rho(x', \eta) t} \alpha(x, t, x', \eta) + \text{smooth}
\]

one gets a formula for the diagonal element.
\[ \langle x | e^{-itp} | x \rangle = \int \frac{d^n x}{(2\pi)^n} \, e^{-itp(x, \frac{x}{2})} \underbrace{a(\frac{x \times x}{4})}_{1 + a_1(x \times x) + a_2(x \times x) + \ldots} + r(t) \]

where \( r(t) \) is smooth. Now from this one can read off an asymptotic expansion in \( t \), e.g., look at
\[ \int \frac{d^n x}{(2\pi)^n} \, e^{-itp(\frac{x}{2})} \, t^k \, a(\frac{x}{2}) \]

where \( a(\frac{x}{2}) \) is homogeneous of degree \( -l \) for \( t \) out.

---

Do this radially:
\[ \int_{0}^{\infty} r^{n-1} dr \, e^{-itr} \, t^k \underbrace{\int a_l(\mu(w)) \, d\mu(w)}_{p(w)=1} \]

\[ \sim c \cdot r^{-l} \]

So this will be the same essentially as
\[ c t^k \int_{0}^{\infty} r^{n-1} dr \, e^{-itr} \, r^{-l} = c t^k \frac{\Gamma(n-l)}{i^n n-l} \]

except when \( l \) becomes \( n, n+1, \ldots \) when one gets log \( t \) terms.

The main virtue of the above expression for \( \langle x | e^{-itp} | x \rangle \) is that it valid in a interval \( |t| < \varepsilon \) and hence tells more about the spectral fn. The point is that it is smooth in the interval except at \( t = 0 \). The asymptotic expansion in \( t \) is equivalent to knowing its structure as a distribution, more precisely the expansion as a F.T. is given.
Finally because of the remainders term
\[ \langle x | e^{-it\mathbf{A}} | x \rangle = \int + R(t) \]
one knows nothing about the constant term in the asymptotic expansion, which makes the method uninteresting for the index them.

Remark: If \( n \) is even, then from the regularity of
\[ J(s) = \frac{1}{\Gamma(2s)} \int_0^\infty \text{Tr} (e^{-tA^{1/2}}) t^{2s} \, dt \]
for \( \text{Re}(s) > 0 \), one concludes that \( \text{Tr} (e^{-tA^{1/2}}) \) has no \( k \)-log\( t \) terms in its asymptotic expansion. Notice that it looks like one should be able to get values for \( J(s) \) at \( 0, \frac{1}{2}, -\frac{1}{2} \) from the terms in the asymptotic expansion, provided one had complete control over \( R(t) \).
November 8, 1982

Today I want to understand why and when \( \text{Tr}(e^{-tA}B) \) has an asymptotic expansion as \( t \downarrow 0 \).

Here \( A \) is a Laplacian and \( B \) is a PDO.

(Question: Is this a distribution in \( t \) with support in \( t > 0 \) in a natural way?)

Let's suppose \( A = \frac{i}{2}(i\partial)^2 \) on \( \mathbb{R}^n \), whence

\[
\langle x | e^{-tA} | x' \rangle = \frac{e^{-\frac{|x-x'|^2}{2t}}}{(2\pi t)^{n/2}} \Theta(t)
\]

and suppose

\[
\langle x | B | x' \rangle = \int \frac{d^n z}{(2\pi)^n} e^{i\frac{1}{2}z(x-x')} b(x, z)
\]

Then

\[
\langle 0 | B e^{-tA} | 0 \rangle = \int dx' \left( \int \frac{d^n z}{(2\pi)^n} e^{-i\frac{1}{2}z(x-x')} \right) \frac{e^{-\frac{|x-x'|^2}{2t}}}{(2\pi t)^{n/2}} \Theta(t)
\]

\[= \int \frac{d^n z}{(2\pi)^n} b(0, z) e^{-\frac{t|z|^2}{2}} \Theta(t) \]

Hence the essential problem is to understand when something like

\[
(*) \quad \int \frac{d^n z}{(2\pi)^n} e^{-\frac{t|z|^2}{2}} b(z)
\]

has an asymptotic expansion in \( t \) as \( t \downarrow 0 \), where \( b(z) \) is assumed to have an asymptotic expansion in \( \xi \) as \( |\xi| \rightarrow \infty \).

Important Observation: Normally when one works with PDO's one works with them modulo smooth operators, which means that one identifies symbols \( b(x, \xi) \) which have the same asymptotic expansion as \( |\xi| \rightarrow \infty \). So if I work in the usual framework, the integral (*) is determined modulo the
same integral with \( b(\xi) = O(1/|\xi|^N) \) for all \( N \).

What is it that one gets, what sort of function of \( t \)?

\[
\int \frac{d^n \xi}{(2\pi)^n} \ e^{-\frac{t}{2} |\xi|^2} b(\xi) = \int \mathcal{H} \ d\nu e^{-\frac{t}{2} \nu^2} \int b(\nu \omega) \ d\mu(\nu) \quad \text{for } |\nu| = 1
\]

Now \( \nu \rightarrow \int b(\nu \omega) \ d\mu(\nu) = \beta(\nu) \) is a smooth fun. of \( \nu \) which in the present case decays as \( \nu \to \infty \). It follows that

\[
f(t) = \int \mathcal{H} \ d\nu e^{-\frac{t}{2} \nu^2} \nu^{n-1} \beta(\nu)
\]

is a smooth function of \( t \) for \( t \geq 0 \), and it has its Taylor series at \( t = 0 \) for the asymptotic expansion.

(Notice that the fact that \( b(\xi) \) is smooth in \( \xi \) is not used.)

The important observation is that the normal ambiguity in \( b(\xi) \) leads to

\[
\int \frac{d^n \xi}{(2\pi)^n} \ e^{-\frac{t}{2} |\xi|^2} b(\xi)
\]

changing by a smooth function on \( t \geq 0 \). This would affect the value of a regularized trace (i.e., the coefficient of \( t^0 \)), but it does not affect the regularization process. Hence maybe these symbol methods can be used to get at things like the curvature.

In general we can write \( \bullet \) as a radial integral. (Change \( \xi \) to \( t \):

\[
\int d\nu e^{-tr^2} r^{n-1} \beta(r)
\]

where \( \beta \) is smooth. (If \( b(\xi) \) is smooth at 0, then \( \beta(r) \) will have a power series expansion in \( r^2 \) at \( r = 0 \).) Also \( \beta(r) \) has an asymptotic expansion as \( r \to \infty \).
Suppose $\beta(n)$ contains $r^k$ in its asymptotic expansion. Then
\[ \int_0^\infty \frac{dx}{n} e^{-tx^2} n^{-r} r^k = \int_0^\infty \frac{dx}{2n} e^{-tx^2} n^{-r} r^{n+k} = \frac{1}{2} \frac{\Gamma\left(\frac{n+k}{2}\right)}{\Gamma(n+k)} e^{t(n+k)/2} \]
which is fine for $k < -n$, but gives trouble for $k = -n$, $-n-2$, \ldots. These are exactly the places where the above formula would give powers $t^0$, $t^1$.

Earlier work on this question: 159-163, 196-200.

In this work I found the following examples where $\text{Tr} (e^{-t A K})$ had a $t^0$ term in the asymptotic expansion. Take $A^2 = A^2$ where $A$ is an odd-dual self-adjoint operator such as $\frac{1}{i} \partial_x + \sigma$.

Then in computing $\delta A(s)$ we encountered
\[ \text{Tr} \left( e^{-t A^2} \frac{1}{|A|^1} \delta A \right) \]
which gives a $t^0$ term. Also from
\[ \eta_A(s) = \text{Tr} \left( \frac{A}{|A|^1} |A|^{-s} \right) = \frac{1}{\Gamma(s)} \int_0^\infty \text{Tr} \left( e^{-t A^2 A} \right) t^{-s} dt \]
there would be no problem in showing $\eta_A(0)$ finite if one were to know for simple reasons that $\text{Tr}(e^{-t A^2 A})$ had only a $t^0$ term in its expansion.

Review: If $A = D^2 = -\nabla^2$, and $B \equiv B(x,0)$ has full symbol $B(x, \xi)$, then I saw that
\[ \langle 0 | B e^{-t A} | 0 \rangle = \int \frac{dx^n \xi}{(2\pi)^n} b(0, \xi) e^{-t |\xi|^2} \]
and I should be able to construct the asymptotic expansion in $t$ as $t \to 0$. To do this, I note the
Integral in the form
\[ \int_0^\infty r^{n-1} dr \ e^{-tr^2} \int b(r\omega) \ d\mu(\omega) \]
\[ |\omega|=1 \]
\[ \beta(n) \]

Now from the asymptotic expansion of \( b(\xi) \) we get,
\[ b(\xi) \sim \sum_{k=0}^{-\infty} b_k(\xi) \]
\( b_k \) homog of deg \( k \)
hence
\[ \beta(n) = \int b(r\omega) d\mu(\omega) \sim \sum_{k=0}^{-\infty} \lambda^k \int b_k(\omega) d\mu(\omega) \]
\[ |\omega|=1 \]
\[ \beta_k \]

Thus
\[ \langle 0 | B e^{-tA} | 0 \rangle = \int_0^\infty r^{n-1} dr \ e^{-tr^2} \beta(n) \]
where \( \beta(n) \sim \sum_{k=0}^{-\infty} \beta_k r^k \)

and from this you must deduce the asymptotic expansion.

Formally
\[ \int_0^\infty r^{n-1} dr \ e^{-tr^2} r^k = \int_0^\infty \frac{dr^2}{2r^2} \ e^{-tr^2} r^{2n+2k} \]
\[ = \frac{1}{2} \Gamma\left( \frac{n+k+1}{2} \right) \frac{1}{(t)^{n+k+1/2}} \]

so the homogeneity degrees giving trouble should be \( k = -n, -n-2, \ldots \) and these correspond to the powers \( t^0, t^1, t^2, \ldots \) which can’t be determined from the asymptotic expansion of \( \beta(n) \). So the conjecture is that I get an asymptotic expansion with terms \( t^k \log t \), \( k = 0, 1, 2, \ldots \) whose coefficients are simply expressed in terms of the \( \beta_k \).
Which kinds of $\psi DO's$ $B$ are going to give these logt terms? In particular I am interested in understanding this regularization method for Greens functions of $\overline{\delta}$-operators.

