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October 9, 1982

Problem: Let $f(x)$ be a distribution on $\mathbb{R}^n$ with compact support and with singular support $\{0\}$. Does

$$\int e^{-t|x|^2} f(x)\, d^n x$$

have an asymptotic expansion in powers of t as $t \to +\infty$? The answer is probably no because one can have $\log t$ contributions.

If I average over the rotation group in $\mathbb{R}^n$ then I get a 1-dimensional integral

$$\int_{-\infty}^{\infty} e^{-tx^2} f(x)\, dx$$

where $f(x) = f(-x)$. For the problems of interest $f(x)$ should, I think, have an asymptotic expansion

$$f(x) = \sum{}' a_k |x|^k \qquad |x| \to 0$$

in terms of homogeneous distributions. Hence it should be [redacted] enough to understand what happens for a homogeneous distribution.

If $k > -1$, then $f(x)dx$ is integrable near 0 and the following calculation is valid

$$\int_{-\infty}^{\infty} e^{-tx^2} |x|^k dx = [redacted] \int_0^{\infty} e^{-tx^2} x^{k+1} \frac{dx^2}{x^2} = \int_0^{\infty} e^{-tu} u^{\frac{k+1}{2}} \frac{du}{u}$$

$$= \Gamma\left(\frac{k+1}{2}\right) t^{-\left(\frac{k+1}{2}\right)}$$

So the interesting phenomena occurs first with the case

$$\int_{-\infty}^{\infty} e^{-tx^2} \frac{1}{|x|} dx,$$

where I have to specify what is meant by the distribution $\frac{1}{|x|}$. First note that a solution of $|x| f = 1$ as a distribution is unique up to adding a multiple of

$\delta(x)$, and this non-uniqueness will hold for $|x|^k$ with $k \leq -1$. Another point is that formally the integral

$$\int_{-\infty}^{\infty} e^{-tx^2} \frac{1}{|x|}\, dx$$

is independent of t, so we are getting another example of a regularization process which will not preserve the symmetries.

The distribution $\frac{1}{|x|}$ can be defined as follows: For $\varphi \in C_0^\infty(\mathbb{R})$ one sets

$$\int \varphi(x) \frac{1}{|x|}\, dx = \int \frac{\varphi(x) - \varphi(0)}{|x|}\, dx . \qquad ??$$

Actually it seems that the simplest general setup is to define the distributions $x^{s-1}\theta(x)$.

In order to make sense out of the distribution $x^{-1}\theta(x)$ it is enough to pick a function $f(x)$ with $f(0) = 1$ and require that

$$\int_{-\infty}^{\infty} f(x)\, x^{-1}\theta(x)\, dx = 0.$$

In other words the distribution is well-defined on the hyperplane of all $\varphi(x)$ satisfying $\varphi(0) = 0$, and hence you are choosing an extension.

Suppose then we have such an extension and put

$$F(t) = \int_0^\infty e^{-tx}\, x^{-1}\, dx$$

using this given extension. Then

$$\frac{d}{dt} F(t) = -\int_0^\infty e^{-tx}\, dx = -\frac{1}{t}$$

so

$$F(t) - F(t') = -(\log(t) - \log(t'))$$

So now it becomes clear that the simplest definition might be the one that gives

$$\int_0^\infty e^{-tx}\, x^{-1}\, dx = -\log t$$

because then one has a simple formula for the Fourier transform.

Let's go back to $A = \frac{1}{i}\partial_x + \sigma$ $\quad\sigma$ constant on $S^1 = \mathbb{R}/L\cdot\mathbb{Z}$. Then

$$\mathrm{Tr}\left(e^{-tB}\,\delta A\right) = \left\{\sum_k{}' \, e^{-t(k+\sigma)^2}\right\}\delta\sigma$$

$$\mathrm{Tr}\left(e^{-tB}B^{-1/2}\,\delta A\right) = \left\{\sum_k{}' \, e^{-t(k+\sigma)^2}\frac{1}{|k+\sigma|}\right\}\delta\sigma$$

Now we know $\displaystyle\sum_k e^{-t(k+\sigma)^2} \sim \int \frac{dk}{2\pi}\, e^{-t(k+\sigma)^2} = \frac{1}{2\sqrt{\pi t}}$

as $t \downarrow 0$; this follows from the transformation formula for the Θ fn. and the error decays exponentially. Analogously we expect

$$\sum_k{}' \, e^{-t(k+\sigma)^2}\frac{1}{|k+\sigma|} \sim \int \frac{dk}{2\pi}\, e^{-t(k+\sigma)^2}\frac{1}{|k+\sigma|}$$

which should diverge logarithmically in t, i.e. const. $\log t$. Actually it would be interesting to see an asymptotic expansion.

In any case if we assume $\mathrm{Tr}(e^{-tB}B^{-1/2}\delta A)$ has an asymptotic expansion with the term $-c_1\log t$, then

$$\delta\eta(s) = -s\,\mathrm{Tr}\left(B^{-\left(\frac{s+1}{2}\right)}\delta A\right) = -s\frac{1}{\Gamma(s/2)}\int_0^\infty \mathrm{Tr}\left(e^{-tB}B^{-1/2}\delta A\right)t^{s/2}\frac{dt}{t}$$

$$\sim -s\frac{(+c_1)}{\Gamma(s/2)}\underbrace{\int_0^1 (-\log t)\,t^{s/2-1}\frac{dt}{t}}_{\int_0^1 \frac{1}{t}\frac{t^{s/2}}{s/2}dt \,=\, \frac{1}{(s/2)^2}} \longrightarrow -\frac{s\,c_1}{\Gamma(\frac{s}{2})\,\frac{s}{2}\,\frac{s}{2}} = -2c_1$$

which resolves the earlier difficulties.

$$\int_0^1 (-\log t)^n t^{a+s} \frac{dt}{t} = \int_0^\infty u^n e^{-u(a+s)} du = \frac{\Gamma(n+1)}{(s+a)^{n+1}}$$

hence a term $(\log t)^n t^a$ in the asymptotic expansion for $f(t)$ as $t \downarrow 0$ produces a pole of order $n+1$ at $s = -a$ in the Mellin transform

$$\int_0^\infty f(t)\, t^s \frac{dt}{t} .$$

Next project: One knows from Atiyah-Singer-Patodi that the η-invariant occurs in the boundary situation. Specifically given a Riemannian $4k$ manifold X with boundary Y, then the η-invariant on the boundary is related to the difference of the integral of the L genus over X and the signature of X. Is this interesting in dimension 2 with the $\bar\partial$-operator?

I first have to understand boundary conditions for an elliptic operator, call it $D: E \longrightarrow F$ and suppose it is of first order. Near a boundary point we can choose coordinates t, x_i so that $t=0$ is the boundary and $t \geq 0$ is the manifold. $\sigma(D, dt)$ is then an isomorphism of E and F in a nbd. of the boundary, so we can suppose $E = F$ and $\sigma(D, dt)$ is the identity. Then

$$\sigma(D, -\omega\, dt + k_i\, dx_i) = -\omega + \sum k_i \sigma_i$$

at a point of the boundary. Ellipticity says this is an isomorphism if $(\omega, k) \neq 0$. This means that $\sum k_i \sigma_i = \sigma(k)$ is invertible for $k \neq 0$ and that for any $k \neq 0$, $\sigma(k)$ has no real eigenvalues.

Fix a $k \neq 0$ and look at the eigenvalues $+\omega$ of $\sigma(k)$. They fall into either the upper or lower half plane. Thinking in terms of plane waves solns. $e^{i(kx-\omega t)} \cdot v$, one sees that for ω in the lower half plane, this plane wave solution has time dependence $e^{at} = ||$ with $a<0$, hence it [redacted] is not L^2 as we move away from X. Thus in order that the index be [redacted] defined this sort of solution must be eliminated by the boundary conditions.

One of the simplest ways to guarantee that $\sigma(k)$ has no real eigenvalues is ~~to~~ require it to be skew-adjoint. Thus the operator at least on the symbol level has the form $\partial_t + A$ where $A = \sum \sigma_j \frac{1}{i}\partial x_j$ is self-adjoint. Following Atyiah, S + P let's suppose the [redacted] differential operator has this form near the boundary where A is independent of t.

I think that a general system of boundary conditions for the elliptic operator D selects for each $k \neq 0$ a subspace complementary to the gen. eigenspace for $\sigma(k)$ belonging to the eigenvalues in the LHP. For example we can take the gen. eigenspace belonging to all eigenvalues in the UHP. But then it seems the homotopy type of the boundary conditions is determined, so the index would be independent of the boundary conditions?? I need some examples.

October 10, 1982

What are appropriate boundary conditions for the operator $\partial_{\bar z} = \frac{1}{2}(\partial_x + i\partial_y)$ in the UHP? They would enable us to solve the inhomogeneous eqn.

$$\partial_{\bar z}\psi = f. \qquad f \in C_0^\infty(\text{UHP})$$

If we use the Fourier transform in x

$$\psi(x,y) = \int \frac{dk}{2\pi} e^{ikx}\psi_1(k,y)$$

then for each k we have to solve

$$\frac{i}{2}(\partial_y + k)\psi_1 = f_1$$

which is an ordinary DE. I use the L^2 bdry condition at $y = +\infty$. If $k > 0$, then the soln. e^{-ky} of the homog. eqn. satisfies this L^2 condition, hence must be eliminated by the boundary conditions. The only possible homogeneous b.c. is then

$$\psi_1(k,y) = 0 \qquad k > 0.$$

For examples take $f = \delta(z-z') = \delta(x)\delta(y-a)$ where $z' = ia$ with $a > 0$. Then

$$(\partial_y + k)\psi_1 = \frac{2}{i}\delta(y-a)$$

so

$$k>0 \qquad \psi_1(k,y) = \begin{cases} 0 & 0 \le y < a \\ \frac{2}{i}e^{-k(y-a)} & y > a \end{cases}$$

$$k<0 \qquad\qquad = \begin{cases} 0 & y > a \\ -\frac{2}{i}e^{-k(y-a)} & 0 \le y < a. \end{cases}$$

So

$$\psi(x,0) = \int \frac{dk}{2\pi} e^{-ikx}\psi_1(k,0) = \int_{-\infty}^{0} \frac{dk}{2\pi} e^{ikx+ka}\left(-\frac{2}{i}\right)$$

$$= -\frac{1}{\pi i}\frac{1}{ix+a} = \frac{1}{\pi}\frac{1}{x-ia} \qquad \text{and by}$$

analytic continuation

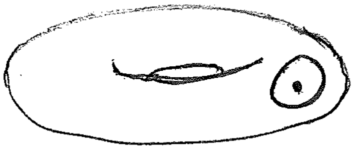
$$G(z,z') = \frac{1}{\pi(z-z')} .$$

Let's go over the programs and goals. I know various pieces of info. about the η invariant which I am trying to make precise and detailed. The first is that it is part of the determinant of a self-adjoint op. A. The second is that it is related to the index thm. in the interior, and possibly to the determinant of the operator in the interior.

Another example. Consider the cylinder $\mathbb{R}/\mathbb{Z}L \times \mathbb{R}$. Better, go back to the situation of the Hamiltonian $\frac{1}{i}\partial_x$ on $L^2(\mathbb{R}/\mathbb{Z}L)$ and the fermion quantization. Then for a Hamiltonian $H = \frac{1}{i}\partial_x + f(x,t)$ with f of compact support we got an amplitude $\langle 0|S|0\rangle$ which could be interpreted as the determinant of

$$\partial_t + \frac{1}{i}\partial_x + f .$$

Consider a Riemann surface (compact) with a small disk taken out. Then the punctured disk is isomorphic to the half cylinder $UHP/z \sim z+L$. I want to think of $\langle 0|S|0\rangle$ above as being a measure of the relative position of $\langle 0|$ which comes from the inside of the disk and $S|0\rangle$ which comes from the outside.



Take a closed Riemann surface M and holom. v.b. E over it. Fix a little disk and put a volume on M and metric on E which are completely flat over this disk. One then has the index thm. over M, namely

$$\chi = \int \text{curv.}$$

Now let's compare this with replacing the interior of the disk with a half-cylinder. This is roughly like writing the $\bar{\partial}$-operator as $\partial_t + \frac{1}{i}\partial_x$. Now the ℓ^2-type boundary conditions that are put on this cylinder are the same as requiring sections of E to vanish at the center of the disk. So the index for the ℓ^2 problem is changed by $\operatorname{rank}(E)$. Now the η invariant for $\frac{1}{i}\partial_x$ on the circle is zero but the zero eigenvalue has $\operatorname{rank}(E)$ for its multiplicity.

October 11, 1982

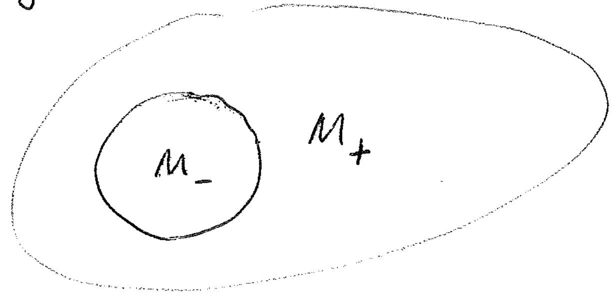
Let us discuss the boundary problem today. The idea here is that we give ourselves a Riemann surface with boundary M_+ and a C^∞ vector bundle E_+ over it. If I give a holom. structure on E_+ and suitable bdry conditions then I get an elliptic boundary value problem which has a finite index and presumably a theory of determinants.

The first thing to work out is the index theorem in this situation, after having made precise what is meant by boundary conditions.

Example: Suppose that M_+ is the complement of a small disk M_- in a closed Riemann surface M, and that E_+ is the restriction of an E given over M. Then a holomorphic structure on E over M_- should give us boundary values for the problem over M_+. Now what is the relation of the index of E over M to the index of E_+ over M_+ with these boundary conditions?

