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1-diml. determinants 167–182 esp. 179–181

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$\mathrm{Tr}(e^{-t\Delta}K)$ does not always have an asymptotic expansion in powers of t p. 196 (also p. 200)

Grassmannian connection see p. 188

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Emphasize the holomorphic viewpoint: Fix the C^∞ surface + vector bundle (M,E) and let $SH(M,E)$ denotes the set of holomorphic structures on this pairs. One then has a map

$$(1) \qquad SH(M,E) \longrightarrow SH(M) = \Gamma\{\mathbb{P}(T^*_M)^+\}$$

whose fibre at a point $T^{1,0} \subset T^*$ is a torsor for $\Gamma\{\mathrm{Hom}(E, E\otimes T^{0,1})\}$. Now if we fix a connection on E then we get a [redacted] holomorphic section of [redacted] the above map. On the other hand we can trivialize the big vector bundle over $SH(M)$ whose fibre is $\Gamma\{\mathrm{Hom}(E,E\otimes T^{0,1})\}$ [redacted] as follows. Choose a basepoint $(T^{1,0})_0$ in $SH(M)$. Then for any point [redacted] we have $(T^{0,1})_0 \subset T^* \to T^*/T^{1,0} = T^{0,1}$ is an isomorphism. Therefore combining a section with this trivialization we get an isomorphism

$$(2) \qquad SH(M) \times \Gamma\{\mathrm{Hom}(E, E\otimes (T^{0,1})_0)\} \xrightarrow{\sim} SH(M,E)$$

which is holomorphic.

Next for the metric viewpoint. Choose a volume on M and an inner product on E. Then

$$SH(M) \xrightarrow{\sim} \{\text{metrics on } \underset{\uparrow \text{ call this set } \mathcal{H}}{M} \text{ with the given volume}\}$$

and if [redacted] $\xi \in SH(M)$, then

$$SH(E)_\xi \xrightarrow{\sim} \left\{\begin{array}{l}\text{connections on } E \text{ compatible}\\ \text{with the inner product}\end{array}\right\}$$

call this set $\mathcal{A}$.

Hence we get an isomorphism

$$(3) \qquad \mathcal{H} \times \mathcal{A} \xrightarrow{\sim} SH(M,E)$$

Notice that if a connection A is fixed, then the sections of (1) obtained from (2) or (3) are the same.

However A by itself [redacted] is just a real vector space. Consequently the product decompositions of (2), (3) appear to be different.

What I should understand now is the curvature form. So we assume to simplify that the [redacted] $\bar\partial$ operator is invertible. Then the line bundle has a canonical section with $|s|^2 = e^{-\zeta'(0)}$ and so the connection form relative to this section s is obtained by differentiating the [redacted] torsion:

$$\theta = \partial \log |s|^2$$

This leads to $i(\delta)\theta = \Big[\mathrm{Tr}_{(reg)}(\Delta^{-1}\delta\Delta)\Big]_{\text{complex linear part}}$

[redacted]

Better version: Any connection [redacted] on [redacted] E gives a section, which is holomorphic, of $SH(M,E) \to SH(M)$. The connections compatible with the inner product give a [redacted] family A of disjoint sections transversal to the fibres, whence the isomorphism (3). [redacted]

The family A is the set of leaves of a foliation of a complex manifold by holomorphic submanifolds. Is it a holomorphic foliation, meaning that this foliation is, locally at least, the fibre foliation of a holomorphic map? If so, then [redacted] A would be a complex manifold isomorphic to each fibre of (1). But this seems wrong, because it's like asking the complex structures on A obtained from different complex structures on M to be the same.

What is a simpler example [redacted] of a foliation by complex submanifolds which is not a holomorphic foliation? Take The bundle $O(1)$ over $\mathbb{P}(T^*)^+$ where T^* is a 2-diml complex vector space. Then each element of $T^*_{\mathbb{R}}$ gives a section, and these sections are disjoint, because if two coincided then the difference would be a real element of $O(-1)$ at that point. Also the different complex structures on $T^*_{\mathbb{R}}$ obtained from the isom. $T^*_{\mathbb{R}} \xrightarrow{\sim} O(1)$

are clearly different.

It's clear that the case of holomorphic structures on (M, E) is just this example spread over M.

[redacted] Let's return to the Laplacean. We fix a hermitian connection on $\mathbf{E}$ and [redacted] just vary the holomorphic structure on $\mathbf{M}$. We work around a holom. structure $(T^{1,0})_0$, describe locally by $(T^{1,0})_0$ spanned by dz. Another holom. structure is then described by a map $h : (T^{1,0})_0 \longrightarrow (T^{0,1})_0$ of norm < 1. Specifically $T^{1,0}_h = \text{span of graph of } h = \text{span of } dz + h\,d\bar{z}$. For the new complex structure we have

$$df = \partial_\omega f \cdot \omega + \partial_{\bar\omega} f \cdot \bar\omega$$

$$= \frac{1}{1-|h|^2}\Big[(\partial_z - \bar{h}\partial_{\bar z}) f\,(dz + h\,d\bar z) + (\partial_{\bar z} - h\partial_z) f\,(d\bar z + \bar h\,dz)\Big]$$

hence the $\bar\partial$ operators on functions is

$$f \longmapsto \frac{1}{1-|h|^2}(\partial_{\bar z} - h\partial_z) f \cdot d\bar z + \bar h\,dz$$

Put another way the projection $T^* \longrightarrow T^{0,1}_h$ is given by inner product with $\partial_{\bar\omega} = \frac{1}{1-|h|^2}(\partial_{\bar z} - h\partial_z)$ times $\bar\omega$.