**Example:** Consider $\text{Tr}(e^{-tA}A^{-1})$ where $A=\Delta$ in the plane. Up to a constant we get

$$
\langle 0 | e^{-tA}A^{-1} | 0 \rangle = \int_0^\infty e^{-\frac{t}{4t}} \frac{1}{2\pi} \log r \, r \, dr \cdot 2\pi = \frac{1}{2\pi t} \int_0^\infty e^{-\frac{u}{4t}} \log u \, du
$$

$$
= \frac{1}{4\pi t} \int_0^\infty e^{-u} \log(u) \, du + \frac{1}{4\pi} \int_0^\infty e^{-u} \log(4u) \, du
$$

$$
= \frac{1}{4\pi} \log t + \text{constant}
$$

**Idea:** The sort of $b(\xi)$ that give a logt term in

$$
\int \frac{d^n k}{(2\pi)^n} e^{-t|k|^2} b(\xi)
$$

are those having a term homogeneous of degree $-n$. Such $b$ also cause log singularities in the Fourier transform

$$
\langle x | B(\omega) | 0 \rangle = \int \frac{d^n \theta}{(2\pi)^n} e^{i \frac{\xi}{\theta} x} b(\xi).
$$

Calculate

$$
F(x) = \int \frac{d^n \xi}{(2\pi)^n} e^{i \frac{\xi}{\theta} x} a(\xi)
$$

$$
= \int_0^\infty \frac{dr}{\pi} \int_{|\omega|=1} e^{i \omega \cdot x} a(\omega) \, d\mu(\omega)
$$

$$
\text{f}(\lambda x)
$$

Formally $F(\lambda x) = F(x)$. Notice that $\frac{\omega \cdot x}{|\omega|^2}$ as a fn.
on \{ |\omega| = 1 \} has two non-degenerate critical points, so by stationary phase

\[ f(x) \sim \frac{e^{in|x|}}{\lambda^{(n-1)/2}} (...) + \frac{e^{-in|x|}}{\lambda^{(n-1)/2}} (...) \]

so if \( n > 1 \), then the integral for \( F(x) \) converges at the \( n \to \infty \) end. I think it holds even for \( n = 1 \) because of the oscillatory factors.

At the \( n = 0 \) end the integral will diverge unless we have

\[ f(0) = \int_{|\omega| = 1} a(\omega) \, d\mu(\omega) = 0 \]

In this case \( F(x) \) should be a well-defined homogeneous function of degree 0.

Check tomorrow that (4) is independent of the norm \( |\lambda| \).
Homogeneous distributions. Let \( E_d \) be the space of tempered distributions on \( \mathbb{R}^n \) which are smooth except at 0, and which are homogeneous of degree \( d \). For example, a homogeneous function of degree \( d \) smooth except at 0 will be integrable near the origin if \( d > -n \), and so will define a homogeneous distribution. Let \( F_d \) be the space of homogeneous functions of degree \( d \) smooth except at 0. Then we have a map

\[
(1) \quad E_d \rightarrow F_d
\]

which is an isomorphism for \( d > -n \). A distribution in the kernel is one having support at 0 and is of the form \( P(0)\delta \), where \( P \) is a homogeneous polynomial. The \( \delta \) distribution has degree \( -n \). Hence the kernel of the above map is non-zero when \( d = -n, -n-1, \ldots \) and its dimension is given by a space of polynomials.

The Fourier transform gives an isomorphism

\[
(2) \quad E_d \sim E_{-n-d}
\]

hence a homogeneous distribution of degree \( d \) can be thought of as a homogeneous function if \( d > -n \) or the F.T. of a homog. fn. of degree \( -n-d \) if \( d < 0 \).

Let's now look at the F.T. by integrating radially.

\[
F(x) = \int \frac{d^n \xi}{(2\pi)^n} e^{i\xi \cdot x} f_d(\xi) = \int \frac{d^{n+d} r}{r} \int_0^\infty e^{in(\omega \cdot x)} f_d(\omega) d\mu(\omega)
\]

The last integral depends only on the function \( f_d \) restricted to \( S^{n-1} \), but \( F(x) \) for \( |x| = r \) also depends on \( d \). However, it is obviously analytic in \( d \), provided there are no convergence problems.
Conjecture: The map \( E_d \rightarrow F_d \) from smooth distributions to homog. fns. is an isomorphism for \( d = -n, -n-1, \ldots \). For \( d = -n-k \) a function \( f \in F_d \) is in the image if for any homogeneous polynomial \( P \) of degree \( k \) one has

\[
\int \frac{P(\omega)f(\omega)}{|\omega|^2} d\mu(\omega) = 0.
\]

Hence \( E_d \rightarrow F_d \) is of index 0 for \( d = -n, -n-1, \ldots \).

(Invariance of \((+)\): \( P(\chi)f(\chi) \frac{d^n\chi}{\chi^n} \) is an \( n \)-form on \( \mathbb{R}^n - \{0\} \) homogeneous of degree 0. \( \mathbb{R}^n - \{0\} \) is a \( \mathbb{R}^n \) principal bundle over \( S^{n-1} \). An analogous situation is a form \( f(x) \, dx \) on \( \mathbb{R}^2 \) which is invariant under translation in the \( y \) direction. I claim \( f(x) \, dx \) is invariantly defined on the base \( \mathbb{R} \). In effect under \( (x,y) \rightarrow (x, y+u(x)) \), the form goes to \( f(x) \, dx(dy + u' \, dx) = f(x) \, dx \, dy \), so the \( f(x) \) part doesn't change. The good way to describe this is to notice that one has the vector field \( dy \) along the fibers which is invariant for the action, hence \( i(dy) \, f(x) \, dx \, dy \) will come from the base.)

Review: A \( \delta \)-D kernel \( \langle x | \delta | 0 \rangle \) is a distribution smooth away from 0, and is given by

\[
\langle x | \delta | 0 \rangle = \int \frac{d^n\chi}{(2\pi)^n} e^{ix\chi} b(\chi)
\]

where \( b \) has an asymptotic expansion as \( \xi \to \infty \)

\[b(\xi) \sim \sum_{n=0}^{\infty} b_n(\xi)\]

where \( n \) = order of \( b \)

On the other hand I can ask if \( \langle x | \delta | 0 \rangle \) has an asymptotic expansion as \( x \to 0 \) in homogeneous fns.
\[ \langle x | B | 0 \rangle \sim \sum_{m=0}^{\infty} \beta_m(x). \]

One might hope that the Fourier transform of the homogeneous function \( b_k(\xi) \) would be \( \beta_m(x) \) for \( m = -n-k \). I have seen however that for \( k = -n, -n-1, -n-2, \ldots \) and hence \( m = 0, 1, \ldots \) there are difficulties with defining the Fourier transform for \( b_k(\xi) \), unless \( b_k(\xi) \) is orthogonal to the homogeneous polynomials of degrees \( m \) (see (+) on the preceding page). In the case that these orthogonality conditions are satisfied, the actual \( \beta_m(x) \) is unspecified up to a homogeneous polynomial of degree \( m \).

**Example:** If I take a \( \partial \)-operator \( \partial_{\xi} + \alpha \), then I can get its parametric in the form

\[ G(\xi, \xi') = \frac{\psi(\xi) \psi(\xi')^{-1}}{\xi - \xi'} + \text{smooth kernel} \]

and hence it is a sum of homogeneous functions of the form \( (\xi - \xi')^n \) for \( n \geq 0 \). On the other hand, the Laplacian in 2-dims has Green's func. \( \propto \log r \) which is not homogeneous.

What I would like to do next is to assume that \( \langle x | B | 0 \rangle \) has an asymptotic expansion as \( x \to 0 \) and show then that \( \langle 0 | e^{+\delta A} B | 0 \rangle \) has an asymptotic expansion in powers of \( \delta \). So I should look at

\[ \int e^{-\frac{|x|^2}{4\delta}} \frac{1}{(4\pi t)^n} \beta_m(x) \, d^n x \]

and begin with the case \( m > -n \) so there is no problem at \( x = 0 \). Clearly we can change variable \( x \to \sqrt{t} x \).
and get

\[
\frac{t^{-\frac{n}{2}} t^{m/2} t^{n/2}}{\Gamma(m/2)} \int e^{-\frac{|x|^2}{4t}} \beta_m(x) \, d^n x
\]

which is a power of \( t \) increasing with \( m \).

For \( m < 0 \) we can use the F.T. picture which is necessary to make sense of \( \beta_m \) as a distribution for \( m < n \) at least. Then

\[
\int e^{-\frac{|x|^2}{4t}} \beta_m(x) \, d^n x = \int \frac{d^m x}{(2\pi)^{m/2}} e^{-|x|^2/2} b_{-n-m}(x)
\]

\[
= t^{-\frac{n}{2}} t^\frac{m}{2} \int \frac{d^m x}{(2\pi)^{m/2}} e^{-|x|^2/2} b_{-n-m}(x)
\]

is also \( t^{n/2} < \text{const} \).

Therefore since heat kernels are Gaussian times smooth functions of \( x/t \), one concludes that \( \text{Tr}(e^{-tA}B) \) will have an asymptotic expansion in powers of \( t \) when \( B \) is a \( \text{Fredholm} \) of the good type.

Possible application: Take \( A \) to be a self-adjoint elliptic operator, say invertible. Then

\[
\eta_A(s) = \text{Tr} \left( \frac{A}{|A|^s} \right) = \text{Tr} \left( \frac{A}{|A|} (A^2)^{-s/2} \right)
\]

\[
= \frac{1}{\Gamma(s/2)} \int_0^\infty \text{Tr} \left( \frac{A}{|A|} e^{-tA^2} \right) t^{s/2} \, dt / A.
\]

Hence if \( \frac{A}{|A|} \) were of the good type, it would follow \( \eta_A(0) \) is defined. So therefore it should be possible to prove this directly by a symbol calculation.

But simpler perhaps is

\[
\eta_A(s) = \text{Tr} (A |A|^{-\frac{s+1}{2}}) = \frac{1}{\Gamma(s+1/2)} \int_0^\infty \text{Tr} (A e^{-tA^2}) t^{s+1/2} \, dt
\]
where the residue of $\eta_A(s)$ at $s=0$ is the coefficient of $t^{1/2}$ in
\[ \text{Tr}(Ae^{-tA^2}) \]

November 10, 1982

Let's try to understand the heat kernel $e^{-tA}$
when $A = -\Delta_x + g(x)$ on $\mathbb{R}$. Seely gets at this via
the contour integral formula
\[ \langle x | e^{-tA} | x' \rangle = \oint \langle x | \frac{1}{\lambda - A} | x' \rangle e^{-t\lambda} \frac{d\lambda}{2\pi i} \]
and a calculation of the resolvent by $\zeta$-00 techniques.
The contour goes around the spectrum of $A$ which
is mainly on the positive real axis, hence is nicely
parameterized by $\lambda = k^2$ where $k$ goes from $\infty + ia$
to $-\infty + ia$ and $a$ is sufficiently positive.

So making this change a variable gives
\[ \langle x | e^{-tA} | x' \rangle = \int \langle x | \frac{1}{k^2 - A} | x' \rangle e^{-tk^2} \frac{i}{\pi} k \, dk \]

\[ \infty + ia \]
Program: I need to understand the heat kernel $\langle x | e^{-tA} | x' \rangle$ well enough to calculate $\langle 0 | B e^{-tA} | 0 \rangle = \int dx \langle 0 | B (x') \rangle \langle x' | e^{-tA} | 0 \rangle$

\[
\int \frac{d^n \xi}{(2\pi)^n} e^{-i\xi x} b(0, \xi)
\]

\[
= \int \frac{d^n \xi}{(2\pi)^n} b(0, \xi) \int dx e^{-i\xi x} \langle x | e^{-tA} | 0 \rangle
\]

So what I want is to work with coordinates so that this last integral has the form $e^{-t(\xi')^2} f(\xi')$, where $f$ has a smooth asymptotic expansion as $|\xi| \to \infty$.