The kernel of the operator $\bar\partial : \Gamma(M_+, E) \longrightarrow \Gamma(M_+, E\otimes T^{0,1})$ with the boundary conditions should be $H^0(M, \Omega(E))$. In effect the kernel of $\bar\partial$ on $\Gamma(M_+, E)$ is all holomorphic sections of E over M_+ and if the section satisfies the b.c. it extends to all of M. Next let's determine the kernel. I know that if the holomorphic bundle E is modified at a finite number of points inside M_-, then we can make the H^0 and H^1 vanish. Hence with these modifications we get a Green's fn. with source at any point of M. This shows that with the modified boundary conditions we get that $\bar\partial : \Gamma(M_+, E)^\sim \longrightarrow \Gamma(M_+, E\otimes T^{0,1})$ is an isomorphism. Now the rest should follow by finite modifications.

The problem I want to understand is the nature of boundary conditions for the $\bar{\partial}$-operator. The point is that the boundary conditions lead to a finite index and probably a theory of determinants. Start with a closed Riemann surface M and C^∞ vector bundle E over it, and suppose that $\deg E / \operatorname{rank} E = g-1$. I take out a small disk M_- from M



Let's put a holomorphic structure $D: E \longrightarrow E \otimes T^{0,1}$ on E and suppose D is invertible. I am going to regard E, D restricted to M_- as giving boundary conditions for E, D over M_+. Thus I will try to solve

$$D\psi = f$$

over M_+, requiring ψ to coincide over ∂M_+ with a holomorphic section of E over $M_- \cup \partial M_+$. I suppose that it is enough to produce the appropriate Green's function at interior points of M_+, and I will ignore the boundary behavior (f supported in $\operatorname{Int} M_+$).

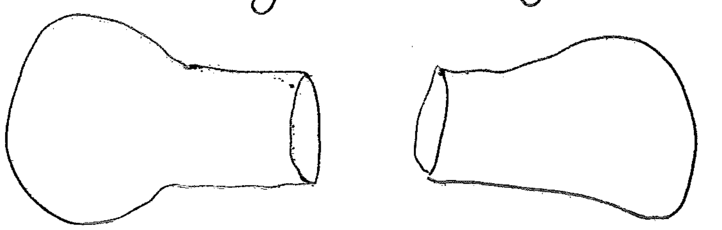
Because D is invertible I know the Green's function $G(z,z')$ is well-defined for all $z' \in M$, and clearly restricting to $z' \in \operatorname{Int}(M_+)$ gives us the Green's function for the b.v. problem.

As I vary the holomorphic structure of E over $\operatorname{Int}(M_-)$, I am changing the boundary conditions. Clearly the boundary conditions I have described are given by a subspace W of $\Gamma(\partial M_+, E)$. I want to describe the essential properties of this subspace which can be used as a generalization. Clearly in this small disk case, where we can choose a local coordinate z with

$$M_- = \{ z \mid |z| < 1 \},$$

we have that M_- is an outgoing subspace. (I can trivialize E over $\overline{M}_-$ in which case $\Gamma(\partial M_+, E)$ = holom. fns. [redacted] $S^1 \to \mathbb{C}^n$, $\Gamma(M_-, \mathbb{C})$ = holomorphic fns. $\{|z| \leq 1\} \to \mathbb{C}^n$.) As the holom. structure of E over M_- is varied, we get different outgoing subspaces of $\Gamma(S^1, E)$. Now we know that the loop group of maps $S^1 \to GL_n$, or better, the group of autos. of E over S^1, acts transitively on the set of outgoing subspaces, and in fact if we choose metrics then the ΩU_n acts simply-transitively.

On the other hand we can complete (M_+, E) in a different way than by a disk:



and this would give a different class of boundary conditions, i.e. subspace $W \subset \Gamma(\partial M_+, E)$. These W will not be stable under multiplication by z.

So the problem is now to pin down the kind of subspaces W that occur. Presumably W is given by the solutions of a pseudo-differential equation, and the symbol of this equation is fixed.

October 12, 1982 (Becky is 16)

I am still trying to understand boundary conditions for the $\bar\partial$ operator on a Riemann surface with boundary. I consider the simplest case of [redacted] removing a disk from a closed Riemann surface, and allowing changes of holom. str. inside the disk.

So I consider a $\bar\partial$ operator $\partial_{\bar z} + \alpha$ with α supported in $|z| < 1$. The boundary conditions [redacted] on S^1 that I get are given by the subspace

$$W = \left\{ f \text{ on } S^1 \;\middle|\; \begin{array}{l} f \text{ extends [redacted] over } |z| \leq 1 \text{ to a soln.} \\ \text{of } (\partial_{\bar z} + \alpha) f = 0 \end{array} \right\}$$

[redacted] Let's solve the equation

$$(*) \qquad \varphi^{-1} \partial_{\bar z} \varphi = \alpha$$

over $|z| \leq 1$. The reason I can do this is that the operator $\partial_{\bar z} + \alpha$ defines a holomorphic structure [redacted] on the trivial (n-diml.) bundle over $\mathbb{C}$, [redacted] and such a bundle is known to be trivial. Then clearly

$$W = \left\{ f \text{ on } S^1 \;\middle|\; \begin{array}{l} \varphi f \text{ extends [redacted] over } |z| \leq 1 \text{ to a} \\ \text{holomorphic fn.} \end{array} \right\}$$

$$= \varphi^{-1} H_+$$

Let us now use the fact that over S^1

$$\varphi^{-1} = \varphi_u \varphi_h$$

where φ_u has unitary values and $\varphi_h H_+ = H_+$ so that φ_h has an invertible holomorphic extension to the interior $|z| < 1$. Thus we conclude that I can solve $(*)$ such that φ is unitary over S^1, and moreover the solution is unique up to left-multiplying by a constant unitary matrix.

Let's return to our earlier treatment of the imaginary time Schroedinger equation

$$(*) \qquad \left(\partial_t + \tfrac{1}{i}\partial_x + f \right)\psi = 0$$

where $f(x,t)$ has compact support. To make the connection with the previous page I should perhaps use periodic boundary conditions in the x-direction. I have discussed how to interpret $\frac{1}{i}\partial_x + f$ as an operator on Fock space, and hence since the spectrum is bounded below, it should be possible always to solve the above equation in Fock space, at least in the t increasing direction.

I am interested in the amplitude $\langle 0|S|0\rangle$ which compares the boundary conditions for $t \ll 0$ and $t \gg 0$. $|0\rangle$ is the vector in Fock space corresponding to the subspace H_+ of $L^2(S^1)^n$, whereas $\langle 0|$ corresponds to H_-. Then $S|0\rangle$ is the subspace you get after propagating from $t \ll 0$ to $t \gg 0$.

The interesting point seems to be the following. One starts with the subspace of $L^2(S^1)^n$ containing all the $|k\rangle$ with $k \leq 0$. This gives me boundary conditions at some time $t \ll 0$, and it singles out a class of functions $\psi(x,t)$ namely those solutions of $(*)$ which are analytic and bounded for $t \ll 0$. For each t this space of global solutions of $(*)$ cuts out in $L^2(S^1)^n$ a subspace W_t which can be described as follows. Choose a φ satisfying

$$\varphi^{-1}\left(\partial_t + \tfrac{1}{i}\partial_x\right)(\varphi) = f \qquad \text{i.e. } \varphi^{-1}\partial_{\bar z}\varphi = \tfrac{2}{i} f$$

and then $W_t = \varphi_t^{-1} H_+$. Also φ should be bdd + analytic as $t \to -\infty$.

This function φ is like the propagator in the usual real-time situation. In the imaginary-time set-up one can't solve the Cauchy problem, hence φ is non-unique. However it becomes essentially unique if one requires that φ have unitary values at a fixed time t.

[redacted]

One of the nice things about the above is that we have associated to an imaginary time Schroedinger equation, something that looks the same as a real-time situation. For each t I get this subspace $W_t \subset L^2(S^1)$ and I know that $W_t = \varphi_t H_+$ where φ_t is unitary on $L^2(S^1)$ commuting with multiplication by e^{ix}. Thus one gets a path of unitary operators. In fact $\varphi_t = \{\varphi_t(x) \in S^1\}$ should be expressible in terms of φ.

What does this actually amount to? Suppose α given with support in $|z| < 1$ and that we choose φ such that

$$\partial_{\bar z} \log \varphi = \alpha$$

Then $W = \varphi^{-1} H_+$ inside $L^2($[redacted] $|z| = e^t)$. Now I have to take $-\log \varphi$ and split it into $\log \varphi_u$ which is purely imaginary and $\log \varphi_h$ which is analytic for $|z| \leq e^t$.

Lets first look at the case where $t > 0$ so that we have passed the support of α. Then $-\log \varphi$ is analytic and so has a Laurent series expansion

$$-\log \varphi = \sum_{n \in \mathbb{Z}} c_n z^n .$$

Clearly $\cancel{\log \varphi_h =}$

$$\log \varphi_u = \sum_{n<0} c_n z^n - \sum_{n<0} \overline{c_n}\, \bar z^n$$

and

$$+\log \varphi_h = -\log \varphi - \log \varphi_u$$

$$= \sum_{n \geq 0} c_n z^n + \sum_{n<0} \overline{c_n}\, \bar z^n$$

Let's go back to $z = x + it$ with $x \in \mathbb{R}/[redacted]\overset{L}{}\mathbb{Z}$, and put $-\log \varphi = \phi$ so that we want to solve

$$\partial_{\bar z}\phi = \frac{i}{2}\left(\partial_t + \frac{1}{i}\partial_x\right)\phi = \alpha .$$

Use F.S. in x.

$$\phi(x,t) = \frac{1}{L}\sum_k \phi_k(t)\, e^{ikx} \qquad \phi_k = \int_0^L e^{-ikx}\phi\, dx$$

whence

$$\frac{i}{2}\left(\partial_t + k\right)\phi_k = \alpha_k$$

so

$$\phi_k = \text{Const.}\, e^{-[redacted]kt} + \frac{2}{i}\int_{-\infty}^{t} e^{-k(t-t')}\alpha_k(t')\, dt'$$

Now [redacted] ϕ is unique up to adding ~~terms~~ $e^{ik(x+it)}$ with $k \le 0$. ($e^{ikx}e^{-kt} \to 0$ at $t \to -\infty$ when $k<0$). So the really important terms are the ϕ_k with $k>0$. The obvious thing to do is to solve $\partial_{\bar z}\phi = \alpha$ so that ϕ decays as $t \to +\infty$ and is bounded as $t \to -\infty$.

Once ϕ is found, then in the factorization $\phi = \log\varphi_u + \log\varphi_h$ we have

$$\log\varphi_u = \frac{1}{L}\sum_{k>0}\phi_k(t)\,e^{ikx} - \frac{1}{L}\sum_{k>0}\overline{\phi_k(t)\,e^{ikx}}$$

assuming $\log\varphi_u$ has no constant term. Now all we have to do is to differentiate $\log\varphi_u$ wrt t, in order to find how φ_u is determined from α.

But it is simpler to keep track of

$$\phi_+ = \frac{1}{L}\sum_{k>0}\phi_k(t)\,e^{ikx}$$

which is equivalent to $\log\varphi_u$. Then

$$\phi_+(x,t) = \frac{1}{L}\sum_{k>0} e^{ikx}\,\frac{2}{i}\int_{-\infty}^{t} e^{-k(t-t')}\underbrace{\alpha_k(t')}_{\int e^{-ikx'}\alpha(x',t')\,dx'}dt'$$

$$\phi_+(x,t) = \int_{t'<t} dx'dt' \; \underbrace{\frac{1}{L}\sum_{k>0} e^{ik(\Delta x)-k\Delta t}\,\frac{2}{i}}\; \alpha(x't')$$

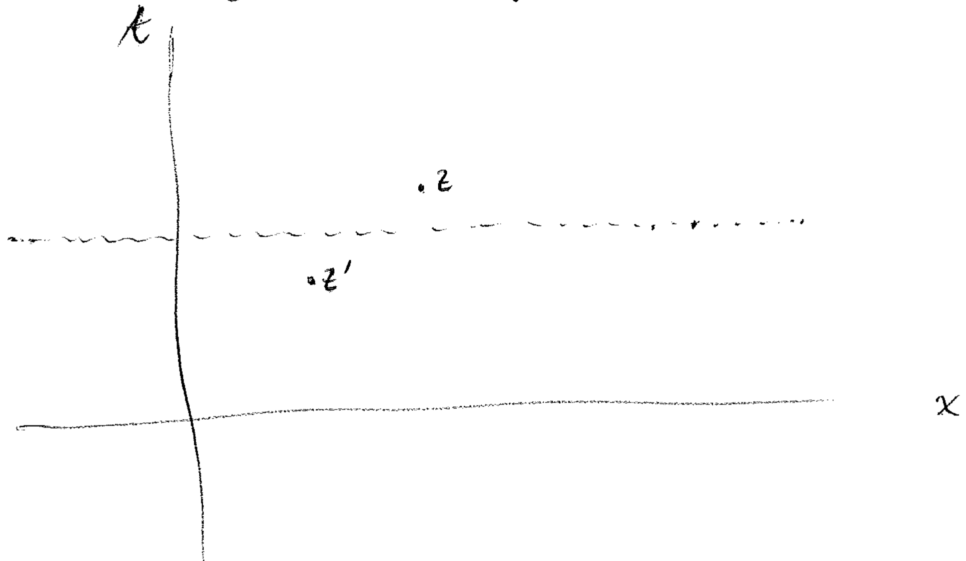
$$\xrightarrow{\text{as } L\to\infty} \int_0^\infty \frac{dk}{2\pi}\,\frac{2}{i}\,e^{ik(\Delta x+i\Delta t)} = \frac{1}{\pi\cdot\Delta z}$$

$$= \int_{t'<t} \frac{d^2z'}{\pi\,(z-z')}\,\alpha(z')$$

This is easy to check because if I combine it with

$$\phi_-(x,t) = \int_{t'>t} \frac{d^2z'}{\pi(z-z')}\,\alpha(z')$$

then I get the solution of $\partial_{\bar z}\phi = \alpha$ which decays [redacted] as $|t|\to\infty$. On the other hand [redacted] $\phi_+(z)$ on the line $\mathrm{Im}(z)=t$ is obviously the boundary value of a function analytic in $\mathrm{Im}(z)>t$.



Question: Does the energy have to be bounded below, or could it be unbounded but too few states exist to make any difference if the system is at a positive temperature? Mathematically it is possible for

$$Z(\beta) = \int e^{-\beta E}\,\sigma(E)\,dE$$

to converge for all $\beta > 0$ without the support of σ being bdd below. Here σ is supposed ≥ 0.