Recall also that the metric on $T^{0,1}_h$ is given by

$$i\,\omega \wedge \bar\omega = |\omega|^2 \rho\, \tfrac{i}{2} dz\,d\bar z$$

$$\shortparallel$$

$$i\,(dz + h\,d\bar z)(d\bar z + \bar h\,dz) = i\,dz\,d\bar z\,(1 - |h|^2)$$

hence $|\omega|^2 = |\bar\omega|^2 = (1-|h|^2)(2/\rho)$.

[redacted] The connection on E can be described in terms of a local orthonormal frame by

$$\nabla = \nabla_z\,dz + \nabla_{\bar z}\,d\bar z \qquad \nabla_z = \partial_z - \alpha^*, \quad \nabla_{\bar z} = \partial_{\bar z} + \alpha$$

hence the $\bar\partial$-operator on E for the complex structure $T^{1,0}_h$ is

$$i(\partial_{\bar\omega})\nabla = \frac{1}{1-|h|^2}(\nabla_{\bar z} - h\nabla_z) \cdot d\bar z + \bar h\,dz$$

Next we compute the Laplacean D^*D.

$$(g|D^*Df) = (Dg|Df) = \int \left[\frac{1}{1-|h^2|}(\nabla_{\bar z}-h\nabla_z)g\right]^* \left[\frac{1}{1-|h|^2}(\nabla_{\bar z}-h\nabla_z)f\right] \times |\bar\omega|^2 \times \text{vol}$$

$$= \int \left[(\nabla_{\bar z}-h\nabla_z)g\right]^* \frac{1}{1-|h|^2}(\nabla_{\bar z}-h\nabla_z)f \; i\,dz\,d\bar z$$

$$= -\int g^* \frac{2}{\rho}\left(\nabla_z - \nabla_{\bar z}\bar h\right)\frac{1}{1-|h|^2}(\nabla_{\bar z}-h\nabla_z)f \; \frac{\rho}{2}\, i\,dz\,d\bar z$$

Hence $\boxed{-D^*Df = \frac{2}{\rho}\left(\nabla_z - \nabla_{\bar z}\bar h\right)\frac{1}{1-|h|^2}\left(\nabla_{\bar z}-h\nabla_z\right)f}$

To first order in h:

$$+\delta\Delta = \frac{2}{\rho}\left(\nabla_{\bar z}\bar h\nabla_{\bar z} + \nabla_z h\nabla_z\right)$$

I'd like to interpret this as $D^*\cdot\delta D + \delta D^*\cdot D$. Now D^* is given by

$$D^*(f\bar\omega) = -\frac{2}{\rho}\left(\nabla_z - \nabla_{\bar z}h\right)f.$$

The real question is how to define δD as an operator from E to $E\otimes(T^{0,1})_0$. The obvious method, since we are using the connection ∇ to give a section of $SH(M,E)\to SH(M)$ is to use ~~this section to lift tangent vectors. So this seems to mean that we~~ the connection ∇. The obvious choice is the operator

$$E \xrightarrow{\nabla} E\otimes T^* \xrightarrow{pr} E\otimes T^{1,0} \xrightarrow{id\otimes h} E\otimes T^{0,1}$$

which gives locally the operator

$$f \longmapsto h\nabla_z f\otimes d\bar z.$$

Actually we want the minus sign since

$$Df = \frac{1}{1-|h|^2}\left(\nabla_{\bar z}-h\nabla_z\right)f\cdot d\bar z + \bar h\, dz$$

$$= (\nabla_{\bar z} f)(d\bar z) - (h\nabla_z f)(d\bar z) + \bar h (\nabla_{\bar z} f) dz + O(h^2) \quad ?$$

September 22, 1982

We have fixed a connection ∇ on E and then lift a holomorphic structure on M to a $\bar\partial$-operator using
$$E \xrightarrow{\nabla} E\otimes T^* \xrightarrow{id\otimes proj} E\otimes T^{0,1}.$$

Fix a basepoint holom. structure $T_0^{1,0}$, then another one is the graph of a map $h: T_0^{1,0} \longrightarrow T_0^{0,1}$ with $|h|<1$. Specifically $T_h^{1,0}$ is spanned by $dz + h\,d\bar z$.

Now we want to understand better $T_h^{0,1}$ as h varies. This is a line bundle over $SH(M)\times M$. If we fix a point m of M, then what we have is a line $T_h^{0,1}|_m$ which depends only on h at m, and is the quotient line $O(1)$ at the point $h(m)\in \mathbb{P}(T^*|_m)^+$. Now $O(1)$ is a holomorphic line bundle over $\mathbb{P}(T^*|_m)^+$ and it has a metric, so there is a connection in it, that is, we have a way to lift vectors.

Let's analyze this situation at a given point of m. We then have a 2-dim'l vector space T^* with oriented real structure, and basis $dz, d\bar z$. We are describing points of $\mathbb{P}(T^*)^+$ by complex numbers h, by the line ($O(-1)$ at h) is spanned by $dz + h d\bar z$. Then by projecting $d\bar z$ into $O(1) = T^{0,1}$ we get a holom. section s of $O(1)$. The orthogonal projection of $d\bar z$ is
$$\left(\frac{1}{1-|h|^2}\, i(\partial_{\bar z} - h\partial_z)\, d\bar z\right)\bar\omega = \frac{1}{1-|h|^2}\,\bar\omega$$

and $|\bar\omega|^2 = (1-|h|^2)(2/\rho)$ we get $|s|^2 = \frac{1}{1-|h|^2}\underbrace{(2/\rho)}_{\text{const. in } h}$. Thus the connection is $\nabla s = s\,\theta$ where
$$\theta = \partial \log |s|^2 = \partial_h \log\frac{1}{1-|h|^2} = \frac{\bar h\, dh}{1-|h|^2}$$

The important point is to notice that $d\bar z$ is flat to the first order at $h=0$.