Now I believe the usual asymptotic expansion for $e^{-tA}$, $A = D^2 + g$ takes the form

\[
\langle x | e^{-tA} | 0 \rangle = \frac{e^{-\frac{\sqrt{x^2}}{4t}}}{(4\pi t)^{n/2}} \sum_{n=0}^{\infty} a_n(x) t^n + \ldots
\]

It maybe is possible in computing the Fourier transform of this to work with the formal power series of $F(tx)$ at $xt = 00$. So therefore I should first understand

\[
\int dx e^{-i\xi x} \frac{e^{-\frac{\sqrt{x^2}}{4t}}}{(4\pi t)^{n/2}} x^a
\]

for a given monomial $x^a$. But this I can calculate via

\[
\int dx e^{-i\xi x} \frac{e^{-\frac{|\xi|^2}{4t}}}{(\sqrt{4\pi t})^n/2} = e^{-t|\xi|^2}
\]

and differentiating with \( \xi \). These are just Gaussian moments.

So we conclude that corresponding to a term
$t^k x^\alpha$ in $F$ we expect a term

$$t^k \ e^{-t|\beta|^2} \ x (\text{polynomial in } t, \beta).$$

Hence what I would like to prove is that

$$\sum \left( \text{some sort of asymptotic formula} \right)$$

$$\int d^n x \ e^{-i \beta x} \langle x | e^{-tA} | 0 \rangle = e^{-t|\beta|^2} \sum t^k \ x^\alpha \ a_{k\alpha}.$$ 

I want then to be able to substitute this in (1) above to get

$$\langle 0 | B e^{-tA} | 0 \rangle = \sum a_{k\alpha} \ t^k \int \frac{d^n \xi}{(2\pi)^n} \ b(0, \xi) \ x^\alpha e^{-t|\xi|^2}.$$ 

However (2) is unreasonable because upon

slavish one gets

$$\langle x | e^{-tA} | 0 \rangle = \int \frac{d^n \xi}{(2\pi)^n} \ e^{i \xi \cdot x - t|\xi|^2} \sum a_{k\alpha} \ t^k \ x^\alpha$$

and the amplitude at the end, being independent of $x,$

is not the sort of this one would get by integrating

the transport equations. (Recall that if $u = e^{i \xi \cdot x - t|\xi|^2}$

then $(\partial_t + A)u = 0$ becomes

$$\left[ \partial_t - |\xi|^2 + (D_x + \xi)^2 + \delta(x) \right] v = 0.$$ 

or

$$\left[ \partial_t + 2 \xi \cdot D_x + D_x^2 + \delta(x) \right] v = 0.$$ 

Hence this can't be solved by a function $v(t, \xi),$ although it can formally be solved with a $v(x, \xi).$
Let's compute the resolvent \( \frac{1}{\lambda - A} \) where \( A = D^2 + q \) on the line. Proceed formally. Put

\[
P(x, D) = \sum_{n \geq 0} a_n(x, D) \frac{1}{(\lambda - D^2)^{n+1}}
\]

\[
= \frac{1}{\lambda - D^2} + a_1 \frac{1}{(\lambda - D^2)^2} + \ldots
\]

and ask that

\[
(\lambda - D^2) P(x, D) = 1.
\]

Then

\[
(\lambda - D^2) a_n \frac{1}{(\lambda - D^2)^{n+1}} = [\lambda - D^2, a_n] \frac{1}{(\lambda - D^2)^{n+1}} + a_n \frac{1}{(\lambda - D^2)^n}
\]

leading to the recursion relations

\[
[\lambda - D^2, a_n] + a_{n+1} = \square \quad 8\quad a_n
\]

or more simply

\[
a_{n+1} = [D^2, a_n] + 8a_n
\]

Notice that if we use

\[
\frac{(-s)(-s-1)\ldots(-s-k+1)}{k!} A^{-s-k} = \frac{1}{2\pi i} \oint \frac{1}{(\lambda - A)^{k+1}} \lambda^{-s} d\lambda
\]

then we get

\[
(D^2 + q)^{-s} = (D^2)^{-s} + (-s) a_1 (D^2)^{-s-1} + \frac{(-s)(-s-1)}{2!} a_2 (D^2)^{-s-2} + \ldots
\]

Thus we relate the expansion of the resolvent to the YDO expansion for \( A^{-s} \). Next use

\[
\frac{(-t)^k}{k!} e^{-tA} = \frac{1}{2\pi i} \oint \frac{1}{(\lambda - A)^{k+1}} e^{-t\lambda} d\lambda
\]

and one gets the expansion for the heat kernel

\[
e^{-\frac{t}{6}(D^2 + q)} = e^{tD^2} + e^{tq}
\]
\[ e^{-t(D^2+b)} = \left( 1 + (-t)q_1 + \frac{(-t)^2}{2!}q_2 + \ldots \right) e^{-tD^2} \]

Let's apply this expansion for the resolvent:

\[ \langle x \left| \frac{1}{\lambda-A} \right| \xi \rangle = e^{i\frac{\xi}{\lambda-A}} \left\{ \frac{1}{\lambda-A} + q_1(x,\xi) - \frac{1}{(\lambda-A)^2} + \ldots \right\} \]

to compute the heat kernel:

\[ \langle x \left| e^{-tA} \right| \xi \rangle = \oint d\lambda e^{-t\lambda} \frac{d\lambda}{2\pi i} \langle x \left| \frac{1}{\lambda-A} \right| \xi \rangle \]

\[ = e^{i\frac{\xi}{\lambda-A}} \left\{ e^{-t\frac{\xi^2}{2}} + q_1(-t)e^{-t\frac{\xi^2}{2}} + \ldots \right\} \]

\[ = e^{i\frac{\xi}{\lambda-A} - t\frac{\xi^2}{2}} \left\{ 1 + a_1(x,\xi)(-t) + a_2(x,\xi)(-t)^2 \right\} \]

which is exact what one obtains using \( \otimes \) above. Thus, Feller's method of constructing for the heat kernel is given by \( \otimes \) integrating the transport equations as a power series in \( t \).
Generalities about pseudo-differential operators: Composition formula:

\( (x) \quad P(x, \xi) \circ Q(x, \xi) = \sum_{\alpha} \frac{1}{\alpha!} \partial_x^\alpha P(x, \xi) \cdot \partial_x^\alpha Q(x, \xi) \)

on the total symbol level. Recall that the symbol of a pseudo-differential operator \( P(x, D) \) is the asymptotic expansion of \( P(x, \xi) \) as \( \xi \to \infty \) where

\( e^{ix \cdot \xi} P(x, \xi) = \langle x | P | \xi \rangle. \)

The total symbol is therefore a formal series of homogeneous functions:

\( P(x, \xi) = \sum \, P_i(x, \xi) \)

such that there are only finitely many \( P_i \) of degree \( \geq N \) for any \( N \). In the above formula, because differentiation \( \partial_\xi \) lowers the degree in \( \xi \) by 1, the right side is such a formal series assuming \( P, Q \) are:

Next start with an elliptic differential operator

\( P(x, D) = P_m + P_{m-1} + \cdots \)

where each \( P_i(x, \xi) \) is a homogeneous polynomial in \( \xi \) of degree \( i \). To construct on the symbol level a

\( Q = \beta_m + \beta_{m-1} + \cdots \)

such that \( P \circ Q = 1 \). I want to understand exactly what the \( \beta_j \) look like:

\[ (P \circ Q)_0 = P_m \beta_{-m} = 1 \]

\[ (P \circ Q)_1 = P_m \beta_{-m-1} + P_{m-1} \beta_{-m} + \partial_\xi P_m \partial_x \beta_{-m} = 0 \]
\[(P \circ Q)_2 = P_{m \xi} \xi_{m-2} + P_{m-1} \xi_{m-1} + P_{m-2} \xi_m\]

\[\frac{1}{2} \frac{D_x}{D_x} = \frac{1}{2} \frac{D_x^2}{D_x} \xi_m\]

Since \(P_m(x, \xi) \neq 0\) for \(\xi \neq 0\), \(\frac{1}{P_m}\) is a nice homogeneous function of degree \(-m\), so one can solve:

\[\xi_m = \frac{1}{P_m}\]

\[\xi_{m-1} = -\frac{1}{P_m} \left( P_{m-1} \xi_m + \xi_{m-1} \right)\]

From the recursion relations one sees that the terms \(\xi_{m-j}\) are obtained by the following processes: 1) differentiating with respect to \(\xi\), 2) multiplying by the \(\frac{D_x}{D_x} P_j(x, \xi)\) which are polynomials in \(\xi\), 3) dividing by \(\frac{1}{P_m}\). Since

\[D_x \left( \frac{1}{P_m} \xi \right) = -\frac{1}{P_m} \frac{D_x}{D_x} \left( D_x \xi_m \right) + \frac{1}{P_m} \frac{D_x}{D_x} \xi\]

it is clear that \(\xi_{m-j}\) is of the form \(\frac{u}{P_m}\) where \(u\) is a polynomial in \(\xi\). Another thing that follows is that if \(P_m\) is of the form \(A \xi_m + q_m\), where the variables are \(x_0, \ldots, x_n\) and everything else \((q_m, p_j)\) is independent of \(x_0, \xi_0\), then the \(\xi_{m-j} = \frac{u}{P_m}\) are independent of \(x_0\) and \(u\) is independent of \(\xi_0\).

Let's go back to the heat kernel \(e^{-tA}\) where \(A\) is a 2nd order operator like the Laplacian, and hence its symbol is a positive-definite form

\[a(x, \xi) = g^{\mu\nu}(x) \xi_\mu \xi_\nu\]
The above calculation gives a formal expansion
\[ \langle x | \frac{1}{\lambda - A} | \xi \rangle = e^{i \xi \cdot x} \left[ \frac{1}{\lambda - \alpha} + \frac{u_1(x, \xi)}{(\lambda - \alpha)^2} + \cdots \right] \]
where \( u_n(x, \xi) \) is a polynomial in \( \xi \). By taking a contour integral in \( \lambda \) we then get a formal expansion
\[ \langle x | e^{-tA} | \xi \rangle = e^{i \xi \cdot x - ta(x, \xi)} \left[ 1 - \frac{t}{2} a_1(x, \xi) + \frac{(t^2)}{2} a_2(x, \xi) + \cdots \right] \]
which in turn gives
\[ \langle x | e^{-tA} | x' \rangle = \frac{d^n x}{(2\pi)^n} e^{i (x-x') \cdot x - ta(x, \xi)} \left[ 1 + \cdots \right] . \]

Because the \( u_n(x, \xi) \) are polynomials in \( \xi \), it follows that the above integral involves moments of a Gaussian measure.

For simplicity suppose \( a(x, \xi) = \xi / |\xi|^2 \), i.e., the metric is conformally equivalent to the standard one. Then
\[ \int \frac{d^n x}{(2\pi)^n} e^{i \Delta x \cdot \xi - t a(x)|\xi|^2} = \frac{e^{-|\Delta x|^2}}{(4\pi t a(x))^n/2} . \]
I can get the moments of this measure by differentiating with \( \Delta x \), which brings down \( t \) factors.