Let's treat the problem using the dominant term approximation:

$$\log Z(\beta) = -\beta E + \log \sigma(E)$$

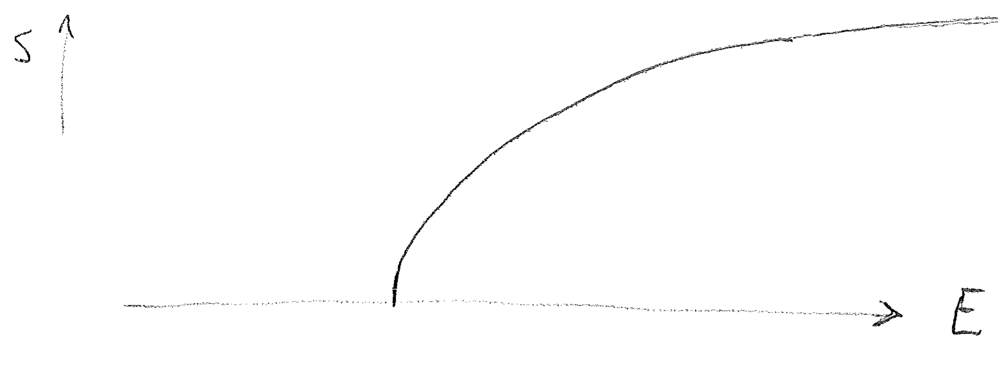
at the point E where the integrand is stationary:

$$\beta = \partial_E \log \sigma(E).$$

Actually we can work with the entropy $S = \beta E + \log Z$ instead of $\log \sigma$, and thus avoid approximations. One has

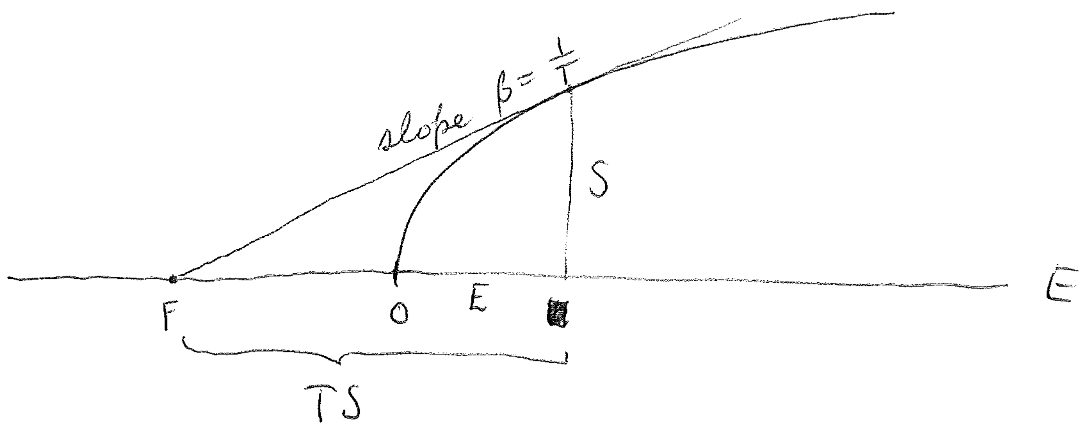
$$\beta = \frac{\partial S}{\partial E}$$

and because the temperature T increases with E we want $\frac{\partial^2 S}{\partial E^2} < 0$, so that S is concave downward. Also $S > 0$ with limit 0 as $T \downarrow 0$. This means that we have the following picture for the graph of S



There is no way for S to stay above 0 and yet be defined for all E. Hence there has to be a smallest energy which I can take to be zero

Picture



So one sees that $F = E - TS$ is negative and decreasing with increasing temperature. In fact

$$dF = \underbrace{dE}_{TdS} - dT\,S - T\,dS = -S\,dT$$

so that
$$S = -\frac{\partial F}{\partial T}$$

October 13, 1982

Let's start with a $\bar\partial$ operator on $\mathbb{C}$

$$\partial_{\bar z} + \alpha = \frac{i}{2}\left(\partial_t + \frac{1}{i}\partial_x + \frac{2}{i}\alpha\right)$$

where α has compact support, and where $z = x + it$. I would like to think of

$$\left(\partial_t + \frac{1}{i}\partial_x + \frac{2}{i}\alpha\right)\psi = 0$$

as being an imaginary-time Schroedinger equation with Hamiltonian (time-dependent)

$$H = \frac{1}{i}\partial_x + \frac{2}{i}\alpha$$

acting on $L^2(\mathbb{R})$. However because the $\bar\partial$ operator is not hyperbolic, one can't solve the Cauchy problem and construct the propagator $U(t,t')$ for this Schroedinger eqn.

If we pass to the Fock space associated to $L^2(\mathbb{R})$ and $\frac{1}{i}\partial_x$, then the propagator for the imaginary time Schroedinger equation should exist, and I think it is possible to describe its effect in the following way. One gets a basic family of vectors in the Fock space belonging to certain subspaces W of $L^2(\mathbb{R})$. I want to think of these subspaces as being boundary conditions for the $\bar\partial$-operator. Let's then consider a subspace W' at time t' and let $t > t'$. Then we look at holom. fns. for the $\bar\partial$-operator in the strip $t' \leq \operatorname{Im} z \leq t$ which belong to W' when restricted to $\operatorname{Im} z = t'$. Then the restrictions of such fns to $\operatorname{Im}(z) = t$ gives a subspace W which I think should be the effect of propagating W' to W by the imaginary-time Schroedinger equation in the Fock space.

Yesterday I had the following idea: The subspace W_t of $L^2(\mathbb{R})$ you get is outgoing in a suitable sense because the solutions of $(\partial_{\bar z} + \alpha)\psi = 0$ form a module over the holomorphic functions. Hence there is a unique unitary automorphism φ_t of $L^2(\mathbb{R})$ up to a scalar which commutes with the multiplication operators and which carries H_+ to W_t. Somehow today this looks less interesting since φ_t will depend on the choice of $W_t = H_+$ for $t \ll 0$. It doesn't seem to be intrinsically related to the $\bar{\partial}$-operator.

Things to understand: I seem to have now some feeling for boundary conditions for the $\bar\partial$-operator as a subspace W of $L^2(S^1)$ which corresponds to a vector in the Fock space. I **still want** to understand the symbol of W and connections with the index + determinants.

The symbol: Let us compute the symbol of the projection operator on W when $W = \varphi H_+$.

$$f(x) = \frac{1}{L}\sum_k e^{ikx}\int e^{-ikx'} f(x')\,dx'$$

$$Pf(x) = \int \Big\{ \frac{1}{L} \sum_{k<0} e^{ik(x-x')} \Big\} f(x')\,dx'$$

$$= \int \frac{1}{L} \frac{e^{-i\Delta x \frac{2\pi}{L}}}{1 - e^{-i\Delta x \frac{2\pi}{L}}} f(x')\,dx'$$

$$= \int \Big\{ \frac{1}{L} \frac{1}{e^{i\frac{2\pi}{L}\Delta x} - 1} \Big\} f(x')\,dx'$$

$$\longrightarrow \int \frac{dx'}{2\pi i} \frac{f(x')}{x - x' - i\varepsilon}$$

So next consider $\varphi : \mathbb{R} \to S^1$ such that $\varphi \to 1$ rapidly as $|x| \to \infty$. Then the kernel of $\varphi P \varphi^{-1}$ is

$$\frac{1}{2\pi i} \frac{\varphi(x)\,\varphi(x')^{-1}}{x - x' - i\varepsilon} = \frac{1}{2\pi i}\Big\{ \frac{1}{x - x' - i\varepsilon} + \varphi'(x')\varphi(x')^{-1} + \cdots \Big\}$$

The highest symbol of the operator $\varphi P \varphi^{-1}$ seems to be the same as that of P. The next term is obviously the one that counts since it gives the degree of φ.

October 14, 1982

Fix a Riemann surface M_+ with boundary and a smooth vector bundle E over it. Now we consider all pairs consisting of a holomorphic structure on E together with a set of boundary conditions. The set of these pairs is a complex manifold (inf. diml.) over which there should be a determinant line bundle [redacted] which is holomorphic. A natural question is whether there is a natural metric on this line bundles.

We consider the case where M_+ is obtained by removing a disk from a closed surface M and where the boundary conditions over M_+ come from different holomorphic structures over the disk. Let's suppose z is a local coordinate so that the disk is $|z|<1$, and trivialize E over the disk as a C^∞ v.b. with metric. [redacted] A holom. structure over the disk will be given by a $\bar\partial$-operator

$$\partial_{\bar z} + \alpha$$

where to simplify we suppose α has support inside $|z|=1$. We then trivial the holomorphic bundle over $|z|\le 1$ by solving

$$(\partial_{\bar z}+\alpha)\varphi = 0$$

with φ invertible. Then

$$\varphi^{-1}(\partial_{\bar z}+\alpha)\varphi = \partial_{\bar z} \qquad \text{or} \qquad \partial_{\bar z}+\alpha = \varphi\,\partial_{\bar z}\varphi^{-1}$$

and hence the subspace of boundary conditions is

$$W = \varphi H_+ \subset L^2(S^1).$$

So we get a map $\alpha \longmapsto W$ by means of which we can hope to "descend" our theory over compact Riemann surfaces to surfaces with boundary. This theory consists of the determinant line bundle $\mathcal{L}$ with canonical section s in the index 0 case, the metric on $\mathcal{L}$ given by analytic torsion, the curvature, the trivialization of the line bundle [redacted] giving rise to determinants, and the anomaly which measures the non-invariance of determinants under gauge transformations.

[redacted] Let's concentrate on the invertible case where the section s trivializes $\mathcal{L}$. Here everything starts from $|s|^2$ being defined as the analytic torsion [redacted] over M. The question is whether this depends only on W and the operator D over M_+. However a gauge transformation ψ of E with support in the disk will not change W [redacted] or D over M_+, yet we know by the anomaly business that $|s|^2$ is changed unless ψ is unitary. So we see that $|s|^2$ doesn't descend.

An idea would be to start with W and lift it to an α which is Yang-Mills [redacted] in some sense. The goal is to have α unique up to a gauge transformation which is unitary and hence preserves $|s|^2$. [redacted]

We know that $W = \varphi H_+$, where φ has unitary values on S^1 and is of degree 0. Thus φ can be extended over the disk to be unitary. Then $\varphi\, \partial_{\bar z}\, \varphi^{-1}$ has the boundary values W. φ is unique up to a unitary gauge transformation supported in the disk, so therefore it is clear that this gives a definition of $|s|^2$ independent of the choice of φ but depending only on W and the $\bar\partial$-operator inside of M_+. [redacted] (Actually φ on S^1 is only unique up to a constant unitary factor on the right, which may have an effect, but not for [redacted] rank 1 bundles.)

(Remark for later: $\bar\partial$-operators of the form $\varphi\, \partial_{\bar z}\, \varphi^{-1}$ [redacted] with φ unitary have curvature 0 because there are the same gauge orbit as $\partial_{\bar z}$. This gives a special class of $\bar\partial$-operators one can study the quantum mechanics of.)

Now I am in a position to make an interesting calculation. Let us take the cylinder $\mathbb{C}/\mathbb{Z}L$ and regard M_+ as the upper [redacted] half $t \geq 0$. Given a [redacted] $\partial_{\bar z} + \alpha$ with α supported compactly away from $t=0$ I have calculated that $\langle 0 | S | 0 \rangle = \det(\partial_{\bar z} + \alpha)$ with suitable conventions. I should really go over these calculations and get formulas for the analytic torsion that go with this determinant, which I think results from a specific trivialization of the determinant line bundle using the basepoint $\alpha = 0$.

One thing that becomes clear is that the determinant [redacted] in the boundary case will depend on more [redacted] than just the subspace W. The line bundle will not be trivial over $\{W\}$ and [redacted] we will need to lift W to a vector u_W in the line attached to W in Fock space before we can expect a determinant number to be defined. This is clear from the formula $\langle 0|S|0\rangle$ for the global determinant. The vector $u_W = U(0,-\infty)|0\rangle$ represents the boundary conditions at $t=0$, and the part $\langle 0|U(\infty,0)$ depends on the $\bar{\partial}$-operator on the part M_+ where $t \geqslant 0$. So the determinant function in the boundary situation is $\langle 0|U(\infty,0)|u_W\rangle$.

October 16, 1982

[redacted] I want to make a list of all the unanswered questions connected with determinants of $\bar\partial$ over closed Riemann surfaces.

The basic problem concerns line bundles where there should be complete formulas for the ζ-function determinant, & Θ function. These are [redacted] needed for Arakelov – Faltings. The idea is that for Yang–Mills connections the answers are easy and the general case can be deduced from this by anomaly formulas.

Then there is the case of $\mathbb{C}$ with an operator $\partial_{\bar z} + \alpha$ with α of compact support where the Θ-determinant is $\langle 0|S|0\rangle$. I need to have formulas for the ζ-fn determinant which is an <u>absolute quantity</u> over a compact [redacted] surface, and its <u>infinite volume</u> limit.

Let's fix a line bundle L of degree $g-1$ over M and equip L with a metric and M with a volume. Then over the space $\mathcal{A}$ of holom. structures on L = connections on L we have the function

$$|S|^2 = e^{-\zeta'_{D^*D}(0)} \qquad D \in \mathcal{A}.$$

which I would like to compute explicitly.

Here is an approach to this problem. [redacted] The complex gauge group acts on $\mathcal{A}$ with quotient the Jacobian of line bundles of degree $g-1$. It acts by translations: $\varphi D \varphi^{-1} = D + \varphi(\bar\partial \varphi^{-1})$. The quotient by the identity component is a complex vector space of dim g, isomorphic to $H^1(\mathcal{O})$. Suppose I fix a basept D_0 for $\mathcal{A}$. Then I can identify $\mathcal{A}$ with $C^\infty(T^{0,1})$ and the action of the [redacted] complex gauge transf. by translation

$$D_0 + \alpha \longmapsto D_0 + \alpha - \bar\partial \log\varphi$$

and $\log \varphi$ for φ in the identity component is any fn. on M. Now we have the formula

$$e^{-\zeta'(0)} = e^{-\|\alpha\|^2} \left| \det(D; D_0) \right|^2$$

and I think I have good control over how $\det(D; D_0)$ behaves under gauge transformations. Hence I should be able to see how $e^{-\zeta'(0)}$ changes under gauge transformations.