Now I should be able to make precise the change in the $\bar\partial$ operator $D_h : E \to E \otimes T_h^{0,1}$ belonging to a change δh in the holomorphic structure h. Use the section $\mathrm{proj}_h(d\bar z) = \dfrac{d\bar z + \bar h\, dz}{1-|h|^2}$ of $T_h^{0,1}$. [redacted] Let's choose an orthonormal frame for E, whence

$$\nabla = \nabla_z\, dz + \nabla_{\bar z}\, d\bar z \qquad \nabla_z = \partial_z - \alpha^*, \quad \nabla_{\bar z} = \partial_{\bar z} + \alpha$$

and then

$$D_h = \mathrm{proj}_h \nabla = \frac{1}{1-|h|^2}\left(\nabla_{\bar z} - h\nabla_z\right)\cdot\left(d\bar z + \bar h\, dz\right)$$

$$D_h f = \left(\nabla_{\bar z} - h\nabla_z\right) f \cdot \underbrace{\frac{d\bar z + \bar h\, dz}{1-|h|^2}}_{s}$$

Next recall that the connection on $T_h^{0,1}$ assigns to δh the change $\delta s = s\,\dfrac{\bar h\,\delta h}{1-|h|^2}$. Hence

$$\delta D_h f = \left(-\delta h\,\nabla_z f\right)s + \left(\nabla_{\bar z} - h\nabla_z\right)f\, s\,\frac{\bar h\,\delta h}{1-|h|^2}$$

$$= \delta h\left\{-\nabla_z f + \frac{\bar h}{1-|h|^2}\left(\nabla_{\bar z} - h\nabla_z\right)f\right\}s$$

$$= \frac{\delta h}{1-|h|^2}\left\{-(1-\cancel{|h|^2})\nabla_z f + \bar h\nabla_{\bar z} f - \cancel{|h|^2\nabla_z f}\right\}s$$

$$= \frac{\delta h}{1-|h|^2}\left\{-\nabla_z f + \bar h\nabla_{\bar z} f\right\}s$$

$$\delta D_h f = -\frac{\delta h}{1-|h|^2}\left(\nabla_z - \bar h\nabla_{\bar z}\right)f \quad \cdot s$$

Compare this with

$$E \xrightarrow{\nabla} E\otimes T^* \xrightarrow{\mathrm{proj}_h} E\otimes T_h^{1,0} \xrightarrow{\delta h} E\otimes T_h^{0,1}$$

This gives $\delta h \cdot \dfrac{1}{1-|h|^2}\left(\nabla_z - \bar h\nabla_{\bar z}\right)\bar\omega$. This probably means you haven't calculated the map $T_h^{1,0} \to T_h^{0,1}$ belonging to δh properly. This map should be the graph of $h+\delta h : T_0^{1,0} \to T_0^{0,1}$ relative to the decomposition $T^* = T_h^{1,0} \oplus T_h^{0,1}$. So

$$dz + (h+\delta h)\,d\bar z = \omega + \delta h\, d\bar z$$

$$= \omega + \delta h \left(\frac{1}{1-|h|^2}\right)\left\{-\bar h\,\omega + \bar\omega\right\}$$

$$= \left(1 + \frac{-\bar h\,\delta h}{1-|h|^2}\right)\omega + \frac{\delta h}{1-|h|^2}\,\bar\omega$$

which is proportional to $\omega + \frac{\delta h}{1-|h|^2}\,\bar\omega$.

Conclusion of the calculation: Let's equip $\{T_h^{0,1}\}$ with the connection in the h-direction coming from the metric and holomorphic structure of $\mathcal{O}(1)$ over $\mathbb{P}(T^*)^+$. Then relative to this connection we can speak about δD_h corresponding to a change δh of h. Then δD_h is minus the operator

$$E \xrightarrow{\nabla} E\otimes T^* \xrightarrow{\mathrm{proj}_h} E\otimes T_h^{1,0} \longrightarrow E\otimes T_h^{0,1}$$

where the last map is $\mathrm{id}\otimes$ the map $T_h^{1,0} \to T_h^{0,1}$ defined by δh.

Simplest case is to work around $h=0$. Then

$$\delta D = -\,\delta h \cdot \nabla_z\,.$$

The problem: I want to think of a tangent vector at a point of $SH(M,E)$ as being a map $J_1^h(E) \to E\otimes T^{0,1}$ whose restriction to $E\otimes T^{1,0}$ is given by a map $\delta h : T^{1,0} \to T^{0,1}$. For example a ∂-operator $E \to E\otimes T^{1,0}$ followed by δh. The point of the above calculation is that if we use the connection ∇ on E to give the ∂-operator, and the connection on $T^{0,1}$, then we get the tangent vector over δh along the section of $SH(M,E) \to SH(M)$ defined by ∇.

However I still expect a metric independent result, namely, that the operator $-\delta h \cdot \nabla'$ restricts to the tangent vector to the section over δh.