Here is a nice observation in seeley’s paper. Suppose the symbol \( p_{m}(x, \xi) \) is a scalar, or more generally commutes with the other terms in the symbol and their derivatives. Then look at \( g^{-m-j}(\lambda, x, \xi) \) which is homogeneous of degree \(-m-j\) in \( (\xi, x)^n \). Also it can be written as a sum of terms
\[ \frac{u_{jk}(x, \xi)}{(\lambda - \rho_m)^k} \]
where \( k \leq 2j + 1 \) and \( u_{jk} \) is a polynomial.
in $i$ of degree $l$, where $l - km = -m - j$.
Now when we do the contour integral
$$\frac{\gamma^k}{(\lambda - \rho_m)^k} \rightarrow e^{-\lambda m \langle i \rangle} (-t)^{k-1} \frac{\gamma^k}{(k-1)!}$$
Now integrate over $i$ to get $\langle x | e^{-tA} | x \rangle$
and you get
$$\int \frac{d^n i}{(2\pi)^n} e^{-\lambda m \langle i \rangle} (-t)^{k-1} \frac{\gamma^k}{(k-1)!}$$
which gives $te^c$ where the exponent is
$$c = -\frac{n+l}{m} + k-1 = \frac{-n + \frac{(km-m-j)}{m} + k-1}{m} = -\left(\frac{n-j}{m}\right)$$
Thus the different homogeneous parts $g_{-m-j}(\lambda, x, \mathbf{i})$ of the symbol for $(\mathbf{A} - \lambda)^{-1}$ contribute the various coeffecs in the asymptotic expansion in powers of $t$.

Review: Viewpoint: I remain fundamentally suspicious of the seeley method because for the laplacean on a curved riemannian manifold, I expect a gaussian factor
$$e^{-\frac{A(x,x')^2}{4t}}$$
instead of the type of factor $e^{-\frac{g(x-x')}{t}}$ where $g$ is a quadratic function. In any case it seems to me that the seeley is limited to an asymptotic expansion for the diagonal part of the heat kernel. However, these objections might not be of any practical importance.
Virtues of this approach. It seems lie equivalent to knowing the full symbol of $A^{-s}$ as an analytic fun. of $s$. Thus from

$$\langle x | \frac{1}{\lambda - A} | \xi \rangle = e^{i \xi \cdot x} \left\{ \frac{1}{\lambda - a} + \frac{u_1(x, \xi)}{(\lambda - a)^2} + \ldots \right\}$$

one gets by contour integration

$$\langle x | A^{-s} | \xi \rangle = e^{i \xi \cdot x} a(\xi)^{-s} \left\{ 1 + (-s) a^{-1} u_1 + \frac{(-s) (-s - 1) a^{-2} u_2 + \ldots}{2} \right\}$$

as well as

$$\langle x | e^{-tA} | \xi \rangle = e^{i \xi \cdot x} a(x, \xi) \left\{ 1 + (-t) a_1 + \frac{(-t)^2 a_2 + \ldots}{2} \right\}$$
\[
\langle x \mid \frac{1}{\lambda - A} \mid \xi \rangle = e^{i \xi x} \left\{ \frac{1}{\lambda - \alpha} + \frac{u_1(x, \xi)}{(\lambda - \alpha)^2} + \ldots \right\}
\]

Plug this into
\[
A^{-s} = \frac{1}{2\pi i} \int \frac{1}{\lambda - A} \lambda^{-s} d\lambda
\]
to get
\[
\langle x \mid A^{-s} \mid \xi \rangle = e^{i \xi x} \left\{ a^{-s} + (-s)a^{-s-1}u_1 + \frac{(-s)(-s-1)a^{-s-2}u_2 + \ldots}{2!} \right\}
\]
Next you try to use this to get
\[
\langle 0 \mid A^{-s} \mid 0 \rangle = \int \frac{d^{n+1}\xi}{(2\pi)^n} \langle 0 \mid A^{-s} \mid \xi \rangle
\]
or more generally
\[
\langle x \mid A^{-s} \mid x' \rangle = \int \frac{d^{n+1}\xi}{(2\pi)^n} e^{i \xi(x-x')} \left\{ a(x, \xi)^{-s} + \ldots \right\}
\]
I want the diagonal terms: x' = x. Since
\[
\begin{aligned}
\frac{d^n}{(2\pi)^n} a(x, \xi)^{-s-k} u_{jk}(x, \xi)
\end{aligned}
\]
formally, \langle x \mid A^{-s} \mid x \rangle is a sum of terms of the form
\[
\int d^{n+1}\xi \ a(x, \xi)^{-s-k} u_{jk}(x, \xi).
\]
(Recall that
\[
\langle x \mid \frac{1}{\lambda - A} \mid x \rangle = \delta^{(2-j)}(\xi, \xi) + \delta^{(2-k)}(\xi, \xi) + \ldots
\]
where
\[
\delta^{(2-j)}(\xi, \xi) = \sum_{k=0}^{2j} \frac{u_{jk}(x, \xi)}{(\lambda - a(x, \xi))^{k+1}}
\]
and \[\deg(u_{jk}) = 2(k+1) = -2-j\] or \[\deg(u_{jk}) = 2k-j\]. Then
we have \[u_k(x, \xi) = \sum_{j=\lfloor k/2 \rfloor}^{2k} u_{jk}(x, \xi)\]. How can this be a polynomial? (Clear, because each \(u_{jk}\) is one.)
so look at

\[(*) \quad \int \frac{dn_{\xi}}{(2\pi)^n} a(x, \xi)^{-s-k} u_{j \xi}(x, \xi) \]

which has degree \(n+2(-s-k)+2k-j = n-2s-j\).

Now one can't integrate a homogeneous function, hence there is something wrong with the formal symbol for \(A^{-s}\) with \(s\). What actually happens is that we have

\[\langle x | A^{-s} | \xi \rangle = e^{i x \cdot \xi} \tilde{A}(x, s, \xi)\]

where \(\tilde{A}\) has an asymptotic expansion

\[\tilde{A}(x, s, \xi) \sim a(x, \xi)^{-s} + (-s) a(x, \xi)^{-s-1} u_1(x, \xi) + \ldots\]

as \(\xi \to \infty\), so it will be true that

\[\langle x | A^{-s} | x \rangle = \int \frac{dn_{\xi}}{(2\pi)^n} \tilde{A}(x, s, \xi)\]

To understand this, I should understand (*) with a cutoff around \(\xi = 0\), and also the case when \(\tilde{A}\) is rapidly decreasing in \(\xi\). In the last case one gets something analytic in \(s\). (?)

If one uses a cutoff in (*) of the form \(f(a(x, \xi))\)

when \(\xi = 0\) near \(0\) and \(\xi = 1\) far out, then we can do the integral radially

\[\int_{|\omega|=1} \int_{r=0} r^{n-1} dr \int_0^{2\pi} \theta_{j \xi}(x, \omega) \rho_{\omega} (a(x, \omega))^{-s-k} \rho_{2k-j} u_{j \xi}(x, \omega) \rho_{\omega} d\theta d\rho_{\omega} d\omega\]

\[= \int_0^{2\pi} \int_{|\omega|=1} r^{n-2s-j} \int_0^{2\pi} f(a(x, \omega)) a(x, \omega)^{-s-k} u_{j \xi}(x, \omega) \rho_{\omega} d\theta d\rho_{\omega} d\omega\]

entire function of \(s\)
Let's compare the Hormander construction of \( e^{-itA^{1/2}} \) with the Seeley construction of \( e^{-tA} \). Thus I first need to get at

\[
\langle x | e^{-tA} | 0 \rangle = \int \frac{d^n \xi}{(2\pi)^n} e^{ix \cdot \xi - tA(x, \xi)} \left\{ 1 + \cdots \right\}
\]

by a WKB method. Seems hard because you can't get \( i\frac{\xi}{2}x - \xi E(\frac{\xi}{2}) \) to be a solution of the H.J. equation, you need \( iW(x, \frac{\xi}{2}) - E(\frac{\xi}{2})x \).

Question for tomorrow:

\[
\langle x | e^{-tA} | x' \rangle = \int \frac{d^n \xi}{(2\pi)^n} e^{i\xi(x-x') - tA(x, \xi)} \left\{ 1 + (-t)u(x, \frac{\xi}{2}) + \cdots \right\}
\]

Doing the \( \xi \) integral in this Gaussian converts the polynomials in \( \xi \) in \( u_n(x, \frac{\xi}{2}) \) into

\[
\frac{e^{-\frac{(x-x')^2}{4tA(x)}}}{(2\pi tA(x))^{n/2}}
\]

which brings down powers of \( \frac{1}{A} \). When does one still get a power series in \( \frac{1}{A} \)?
Given an elliptic operator on a compact manifold $M$ one can tensor it with a vector bundle over $M$ and take the index. This gives us a map

$$K^0(M) \longrightarrow \mathbb{Z}$$

which one can interpret as capping with a class in $K_0(M)$ determined by the elliptic operator.

More generally if I have a family of elliptic operators on the fibres of $X \to Y$, then I get a map

$$K^0(X) \longrightarrow K^0(Y).$$

I feel that these ideas are susceptible to various refinements of a non-topological character involving connections and differential forms. What are some examples?

Let's begin with

$$K^{-1}(X) = K^0(S^1 X) = [X, U]$$

and try to describe the map $\otimes$ when $q = -1$.

Let's suppose then we have an $f: X \to U_1$. Then over $S^1 \times X$ we can form a twisted version of the symbol of the operator $D: E \to F$ over $X$ we begin with. Thus if $S' = [0, 1]/0 = 1$, we use $f$ to form M"obius bundle versions of $E, F$ over $S' \times X$. Now try to compute the index of this family. The index is a virtual bundle over $S' \times Y$. What I might try to do is to make a slightly more rigid version of this virtual bundle, which first suffers from the fact that only the symbol is really defined over $S' \times X$. If I really want a bona fide operator on $S' \times X$, I must join together $D$ and $f D f'$. So what might this mean? Take a linear path $D + t B$, understand how...
the index of $D + tB$ isn't changing, and combine this with the isomorphism of the index of $D$ with that of $fDF^{-1}$ given by $f$. Here the index means the virtual bundle $(\text{Ker}) - (\text{Cok})$ in some sense, and in particular one has the determinant line associated to it, which is precisely defined.

So it's clear now that I am getting exactly the stuff I looked at before.
Let us go over the heat kernel formulas with a view toward obtaining an expression for the anomaly. The idea I have is that I should be able to write down an expression for the current \( J_0 \) defined by

\[
\text{Tr} \left( (D^*D)^{-1} D^* \partial D \right) \bigg|_{s=0} = \int \text{tr}(J_0 \partial D)
\]

and then \([D, J_0]\) should be the measure which gives the index.