The idea is that

$$\log \left(\det(\underbrace{\varphi D \varphi^{-1}}_{D - \bar\partial \log \varphi}; D_0) \Big/ \det(D; D_0) \right) = \text{linear fn. of } \bar\partial \log \varphi.$$

so if I put $F(D) = \log \det (D; D_0)$ I see that

$$F(D - \bar\partial \log \varphi) - F(D) = \text{linear fn. of } \bar\partial \log \varphi$$

and hence $F(D)$ is a quadratic fn. of D.

Formally if

$$F(\alpha) = \tfrac{1}{2} \alpha^2$$

then

$$F(\alpha) + \overline{F(\alpha)} - |\alpha|^2 = \frac{1}{2}\alpha^2 + \frac{1}{2}\bar\alpha^2 - \alpha\bar\alpha$$

$$= \frac{1}{2}(\alpha - \bar\alpha)^2$$

which is constant in the real directions.

This is the sort of situation you want to see on any complex gauge orbit. The idea is that the complex gauge orbit is a complex affine space and the determinant function $F(D) = \log\det(D; D_0)$ is quadratic. Then $F(D) + \overline{F(D)} - \|D\|^2$ is constant in the "real" directions with are the orbits of the gauge group.

Notice that this picture completely determines the behavior of $e^{-\zeta'(0)}$ up to a constant on the complex gauge orbit. It is completely equivalent to variational calculations based on the anomaly formula, and so makes sense even when the eigenvalues are not discrete.

So let's see what this means over $\mathbb{C}$ for the operators $\partial_{\bar z} + \alpha$ where $\alpha \in C_0^\infty(\mathbb{C})$. I recall my calculation

$$\log\det(\partial_{\bar z} + \alpha) = \log\det(1 + \partial_{\bar z}\alpha)$$

$$= \cancel{\mathrm{Tr}(\partial_{\bar z}\alpha)} - \frac{1}{2}\mathrm{Tr}(\partial_{\bar z}\alpha)^2 + \underbrace{\frac{1}{3}\mathrm{Tr}(\quad)^3}_{0} - \cdots$$

$$= \frac{1}{2}\int \frac{\alpha(z)\,\alpha(z')\,d^2z\,d^2z'}{(z-z')^2 \quad \pi^2}$$

so that $$F(\alpha) = \frac{1}{2}\alpha^t K \alpha \qquad \text{where} \qquad K(z,z') = \frac{1}{\pi^2(z-z')^2}.$$

Let's try now to compute $F(\alpha) + \overline{F(\alpha)} - \|\alpha\|^2$ so as to get $-\zeta'(0)$. Maybe simpler is to show this doesn't change under unitary gauge transformations. So consider

$$\delta\alpha = [f, \partial_{\bar z} + \alpha] = -\partial_{\bar z} f.$$

Then

$$\delta F(\alpha) = \int \frac{d^2z\,d^2z'}{\pi^2}\,(-\partial_{\bar z} f)(z)\,\underbrace{\frac{1}{(z-z')^2}}_{\partial_z \frac{-1}{z-z'}}\,\alpha(z')$$

$$= \int \frac{d^2z\,d^2z'}{\pi^2}\,(\partial_z f)(z)\,\underbrace{\partial_{\bar z}\left(\frac{1}{z-z'}\right)}_{\pi\,\delta(z-z')}\,\alpha(z')$$

$$= \int \frac{d^2z}{\pi}\,\partial_z f\;\alpha(z) = -\int \frac{d^2z}{\pi}\,f\,\partial_z\alpha$$

$$\delta\overline{F(\alpha)} = -\int \frac{d^2z}{\pi}\,\bar f\,\partial_{\bar z}\bar\alpha$$

$$\delta\|\alpha\|^2 = \int \frac{d^2z}{\pi}\left[(-\partial_{\bar z} f)\,\bar\alpha + \alpha\,(-\partial_z \bar f)\right]$$

$$= \int \frac{d^2z}{\pi}\left[f\,\partial_{\bar z}\bar\alpha + \bar f\,\partial_z\alpha\right]$$

$$\delta\left[F(\alpha) + \overline{F(\alpha)} - \|\alpha\|^2\right] = -\int \frac{d^2z}{\pi}\left\{f\left[\partial_z\alpha + \partial_{\bar z}\bar\alpha\right] + \bar f\left[\partial_z\alpha + \partial_{\bar z}\bar\alpha\right]\right\}.$$

So clearly if f is purely imaginary then $\delta\left[F(\alpha)+\overline{F(\alpha)}-\|\alpha\|^2\right]=0$, because $\partial_z\alpha+\partial_{\bar z}\bar\alpha$ is real.

Now I should ask how simple I can make α using unitary gauge transformations. In this setup we have a single complex gauge orbit so that

$$\varphi\,\partial_{\bar z}\,\varphi^{-1}=\partial_{\bar z}+\alpha \qquad \text{where} \quad -\log\varphi=\int\frac{\alpha(z')\,d^2z'}{\pi(z-z')}$$

Modulo unitary gauge transformations, i.e. φ such that $\log\varphi$ is purely imaginary one can suppose $\log\varphi$ real. Consider $\alpha=-\partial_{\bar z}\log\varphi$ [redacted] real and a change

$$\delta\alpha=-\partial_{\bar z}\underbrace{\delta\log\varphi}_{f} \qquad \text{so that } f \text{ is real.}$$

We have $\quad \partial_z\alpha+\partial_{\bar z}\bar\alpha=-2\partial^2_{z\bar z}\log\varphi \quad$ and hence

$$\delta\left[F(\alpha)+\overline{F(\alpha)}-\|\alpha\|^2\right]=-\int\frac{d^2z}{\pi}\left\{\delta\log\varphi\left[-2\partial^2_{z\bar z}\log\varphi\right]\right\}2$$

$$=\int\frac{d^2z}{\pi}\,\delta\log\varphi\;\underbrace{4\partial^2_{z\bar z}}_{\partial_x^2+\partial_y^2}(\log\varphi)$$

$$\therefore\quad F(\alpha)+\overline{F(\alpha)}-\|\alpha\|^2=\frac{1}{2}\int\frac{d^2z}{\pi}(\log\varphi)\,\partial_x^2+\partial_y^2(\log\varphi)$$

$$=-\int\frac{d^2z}{2\pi}\underbrace{\left[(\partial_x\log\varphi)^2+(\partial_y\log\varphi)^2\right]}_{4\left|\partial_{\bar z}\log\varphi\right|^2}$$

$$=-2\int\frac{d^2z}{\pi}|\alpha|^2=-2\|\alpha\|^2$$

Simpler: Suppose $\partial_z\alpha$ is real e.g. if $\alpha=-\partial_{\bar z}\log\varphi$ with $\log\varphi$ real, and that $\delta\alpha=-\partial_{\bar z}f$ also preserves this condition: $\delta(\partial_z\alpha)=-\partial^2_{z\bar z}f$ real, e.g. f real. Then from the above

$$\delta F(\alpha)+\delta\overline{F(\alpha)}=-\int\frac{d^2z}{\pi}\left(f\,\partial_z\alpha+\bar f\,\partial_{\bar z}\bar\alpha\right)=$$

$$=-\int\frac{d^2z}{\pi}\left(f\,\partial_{\bar z}\bar\alpha+\bar f\,\partial_z\alpha\right)=\int\frac{d^2z}{\pi}\left(\partial_{\bar z}f\,\bar\alpha+\overline{\partial_{\bar z}f}\,\alpha\right)$$

$$= -\int \frac{d^2 z}{\pi}\left(\delta\alpha\,\bar{\alpha} + \overline{\delta\alpha}\cdot\alpha\right) = -\delta\|\alpha\|^2 .$$

Notice that if $-\log\varphi = \int \frac{\alpha(z')\,d^2z'}{\pi(z-z')}$ and $\partial_z\alpha$ is real, then $\partial^2_{z\bar{z}}(-\log\varphi) = \partial_{z}\alpha$ ~~real~~ so $\log\varphi$ should be real. Simpler

$$-\log\varphi = \int \alpha(z')\,\partial_{z'}\log|z-z'|^2\,\frac{d^2z'}{\pi}$$

$$= \int -\partial_{z'}\alpha(z')\,\log|z-z'|^2\,\frac{d^2z'}{\pi} \quad \text{is real.}$$

Thus the class of α with $\partial_z\alpha$ real is the same as the class of $\alpha = -\partial_{\bar{z}}\log\varphi$ with $\log\varphi$ real.
Hence $\delta\alpha = -\partial_{\bar{z}}\underbrace{\delta\log\varphi}_{f}$ so f will be real, hence

$$\delta F(\alpha) = -\int\frac{d^2z}{\pi}\, f\,\partial_z\alpha \quad \text{will be real.}$$

<u>Conclude:</u> If $\partial_z\alpha$ is real, then

$$F(\alpha) = -\frac{1}{2}\|\alpha\|^2 \qquad \text{i.e.}$$

$$\frac{1}{2}\int\frac{\alpha(z)\,\alpha(z')}{(z-z')^2}\,\frac{d^2z\,d^2z'}{\pi^2} = -\frac{1}{2}\int\frac{d^2z}{\pi}\,|\alpha|^2$$

$$\Vert$$

Direct proof:

$$\frac{1}{2}\int\alpha(z)\,\partial_z\!\left(\frac{-1}{z-z'}\right)\alpha(z')\,\frac{d^2z\,d^2z'}{\pi^2} = \frac{1}{2}\int\partial_z\alpha(z)\,\frac{1}{z-z'}\,\alpha(z')\,\frac{d^2z\,d^2z'}{\pi^2}$$

$$= \frac{1}{2}\int\frac{d^2z}{\pi}\,\underbrace{\partial_z\alpha}_{\partial_{\bar{z}}\bar{\alpha}}\,\underbrace{\int\frac{d^2z'}{\pi}\,\frac{1}{z-z'}\,\alpha(z')}_{-\log\varphi}$$

$$= \frac{1}{2}\int\frac{d^2z}{\pi}\,\bar{\alpha}\,\underbrace{\partial_{\bar{z}}\log\varphi}_{-\alpha} = -\frac{1}{2}\int\frac{d^2z}{\pi}\,|\alpha|^2$$

October 17, 1982

The goal is to work out thoroughly the case of line bundles over a Riemann surface of degree $g-1$. This is possible because of the regularization anomaly.

Lemma: Let $F(\alpha) = \frac{1}{2}\alpha^t K \alpha$ be a quadratic form on $\mathbb{C}^n$ such that $F(\alpha) + \overline{F(\alpha)} - |\alpha|^2$ is constant in real directions (i.e. under $\alpha \mapsto \alpha + \beta$ where $\beta \in \mathbb{R}^n$). Then $K = I$.

Proof: If $\delta\alpha \in \mathbb{R}^n$, then

$$0 = \delta F(\alpha) + \delta\overline{F(\alpha)} - \delta|\alpha|^2$$

$$= \delta\alpha^t \cdot K\alpha + \overline{\delta\alpha}^t \bar{K}\bar{\alpha} \text{ [redacted] } - \overline{\delta\alpha}^t \alpha - \bar{\alpha}^t \delta\alpha$$

$$= \delta\alpha^t \left[K\alpha + \bar{K}\bar{\alpha} - \alpha - \bar{\alpha} \right]$$

and so $\quad K\alpha + \bar{K}\bar{\alpha} - \alpha - \bar{\alpha} = 0.\quad$ This must hold for any $\alpha \in \mathbb{C}^n$, hence $K\alpha = \alpha$. QED.

Cor. If $\alpha \in \mathbb{R}^n$ then $F(\alpha) = \frac{1}{2}|\alpha|^2$

If $\alpha \in i\mathbb{R}^n$ then $F(\alpha) = -\frac{1}{2}|\alpha|^2$

This lemma should describe the basic behavior of $F = \delta \log \det(D)$ and $F + \bar{F} - |\alpha|^2 = -\zeta'(0)$ on a complex gauge gp. orbit, the real directions being the gauge group orbits.

The general setting is [redacted] as follows. I fix a C^∞ line bundle of degree $g-1$ with a metric over M and a volume on M and then consider all $\bar\partial$-operators D on this line bundle. [redacted] Generically the $\bar\partial$-operator is invertible and the ζ fn determinant of D^*D is defined. I fix an invertible D_0 and consider its orbit under the [redacted] identity component of the ex. gauge group. This means I am looking at the operators

$$D = e^f D_0 e^{-f} = D_0 - \bar\partial f$$

where f is a function on M. So we have an affine space under the complex vector space of exact $(0,1)$ forms. The real directions come from purely imaginary f.

I want to prove that $-\zeta'(0)$ is a real quadratic function on the orbit, and then for purposes of calculations I will take the basepoint D_0 to be a stationary point. To show something is quadratic, one can show it's derivative in any given direction is linear. So take the direction δf whence $\delta D = [\delta f, D]$ and

$$\delta\left[-\zeta'_{D^*D}(0)\right] = \int J_D'' \underbrace{\delta D}_{[\delta f, D]} \quad + \text{c.c.}$$

(minus due to D, J_D being 1-forms)
$$= \int -[D, J_D''] \, \delta f \quad + \text{c.c.}$$

Here J_D'' is the finite part of the Green's fn. for D defined using the hermitian connection belonging to D. Let's recall the local formulas.

$$D = (\partial_{\bar z} + \alpha) d\bar z$$

$$\nabla = (\partial_z - \alpha^*) dz + (\partial_{\bar z} + \alpha) d\bar z$$

$$\nabla^2 = \left(\partial_z \alpha + \partial_{\bar z} \alpha^* + [\alpha, \alpha^*]\right) dz\, d\bar z$$

$$G = \frac{i}{2\pi}\left\{ \frac{1}{z - z'} + \beta_{z'} - \alpha_{z'} \frac{\bar z - \bar z'}{z - z'} + \ldots \right\} dz'$$

$$J_D'' = \frac{i}{2\pi}\left[(\beta - \alpha^*) + \frac{1}{2} \partial_z \log |\partial_z|^2 \right] dz$$

$$[D, J_D''] = \frac{i}{2\pi}\Big[\underbrace{\left(\partial_z \alpha + \partial_{\bar z} \alpha^* + [\alpha, \alpha^*]\right) dz d\bar z}_{\text{curvature of } \nabla \text{ on } E} + \underbrace{\frac{1}{2} \bar\partial \partial \log |\partial_z|^2}_{\frac{1}{2} \text{curv. of } T} \Big]$$

With line bundles it is clear that $[D, J_D'']$ is linear in D.