Situation: I have fixed a hermitian connection in E and hence can convert holomorphic structures on M into ∂-operators. So I get a family of Laplaceans Δ_h and want to have

$$\circledast \qquad \delta \Delta_h = D_h^* \, \delta D_h + (\delta D_h)^* D_h$$

where δD_h is complex linear in δh and $(\delta D_h)^* = \delta(D_h^*)$ is conjugate-linear. ~~To see this is true we work~~ ~~at a p~~ δD_h is defined as above using the ∂-part of the connection on E, namely

$$\delta D_h = - \delta h \, \nabla' .$$

The notation is lousy. We have at the point of SH(M) $D = \nabla''$ and $\delta D = -\delta h \, \nabla'$, where $\nabla = \nabla' + \nabla''$. Now to check $\circledast$ we can work locally using local coordinates. We have

$$-\Delta = \frac{2}{\rho}\left(\nabla_z - \nabla_{\bar z} \bar h\right) \frac{1}{1-|h|^2}\left(\nabla_{\bar z} - h \nabla_z\right)$$

$$Df = (\nabla_{\bar z} - h\nabla_z) f \cdot s \qquad\qquad s = \frac{d\bar z + \bar h \, dz}{1-|h|^2}$$

$$-D^*(fs) = \frac{2}{\rho}\left(\nabla_z - \nabla_{\bar z} \bar h\right) \frac{1}{1-|h|^2} f$$

and we are working around $h = 0$. Thus

$$\delta \Delta = \frac{2}{\rho}\left(\nabla_{\bar z} \overline{\delta h} \, \nabla_{\bar z} + \nabla_z \delta h \, \nabla_z\right)$$

$$(\delta D) f = -\delta h \, (\nabla_z f) \cdot s$$

$$-D^*(fs) = \frac{2}{\rho}\left(\nabla_z - \nabla_{\bar z} \overline{\delta h}\right) f$$

Therefore $$D^* \delta D f = \frac{2}{\rho} \nabla_{\bar z}\ \delta h\ \nabla_z f$$

and so $\circledast$ will will work.

Now we come to the critical problem, namely, how to compute

$$\mathrm{Tr}^{(\mathrm{reg})}(D^{-1}\delta D) = 0\text{th coeff. of } \mathrm{Tr}(e^{-t\Delta} D^{-1}\delta D).$$

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The problem is to calculate the regularized $\mathrm{Tr}(D^{-1}\delta D)$ where $D : E \longrightarrow E \otimes T^{0,1}$ is the $\bar\partial$-operator ∇'' belonging to a hermitian connection $\nabla : E \to E\otimes T^*$ on E, and a complex structure $T^{1,0}$ on M, and where $\delta D = -\delta h\, \nabla'$ where ∇' is the ∂-operator $\nabla' : E \longrightarrow E \otimes T^{1,0}$ belonging to ∇ and $\delta h : T^{1,0} \longrightarrow T^{0,1}$ is an infinitesimal change in the complex structure.

It seems that the first thing to do is to describe carefully the singularities of $D^{-1}\delta h\,\nabla'$ along the diagonal, and then to discuss their regularization. Let's avoid assuming the connection is hermitian, and let's choose a local holomorphic frame in E so that we have

$$\nabla = \underbrace{(\partial_z + \gamma)dz}_{\nabla'} + \underbrace{\partial_{\bar z}\, d\bar z}_{\nabla'' = D}$$

Then we have that the kernel $G(z,z')dz'$ for D^{-1} is of the form

$$\left(\frac{2\pi}{i}\right) G(z,z') = \frac{1}{z-z'} + \text{smooth}.$$

Now $D^{-1}\delta h\,\nabla'$ is the operator

$$f \longmapsto \int G(z,z')\,dz'\;\delta h(z')\frac{d\bar z'}{\cancel{dz'}}\left(\partial_{z'} + \gamma\right) f\, \cancel{dz'}$$

$$= \int G(z,z')\,\delta h(z')\left(\partial_{z'}+\gamma\right) f(z')\,dz'd\bar z'$$

$$= \int \left[-\partial_{z'}\big(G(z,z')\delta h(z')\big) + G(z,z')\delta h(z')\gamma(z')\right] f(z')\,dz'd\bar z'$$

So the kernel of $D^{-1}\delta h\,\nabla'$ is

$$\left[-\partial_{z'}\big(G(z,z')\,\delta h(z')\big) + G(z,z')\,\delta h(z')\,\gamma(z')\right] \quad dz'd\bar z'$$

We can write this

$$[\text{redacted}]\left[-\partial_{z'}G(z,z')_{\wedge}\,\delta h(z') + G(z,z')\left(-\partial_{z'}\delta h(z') + \delta h(z')\gamma\right)\right] dz'd\bar{z}'$$

We know that when we regularize the kernel $G(z,z')\,\delta\alpha(z')\,dz'd\bar{z}'$ using the heat kernel, we get $J(z)\,\delta\alpha(z)\,dz\,d\bar{z}$, where J is the finite part of G constructed using the connection ∇ and the metric on M. It's clear now that we must review the derivation of this result, in order to generalize to include the $\partial_{z'}G(z,z')$ term.

We need the asymptotic expansion of the heat kernel $e^{-t\Delta}$, which has the form

$$\frac{e^{-\frac{u(z,z')}{2t}}}{2\pi t}\,\sigma(z,z')\left[I + tA_1(z,z') + t^2A_2(z,z') + \cdots\right]\rho(z')\tfrac{i}{2}dz'd\bar{z}'$$

where $u(z,z')$ = distance squared between z, z' and $\sigma(z,z')$ is an appropriate determinant. [redacted] Because Δ is self-adjoint the kernel should be symmetric except for the volume element.