I will assume that \( D \) is a kind of Dirac operator, that is, its symbol is the same as

\[
\sum' g^k \xi_{k}^e
\]

where

\[
(\sum' g^k \xi_{k}^e)^* (\sum' g^k \xi_{k}^e) = \frac{i}{2} \gamma^e \quad \text{hence}
\]

\[
\sum' (g^k)^* g^e = \delta_{k}^e.
\]

For example, \( D = i \partial_x + \partial_y \) in 2dims. has symbol

\[
\begin{pmatrix}
\xi_1 + i \xi_2 \\
\xi_1 + i \xi_2
\end{pmatrix}
\]

and

\[
(\xi_1 + i \xi_2)(\xi_1 + i \xi_2) = \xi_1^2 + \xi_2^2.
\]

Let's now go over the calculation of the heat kernel for a 2nd order positive elliptic operator \( A \), whose top symbol is \( a_2(x, i) \).

\[
\langle x | A | \xi \rangle = e^{ix^2} \left\{ a_2(x, i) \right\} + \langle x | a_0(x, i) \rangle + \ldots
\]

We calculate the symbol of the resolvent formally

\[
\langle x | \frac{1}{\lambda - A} | \xi \rangle = e^{ix^2} \left\{ a_{-2}(x, \lambda, i) \right\} + \langle x | a_{-1}(x, \lambda, i) \rangle + \ldots
\]
where \( g_{-2-j}(x, i) \) is homogeneous of degree -2-j in \( x^{1/2}, \xi \). Recursion formula:

\[
a_2(x, i) g_{-2}(x, i) = 1
\]

\[
a_2 g_{-3} + a_1 g_{-2} + \partial_x^2 a_2 \cdot D_x g_{-2} = 0
\]

In general

\[
(\lambda - \sum a_i) \sum g_j = \sum \frac{1}{x!} \partial_x^i (x) \cdot D_x^{i+j} g_j = 1
\]

homog. of degree \( i+j-k+1 \).

So what is the structure of \( g_{-2-j} \)? You differentiate with \( x \) the earlier \( g_{-2-j} \) multiply by \( \xi \)-derivatives of the \( a \) and divide by \( \lambda - a(x, i) \). Hence

\[
g_{-2-j} = \sum \frac{u_{jk}(x, i)}{(\lambda - a)^{k+1}}
\]

\[\text{deg} u_{jk} = -2-j + 2(k+1) = 2k-j\]

Do the contour integral

\[
\langle x | e^{-tA} | x \rangle = e^{izx} \sum_{j,k} u_{jk}(x, i) e^{-t a(x, i)} \frac{(-t)^k}{k!}
\]

and then

\[
\langle x | e^{-tA} | x \rangle = \int \frac{d^n \xi}{(2\pi)^n} \sum_{j,k} u_{jk}(x, i) e^{-t a(x, i)} \frac{(-t)^k}{k!}
\]

\[
= \sum \frac{\kappa^{\frac{-n-j}{2} + \frac{1}{2}}}{\kappa^{\frac{n}{2} - \frac{1}{2}} (2k-j)} (t)^k \frac{(-1)^k}{k!} \int u_{jk}(\xi) e^{-a(\xi)}
\]

\[
= \sum \frac{\kappa^{\frac{-n+j}{2}}}{\kappa^{\frac{n}{2} + \frac{1}{2}}} \int \frac{d^n \xi}{(2\pi)^n} \sum_{k} \frac{(-1)^k}{k!} u_{jk}(\xi) e^{-a(\xi)}
\]

A question I had is whether \( \ldots \) in order to calculate this series one needs to know the \( \ldots \)
symbol of $\frac{1}{\lambda - A}$ or whether one can do with the symbol of $\frac{1}{A}$. So can $\langle x | e^{-tA} | x \rangle$ or at least the constant term be expressed in terms of $\langle x | \frac{1}{A} | x \rangle = e^{i\frac{x}{A}} \left\{ \sum (-1)^k u_{jk}(x, i) \right\}_{a^{k+1}}$

Here the $u_{jk}$ are not well defined, because we could multiply $\frac{u_{jk}}{a^{k+1}}$ in the numerator and denominator by $a$. As a test case try

$$\frac{(-1)^k a^{k-\frac{i}{2}}}{(\lambda - a)^{k+1}} = \frac{u_{jk}}{(\lambda - a)^{k+1}}$$

and note this is independent of $k$ when $\lambda = 0$. Then

$$\int \frac{d^n \xi}{(2\pi)^n} \frac{(-1)^k}{k!} u_{jk} e^{-a} = \int \frac{d^n \xi}{(2\pi)^n} \frac{1}{k!} a^{k-\frac{i}{2}} e^{-a}$$

$$= \int r^{n-1} dr \frac{1}{k!} r^{2k-j} e^{-r^2} \int d\mu(\omega)$$

$$= \frac{1}{k!} \int \frac{dr}{2\pi} r^{\frac{n+2k-j}{2}} e^{-r^2} c_j = c \frac{1}{k!} \Gamma \left( \frac{n+2k-j}{2} \right)$$

This obviously depends on $k$, unless $j = n-2$.

Now that I understand something about the heat kernel maybe I can understand the anomaly formula in general. Also I think it's true that the whole heat kernel can now be excised from the index thm. proof. Not quite - you still can't make sense of the diagonal values $\langle x | \frac{1}{\lambda - A} | x \rangle$. 

\[\]
Here's a little check with Seeley's work. If \( A \) were a \( \psi \)-DO, then in the recursion relation you get

\[
(a_2 \psi_{-2-j} + a_0 \psi_{-1-j} + a_1 \psi_{-j} + a_{-1} \psi_{1-j} + \cdots + a_{-j} \psi_{-2}) + \ldots
\]

and since \( \psi_{-2} = \frac{1}{(a-a)^2} \), \( \psi_{-2-j} \) can contain a term \( \frac{1}{(a-a)^2} \) term i.e.

\[
\psi_{-2-j} = \sum_{k=1}^{j} \frac{u_{ijk}}{(a-a)^{k+1}}
\]

instead of the \( 2k-j \geq 0 \). So the integral

\[
\int d^{n}\tilde{q} \ u_{ijk}(\tilde{q}) \ e^{-a(\tilde{q})}
\]

which gives trouble when \( -n = 2k-j \) or \( -\frac{n+j}{2} = k \), leads to a \( k \)-log \( k \) term in the asymptotic expansion for \( k \geq 1 \).
Proof of the anomaly formula: First suppose \( D \) is an invertible elliptic operator. I suppose we can regularize \( D^{-1} \) along the diagonal, i.e. \( \exists J \) such that

\[
\int J \cdot \delta D = \left[ \text{Tr} \left( e^{-tD^* D} D^{-1} \delta D \right) \right] \text{const. term}.
\]

Then take \( \delta D = [f, D] \). Then

\[
\int J \cdot [f, D] = \int [D, J] f
\]

\[
\text{Tr} \left( e^{-tD^* D} D^{-1} [f, D] \right) = \text{Tr} \left( D e^{-tD^* D} D^{-1} f - e^{-tD^* D} f e^{-tD^* D} \right)
\]

\[
= \text{Tr} \left( (e^{-tDD^*} - e^{-tD^*D}) f \right)
\]

On the other hand we know from the proof of the index thm. that

\[
\left[ \langle x | e^{-tDD^*} | x \rangle - \langle x | e^{-tDD^*} | x \rangle \right] = C_D(x)
\]

cst. term in \( t \)

is a measure one integrates to get the index of \( D \).

Hence combining one gets

\[
[D, J] = -C_D
\]

which is the anomaly formula.

Now there has to be a simple relation between the curvature of the line bundle and the anomaly formula given via the moment map.

Recall in general that if a Lie alg. of acts on \((M, h)\) preserving a connection on \( L \), then the action on \( L \) can be given by first order operators on sections of \( L \). Then \( K \) of
becomes the operator

\[ D_X \mathcal{F} = \nabla_X + \xi_X \]

where \( \nabla \) is the connection and \( \xi \) is the vector field induced by \( X \) on \( M \). The fact that \( X \) preserves the connection is

\[ [\nabla_X, \nabla_Y] = \nabla_{[X,Y]} \quad \forall \in \Gamma(T_M) \]

or simply

\[ Y\xi_X = \Omega(X,Y) \]

where \( \Omega \) is the curvature. This last formula says that

\[ d\xi_X = i(\xi)\Omega \]

so that \( \xi_X \) is a moment map for the action.

Now take the line bundle over operators \( D + B \) and the action of gauge transformations. The anomaly formula gives

\[ \psi_X(D) = + \text{Tr} \left( \left[ (D^*D)^{-s} - (DD^*)^s \right] X \right) \bigg|_{s=0} 
= + \int \omega_X \cdot X. \]

Where does this come from? If \( s \) is the canonical section, then

\[ (\nabla_Y s) \mathcal{F} = \Theta(Y)s \quad \Theta(Y)(0) = \text{Tr}^{\text{reg}}(0^{-1}Y) \]

Since \( s \) is preserved under the gauge action

\[ 0 = D_X s = (\nabla_X + \xi_X)s = (\Theta(X) + \xi_X)s \]

and hence

\[ \psi_X(D) = -\text{Tr}^{\text{reg}}(D^{-1}[X,D]) \]

\[ = \int \omega_X \cdot X \quad \text{by previous page} \]
To therefore the moment map is given by the measure $\omega_D$. Since I know $\varphi_X$, then I can use the formula

$$\gamma \varphi_X = \Omega(X, Y)$$

perhaps to determine $\Omega$. One gets

$$\Omega(X, Y)(0) = \varphi_X(0 + Y) - \varphi_X(0) \mod Y^2$$

$$= \int (\omega_{D+Y} - \omega_D) \cdot X \mod Y^2$$

but this would have to be written in terms of $X = [x, \omega]$. Not useful as it stands, because we lack information on $\omega_D$.

Next project: Find $\omega_D$ for a Dirac equation in $\mathbb{R}^n$ (in even) with arbitrary gauge potential.

Written: Consider a Riemannian manifold $M$. Then one can use a path integral over paths in $M$ to compute $e^{-tH}$ where $H$ is the energy ($= \frac{1}{2} \Delta$). Hence a path integral over all maps $S^1 \rightarrow M$ will give $\text{Tr} \left( e^{-tH} \right)$, however one can't hope this to be an integral over the fixed points like in the Duistermaat formula. The reason is that the Duistermaat formula involves

$$\int e^{-tH} \cdot \omega^n$$

where $\omega^n$ is the symplectic volume, whereas the path integral uses the Riemannian volume. The two are related by a Pfaffian.

So in the space of loops $S^1 \rightarrow M$ one considers a closed 2-form $\Omega(x, y) = \int (x, y) \, dx$
where \( \frac{\delta}{\delta x} \) denotes covariant differentiation. Then the vector field \( X : g(x) \to \dot{g}(x) \) gives rise to the energy function \( H(g) = \int_{\Omega} \frac{1}{2} |\dot{g}|^2 \, dx \) in the sense that
\[
dH = \iota(x) \Omega.
\]

Let \( Y \) be a tangent vector at \( g \), then
\[
(i(Y) \, dH)(g) = \int_{\Omega} H(g + Y) - H(g) \mod Y^2
= \int_{\Omega} \dot{g} \cdot Y = \Omega(X_g, Y).
\]

(Notice that in this example, because we start with the vector field \( X \), the form \( \Omega \) doesn't have to be non-degenerate.)