So we have proved that $-\zeta'_{D^*D}(0)$ is a quadratic function on the orbit and that it is stationary when the curvature of D exactly cancels $\frac{1}{2}\times$ curvature of T. So let's pick such a point D_0 as our basepoint in the orbit. Now we construct the analytic determinant with this basepoint and put

$$F(D) = \log\det(D; D_0) \qquad\qquad D = e^{f} D_0 e^{-f} = D_0 - \bar\partial f$$

Then

$$\delta F = \int J_D\, \delta D = -\int [D, J_D]\,\delta f$$

where J_D is constructed using the connection with $\bar\partial$ half D and ∂-half $= \nabla_0'$. Thus

$$\nabla = \nabla_0 + (D - D_0) = \nabla_0 - \bar\partial f$$

$$\nabla^2 = \nabla_0^2 - \partial\bar\partial f$$

$$[D, J_D] = \frac{i}{2\pi}\left[\nabla_0^2 + \frac{1}{2}\bar\partial\partial \log\rho\right]$$

By construction $\nabla_0^2 + \frac{1}{2}\bar\partial\partial\log\rho = 0$. Thus

$$[D, J_D] = \frac{i}{2\pi}(-\partial\bar\partial f) = \frac{i}{2\pi}\bar\partial\partial f$$

$$\delta F = -\int \frac{i}{2\pi}\bar\partial\partial f\ \delta f = \int \frac{i}{2\pi}\partial^2_{z\bar z} f\ \delta f\ dz\,d\bar z$$

$$F = \frac{-1}{2}\int \frac{d^2 z}{\pi}\,\partial_{\bar z} f\,\partial_z f$$

Notice that if we put $\alpha = -\partial_{\bar z} f$, so $D - D_0 = \alpha\, d\bar z$ then for f real we have

$$F = -\frac{1}{2}\|\alpha\|^2$$

which checks with yesterday's calculations.

Summary: I am considering a complex gauge orbit on the space of $\bar\partial$-operators on a line bundle of degree $g-1$, and I suppose the operators are invertible. Then the orbit is an affine space for the vector space of $(0,1)$-forms of the form $\bar\partial f$ with f a fn. I choose a point D_0 of the orbit where the curvature cancels $\frac{1}{2}\times$ curvature of T, equivalently a point where $\zeta'(0)$ is stationary on the orbit, or where $|s|^2$ is [redacted] stationary. Then [redacted] the points of the orbit are [redacted] of the form $D = D_0 + \bar\partial f$. Then

$$\log\det(D;D_0) = -\frac{1}{2}\int \frac{i}{2\pi}\,\partial f \wedge \bar\partial f = -\frac{1}{2}\int \partial_z f\, \partial_{\bar z} f\, \frac{d^2 z}{\pi}$$

$$\|\bar\partial f\|^2 = \int \frac{i}{2\pi}\,\overline{\bar\partial f} \wedge \bar\partial f = \int [redacted] |\partial_{\bar z} f|^2 \frac{d^2 z}{\pi}$$

so that

$$\log\det(D;D_0) = \begin{cases} \frac{1}{2}\|\bar\partial f\|^2 & \text{if } f \text{ purely imag} \\ -\frac{1}{2}\|\bar\partial f\|^2 & \text{if } f \text{ real} \end{cases}$$

<u>Picture</u>: Lets put $A = \bar\partial f$ and split

$$A = A_\ell + A_t$$

$\partial f = \overline{\bar\partial \mathrm{Re} f} + \overline{\bar\partial\, i \,\mathrm{Im} f}$

[redacted] $= \bar A_t - \bar A_\ell$

$$A_\ell = \text{longitudinal part} = \bar\partial\, i\,\mathrm{Im} f$$
$$A_t = \text{transverse part} = \bar\partial\, \mathrm{Re} f$$

Then

$$\boxed{-\zeta'_{D^*D}(0) = -\zeta'_{D_0^* D_0}(0) - 2\|A_t\|^2}$$

$$\log\det(D;D_0) = -\frac{1}{2}\int \frac{i}{2\pi}\,\partial f \wedge \bar\partial f = -\frac{1}{2}\int \frac{i}{2\pi}\left(\bar A_t - \bar A_\ell\right)\wedge\left(A_t + A_\ell\right)$$

$$= -\frac{1}{2}\left\{\|A_t\|^2 - \|A_\ell\|^2 + (A_t|A_\ell) - (A_\ell|A_t)\right\}$$

$$= -\frac{1}{2}\left\{\|A_t\|^2 - \|A_\ell\|^2 + 2i\,\mathrm{Im}(A_t|A_\ell)\right\}$$

But $\displaystyle (A_t|A_\ell) = \int \frac{i}{2\pi}\, \bar{A}_t \wedge A_\ell = \int \frac{i}{2\pi}\, \partial \operatorname{Re} f \wedge \bar\partial\, i \operatorname{Im} f$

$$= \int \frac{d^2 z}{\pi}\, \partial_z \operatorname{Re} f\; \partial_{\bar z}\, i \operatorname{Im} f \qquad \text{purely Im.}$$

So

$$\log\det(D;D_0) = +\frac{1}{2}\Big\{ \|A_\ell\|^2 - \|A_t\| + \underbrace{2(A_\ell|A_t)}_{\text{purely imag}} \Big\}$$

$$\|A\|^2 = \|A_\ell\|^2 + \|A_t\|^2$$

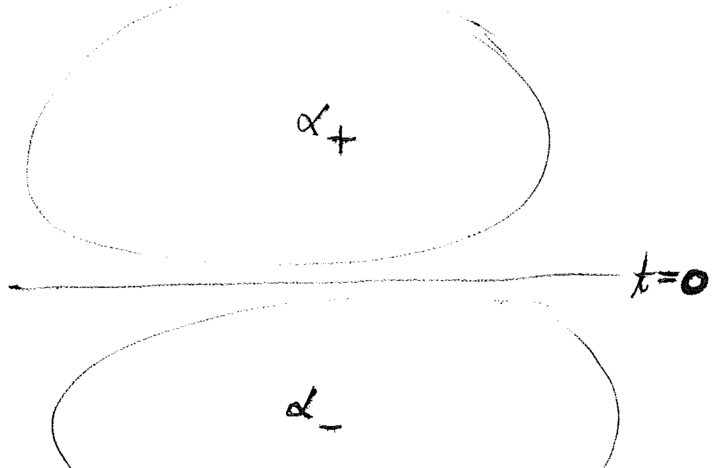
So that $\quad 2\operatorname{Re}\log\det(D;D_0) - \|A\|^2 = -2\|A_t\|^2$

consistent with the formula for ζ.

Let's now go back to the boundary value problem. Let's take the situation where $z = x + it \in \mathbb{C}$ and the surface M_+ is $\operatorname{Im}(z) = t \geq 0$. Let a subspace W of boundary conditions $W \subset L^2(\mathbb{R})$ be given, say it is of the form φH_-, where $\varphi : \mathbb{R} \to U(1)$ is a suitable map and H_- is the Hardy space of fns with analytic decaying extension to the LHP $t < 0$. I would like W to come from a $\bar\partial$-operator

$$\partial_{\bar z} + \alpha_-$$

in the LHP. So assume this. Now associated to $\alpha = \alpha_+, \alpha_-$ is a certain determinant:

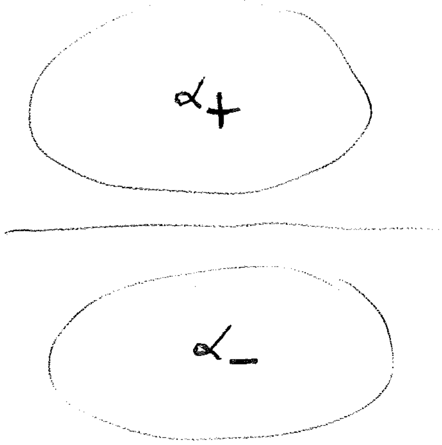


$$\langle 0|S|0\rangle = \exp\Big[-\frac{1}{2}\int \partial_z f\, \partial_{\bar z} f\, \frac{d^2 z}{\pi}\Big]$$

and also a ζ-fn determinant. These depend on α_-, but what I want to do is to obtain things depending only on α_+ and the boundary conditions. I can do this for the determinant namely $\langle 0|S|0\rangle = \langle 0|\underbrace{U(\infty,0)}_{\substack{\text{depends}\\ \text{on } \alpha_+}} \cdot \underbrace{U(0,-\infty)}_{\substack{\text{depends on}\\ \alpha_-}}|0\rangle$

October 18, 1982

I want next to discuss the boundary value problem: I take the $M_+ = UHP$ and suppose $\alpha = (\alpha_+, \alpha_-)$ [redacted] is a given $\bar\partial$ operator: $\partial_{\bar z} + \alpha$. The



operator in the LHP gives boundary values $W \subset L^2(\mathbb{R})$ for the operator in the UHP. The goal is to assign to $\{(W, \alpha_+)\}$, or something slightly bigger, a line bundle $\mathcal{L}$ with section s, metric & trivialization yielding a determinant.

The line bundle with section $(\mathcal{L}, s)$ is the same as a divisor in the space of (W, α_+) and that is clearly given, so what really is to be defined is the metric $|s|^2$ and the trivialization yielding the determinant.

I consider the pairs (W, α_+) as forming a complex manifold, and restrict to the open set where the $\bar\partial$ operator is invertible, which means that W is complementary to W_+ = boundary values for $\partial_{\bar z} + \alpha_+$ in UHP. Over the open set I want to define

a fn. $|s|^2 > 0$

a 1-form $\theta = \partial \log |s|^2$

a closed 2-form $d\theta = \bar\partial\partial \log |s|^2$.

Each of these determines it's successor, but it might happen that θ is intrinsically defined without $|s|^2$ being so, or $d\theta$ without θ. For example $d\theta$ should be the Kähler 2-form on [redacted] $\{(W, \alpha_+)\}$.

Let's fix α_+ and allow W to vary. Then the open set is the W which are complementary to W_+ and we can ask to define $\log|s|^2$, Θ, or $d\Theta$. I think I know what $d\Theta$ is; [redacted] it is the Kähler form on the Grassmannian. I want to write it as $\bar\partial\partial\log|s|^2$ which determine $\log|s|^2$ up to [redacted] a term $\log|f|^2$ where f is analytic. The hope is that over the open set there is an obvious inner product $-\|\gamma\|^2$ with $\bar\partial\partial(-\|\gamma\|^2) = d\Theta$ which can then be [redacted] combined with an [redacted] analytic determinant to define $\log|s|^2$.

Here is how to describe $(\mathcal{L}, s)$ over $\{(W,\alpha_+)\}$. The fibre of $\mathcal{L}$ at (W,α_+) can be identified with the dual of the line belonging to W in Fock space and s is the linear function on that line given by $\langle 0|U(\infty,0)$.

[redacted]

Question: To each α_- we get a vector $U(0,-\infty)|0\rangle$ in Fock space depending analytically on α_-. Is there a simple [redacted] formula for [redacted] the norm of this vector?

Question: Consider a Riemann surface M_+ with bdry. and volume and a C^∞-bundle with metric E over M_+. Say the bdry is connected. For any holom. structure on E we associate the subspace W of $\Gamma(\partial M_+, E)$ of boundary values. Can we lift $D \longmapsto W$ to a map $D \mapsto u_W$ in Fock space in general? (Is it clear that W always determines an element in the Fock space? This would seem to follow from the fact that $\mathrm{Diff}_0(S^1)$ acts on Fock space.)

In any case I should perhaps try to get at the infinitesimal variation for vector bundles, as opposed to just line bundles.

Problem: Take (M_+, E) a Riemann surface (compact) with boundary and smooth vector bundle over it. Then let $\mathcal{a} = \bar\partial$-operators on E. Then for any D in $\mathcal{a}$ we have a line in the Fock space of $\Gamma(\partial M_+, E)$ associated to the subspace W_D of boundary values of holomorphic sections over M_+. This gives a line bundle $\mathcal{L}$ over $\mathcal{a}$. Construct a trivialization of this line bundle. Can one use the technique of the closed case, namely define a metric, calculate curvature, etc.

October 19, 1982

Yesterday I found a good way to think of the boundary problem which goes as follows. To each holomorphic structure on E over M_+ we associate the line in the Fock space of $\Gamma(\partial M_+;E)$ which belongs to the subspace W_+ of boundary values of holomorphic sections. This gives us a holomorphic line bundle $\mathcal{L}$ over the space $\mathcal{A}$ of holomorphic structures which I can try to trivialize by the curvature method. Notice that if I give a dual vector on the Fock space belonging to a subspace W_- of boundary conditions, then from the trivialization of $\mathcal{L}$ I actually get a determinant fn. on $\mathcal{A}$.

Now it is essential to work out this program thoroughly. First you should understand the holom. setting. Suppose I have a $D: E \to E\otimes T^{0,1}$ and I consider an infinitesimal variation $\delta D \in \Gamma\{Hom(E, E\otimes T^{0,1})\}$. To D we have a subspace $W \subset \Gamma(\partial M_+, E)$, hence to δD should belong a tangent vector to W in the Grassmannian, i.e. a map $W \longrightarrow V/W$.

Linear algebra: Suppose we have $A: V \to W$ changing by δA. Then how can we compute the corresponding tangent vector to $\mathrm{Ker}(A)$ in the Grassmannian? If $\psi \in \mathrm{Ker}(A)$ we want to lift the change δA to a change $\delta\psi$ satisfying

$$(A+\delta A)(\psi + \delta\psi) = 0$$

or

$$\delta A\cdot\psi + A\cdot\delta\psi = 0$$

Let $B: W \to V$ satisfy $AB = \mathrm{Id}$. Then

$$\delta\psi = -B\,\delta A\,\psi$$

works, and changing B changes $\delta\psi$ by something in $\mathrm{Ker} A$. Thus $\psi \longmapsto -B\delta A\psi \mod \mathrm{Ker} A$

gives a well-defined map

$$\mathrm{Ker}\,A \longrightarrow V/\mathrm{Ker}\,A$$

which is the desired tangent vector. Maybe a simpler formula is

$$\psi \longmapsto -A^{-1}\delta A\psi \quad \text{mod } \mathrm{Ker}\,A$$

where A^{-1} is any half-inverse for A. In practice one chooses a subspace complementary to $\mathrm{Ker}\,A$ to define A^{-1}.