Let's review what we know [redacted] about σ. The setting is a Riemannian manifold and we look at the Lagrangian giving the kinetic energy of a curve. The Hamiltonian is $H = \frac{1}{2}|p|^2$ and the Hamilton-Jacobi equation is $\partial_t S + \frac{1}{2}|\nabla_x S|^2 = 0$. A simple time dependence is $S(xt, x't') = \dfrac{u(x,x')}{t-t'}$ in which case we get $|\nabla u|^2 = 2u$ which has the solution $u(x,x') = \frac{1}{2}$ distance squared between x and x'. It will be simpler to change u by a factor of 2 and put

$$S(xt, x't') = \frac{u(x,x')}{2(t-t')}$$

whence u = distance squared satisfies $|\nabla u|^2 = 4u$.

Now let us consider a linearized problem around a trajectory

The linearized problem gives a symplectic transformation from $\delta x', \delta p'$ to $\delta x, \delta p$. This symplectic transformation is described by the action $x, x' \longmapsto S(tx, t'x')$ by the formulas
$$p = \frac{\partial S}{\partial x} \qquad -p' = \frac{\partial S}{\partial x'} .$$
(Strictly one must take these formulas and [redacted] allow changes $\delta x, \delta x'$ and compute $\delta p, \delta p'$).

~~Somehow~~ we now quantize, i.e. we consider the kernel $U(xt, x't')$ which computes the transformation from a wave function at time t' to a wave function at time t. This has the form $e^{iS(xt,x't')}$ for the most part. Actually better would be
$$e^{i/\hbar\, S(xt, x't')}\, \sigma(xt, x't')\, dx'$$
with an error $1 + O(\hbar)$. To determine σ one [redacted] works along the trajectory from $x't'$ to xt and linearizes the problem. This [redacted] gets you to the case where S is quadratic in x, x' (which now correspond to the old $\delta x, \delta x'$.) Then the factor σ should be determined so that one gets a unitary transformation. But we know that the kernel
$$e^{i\left(\frac{ax^2}{2} + bxx' + c\frac{x'^2}{2}\right)}$$
gives a unitary transformation when multiplied by a fn. of b, namely $\sqrt{\frac{b}{2\pi}}$, and in n-dimensions $\sqrt{\det\left(\frac{b}{2\pi}\right)}$. However $b = \frac{\partial^2 S}{\partial x \partial x'}$. Thus σ is determined by unitarity of the Schroedinger operator.

There's also the heat operator viewpoint. One wants the (approximate) heat kernel

$$\frac{e^{-\frac{u(x,x')}{2t}}}{(2\pi t)^{n/2}}\,\sigma(x,x')\,d^n x'$$

to have total integral one. [redacted] One way to arrange this is to take the Gaussian measure on the tangent space T_x at x[redacted] and transport it to the manifold via the exponential map. In fact that is what the above approximate heat kernel maybe is. $u(x,x')$ is the distance squared of x' from x and $\sigma(x,x')\,d^n x'$ is the image of the measure on T_x at the point x'. Hence $\sigma(x,x')$ is the Jacobian of the map exp: $y \mapsto x'$ where $y \in T_x$.

Let's check this: y is essentially [redacted] the initial velocity of the trajectory from x to x'. Let's interchange x and x' so that x' is fixed. Then $y = p' = -\frac{\partial S}{\partial x'}$ and hence the Jacobian is the matrix $-\frac{\partial^2 S}{\partial x \partial x'}$. So the Jacobian determinant of $y \mapsto x$ is $\det\left(-\frac{\partial^2 S}{\partial x \partial x'}\right)$. This is missing a square root. ??

So there is a mistake, probably in assuming the approximate heat kernel should be the image of the Gaussian measure.

September 24, 1982

The problem is to calculate the regularized trace of $D^{-1}\,\delta h\,\nabla'$. For this we need to evaluate asymptotically as $t \downarrow 0$ an integral

$$\int d^2z \; e^{-\frac{u(z)}{t}} \frac{F(z)}{tz^2}$$

where F is smooth near $z=0$ and $u(z) = |z|^2 v(z)$ where v is smooth and >0 near $z=0$. We can evaluate the coefficients formally using the power series of v and F around $z=0$. The above integral is equal to

$$\int d^2z \; e^{-|z|^2 v(\sqrt{t}z)} \frac{F(\sqrt{t}z)}{tz^2}$$

$$= \int \frac{d^2z}{tz^2} \, e^{-|z|^2 v_0 - |z|^2\left[v_1\sqrt{t}z + v_{\bar 1}\sqrt{t}\bar z + v_2\frac{tz^2}{2} + v_{1\bar 1}tz\bar z + v_{\bar 2}\frac{t\bar z^2}{2} + \cdots\right]} \times \left(F_0 + F_1\sqrt{t}z + F_{\bar 1}\sqrt{t}\bar z + \cdots\cdots\right)$$

By S^1 symmetry the moments $\int d^2z \; e^{-|z|^2 v_0} z^m \bar z^n$ vanish unless $m=n$. In this case we have

$$\int d^2z \; e^{-|z|^2 v_0} |z|^{2n} = n!\,(v_0)^{-n} \underbrace{\int d^2z \, e^{-|z|^2 v_0}}_{\frac{\pi}{v_0}}$$