The generalization of the Duistermaat formula thus becomes
\[
\int e^{-tH(g)} \cdot Pfaff_\Omega \cdot (\text{Riem. volume})
\]
and the Pfaffian is to be realized by a fermion integral. Thus ghost fermions are to be put into the Lagrangian before getting the good quantum theory.

Next the restriction of \( H \) to the fibres is 0 and hence the result must be independent of \( t \).

Somehow what Witten gets is
\[
\text{Tr}_2 \left( e^{-tD^2} \gamma_5 \right) = \text{Index of Dirac operator}.
\]

In some general way, he is able to use super-symmetric ideas to view the standard elliptic complexes \((d, \delta, \text{maybe signature + Dirac})\) with the difference of traces in quantum mech. terms.
Review the Distermaat theorem: We have a circle action on \( M \) with infinitesimal generator \( X \), a closed 2-form \( \omega \) on \( M \) invariant under the action, and the moment map \( f: M \to \mathbb{R} \) defined by

\[
    df = i(X) \omega
\]

(assume the closed form on the right is exact). On \( S^{2n-1} \) we have the connection form \( \eta \) with \( i(X)\eta = 1 \) and \( \Theta(X)\eta = 0 \), and so \( d\eta \) is the generator for \( H^2(S^{n-1}) \). Then extend \( \omega \) to the equivariant form \( \tilde{\omega} = \omega + d(f\eta) = \omega + df\eta + f d\eta \); \( \tilde{\omega} \) reps a class in \( H^2_{S^1}(M) \). Cohomological first formula:

\[
    M^{S^1} \to M \xrightarrow{\pi} \quad \pi^* \omega = \sum \frac{i_p^*\omega}{e(\lambda_p)} \quad \text{in} \quad H^*_{S^1}([u^{-1}]).
\]

Apply this to \( \alpha = \tilde{\omega}^{n+k} = (\omega + df\eta + f d\eta)^{n+k} \). Since \( \eta^2 = 0 \) and \( M \) has even dimension

\[
    \pi^* \omega = \left[ \int_M \frac{f^k}{k!} \omega^n \right] (d\eta)^k
\]

so we get

\[
    (\int_M \frac{f^k}{k!} \frac{\omega^n}{n!} (d\eta)^k) = \sum \frac{f(p)^{n+k}}{(n+k)!} \frac{(d\eta)^{n+k}}{e(\lambda_p)}
\]

or using a generating fn:

\[
    \int_M e^{tf} \frac{\omega^n}{n!} = \sum \frac{e^{-t f(p)}}{e(\lambda_p)^n} (d\eta)^n
\]

Next we want to look at an infinite diml...
version of this. Let's take the examples we get from loop groups. Here one takes a compact connected Lie group $G$ and looks at connections on the principal bundle $S^1 \times G \to G$ with a given monodromy. This gives me a symplectic manifold, and we can ask about fixpts. For this example we should want to take a torus action. The trouble with this example is that the fixpoints will correspond to an affine Weyl group orbit, so I won't see the relation with the index, but rather with the Lefschetz fixpt. formula.

So let's try to bring in the Borel-Weil theorem. Over this symplectic manifold $M$ will be a line bundle with a holomorphic polarization, and the representation is the space of holomorphic sections. Now you want to apply the ABL-formula to determine the character of this representation. Now I know that the fixpts $P$ are all conjugate under the Weyl group, and the contribution of a fixpt should have the Weyl character formula denominator. Hence I should see what $\chi(y) \frac{1}{\text{det}(\tau)}$ amounts to in this case.

Setup: $T$ maximal torus inside of $G$, then $S^1 \times T$ acts on the connections with a given generic monodromy. A connection $\frac{dx}{2\pi i} A$ will be a fixpt where $A$ is constant and centralized by $T$, hence $A$ must be in $\text{Lie}(T)$, and $\exp(2\pi A) = \text{monodromy} \mu$. So if $\mu=(\mu)$ is diagonal, then the possible $A$ are $\left(\frac{1}{2\pi \log \mu_j} + \frac{1}{2} i\mathbb{Z}\right)$. Now I want to determine the tangent space to such a fixpoint. I am working with an orbit under the group $\mathbb{Y} = \text{Map}(S^1, G)$ with stabilizer $T$. So I know that the tangent space is $\text{Lie}(\mathbb{Y})/\text{Lie}(T)$, and so the different characters on $S^1 \times T$ are the affine roots essentially.
November 19, 1982

Last night I thought about the fact that field theory doesn't have a covariant Hamiltonian formulation. I know that single particle motion can be given a covariant Hamiltonian formulation as follows. Inside of the cotangent bundle is a hypersurface \( Y \)

\[
H(q, p, t) = \omega
\]

where the canonical 1-form on the cotangent bundle is
\[
\eta = p \, dq - \omega \, dt.
\]

If space-time has dimension \( d \), then the hypersurface has dimension \( 2d - 1 \). Inside the hypersurface is a 1-dimensional distribution which is the kernel for the symplectic 2-form \( \omega \) restricted to \( Y \). Integral curves of this distribution are the solutions of Hamilton's equations. Finally, Lagrangian submanifolds contained in \( Y \) are given by solutions of the Hamilton-Jacobi equation.

Let's next consider 2 independent particles moving in space-time. Think of the hypersurface (1) as giving the allowed possibilities of local energy-momentum for the particle at a given space-time point. So with 2 independent particles the allowed possibilities are \( Y_1 \times Y_2 \subseteq T^* \times T^* \).

In this situation, the Hamilton distribution is 2 dimensional and solutions of the Hamilton-Jacobi equations are generically described by Lagrangian submanifolds \( \mathcal{T}^* \times \mathcal{T}^* \) contained in \( Y_1 \times Y_2 \).

An interesting point is that upon quantizing this situation one gets a wave function \( \Psi(x_1, x_2) \) satisfying two Schrödinger type equations. In other
An interesting question is what an interaction between 2-particles looks like from this viewpoint. We can define an interacting system of 2-particles to be a codimension 2 submanifold \( Y \) of \( T^* \times T^* \) which is integrable. Integrable means that the space of functions vanishing on \( Y \) is closed under Poisson bracket, and then the vector fields belonging to these functions give rise to an integrable 2-diml distribution in \( Y \).

I need some simple examples of interacting particles and then I could decide perhaps whether it is possible to reduce things to the standard form of a single Hamiltonian \( H(x_i, p_i, x_j, p_j, t) \). Thus I should think of a \( Y \subset T^* \times T^* \) as being the zero set of a pair of Hamiltonian functions \( \tilde{H}_1, \tilde{H}_2 \) on \( T^* \times T^* \) with \( \{ \tilde{H}_1, \tilde{H}_2 \} = \tilde{f}_1 \tilde{H}_1 + \tilde{f}_2 \tilde{H}_2 \). These functions \( \tilde{H}_2 \) depend on \( x_i, p_i, t_i, \omega_i \). We assume that the distribution generated by the vector fields \( X_{\tilde{H}_i}, i = 1, 2 \) projects non-singularly on \( t_1, t_2 \) space. This means that given a trajectory of our system, which is a 2-diml manifold, we get two curves (world lines) in space-time representing the trajectories of the two particles. Because the vector fields \( X_{\tilde{H}_j} \) have indep. projections in the \( t_1, t_2 \) space, it follows that \( \omega_i \tilde{H}_j \) is non-singular, thus the normal bundle to \( Y \) projects non-singularly on the \( \omega_1, \omega_2 \) plane.
Consequently we can describe $Y$ by two Hamiltonian functions:

$$\omega_i = H_i(x_i, t; \dot{y}, p_j).$$

Next, I should write down the integrability condition. Maybe a better thing to do first is to look at the Hamilton-Jacobi equations:

$$(\star) \quad \frac{\partial S}{\partial t_k} + H_i\left(x_1, t_2, \frac{\partial S}{\partial x_1}, \frac{\partial S}{\partial x_2}\right) t_1, t_2 = 0$$

which are two first-order equations for $S(x_1, x_2, t_1, t_2)$. (Here $S$ is something like a wave function. Ultimately it should be the phase in a wave function.)

The simplest way to get two equations like $(\star)$ is to use vector fields. Thus I am going to assume $H_i$ is linear in the $p_j$.

Thus I get two vector fields

$$X_i = \frac{\partial}{\partial t_i} + \sum a_j(x_i, t_1, t_2) \frac{\partial}{\partial x_j}$$

which generate an integrable distribution, i.e. a foliation in $M \times M$.

I should first understand a foliation in space-time $M$. This is the thing you get from a 1-parameter family of diffeomorphisms of space

Like other Hamiltonians which are homogeneous of degree 1, this doesn't correspond to a typical mechanical system, because $S(x, t) \in M$, you can't prescribe
The momentum arbitrarily. Anyway this is the sort of trajectories we get when we look at the space-time Dirac equation.

Now the next point is that the leaves of the foliation in $M \times M$ must project into curves in the two factors in order that a leaf be considered a trajectory. So a leaf must be a product of two curves in $M$, which are time-like curves. I find that the leaves have the following description. Each leaf contains a unique point, such that $t=t'=0$. The leaf projects into two curves in $M$. One curve is $x = f(t)$ and the other is $x' = g(t')$, and so the leaf consists of all $(f(t), g(t'), t, t')$. Hence if I understand the trajectory $(f(t), g(t), t, t)$, then I recover the whole leaf.

Question: Does $I$ any relation with the eigenvalues of $g^{2n+1} = g \ldots g^{2n}$ on the Clifford module and the Gauss sign?
Local index formula over $\mathbb{R}^{2n}$ for the Dirac operator with coefficients in a vector bundle. Start with the space of spinors $S = S^+ \oplus S^-$ and the Dirac matrices $\sigma^\mu$ which are hermitian, of square $1$, and which anti-commute. Put $Q_5 = \pm 1$ in $S^\pm$ so that the $\sigma^\mu$ anti-commute also with $Q_5$ and hence have the form

$$\sigma^\mu = \begin{pmatrix} 0 & (\sigma^\mu)^* \\ (\sigma^\mu) & 0 \end{pmatrix}$$

where $\sigma^\mu : S^+ \rightarrow S^-$. Next take a vector bundle $E$ over $\mathbb{R}^{2n}$ with a hermitian metric and connection $\nabla$. Then the Dirac operator with coefficients in $E$ is

$$D = \frac{i}{2} \sigma^\mu \nabla_\mu \quad \text{on} \quad S \otimes E$$

and since it anti-commutes with $Q_5$ we have

$$D = \begin{pmatrix} 0 & D^* \\ D & 0 \end{pmatrix} \quad D : S^+ \otimes E \rightarrow S^- \otimes E.$$ 

I propose now to compute the local index expression

$$\left< x \right| e^{-tD^2} Q_5 \left| x \right>_{\mathbb{R}^{2n}} \bigg|_{t=0} = \left< x \left| e^{-tD^*D} \right| x \right> - \left< x \left| e^{-tD^*D} \right| x \right>_{t=0}$$

which should involve the character of $E$ calculated via differential forms from the connection $\nabla$. This should be done exactly by the feebly method using the fact that the top symbol of $D^2$ is $13/2$.