Now I apply this to the case of the $\bar\partial$-operator

$$D: \Gamma(M_+, E) \longrightarrow \Gamma(M_+, E\otimes T^{0,1})$$

which we know is onto. Then if I define an inverse G for D by choosing boundary conditions W_- complementary to $\mathrm{Ker}\,D\,|\,\partial M_+ = W_+$, I get the map

$$\begin{array}{ccc} \mathrm{Ker}\,D \xrightarrow{\sim} W_+ & \longrightarrow & W_- \\ \ni \psi & \longmapsto & -G\delta D\psi \,|\, \partial M_+ \end{array}$$

describing the tangent vector in the Grassmannian.

Next point: lift this tangent vector into Fock space. Think generally first and put $V = \Gamma(\partial M_+, E)$ and suppose we have a $\neq 0$ vector in the line in Fock space attached to W_+. Now corresponding to $V = W_- \oplus W_+$ is

$$\tilde\Lambda V = \Lambda W_- \otimes \Lambda (W_+)^*$$
$$\cup$$
$$W_- \otimes (W_+)^* = \mathrm{Hom}(W_+, W_-)$$

so there is a way to lift tangent vectors in the Grassmannian to vectors in $\tilde\Lambda V$. If I always take $W_- = (W_+)^\perp$ for the L^2-structure, then I am getting the standard hermitian connection on $O(-1)$ on the Grassmannian.

In finite dimensions at least we have

$p = \dim W$

Tang. sp. to $\Lambda^p W$ in $\mathbb{P}(\Lambda^p V)$

$$= \mathrm{Hom}(\Lambda^p W, \Lambda^p V / \Lambda^p W) \supset \mathrm{Hom}(\Lambda^p W, V/W \otimes \Lambda^{p-1} W)$$

$$= V/W \otimes \Lambda^{p-1} W \otimes (\Lambda^p W)^* = V/W \otimes W^* .$$

$$= \text{Tang. sp. to } W \text{ in } \mathrm{grass}_p(V).$$

In fact, we can interpret this embedding as induced by the action of the Lie algebra $\mathrm{End}(V) = V \otimes V^*$. In the infinite dimensional case I'm involved with the essential new phenomenon is the fact that the lifting of suitable endos. A of V to the Fock space is defined up to an additive scalar. This should not affect the above embedding: It still should be true that the tangent space to W in the Grassmannian embeds in the above way into the tangent space to the line of W in Fock space.

The key point to understand then is how we are going to move a given vector Ψ in the line of W_+ corresponding to the change δD. The obvious $\delta\Psi$ takes place $\perp$ to the line of W_+; this is just the standard connection from the L^2-structure, and it doesn't <u>change</u> the norm of Ψ in the Fock space.

Review: Over the space $\mathcal{A}$ of $\bar\partial$-operators we have this line bundle $\mathcal{L}^*$ defined whose fibre at a point is the line in Fock space given by boundary values. I want to put a metric on $\mathcal{L}^*$ or at least a connection. This metric should somehow be the product of the Fock space metric with a ζ-determ. factor. I know what the curvature should be for this metric namely [redacted] $\bar\partial\partial\|\alpha\|^2$ where

$$\|\alpha\|^2 = \int_{M_+} \frac{i}{2\pi}\, tr(\alpha^* \wedge \alpha) .$$

What does the connection on $\mathcal{L}^*$ belonging to a multiple of the usual metric look like? The differences of two connections is a 1-form, in this case ∂f, where [redacted] e^f is the ζ-determinant factor. So I am looking for a real fn. on $\mathcal{A}$ such that $\bar\partial\partial f$ will be a sum of $\bar\partial\partial\|\alpha\|^2$ and the usual curvature form [redacted] for $\mathcal{L}^*$ pulled back by the boundary-values map $D \longmapsto W_+(D)$.

Candidate: ∂f is $Tr_{reg}(D^{-1}\delta D)$

where D^{-1} is the [redacted] Green's fn. for D with boundary values in $W_+(D)^\perp$ and where the regularization is computed using the unitary connection associated to D.

This should work because the resulting J will be non-analytic for two reasons, because of the regularization process and because of the way the Green's function changes. The last should follow in the same way we handled non-zero index.

It seems that the resulting connection on $\mathcal{L}$ is independent of the inner product put on the Fock space. Perhaps the simplest way to define the connection near a point D_0 of $\mathcal{A}$ is to choose W_- complementary to the b.v. subspace $W_+(D_0)$. Now use the Green's function with b.values in W_- for D near D_0 to define ∂f as $Tr_{reg}(D^{-1}\delta D)$, and combine ∂f with the flat connection on the line subbundle which comes from the subspace W_-.

October 20, 1982

$\mathcal{A}$ = space of $\bar\partial$-operators on E over M_+, $\mathcal{L}^*$ = line bundle over $\mathcal{A}$ whose fibre at A is the line in the Fock space of $V = \Gamma(\partial M_+, E)$ attached to the subspace $\operatorname{Ker}(\bar\partial_A) = W_A$ restricted to the boundary. Problem: Define a flat connection on $\mathcal{L}$. Method: For each subspace W_- of boundary values of index 0 we get an open subset of $\mathcal{A}$ and a trivialization of $\mathcal{L}$ by choosing a vector $\langle u_{W_-}|$ in the Fock line of W_- and considering this as a linear functional on $\mathcal{L}^*$. Thus I get a family of sections s_U of $\mathcal{L}$ where $U = (W_-, u_{W_-})$. To define a connection I have to give a family of 1-forms θ_U, [redacted] θ_U defined on the open set where $s_U \neq 0$ such that

$$(*) \qquad \theta_U - \theta_V = d\log f_{UV}$$

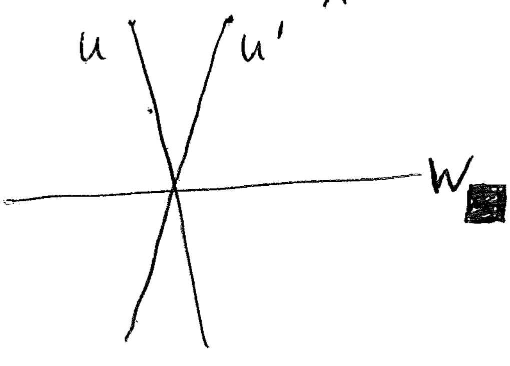
where f_{UV} is the transition fn: $s_V = f_{VU}\, s_U$

The open set where $s_U \neq 0$ is the set of A such that $\operatorname{Ker}(D_A)$ is transversal to W_-, and then one can define ~~an~~ inverse D_A^{-1} with the boundary conditions W_-. Put

$$\theta_U = \operatorname{Tr}^{\mathrm{reg}}(D^{-1}\delta D)$$

where the regularization is performed using a fixed ∂-operator over M_+ and volumes. The problem is now to check $(*)$.

First what are the transition fns? Change notation to where $U, U' \subset V = \Gamma(\partial M_+, E)$ are two boundary subspaces and $W = W_A \subset V$ is complementary to both. [redacted] I pick generators $\langle u|$, $\langle u'|$ for the Fock lines of U, U' and then consider the fn.



$$f(W) = \frac{\langle u|W\rangle}{\langle u'|W\rangle}$$

which is independent of the generator $|w\rangle$ for the Fock line of W. Now I want to calculate the $\delta \log f$ corresponding to a change in W.

Curious: If W' is also transversal to U, U' then the quantity (kind of cross-ratio)

$$\frac{f(W)}{f(W')} = \frac{\langle u|W\rangle}{\langle u'|W\rangle} \Big/ \frac{\langle u|W'\rangle}{\langle u'|W'\rangle}$$

is independent of choices of the generators $\langle u|$, $\langle u'|$.

Let's try to understand what is happening in finite dimensions. Let V be finite-dimensional and fix a volume in it, i.e. an isom $\int : \Lambda^n V \simeq \mathbb{C}$. Then $|W\rangle \in \Lambda^p V$ $\quad \langle u| \in \Lambda^{n-p} V$ and

$$\langle u|W\rangle = \int \langle u| \wedge |W\rangle .$$

Another way to describe this is to combine the volume in V and $|W\rangle$ to get a volume in V/W. Then $\langle u|W\rangle$ is the ratio of the volumes under the [redacted] isomorphism $U \xrightarrow{\sim} V/W$. Hence

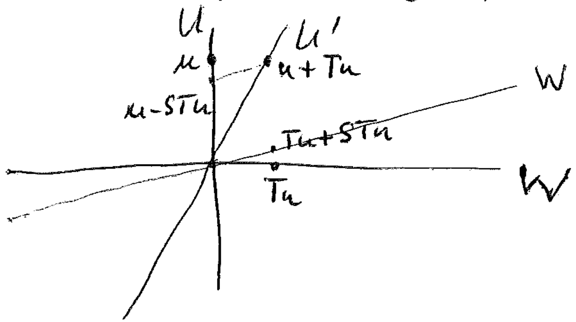
$$f(W) = \frac{\langle u|W\rangle}{\langle u'|W\rangle}$$ is the determinant of the map

$$U \xrightarrow{\sim} V/W \xleftarrow{\sim} U' \quad \left(\begin{matrix}\text{projection}\\ \parallel W\end{matrix}\right)$$

relative to the volumes in U, U'. Finally $f(W)/f(W')$ is the absolute determinant of

$$U \xrightarrow[\text{proj} \parallel W]{\sim} U' \xleftarrow[\text{proj} \parallel W']{\sim} U$$

Here is how to make this more explicit.



Let $W' = \Gamma_S \qquad S : W \to U$

$U' = \Gamma_T \qquad T : U \to W$

Then

$$\begin{array}{ccccccccc}
U & \xrightarrow{\sim} & V/W & \xleftarrow{\sim} & U' & \xrightarrow{\sim} & V/W' & \xleftarrow{\sim} & U \\
u & \longmapsto & \underset{\text{mod } W}{u} & \longleftarrow\!\shortmid & u+Tu & \longmapsto & \underset{\text{mod } w'}{u+Tu} & \longleftarrow\!\shortmid & \begin{array}{c} u-STu \\ Tu+STu \end{array}
\end{array}$$

so we have the following effect on volumes

$$\frac{f(W)}{f(W')} = \langle U|W\rangle \langle U'|W\rangle^{-1} \langle U'/W'\rangle \langle U|W'\rangle^{-1} = \det(1-ST)$$

A basic ingredient of Fock space is that the operators S, T are Hilbert-Schmidt, hence ST is of trace class and so $\det(I-ST)$ is always defined.

Situation: M_+ is a Riemann surface with boundary, E is a C^∞ vector bundle over it, $\mathcal{A}$ = space of $\bar\partial$-operators $D: \Gamma(E) \to \Gamma(E\otimes T^{0,1})$. I am assuming known that for any such D the restriction homomorphism

$$\mathrm{Ker}(D) \longrightarrow \Gamma(\partial M_+, E)$$

is injective and the image is the kind of "half-space" which has a Fock line. [redacted]

It seems advantageous to do as much as possible inside $\Gamma(E)$ and to separate out the necessary properties of the restriction homomorphism. A set of b.c. for D will be a subspace U of $\Gamma(E)$ complementary to $\mathrm{Ker}\,D$ and it ~~determines~~ an inverse $G_U : \Gamma(E\otimes T^{0,1}) \xrightarrow{\sim} U \subset \Gamma(E)$. To such a U I am planning to associate a differential form Θ_U on the open set of $\mathcal{A}$ where $\mathrm{Ker}\,D$ is complem. to U by

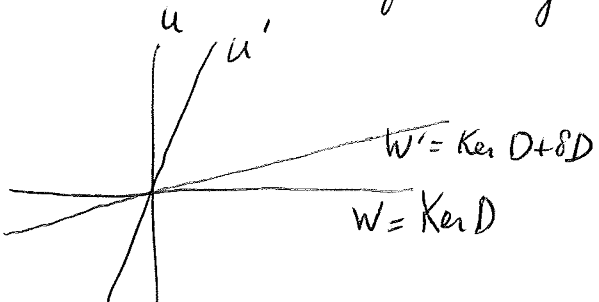
$$\Theta_U(\delta D) = \mathrm{Tr}^{(\mathrm{reg})}(G_U \cdot \delta D)$$

This is to be the connection form associated to the dual section of the Fock line bundle over $\mathcal{A}$ [redacted] obtained by picking a generator $\langle U|$ in the Fock line for U. We have

$$\Theta_{U'} - \Theta_U \ : \quad \mathrm{Tr}\,[redacted]\big((G_{U'} - G_U)\cdot\delta D\big)$$

because $G_{U'} - G_U$ has a smooth kernel. Note that the trace can be taken over $\mathrm{Ker}(D)$ because $\mathrm{Im}(G_{U'} - G_U) \subset \mathrm{Ker}\,D$.

Recall from yesterday the following:



U' = graph of $T: U \to W$

W' = graph of $S: W \to U$

One has $S = -G_U\,\delta D$. Then

$$\delta \log \frac{\langle U'|W\rangle}{\langle U|W\rangle} = [redacted]\,\log\det(1 - TS) = -\mathrm{tr}(TS)$$

$$= \mathrm{tr}(TG_U\,\delta D) = \mathrm{tr}\big((G_{U'} - G_U)\,\delta D\big)$$

Here if $\psi \in \Gamma(E \otimes T^{0,1})$, then $G_{\mathcal{U}'}\psi = \varphi + T\varphi$ with $\varphi \in$, so $\psi = D\varphi \Rightarrow \varphi \doteq G_{\mathcal{U}}\psi$. Hence

$$G_{\mathcal{U}} + TG_{\mathcal{U}} = G_{\mathcal{U}'} .$$

The above calculation shows that for the transition function $f_{\mathcal{U}'\mathcal{U}} = \dfrac{\langle \mathcal{U}'|W\rangle}{\langle \mathcal{U}|W\rangle}$ between the sections $\langle \mathcal{U}'|$ and $\langle \mathcal{U}|$, we have

$$d \log f_{\mathcal{U}'\mathcal{U}} = \Theta_{\mathcal{U}'} - \Theta_{\mathcal{U}}$$

as desired.