Let's look at the terms in the integral involving $1, \sqrt{t}, t$

$$\boxed{\cancel{e^{-|z|^2 v_1\sqrt{t}z} = 1 - |z|^2 v_1\sqrt{t}z + \tfrac{1}{2}|z|^4 v_1^2 t z^2 + \cdots}}$$

$$\boxed{\cancel{e^{-|z|^2 v_{\bar 1}\sqrt{t}\bar z} = 1 - |z|^2 v_{\bar 1}\sqrt{t}\bar z + \tfrac{1}{2}|z|^4 v_{\bar 1}^2 t \bar z^2 + \cdots}}$$

$$\boxed{\cancel{e^{-|z|^2(v_1\sqrt{t}}}}$$

$$e^{-|z|^2\sqrt{t}\,(v_1 z + v_{\bar 1}\bar z) - |z|^2 t\left(v_2\frac{z^2}{2} + v_{1\bar 1} z\bar z + v_{\bar 2}\frac{\bar z^2}{2}\right) + \cdots}$$

$$= \left(1 - |z|^2\sqrt{t}\,(v_1 z + v_{\bar 1}\bar z) + \tfrac{1}{2}|z|^4 t\,(v_1 z + v_{\bar 1}\bar z)^2 + \cdots\right)$$

$$\times\left(1 - |z|^2 t\left(v_2\frac{z^2}{2} + v_{1\bar 1} z\bar z + v_{\bar 2}\frac{\bar z^2}{2}\right) + \cdots\right)$$

Also $\quad F(\sqrt{t}\,z) = F_0 + \sqrt{t}\,(F_1 z + F_{\bar 1}\bar z) + t\left(F_2\frac{z^2}{2} + F_{1\bar 1} z\bar z + F_{\bar 2}\frac{\bar z^2}{2}\right) + \cdots$

When these are multiplied, we look for monomials $|z|^{4n} z^2$. Thus there is no $\frac{1}{t}$, or $\frac{\sqrt{t}}{t}$ contribution. The first contribution occurs for $\frac{t}{t}$ and the coefficient is

[redacted] $\frac{1}{2}v_1^2 F_0\langle|z|^4\rangle - \frac{1}{2}v_2 F_0\langle|z|^2\rangle + \frac{1}{2}F_2\langle 1\rangle -$ [redacted] $v_1 F_1\langle|z|^2\rangle$

where $\langle|z|^{2n}\rangle = \int e^{-|z|^2 v_0}|z|^{2n}\,d^2z = \frac{\pi}{v_0}\frac{n!}{v_0^n}$. Therefore we end up with the formula:

$$\boxed{\begin{aligned}&\int d^2z\, e^{-\frac{u(z)}{t}}\frac{F(z)}{t z^2} \sim \frac{\pi}{v_0}\left\{\frac{v_1^2}{v_0^2}F_0 - \frac{1}{2}\frac{v_2}{v_0}F_0 + \frac{1}{2}F_2 - \frac{v_1}{v_0}F_1\right\}\\ &\text{where } u(z) = |z|^2\left(v_0 + v_1 z + v_{\bar 1}\bar z^2 + v_2\frac{z^2}{2} + \cdots\cdots\right)\\ &\qquad F(z) = F_0 + F_1 z + F_{\bar 1}\bar z + F_2\frac{z^2}{2} + \cdots\cdots\end{aligned}}$$

Special case: $\quad F(z) = z f(z) = z(f_0 + f_1 z + f_{\bar 1}\bar z + \cdots)$

Then $\quad F_0 =$ [redacted] $0,\ F_1 = f_0,\ F_2 = 2f_1 \quad$ and we get

$$\int d^2z\,\frac{e^{-\frac{u(z)}{t}}}{t}\,\frac{f(z)}{z} \sim \frac{\pi}{v_0}\left\{f_1 - \frac{v_1}{v_0}f_0\right\}$$

which should check with our previous work.

September 27, 1982

$$u(z) = |z|^2 (v_0 + v_1 z + v_{\bar 1}\bar z + \dots) \qquad |\partial_z u|^2 = \rho u$$

I need to know v to second order.

$$\partial_z u = \bar z v + |z|^2 \partial_z v = \bar z (v + z\partial_z v)$$

$$= \bar z \left(v_0 + v_1 z + v_{\bar 1}\bar z + v_2 \frac{z^2}{2} + v_{1\bar 1} z\bar z + v_{\bar 2}\frac{\bar z^2}{2} + \dots \right)$$

$$v_1 z \qquad\qquad + v_2 z^2 + v_{1\bar 1} z\bar z$$

$$\partial_z u = \bar z \left(v_0 + 2v_1 z + v_{\bar 1}\bar z + \tfrac{3}{2} v_2 z^2 + 2v_{1\bar 1} z\bar z + \tfrac12 v_{\bar 2}\bar z^2 + \dots \right)$$

$$\partial_{\bar z} u = z \left(v_0 + v_1 z + 2v_{\bar 1}\bar z + \tfrac12 v_2 z^2 + 2v_{1\bar 1} z\bar z + \tfrac32 v_{\bar 2}\bar z^2 + \dots \right)$$

$$|\partial_z u|^2 = |z|^2 \Big\{ v_0^2 + 3v_0 v_1 z + 3 v_0 v_{\bar 1}\bar z + (2 v_0 v_2 + 2v_1^2) z^2$$
$$+ (4 v_0 v_{1\bar 1} + 5 v_1 v_{\bar 1}) z\bar z + \dots \Big\}$$

$$\rho u = |z|^2 \Big\{ \rho_0 v_0 + (\rho_0 v_1 + \rho_1 v_0) z + (\rho_0 v_{\bar 1} + \rho_{\bar 1} v_0)\bar z$$
$$+ \left(\rho_0 \frac{v_2}{2} + \rho_1 v_1 + \frac{\rho_2}{2} v_0\right)\frac{z^2}{2} + \left(\begin{matrix}\rho_0 v_{1\bar 1} + \rho_{1\bar 1} v_0 \\ \rho_1 v_{\bar 1} + \rho_{\bar 1} v_1\end{matrix}\right) z\bar z + \dots$$