We must first understand traces of products of the $\sigma$-matrices. Think of $\mathbb{R}^{2n}$ as being $\mathbb{C}^n$. Better still to take the even dimensional Euclidean space $V = \mathbb{R}^{2n}$ and to describe the spinor representation of $\text{Cl}(V)$ we choose a "polarization" i.e. complex structure on $V$. Then the
spinor module is $\Lambda V$ and

$$C(V) = \text{End} (\Lambda V)$$

$$\mathcal{F} : \nu \mapsto e^\nu + \mathbf{i}^\nu$$

Now choose an orthonormal basis for $V$ and let the corresponding $\mathcal{F}$ operators be $\mathcal{F}^1, \ldots, \mathcal{F}^{2n}$. Then

$$\mathcal{F}^5$$

is

$$\mathcal{F}^{2n+1} = \mathcal{F}^1 \cdots \mathcal{F}^{2n}$$

Changing the orientation changes $\mathcal{F}$ to $-\mathcal{F}$ because the $\mathcal{F}^i$ anti-commute. Also

$$(\mathcal{F}^{2n+1})^2 = (-1)^{2n(2n-1)/2} = (-1)^n$$

One can check that eigenspaces of $S = \Lambda V$ under $\mathcal{F}^{2n+1}$ are $S^+ = \Lambda^{2n}$ even and $S^- = \Lambda^{2n}$ odd.

I want to know about traces of the form

$$\text{tr } (\mathcal{F}^1 \cdots \mathcal{F}^k \mathcal{F}^{2n+1})$$

and most of these should be zero. One can reduce to the case when $\mathcal{F}^1, \ldots, \mathcal{F}^k = 1, \ldots, 2k$. An odd $l$ gives an operator $\mathcal{F}^{2k+1}$ carrying $S^+$ to $S^-$ and hence 0 trace. Also because of the fact that the problem depends on the orthogonal structure of $V$ the actual values of $k, k_1, \ldots, k_k$ are irrelevant.

So use the decomposition $V = V_k \oplus V_{n-k}$ where $\mathcal{F}^1, \ldots, \mathcal{F}^{2k}$ come from an orthonormal basis of $V_k$. Then

$$\Lambda V = \bigoplus \Lambda^p V_k \otimes \Lambda^q V_{n-k}$$

the operator $\mathcal{F}^1 \cdots \mathcal{F}^{2k} \mathcal{F}^{2n+1}$ is essentially $(-1)^k \otimes \text{id}$ while $\mathcal{F}^{2n+1}$ is essentially $(-1)^{p+q}$ (essentially means factor of $\pm i$). Thus the product has the trace

$$\text{tr } (\mathcal{F}^1 \cdots \mathcal{F}^{2k} \mathcal{F}^{2n+1}) = \sum_{p} (-1)^p \dim \Lambda^p V_k \otimes \Lambda^q V_{n-k}$$

$$= \left( \frac{(-1)^k}{2^p} \right) \left( \frac{\text{vol}(S^n)}{2^q} \right) = 2^p \cdot 0$$
which gives 0 unless $k=n$. Thus

$$\text{tr} \left( y_{1j} \cdots y_{2n+1} \right) = 0 \quad \text{for} \quad 1 \leq i \leq 2n,$$

when $k \leq 2n$.

Here is an alternative way to understand the lemma. The Clifford algebra $C(V)$ of the real Euclidean space $V$ has a canonical increasing filtration such that $gr C(V) = \Lambda(V \otimes \mathbb{C})$ so what we are proving is that $\alpha \mapsto \text{tr}(\alpha Q)$, $Q = r_{2n+1}$, kills $F_{2n-1} C(V)$. So instead of using the basis $y^1, \ldots, y^{2n}$ for the space of operators $\mathcal{O}(V) \otimes \mathbb{C} < C(V)$ we can use the creation and annihilation operators, so I write the operators in normal product form

$$\psi_1^* \cdots \psi_a^* \psi_b \cdots \psi_{2n}$$

and want to compute the $Q$-trace, which after all is the alternating sum $\Lambda^k C(V)$. Clearly I need $a = b$ to have something of degree 0. Then I should think of $\psi_1^* \cdots \psi_a^*$ as giving an $a$-dim $W \subset V^*$ while $\psi_1 \cdots \psi_b$ gives an $a$-dim $W \subset V$. Generically I can suppose $W' \cong W^*$. In this case I get a splitting $V = W \oplus (W')^\perp$ and the computation of the trace proceeds as before.

Perhaps what’s important for future work is that Clifford algebra traces correspond to intersections. I’ve seen something like this in the Japanese work, and it also is part of the Lefschetz fixed-point business. Let’s review the Sato stuff.

We are interested in endomorphisms of $\Lambda V$. 
that is, elements of
\[ C(V_{ir}) = \text{End} (\Lambda V). \]

An interesting way to obtain such endomorphisms is by means of a correspondence:
\[ W \subset V \times V. \]

It would be better to write this
\[
\begin{array}{ccc}
  W & \xrightarrow{p_2} & V \\
  \downarrow{p_1} & & \\
  V
\end{array}
\]

and to assume \( p_1 \) is "oriented" in the sense that we trivialize \( \Lambda(p_1) = \Lambda(W)^* \otimes \Lambda(V). \) Then pull-back via \( p_1 \) leads to a map
\[ p_2^* : \Lambda V \longrightarrow \Lambda W \]

which can be followed by the usual \( (p_2)_* = \Lambda(p_2) \)

More generally, I can use the formula
\[ (p_2)_* \circ K \circ p_2^* \]
to associate an endomorphism to any \( K \in \Lambda W. \) But then one might as well make \( W = V \times V. \) In this way I am thinking of an isomorphism
\[ \Lambda(V \times V) \longrightarrow \text{End} (\Lambda V), \]

whereby subspaces correspond to endos.

It should be true then that the alternating sum of traces of all degree 0 endoms of \( \Lambda V \) corresponding to intersection of the subspaces with the diagonal. This relates a non-geometric trace operation to a geometric intersection.
Let's apply the lemma to prove a local index formula for the Dirac operator in flat space. We have

$$\psi = \frac{i}{2} \sum g^\mu \nabla_\mu$$

$$\psi^2 = - \sum_{\mu, \nu} g^\mu \nabla_\mu \sigma^{\nu \mu} \nabla_\nu = - \left[ \nabla^2 + \frac{1}{2} \sum g^\mu g^\nu \{\nabla_\mu, \nabla_\nu\} \right]$$

$$= - \nabla^2 - \frac{1}{2} \sigma F$$

$$\sigma^{\mu \nu} = \begin{cases} 0 & \mu = \nu \\ g^\mu g^\nu & \mu \neq \nu \end{cases}$$

Now look at the resolvent

$$\frac{1}{\lambda - \psi^2} = \frac{1}{\lambda + \nabla^2 + \frac{1}{2} \sigma F}$$

$$= \frac{1}{\lambda + \nabla^2} + \frac{1}{\lambda + \nabla^2} \left( -\frac{1}{2} \sigma F \right) \frac{1}{\lambda + \nabla^2} + \cdots$$

Let \( tr_\delta \) denote the spinorial trace. All these operators work on \( S \otimes E \) and we have

\[ \text{End} (S \otimes E) = \text{End}(S) \otimes \text{End}(E) \xrightarrow{tr_\delta} \text{End}(E) \]

Thus,

$$tr_\delta \left( \frac{1}{\lambda - \psi^2} Q \right) = \frac{1}{\lambda + \nabla^2} tr_\delta (Q) + \frac{1}{\lambda + \nabla^2} \left( -\frac{1}{2} \right) tr_\delta (\sigma Q) F \frac{1}{\lambda + \nabla^2}$$

and the first term to give a possible non-zero contribution is the \( n \)th. This is a \( \delta D \) of order \(-2(n+1)\) in \( \mathbb{R}^{2n} \), so it has a continuous kernel, and the diagonal values make sense. So I can at least conclude that

$$tr_\delta (\sigma^+) = tr_\delta (\sigma^-)$$

$$\langle x | tr_\delta \left( \frac{1}{\lambda - \psi^2} Q \right) | x \rangle = \langle x | \frac{1}{\lambda - \psi^2} | x \rangle - \langle x | \frac{1}{\lambda - \psi^2} | x \rangle$$

is a well-defined endomorphism of \( E \).

To understand the next point, let us put \( x = 0 \) and assume that \( \nabla^2 = \partial^2 \), \( F \) constant.
Then I have to understand
\[ (-\frac{1}{2})^n \text{tr}_S (\sigma F)^n Q \langle 0 \mid \frac{1}{(\lambda + q^2)^{n+1}} \mid 0 \rangle. \]

But
\[ \langle 0 \mid \frac{1}{(\lambda + q^2)^{n+1}} \mid 0 \rangle = \int_0^\infty \frac{d\xi}{(2\pi)^n} \frac{1}{(\lambda + |\xi|^2)^{n+1}} \]
\[ = \frac{\text{vol}(S^{n-1})}{(2\pi)^{2n}} \int_0^\infty \frac{\xi^{2n-1}}{\xi^2} d\xi \frac{1}{(\lambda + \xi^2)^{n+1}} \]
\[ = \frac{\text{vol}(S^{2n-1})}{(2\pi)^{2n}} \frac{1}{n} \int_0^\infty \frac{u^{n-1}}{(\lambda + u)^{n+1}} \frac{du}{u} \]
Atiyah talks on Witten's mod 2 anomaly.

Background: If $A$ is a skew-symmetric real matrix, then its eigenvalues are of the form $\pm i \lambda$ with $\lambda \in \mathbb{R}$. If $D$ is a real skew-symmetric elliptic operator (more generally Fredholm operator), then $\dim(\ker D) \mod 2$ is a homotopy invariant. This is because non-zero eigenvalues must disappear in pairs under specialization.

To next take a real skew-symmetric elliptic operators $
abla$ over the parameter space a real line bundle whose fibre at $D_A$ is the highest exterior power of $\ker (D_A)$. (Moreover, analytic torsion, better the $5$-form, $F^{-2}_A$ should give a metric on this line bundle.) I forgot to mention that this line bundle has a canonical section which vanishes when $D_A$ has $\ker \neq 0$.

Now we take the Dirac operators as examples. The Dirac operator written as a real operator looks like

$$D = \sum (\frac{i}{\sqrt{2}} \gamma^\mu) \gamma^\mu$$

where $\frac{i}{\sqrt{2}} \gamma^\mu$ are symmetric anti-commuting matrices of square $-1$, so that $D^2 = - \partial^2$. Now modify this by a gauge potential: $\partial^\mu \to \partial^\mu + A^\mu$ where $A^\mu$ is skew-symmetric, and then one can ask questions about gauge invariants.

General example: Given $D : E \to F$ an complex vector spaces I can consider the self-adjoint of $D^\dagger D$ in $E \otimes \bar{F}$ with the $Q_5 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ operator. Because $Q_5$ anti-commutes with $D$ it establishes isomorphisms between the $+1$ and $-1$ eigenspaces for $Q_5 \neq 0$. Also one knows that $\ker D \cong \ker D \oplus \ker D^\dagger$, so that $\dim(\ker D) = \text{Index} D \mod 2$.