The above proof fudges slightly in that the restriction map $\Gamma(E) \twoheadrightarrow \Gamma(\partial M_+, E)$ is not mentioned. What happens is that

$$\Theta_{\mathcal{U}'} - \Theta_{\mathcal{U}} : \qquad \operatorname{Tr}_{\Gamma(E)}\big((G_{\mathcal{U}'} - G_{\mathcal{U}})\delta D\big) = \operatorname{Tr}_{\operatorname{Ker} D}\big((G_{\mathcal{U}'} - G_{\mathcal{U}})\delta D\big)$$
$$= -\operatorname{Tr}_W (TS)$$

where the last step uses the isom. $\operatorname{Ker} D \xrightarrow{\sim} W$. This is more or less clear.

Review τ-functions. Start with a Schroedinger equation

$$\left[-\partial_x^2 + q(x)\right]\psi = k^2\psi$$

on the line where $q \to 0$ fast as $|x| \to \infty$. Normally we define the solution $\phi_k(x)$ for $k \in UHP$ such that $\phi_k(x) \sim e^{-ikx}$ as $x \to -\infty$. Similarly define $\psi_k(x) \sim e^{ikx}$ as $x \to +\infty$ analytic for $k \in UHP$. On the real axis

$$\phi_k = A(k)\psi_{-k} + B(k)\psi_k$$

where $\quad W(\phi_k, \psi_k) = 2ik\, A(k)$ is analytic in UHP and $A(k) = 1 + O(\tfrac{1}{k})$ as $|k| \to \infty$ in the UHP.

Also from $\quad \phi_{-k} = B(-k)\psi_{-k} + A(-k)\psi_k \qquad k$ real

we get by looking at Wronskians

$$A(k)\,A(-k) - B(k)\,B(-k) = 1 \qquad " \; .$$

So next there is no reflection: $B = 0$. Then

$$A(k) = \frac{1}{A(-k)}$$

shows A extends meromorphically to the whole k-plane. The same is true for

$$\phi_k = A(k)\,\psi_{-k} .$$

In fact A has to be rational with a finite number of zeroes in the UHP and their negatives as poles in the LHP. It turns out that the function $\psi_k(x)$ has the form

$$\psi_k(x) = e^{ikx}\underbrace{\left\{1 + \frac{a_1(x)}{k} + \frac{a_2(x)}{k^2} + \cdots\right\}}_{\text{convergent in nbd of } \infty .}$$

To get to the algebraic approach change ik to z. and $q(x)$ to $-q(x)$, so that

$$\left[\partial_x^2 + q(x)\right]\psi_z(x) = z^2\psi_z(x)$$

$$\psi_z(x) = e^{xz}\left(1 + \frac{a_1(x)}{z} + \frac{a_2(x)}{z^2} + \cdots\right).$$

I should think of z as the eigenvalues of the Schroedinger operator.

Algebra vieivpoint: Work in $L^2(S^1)$. We consider a half-space W which gives a line in the Fock space and which has index zero relative to $H_+ =$ span of $1, z, z^2, \ldots$ Then define the BA fn. $\psi_z(x_1, x_2, \ldots)$ by

$$\psi_z(\underline{x}) = \text{unique elt. in } W \cap e^{\sum x_k z^k}(1 + H_-)$$

where $H_- =$ span of $z^{-1}, z^{-2}, \ldots$

It is easy to check that when $z^2 W \subset W$, then one gets a $q(\underline{x})$ such that

$$\left[\partial_{x_1}^2 + q(\underline{x})\right] \psi_z(\underline{x}) = z^2 \psi_z(\underline{x}).$$

What is the τ-fn? Let's go over the formula for the BA fn. which I will denote β_z to avoid confusion with the field operators. We have a half-space

$$W_x = e^{-\sum x_k z^k} W$$

and we want to get the element β_z in $W \cap (1 + H_-)$. Let $|W_x\rangle$ generate the Fock line of W_x. Formally

$$|W_x\rangle = w_1 \wedge w_2 \wedge \cdots\cdots \qquad \langle \zeta | n \rangle \langle n |$$

where w_n is a basis for W_x. Thus for $\psi_\zeta = \sum \zeta^n \psi_n$ one has

$$\langle \alpha | \; \psi_\zeta | W_x \rangle = \sum (-1)^{n-1} w_n(\zeta) \langle \alpha | w_1 \wedge \ldots \wedge \hat{w}_n \wedge \ldots \rangle \in W$$

for any $\langle \alpha |$. Now take

$$\langle \alpha | = \langle 0 | \psi_0^*$$

to correspond to the "dual" subspace $\mathbb{C} + H_-$. One has

$$\langle 0 | \psi_0^* \psi_n = \Big(\psi_n^* \underbrace{\psi_0 | 0 \rangle}_{z^1 \wedge z^2 \wedge \ldots} \Big)^* = 0 \qquad n > 0,$$

hence $\langle 0 | \psi_0^* \psi_\zeta$ contains ζ^n with $n \le 0$. Thus

$$\langle 0 | \psi_0^* \psi_\zeta | W_x \rangle \in (\mathbb{C} + H_-) \cap W$$

And as $\zeta \to \infty$ we get the value $\langle 0 | \psi_0^* \psi_0 | W_x \rangle = \langle 0 | W_x \rangle$. Thus

$$\beta(\zeta) = \frac{\langle 0 | \psi_0^* \psi_\zeta | W_x \rangle}{\langle 0 | W_x \rangle}.$$

Now the denominator is essentially the τ-function.

George's action: Suppose $G = G_+ G_-$, $G_+ \cap G_- = 1$.
Then G acts on itself by
$$g * (h) = (hgh^{-1})_-^{-1}\, h$$

Here's a possible interpretation. Note that this action carries each coset $G_- h$ into itself. ~~The coset~~ One has a bijection $G_- \simeq G_+ \backslash G$, hence also
$$G_- h \simeq G_+ \backslash G$$
and G acts naturally on $G_+ \backslash G$.

To derive a formula for this action, use
$$\begin{aligned} G &\xrightarrow{\ \sim\ } G_+ \backslash G \times G_- \backslash G \\ g &\longmapsto (G_+ g\ ,\ G_- g) \end{aligned}$$
whose inverse is $(G_+ x, G_- y) \longmapsto G_+ x \cap G_- y = (G_+ x y^{-1} \cap G_-) y =$
$= (G_+ (xy^{-1})_- \cap G_-) y = (xy^{-1})_- y$. The action is then given by

$$\begin{tikzcd}
G \arrow[r, "\sim"] \arrow[d, "g*"] & G_+ \backslash G \times G_- \backslash G \arrow[d] \\
G \arrow[r, "\sim"] & G_+ \backslash G \times G_- \backslash G
\end{tikzcd}
\qquad
\begin{tikzcd}
h \arrow[d, mapsto] & (G_+ h\ ,\ G_- h) \arrow[d, mapsto] \\
g*(h) & (G_+ h g^{-1}, G_- h)
\end{tikzcd}$$

so
$$\boxed{g_*(h) = (hg^{-1}h^{-1})_-\, h}$$

which probably agrees with George's formula because I use $g = g_+ g_-$ instead of $g = g_- g_+$ to define g_-.

October 23, 1982

On τ-functions:

Notation: We work in the Fock space of $L^2(S^1)$ with $|0\rangle = z^0 \wedge z^1 \wedge \cdots$ and put

$$\rho_n = \sum \psi^*_{n+k} \psi_k = \text{derivation assoc. to multiplication by } z^n.$$

Then $\rho_n |0\rangle = 0$ for $n \geq 0$ (which defines precisely ρ_0). Note that if $|W\rangle$ is a vector $\neq 0$ in the Fock line belonging to a half-space W and if $z^m W \subset W$ (hence $m \geq 0$) then

$$\rho_m |W\rangle = \begin{cases} 0 & m > 0 \\ \text{index of } W \text{ relative to } H_+ & m = 0. \end{cases}$$

From now on consider only W of index 0, i.e. we work in the piece of Fock space where $\rho_0 = 0$. We know this ~~has an orthonormal basis~~ piece $\mathcal{F}_0$ is an L^2 symmetric algebra. Precisely because the $\{\rho_n \mid n \geq 1\}$ is a family of destruction operators we get an orthogonal basis for $\mathcal{F}_0$ given by monomials

$$\rho^*_\alpha |0\rangle \qquad \alpha = (\alpha_1, \alpha_2, \ldots) \leftarrow \text{finite support}.$$

Now a τ-function is something of the form

$$\tau(\underline{x}) = \langle 0 | e^{\sum_1^\infty x_n \rho_n} | W\rangle \qquad \underline{x} = (x_1, x_2, \cdots)$$

for some half-space W of index 0. It is known this is an entire function of $\underline{x}$. If one writes it in the form

$$\tau(\underline{x}) = \sum_{\alpha \geq 0} \frac{x^\alpha}{\alpha!} \langle \rho^*_\alpha |0\rangle \mid W\rangle$$

one sees this function completely determines the vector $|W\rangle$.

October 24, 1982

Let's concentrate on the idea of the τ-function being a determinant. For example suppose W comes from an algebraic curve. This means we have a small disk M_- in a Riemann surface M, a line bundle L over M trivialized over M_- with $h^0 = h^1 = 0$. Then $W = \Gamma(M_+, L) \subset \Gamma(\partial M_+, L)$, and $\langle 0|$ represents the space $\mathbb{C} + H_-$ of holomorphic sections over the disk. Next we take the clutching fn. $e^{\sum x_n z^n}$ over ∂M_+ and form a new line bundle $L_{\underline{x}}$ whose sections over M_+ are W and over M_- are $e^{-\sum x_n z^n}(\mathbb{C} + H_-)$. Then

$$\tau(\underline{x}) = \langle 0 | e^{\sum x_n p_n} | W \rangle$$

is a determinant function for the family of line bdles $L_{\underline{x}}$ in the sense that it vanishes exactly when cohomology appears.

Recall that the clutching fn. construction gives an action of the group of invertible fns. over ∂M_+ on the iso classes of line bundles. Specifically we have

$$\Gamma(M_-, \mathcal{O}^*) \times \Gamma(M_+, \mathcal{O}^*) \longrightarrow \Gamma(\partial M_+, \mathcal{O}^*) \xrightarrow{\delta} \underset{\text{Pic}(M)}{\underset{\shortparallel}{H^1(M, \mathcal{O}^*)}} \longrightarrow 0$$

where the coboundary is a homomorphism. In other words we have a homomorphism $\varphi \longmapsto \mathcal{O}_\varphi$ from $\Gamma(\partial M_+, \mathcal{O}^*)$ to Pic and then $L_\varphi = \mathcal{O}_\varphi \otimes L$. Hence I should think of the x_n are linear coordinates on the ~~Grassmannian~~ Jacobian. The analogous thing for $\bar\partial$-operators is to use

$$(C^\infty)^\times \xrightarrow{\bar\partial \log} C^\infty(\Omega^{0,1}) \longrightarrow H^1(M, \mathcal{O}^*) \longrightarrow 0,$$

hence for line bundles the class of the line bundle depends linearly on the $\bar\partial$-operator.

Therefore I should think of $\underline{x}$ as the B in $D = D_0 + B$ and in analogy with the theory of determinants look at

$$\delta \log \tau(\underline{x}) = \sum_n \frac{\langle 0 | \rho_n e^{\sum x_n \rho_n} | W \rangle}{\langle 0 | e^{\sum x_n \rho_n} | W \rangle} \delta x_n$$

(Note the operators ρ_n $n \geq 1$ commute). Recall

$$\psi^*(\zeta)\psi(\zeta) = \sum \zeta^{-m} \psi^*_{\substack{m \\ k+n}} \sum \zeta^k \psi_k$$

$$= \sum \zeta^{-n} \sum_k \psi^*_{k+n} \psi_k = \sum \zeta^{-n} \rho_n$$

hence what is involved in $\delta \log \tau(\underline{x})$ is

$$\frac{\langle 0 | \psi^*(\zeta)\psi(\zeta) | W_x \rangle}{\langle 0 | W_x \rangle}$$

which certainly resembles the diagonal part of a Green's function.

<u>Question</u>: Consider the kernel

$$\frac{\langle 0 | \psi^*(\zeta)\psi(\zeta') | W \rangle}{\langle 0 | W \rangle}$$

over S^1. What sort of operator is it?

$$\langle 0 | \psi^*(\zeta)\psi(\zeta') | 0 \rangle = \sum \zeta^{-m} \zeta'^{n} \langle 0 | \psi^*_m \psi_n | 0 \rangle$$

$$= \sum_{n \geq 0} \zeta^{-n} \zeta'^{n}$$

$$= \sum_{n \leq 0} \langle \zeta | n \rangle \langle n | \zeta' \rangle = \text{projection operator on } \mathbb{C} + H_-$$

<u>Better</u>: Look at the kernel

$$\frac{\langle 0 | \psi(\zeta)\psi^*(\zeta') | W \rangle}{\langle 0 | W \rangle} = \delta(\zeta - \zeta') - \frac{\langle 0 | \psi^*(\zeta')\psi(\zeta) | W \rangle}{\langle 0 | W \rangle}$$

~~We know already~~ which looks like it comes from a $\bar\partial$-operator Green's function as $t \to (t')^+$. The limit as $t \to (t')^-$ is just minus

$$K(\zeta, \zeta') = + \frac{\langle 0 | \psi^*(\zeta')\psi(\zeta) | W \rangle}{\langle 0 | W \rangle}$$

and you see the Green's fn. jumping by the identity as t crosses t'. Recall if $|W\rangle = w_1 \wedge w_2 \wedge \cdots$

$$\langle \alpha | \psi(\zeta) | W \rangle = \sum (-1)^{n-1} w_n(\zeta) \langle \alpha | w_1 \wedge \cdots \wedge \hat{w}_n \wedge \cdots \rangle$$

and so we see that the image of K is contained in W. On the other hand

$$\overline{\langle 0 | \psi^*(\zeta') | \beta \rangle} = \overline{\sum_n \langle 0 | (\zeta')^{-n} \psi_n^* | \beta \rangle}$$

$$= \sum_{n \geq 0} \langle \beta | \psi_n | 0 \rangle (\zeta')^n$$

~~[illegible] should integrate against [illegible]~~ so that integrating $\langle 0 | \psi^*(\zeta') | \beta \rangle$ against $f(\zeta')$ is like taking the inner product with an elements of H_+. Consequently the kernel of K contains H_-.