$$v_0^2 = \rho_0 v_0 \implies \boxed{v_0 = \rho_0}$$

$$3 v_0 v_1 = \rho_0 v_1 + \rho_1 v_0 \qquad 2v_1 = \rho_1 \qquad \boxed{v_1 = \tfrac12 \rho_1} \quad \boxed{v_{\bar 1} = \tfrac12 \rho_{\bar 1}}$$

$$2\underbrace{v_0}_{\rho_0} v_2 + 2 v_1^2 = \rho_0 \frac{v_2}{2} + \rho_1 v_1 + \frac{\rho_2}{2} v_0$$

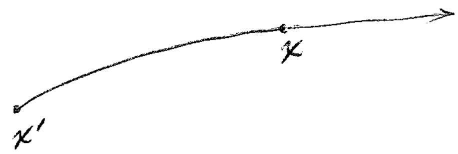
$$4\rho_0 v_2 + 4 \overbrace{v_1^2}^{\frac14 \rho_1^2} = \rho_0 v_2 + 2\rho_1 \overbrace{v_1}^{\frac12 \rho_1} + \rho_2 \rho_0$$

$$3\rho_0 v_2 = \rho_0 \rho_2 \qquad \boxed{v_2 = \tfrac13 \rho_2}$$

$$4 \underbrace{v_0}_{\rho_0} v_{1\bar 1} + 5\underbrace{v_1 v_{\bar 1}}_{\frac14 \rho_1\rho_{\bar 1}} = \rho_0 v_{1\bar 1} + \rho_{1\bar 1}\underbrace{v_0}_{\rho_0} + \underbrace{\rho_1 v_{\bar 1} + \rho_{\bar 1} v_1}_{\rho_1 \rho_{\bar 1}}$$

$$3\rho_0 v_{1\bar 1} = -\frac14 \rho_1 \rho_{\bar 1} + \rho_0 \rho_{1\bar 1} \qquad \boxed{v_{1\bar 1} = \frac13 \rho_{1\bar 1} - \frac1{12}\frac{\rho_1 \rho_{\bar 1}}{\rho_0}}$$

Normal coordinates: Let $S(x,x') = \frac{1}{2} r(x,x')^2$ in a Riemannian manifold. We know that $\nabla_x S(x,x')$



is a tangent vector at x to the geodesic from x to x' and it has length $r(x,x')$. Similarly $-\nabla_{x'} S(x,x')$ is tangent to the geodesic at x' and has length $r(x,x')$. Hence under the exponential map $-\nabla_{x'} S(x,x')$ goes to x. Thus

$$x \longmapsto -\nabla_{x'} S(x,x') \in T_M|_{x'}$$

is the inverse of the exponential map at the point x'.

Go back to our surface with the complex coordinate z and the volume $\rho \frac{i}{2} dz d\bar{z}$. $dS(z,z') = \partial_x S\, dx + \partial_y S\, dy$ corresponds to $\nabla S = \frac{1}{\rho}\partial_x S\, \partial_x + \frac{1}{\rho}\partial_y S\, \partial_y$. In general we like to describe tangent vectors as

$$\begin{aligned} a\,\partial_z + \bar{a}\,\partial_{\bar{z}} &= a\,\tfrac{1}{2}(\partial_x - i\partial_y) + \bar{a}\,\tfrac{1}{2}(\partial_x + i\partial_y) \\ &= \mathrm{Re}(a)\,\partial_x + \mathrm{Im}(a)\,\partial_y \end{aligned}$$

Thus ∇S corresponds to $a = \frac{1}{\rho}(\partial_x S + i\,\partial_y S) = \frac{2}{\rho}\partial_{\bar{z}} S$:

$$\boxed{\nabla S = \frac{2}{\rho}\partial_{\bar{z}} S \cdot \partial_z + \frac{2}{\rho}\partial_z S \cdot \partial_{\bar{z}}}$$

Thus with $u = r(z,z')^2$ we know that the radial vector of length r is

$$\frac{1}{\rho}\left(\partial_{\bar{z}} u \cdot \partial_z + \partial_z u \cdot \partial_{\bar{z}}\right)$$

September 26, 1982

It seems desirable to work out what you can expect from a calculation in normal coordinates, before you immerse yourself in calculations.

Review the setup. I have a surface M with metric and a hermitian vector bundle with connection $\nabla : E \to E \otimes T^*$. Then $\nabla = \nabla' + \nabla''$ and we write $D = \nabla''$. The boundary conditions are such that D is invertible. Given any $\delta h : T^{1,0} \to T^{0,1}$ I can form the operator $\underbrace{D^{-1}\delta h \nabla'}_{K}$:

$$E \xrightarrow{\nabla'} E \otimes T^{1,0} \xrightarrow{\delta h} E \otimes T^{0,1} \xrightarrow{D^{-1}} E$$

and the problem is to calculate the regularized trace of this operator. This is defined by the constant term in the asymptotic expansion for

$$\mathrm{Tr}\left(e^{-t\Delta} K\right). \qquad \Delta = D^* D$$

[redacted] In the present situation, because δh represents a change in the metric which doesn't change the volume and hence the value of ζ at $s = 1$, [redacted] one expects the asymptotic expansion to begin with the constant term.