Idea: We have the gauge group $G$ acting in the space of connections and we want to construct gauge invariant determinants for the operator $D_A$ associated to
a connection $A$. A determinant function is a trivialization of the determinant line bundle $L$ over $A$. A gauge invariant determinant is a trivialization of the descended line bundle over $A/\mathbb{G}$. The point is that $L$ is defined over $A/\mathbb{G}$ (modulo $J$ acts freely). So one can see that gauge invariant determinants don't exist by showing $L$ over $A/\mathbb{G}$ is non-trivial. Because of

$$H^0(\mathbb{G}, \mathbb{Z}/2) = \text{vanishing} \quad H^1(A/\mathbb{G}, \mathbb{Z}/2)$$

one might construct an example using a disconnected gauge group.

Actual talk:

Anomaly = quantity conserved classically but not when quantized.

1. Index theory: $M$ compact, $D: C^\infty(M, E) \rightarrow C^\infty(M, F)$

$$\text{index}(D) = \text{top formula}$$

family $D_t$ on compact space $T$. Then

$$\text{index}(D) \in K_0(T)$$

Real operators $D_t \in KO(T)$

Examples:

(a) $T = S^1$

$$\text{ind} \in KO(S^1) = (\mathbb{Z}/2) \oplus \mathbb{Z}$$

Assume $D$ gen. invertible

$$\omega_t(\text{Ind } D)$$

$a_t = \text{no. mod } 2$ of $t$, $D_t$ not invertible

(b) $T = S^2$

$$\text{ind} \in K_0(S^2) = \mathbb{Z} \oplus \mathbb{Z}$$

$$c_1(\text{Ind } D)$$

$c_1 = \text{alg. no.}$ of $t$, $D_t$ not invertible
more sophisticated
\[ \text{ind } D \in K_0(T) = \text{det}(\text{ind } D) \text{ real line bundle} \]

**\(\infty\)-determinants:** 
\[ \Delta \text{ Laplace type op. elliptic s.a. } > 0 \]

\[ \text{det } \Delta = \prod \lambda \text{ Regularize } \]

\[ j(s) = \text{Tr} (\Delta^{-s}) = \sum \lambda^{-s} \text{ Res } > \frac{\dim M}{\text{order } \Delta} \]

\[ j(0) = \text{dim Hilbert space} \]

\[ j'(0) = -\log \text{det } \Delta \]

Ex: 
\[ -\frac{d^2}{dx^2} \text{ on } S^1 \]

\[ j(s) = 2 \sum \frac{1}{n^{2s}} \]

\[ \frac{-d^2}{dx^2} + \alpha^2 \quad u^2 - x^2 \quad \frac{\text{Tr}(u_n^2 \cdot x^2)}{\text{Tr}(u^2)} = \frac{\sin \pi u}{\pi u} \]

\[ \text{det } D = (\text{det } D^\dagger D)^{1/2} e^{i\theta} \text{ can thus be done} \]

Assume \( \omega (\text{index}) \neq 0 \). Claim 1 no possible \( \text{det } (D_\xi) \in \mathbb{R} \) varying smoothly with \( \xi \).

Also complex case if \( \omega (\text{index}) \neq 0 \) over \( S^2 \) can't construct smooth \( \text{det } (D_\xi) \in \mathbb{C} \).

\[ M = S^4 \quad \text{Basic operator } \quad D : S^+ \rightarrow S^- \]

\[ G \rightarrow \text{Aut}(V) \quad G = SU_2, SU_3 \]

\[ M = S^4 \quad V = P \times^G V \]

Need connection \( \omega \) on \( P \)

\[ D_A : S^+ \otimes V \rightarrow S^- \otimes V \]

\( \omega \) = gauge transformations = id at \( \infty \).

\[ P = S^4 \times G \Rightarrow G = \Omega^4 G \]
\( a \to a/3 \quad \text{free action} \)

index \((D_{\alpha}) \in K(a/3)\)

\[ \beta_{\alpha} = \Omega^3 G \]

What does index mean? Say? Yet a map \( \Omega^3 G \to \mathbb{Z} \times \mathbb{R} \)

\( a \)

\[ G = SU(2) \quad V = \mathbb{C}^2 \quad P = M \times G \]

\[ \Omega^3(SU(2)) = \Omega^3(S^3) \to \Omega^3(S^3) \sim \mathbb{Z} \times \mathbb{R} \]

Claim the index of this family of Dirac operators is this Bott map. We know

\[ \pi_1(\Omega^3 SU(2)) = \pi_4(SU(2)) = \pi_4(S^3) = \mathbb{Z}_2 \sim \pi_1(BO) \]

In this example the Dirac op. over \( S^4 \) is quaternionic, but also \( SU(2) \) on \( \mathbb{C}^2 \) is quaternionic so the Dirac on \( S^4 \otimes V \to S^4 \otimes V \) is real.

\( b \)

\[ G = SU(3) \quad V = \mathbb{C}^3 \]

\[ \pi_2(\Omega^3 SU(3)) \to \pi_2(BU) \]

\[ \pi_5(SU(3)) \]

Condition for anomaly to disappear

\[ \sum \lambda_i^3 = 0 \quad \text{eigenvalue} \]
Consider a compact Kähler manifold $M$, a $\mathbb{C}^\infty$ vector bundle $E$ with metric over $M$, and let $\mathcal{A}$ be the space of holomorphic structures on $E$. These can be identified with certain connections of $E$ preserving the metric, namely, the ones whose curvature $F = \nabla^2$ is of type $(1,1)$, so that $\nabla$ induces a $\overline{\partial}$ operator.

Over $A \times M$ we obtain then a holomorphic bundle $\tilde{E}$ which is $pr_2^*(E)$ equipped with the appropriate varying holomorphic structure. Let's compute $c_\ast(pr_1, \tilde{E})$ using GRR. The conjecture I have is that this determinant line bundle equipped with the analytic torsion metric has curvature form given by the GRR formula on the differential form level.

I want to compute $\text{ch}(\tilde{E})$ on the differential form level. It's enough for calculating $c_\ast(pr_1, E)$ to work with a 1-ex. family. So I will suppose given a 1-parameter family of holomorphic structures on $E$ depending holom. in $t$.

Locally these can be described by $\tilde{\partial}$ as in the past

$$\tilde{\partial}_t : E \to E \otimes T^0_m.$$

Locally using coords $z^\mu$ on $M$ and a local trivialization of $E$ with metric we have

$$\tilde{\partial}_t = \left[ \partial_{\bar{z}}^\mu + \alpha^\mu_{\bar{z}}(t, z) \right] d\bar{z}^\mu$$

The $\tilde{\partial}$ on $\tilde{E}$ is going to be

$$\tilde{\partial} = \partial_t dt + \left[ \partial_{\bar{z}}^\mu + \alpha^\mu_{\bar{z}} \right] d\bar{z}^\mu$$

because we want holom. sections of $\tilde{E}$ to be a module over holom. funs. in $t$. The connection on $\tilde{E}$ is then

$$\nabla_{\tilde{E}} = \partial_t dt + \partial_{\bar{z}} d\bar{z}^\mu + \left[ \partial_{\bar{z}}^\mu - \alpha^\mu_{\bar{z}} \right] d\bar{z}^\mu + [\partial_{\bar{z}}^\mu, \omega] d\bar{z}^\mu$$
which can be written

\[ \nabla_{\tilde{E}} = d_t + \nabla_E \quad \text{on} \quad \tilde{E} = \rho_{t^*} E \]

Hence the curvature is

\[ \nabla_{\tilde{E}}^2 = \nabla_{\tilde{E}}^2 + (d_t \nabla_{\tilde{E}} + \nabla_{\tilde{E}} d_t) \]

or

\[ F_{\tilde{E}} = F_E + d_t A. \]

In local coords,

\[ \nabla_{\tilde{E}}^2 = (\partial_{\tilde{t}^\mu} x_\nu + \varphi_{\tilde{t}^\mu} x^*_\nu - [x^*_\mu, x_\nu]) \, dz^\mu \, d\tilde{z}^\nu \]

\[ + \partial_{\tilde{t}} x^*_\mu \, dt \, d\tilde{z}^\nu - \partial_{\tilde{t}^\mu} x^*_\nu \, dt \, d\tilde{z}^\nu \]

where \( \partial_{\tilde{t}^\mu} = 0 \) as we assume \( x \) is holomorphic in \( t \).

Then

\[ \text{ch} (\tilde{F}) = \text{ch}(e^F) = \text{ch}(F) \, \text{ch}(d_t A) \]

except maybe we should be careful because these are matrices.

Let's suppose first that we are working with line bundles. Then the curvature form takes values in the commutative ring of even forms; and so the above works. Thus we get in general

\[ e^{d_t A} = 1 + d_t A + \frac{1}{2} (d_t A)^2 + \ldots \]

\[ = 1 + d_t A + \frac{1}{2} \left( \partial_{\tilde{t}^\mu} x^*_\nu \partial_{\tilde{t}^\nu} x^*_\mu \right. \, dt \, d\tilde{t} \, d\tilde{z}^\mu \, d\tilde{z}^\nu \]

\[ + \left. \partial_{\tilde{t}} x^*_\mu \partial_{\tilde{t}} x^*_\nu \, dt \, d\tilde{t} \, d\tilde{z}^\mu \, d\tilde{z}^\nu \right] \]

so for line bundles

\[ e^{d_t A} = 1 + d_t A + \partial_{\tilde{t}^\mu} x^*_\nu \partial_{\tilde{t}^\nu} x^*_\mu \, dt \, d\tilde{t} \, d\tilde{z}^\mu \, d\tilde{z}^\nu \]
GRR gives

\[ c_1^b((p_1)_* F_{\tilde{E}}) = (p_1)_* \left( c_1(\tilde{E}) \cdot \text{Todd}(M) \right) \]

where \((p_1)_*\) is now to be interpreted as integration over the fibre \(M\). \(\text{Todd}(M)\) is a collection of forms of type \((p,p)\) over \(M\) in \(d\varepsilon, d\bar{\varepsilon}\). For line bundles, we have

\[ e^{\tilde{F}} = e^F e^{d\varepsilon A} \]

forms of type \((p,p)\) in \(d\varepsilon, d\bar{\varepsilon}\)

so it's clear that we get a contribution only from the \((d\varepsilon A)^2\) term, because an odd product of \(d\varepsilon, d\bar{\varepsilon}\) integrates to 0. Thus

\[ c_1((p_1)_* F_{\tilde{E}}) = \left\{ \int_M \frac{d\varepsilon^\mu}{\varepsilon^\mu} \frac{d\bar{\varepsilon}^\nu}{\bar{\varepsilon}^\nu} \frac{d\bar{\varepsilon}^\rho}{\bar{\varepsilon}^\rho} \right\} \cdot \left[ e^F \text{Todd}(M) \right]^{\frac{1}{2}} \frac{d\varepsilon}{(d\varepsilon A)^2} \]