Notice that

$$I - K : \quad \frac{\langle 0 | \psi(\zeta)\, \psi^*(\zeta') | W \rangle}{\langle 0 | W \rangle}$$

has W contained in its kernel and its image contained in H_-.

$$\langle 0 | \psi(\zeta) = \sum \zeta^n \langle 0 | \psi_n = \sum_{n<0} \zeta^n \langle 0 | \psi_n$$

$$\int \psi^*(\zeta') | W \rangle\, w(\zeta') = \Big\{ \int \sum_n \psi_n^* \langle n | \zeta' \rangle \langle \zeta' | w \rangle \Big\} | W \rangle$$
$$= \Big(\sum_n \psi_n^* \langle n | w \rangle \Big) | W \rangle = w \wedge | W \rangle = 0.$$

Thus $K(I-K) = 0$ and so K is a projection, clearly the projection relative to $H = H_- \oplus W$. Therefore we have proved:

<u>Prop</u>: Let $H = H_- \oplus W$ and let $\langle 0|$, $|W\rangle$ denotes Fock vectors belonging to H_- and W. Then

$$\delta(\zeta-\zeta') = \underbrace{\frac{\langle 0|\psi(\zeta)\psi^*(\zeta')|W\rangle}{\langle 0|W\rangle}}_{\substack{\text{proj on } H_- \text{ with}\\ \text{kernel } W}} + \underbrace{\frac{\langle 0|\psi^*(\zeta')\psi(\zeta)|W\rangle}{\langle 0|W\rangle}}_{\substack{\text{proj. on } W \text{ with}\\ \text{kernel } H_-}}$$

The Baker-Akhieser fn. comes out as follows: You want $\beta \in (1+H_-)\cap W$, so take 1 and split it

$$1 = f_- + w \quad \in H_- \oplus W$$

Then $\beta = w = 1 - f_-$. So

$$\beta = \underset{\text{on } W}{\text{proj of}}\ 1 = \int \frac{\langle 0|\psi^*(\zeta')\psi(\zeta)|W\rangle}{\langle 0|W\rangle}\, 1 \,\frac{d\zeta'}{2\pi i\zeta'}$$

$$= \frac{\langle 0|\psi_0^*\psi(\zeta)|W\rangle}{\langle 0|W\rangle}$$

Conclusion: I now am beginning to understand how the τ-fns. fit into the theory of determinants. So what remains is to work out the connection:

regularization process
gauge transformations and anomalies.

Actually I can interpret

$$G(\zeta,\zeta') = \frac{\langle 0|\psi(\zeta)\psi^*(\zeta')|W\rangle}{\langle 0|W\rangle}$$

$$= -\frac{\langle 0|\psi^*(\zeta')\psi(\zeta)|W\rangle}{\langle 0|W\rangle}$$

$|\zeta| = |\zeta'|^+$

$|\zeta| = |\zeta'|^-$

as the Green's function of a $\bar\partial$-operator in the following way. Think of

$$\zeta [redacted] = e^{-i(x+it)} = e^{-ix+t},$$

so that $|\zeta| > |\zeta'|$ when $t > t'$. In the above expression suppose that $|W\rangle$ is the element of the Fock space of $t=0$ coming from boundary conditions at ~~some~~ time $t_1 < 0$. The idea I have is that

$$\psi(\zeta) = \psi(e^{-ix+t}) = e^{-H_0 t}\psi(e^{-ix})e^{H_0 t}$$

and that $e^{H_0 t}|W\rangle$ makes sense for $t \leq t_1$. Thus the above Green's function is a time-ordered Green's fn.

Now the analogy with $\bar\partial$-operators is even better. [redacted] I am working with $\bar\partial$-operators of the form $\partial_{\bar z} + \alpha$, where α is of the form $\alpha(x)\delta(t)$. To compute the determinant I have to consider a change $\delta\alpha$. To form the Green's fn. for $\partial_{\bar z}+\alpha$ I use the operator to transport the boundary conditions for $t<0$ past $t=0$ to obtain W. Then $tr(G\,\delta\alpha)$ involves restricting G to $t=0$ in which case the projection operator is obtained. [redacted] This is sufficiently imprecise that one has to specify how the regularization is taking place.

So the problem becomes to make explicit

$$\frac{\langle 0|\psi^*(\zeta)\psi(\zeta)|W\rangle}{\langle 0|W\rangle} = \sum_{n\geq 1}\frac{\langle 0|p_n|W\rangle}{\langle 0|W\rangle}\zeta^{-n}$$

as a kind of regularized-diagonal-value for the function $-G(\zeta,\zeta')$. The obvious thing to try is to remove

$$\langle 0 \,|\, \psi^*(\zeta')\,\psi(\zeta)\,|\, 0 \rangle = \sum_{n\geq 0} \zeta'^{-n}\zeta^{n} = \frac{1}{1-\zeta/\zeta'}$$

and then take the limit as $\zeta \to \zeta'$. This clearly works for $W = H_+$.

When the [redacted] subspace W comes from a $t_1 < 0$ then G is a Green's function for $\partial_{\bar\zeta}$ in an annulus around $|\zeta| = 1$, hence I know that $-G(\zeta,\zeta') - \frac{1}{1-\zeta/\zeta'}$ is smooth (even analytic near $\zeta = \zeta'$) and so this is a valid regularization. Let's assume it works and go on.

Review: For each $\varphi : S^1 \longrightarrow \mathbb{C}^*$ of degree 0 I have determinant line determined by [redacted] H_- and by φW, so I get a line bundle over the set of these φ along with a canonical section. The τ-function is a determinant for this family (i.e. it is the image of the canonical section under a trivialization of this line bundle). We have

$$\tau(\varphi) = \langle 0 | \tilde\varphi | W \rangle$$

for some way of lifting φ into an operator on the Fock space. Hence

$$\delta \log \tau = \frac{\langle 0 |\, \widetilde{\delta\varphi\,\varphi^{-1}}\; \tilde\varphi | W \rangle}{\langle 0 | \tilde\varphi | W \rangle}.$$

This doesn't depend on [redacted] the choice of $\tilde\varphi$ and so reduces us to defining the operator $\widetilde{(\delta\varphi)\varphi^{-1}}$. In fact we can interpret as follows:

$$\delta \log \tau = \operatorname{tr}_{(reg)}(\delta\varphi\,\varphi^{-1} \cdot E)$$

where E is the projection [redacted] onto φW with kernel H_-. I think that the regularization process amounts to

$$tr_{(reg)}\left(\delta\varphi\varphi^{-1}E\right) = tr\left((\delta\varphi\varphi^{-1})(E-E_0)\right)$$

where E_0 = projection onto H_+ with kernel H_-.

Next I want to understand what gauge transformations might be in this setup. From the viewpoint of a family L_φ of holom. line bundles we have to give an isomorphism of the holom. family, that is for each φ an assoc. φ' and a holom. isomorphism of $L_\varphi \simeq L_{\varphi'}$. In the clutched case we therefore give an isomorphism φ^+ over M_+ and a φ^- over M_- and then φ' is roughly $\varphi^+\varphi(\varphi^-)^{-1}$.

This is also consistent with the $\bar\partial$ viewpoint, where we think of the clutched setup as given by singular $\bar\partial$ operators

$$\partial_{\bar z} + \alpha(x)\delta(t).$$

A gauge transformation φ changes $\alpha(x)\delta t$ by $\partial_{\bar z}\varphi\cdot\varphi^{-1}$, hence in order for it to preserve the form, φ must be a pair φ^+, φ^- of holomorphic autos. in the pieces M_+, M_- with a jump allowed across ∂M_+.

Hence I conclude that when I have a H_-, W given a gauge transformation is to be a pair of maps φ_-, φ_+ in the group of maps $S^1 \to \mathbb{C}^*$ of degree 0 such that φ_- preserves H_- and φ_+ preserves W. Then the effect on φ is $\varphi \mapsto \varphi_-\varphi\varphi_+$. Perhaps an interesting example occurs when W is a half-space such that $z^2W \subset W$. Then clearly

$$\varphi_+ = e^{\sum_1^\infty a_{2n}z^{2n}} \qquad \varphi_- = e^{\sum_0^\infty a_n z^{-n}}$$

will give gauge transformations.

It seems to be useful to eliminate φ_-

entirely by restricting the clutching functions to have the form $\varphi = e^{\sum_1^\infty x_n z^n}$. So our family consists of just these φ and our gauge transf. are given by things of this type preserving W.

Is there any anomaly? It seems not. For example if $z^2 W \subset W$, then it seems one has

$$\rho_2 |W\rangle = \rho_4 |W\rangle = \rho_6 |W\rangle = \cdots = 0.$$

In effect because $z^2 W \subset W$, we know that $\rho_2|W\rangle = c|W\rangle$ where c is [redacted] some sort of trace of z^2 acting on W, [redacted] and $c = 0$ because $z^2 = 0$ mod a certain filtration. This last point one sees by choosing a basis $w_1, w_2, \cdots$ for W with $z^2 w_n = w_{n+2}$ and then if one takes $|W\rangle = w_1 \wedge w_2 \wedge \cdots$ it's clear.

More convincing would be to consider the subgroup of the restricted general linear gp (this acts on Fock space up to scalar factors) which preserves W and show that z^2 is a bracket somehow.

However these arguments are all fallacious. Look at the simplest case where W is a half-space such that $zW \subset W$ and of index 0. Then I know that $W = f H_+$ where $f : S^1 \to S^1$ has degree 0 say

$$f = e^{\sum c_n z^n} \qquad c_{-n} = -\bar{c}_n, \quad c_0 = 0.$$

Then I can take $|W\rangle = e^{\sum_1^\infty c_{-n}\rho_{-n}} e^{\sum_1^\infty c_n \rho_n} |0\rangle = e^{\sum_1^\infty c_{-n}\rho_{-n}}|0\rangle$

and the τ-fn. is

$$\tau(x) = \langle 0 | e^{\sum_1^\infty x_n \rho_n} e^{+\sum_1^\infty c_{-n}\rho_n} | 0 \rangle$$

$$= e^{\sum_1^\infty x_n c_{-n} n} \qquad \text{since } [\rho_n, \rho_{-n}] = [redacted]\, n$$

Thus the anomaly exists.

October 27, 1982

One aspect of the KdV theory which I have never understood is the fact that the non-linear equation can be expressed as [redacted] an integrability condition among [redacted] operators depending on an extra spectral parameter z. The operators are in the form $\partial_x - U$, $\partial_t - V$ and the integrability condition is

$$0 = -[\partial_x - U, \partial_t - V] = \partial_x V - \partial_t U - [U, V].$$

When this is satisfied, there exists an invertible matrix fn. $\psi(x,t,z)$, the Baker fn, satisfying

$$\partial_x\psi\,\psi^{-1} = U \qquad \partial_t\psi\,\psi^{-1} = V.$$

Ex. 1: The chiral model. Here the operators [redacted] U, V are of the form $U = \dfrac{u(x,t)}{z-1}$ $V = \dfrac{v(x,t)}{z+1}$

so

$$\frac{\partial_t u}{z-1} - \frac{\partial_x v}{z+1} + \underbrace{\frac{[u,v]}{(z-1)(z+1)}}_{[u,v]\frac{1}{2}\left(\frac{1}{z-1} - \frac{1}{z+1}\right)} = 0$$

or

$$\partial_t u + \tfrac{1}{2}[u,v] = 0$$
$$\partial_x v + \tfrac{1}{2}[u,v] = 0$$

Ex. 2: Modified KdV. KdV arises from the Schroedinger equation $(\partial_x^2 - q)\psi = z^2\psi$, and Modified KdV arises from the system

$$\partial_x\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix} = \begin{pmatrix} z & p \\ p & -z\end{pmatrix}\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix}$$

The relation is that if p satisfies MKdV, then $q = p' + p^2$ satisfies KdV. $p \mapsto p' + p^2$ is the Miura transf. (Signs are probably all wrong.)

The above equation can be written

$$\partial_x \psi = U\psi \qquad U = p\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + \begin{pmatrix} 1 & \\ & -1 \end{pmatrix} z$$

so there is a singularity at $z = \infty$ which is to be preserved.

It should be possible for me to proceed from ~~the refle~~ the scattering analysis of Schr. eqn. when there is no reflection and generalize to the MKdV case.

Thus consider

$$\partial_x \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = \begin{pmatrix} ik & u \\ u & -ik \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$$

where $u(x) \to 0$ as $|x| \to \infty$. In general we have solutions analytic for k in the UHP

$$\varphi^+ \sim \begin{pmatrix} 0 \\ e^{-ikx} \end{pmatrix} \quad x \to -\infty \quad , \quad \psi^+ \sim \begin{pmatrix} e^{ikx} \\ 0 \end{pmatrix} \quad x \to +\infty$$

Also $\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}, k \longmapsto \begin{pmatrix} \psi_2 \\ \psi_1 \end{pmatrix}, -k$ is a symmetry taking φ^+, ψ^+ into solutions φ^-, ψ^- analytic in the LHP. We have

$$\begin{aligned} \varphi^+ &= A\psi^- + B\psi^+ \\ \varphi^- &= \tilde{B}\psi^- + \tilde{A}\psi^+ \end{aligned} \qquad \begin{aligned} \tilde{B}(k) &= B(-k) \\ \tilde{A}(k) &= A(-k) \end{aligned}$$

where $A(k) = |\psi^+ \ \varphi^+|$ is analytic in UHP and $\to 1$ as $|k| \to \infty$. Also $A\tilde{A} - B\tilde{B} = 1$.

Now assume no reflection: $B = \tilde{B} = 0$. Then

$$A(k) = \frac{1}{A(-k)}$$

shows A extends meromorphically into the LHP, hence it must be rational of the form

$$A(k) = \prod \frac{k - K_i}{k + K_i} \qquad \operatorname{Im}(K_i) > 0.$$

(Just a repeat of old stuff for Schroed. eqn., so I stopped.)