The operator $e^{-t\Delta} K$ has a smooth Schwartz kernel, hence its trace is calculated by restricting to the diagonal and integrating. Let K have kernel $K(x, x')|d^2x'|$ and $e^{-t\Delta}$ have kernel $L_t(x, x')|d^2x'|$. Then $e^{-t\Delta}K$ has the kernel

$$\left[\int L_t(x, x'')|d^2x''| K(x'', x')\right] |d^2x'|$$

and [redacted] to take the trace we put $x' = x$ and integrate. One knows

$$L_t(x, x') = \frac{e^{-\frac{r(x,x')^2}{2t}}}{(2\pi t)^{n/2}} \tilde{L}_t(x, x')$$

x, x' have to be in a nbd. of the diagonal

where $\tilde{L}$ is smooth in t, x, x' and where it has a nice asymptotic expansion as $t \to 0$:

$$\tilde{L}_t \sim \tilde{L}_0 + t\tilde{L}_1 + \cdots$$

So we end up needing an asymptotic expansion for

$$\int |dy| \; [redacted] \; \frac{e^{-\frac{r(x,y)^2}{2t}}}{(2\pi t)^{n/2}} \tilde{L}_t(x,y) K(y,x).$$

Presumably this will be calculated separately for each term $t^m L_m$ and then added together, so it is sufficient to understand the case where $\tilde{L}_t = L_{[redacted]}(x,y)$ is a smooth function. Also the asymptotic expansion will depend only on the power series expansion of L and K around $y = x$. [redacted] Since K is singular, I should really say the symbol of K as a pseudo-differential operator.

[redacted] Therefore we reach the following analytical question: To show that

$$\int \underbrace{d^n y \; \frac{e^{-\frac{|y|^2}{2t}}}{(2\pi t)^{n/2}}}_{\text{Gaussian measure}} \; \underbrace{L(y)}_{\text{smooth}} \; \underbrace{K(y)}_{\substack{\text{singular at } y=0 \\ \text{with Fourier transf. having} \\ \text{asymptotic expansion in} \\ \text{homogeneous functions of } k.}}$$

has an asymptotic expansion in t as $t \downarrow 0$. Clearly we should replace LK by K alone, using the composition thm. for ΨDO's. Put

$$K(y) = \int \frac{d^n k}{(2\pi)^n} e^{iky} \hat{K}(k)$$

Then

$$\int d^n y \; \frac{e^{-\frac{|y|^2}{2t}}}{(2\pi t)^{n/2}} K(y) = \int \frac{d^n k}{(2\pi)^n} \underbrace{\left(\int d^n y \; \frac{e^{-\frac{|y|^2}{2t}}}{(2\pi t)^{n/2}} e^{iky} \right)}_{e^{-\frac{t|k|^2}{2}}} \hat{K}(k)$$

$$= t^{-n/2} \int \frac{d^n k}{(2\pi)^n}\, e^{-\frac{1}{2}|k|^2}\, \hat{K}\left(\frac{k}{\sqrt{t}}\right)$$

But now we are assuming that $\hat{K}$ has an asymptotic exp.

$$\hat{K}(k) \sim \sum_{m \leq N} f_m(k) \qquad \text{as} \quad k \to \infty$$

where the f_m are homogeneous of degree m. Hence the above becomes formally

$$t^{-n/2} \int \frac{d^n k}{(2\pi)^n}\, e^{-\frac{1}{2}|k|^2} \sum_{m \leq N} t^{-m/2} f_m(k).$$

One must be a bit more careful. If $\hat{K}$ is smooth and rapidly decreasing, then it has zero asymptotic expansion, yet

$$\int \frac{d^n k}{(2\pi)^n}\, e^{-t\frac{|k|^2}{2}}\, \hat{K}(k) \longrightarrow \int \frac{d^n k}{(2\pi)^n}\, \hat{K}(k) = K(0).$$

So what happened? Look first at the case where K is smooth at $y = 0$. Then expand

$$K(y) = \sum K_\alpha \frac{y^\alpha}{\alpha!}$$

and

$$\int d^n y\, \frac{e^{-\frac{|y|^2}{2t}}}{(2\pi t)^{n/2}}\, K(y) = \sum_{\alpha \text{ even}} \frac{K_\alpha}{\alpha!}\, t^{|\alpha|/2} \left(\int d^n y\, \frac{e^{-\frac{|y|^2}{2}}}{(2\pi)^{n/2}}\, y^\alpha \right)$$

is the asymptotic expansion. The same result is obtained if one does

$$\int \frac{d^n k}{(2\pi)^n}\, \underbrace{e^{-t\frac{|k|^2}{2}}}_{\sum \frac{t^m}{m!\, 2^m} |k|^{2m}}\, \hat{K}(k)$$

$$\sum \frac{t^m}{m!\, 2^m} |k|^{2m}\, \hat{K}(k)$$

and relates the moments of $\hat{K}$ to derivatives of $K(y)$ at $y = 0$.

Next I have to work in the singularities of K at $y = 0$. These have a different effect than the derivatives.

So we seem to have found an interesting analytical question, namely under what conditions on a distribution $K(y)$ we get an asymptotic expansion for

$$\int d^n y \, \frac{e^{-|y|^2/2t}}{(2\pi t)^{n/2}} K(y)$$

as $t \downarrow 0$. This obviously depends only on K in the nbd. of 0. We can do the integration over the angular directions $|y| =$ [redacted] r first, hence we must first understand the case of a radial integral.

$$\int_0^\infty \mathrm{vol}(S^{n-1}) \, r^{n-1} dr \, \frac{e^{-\frac{r^2}{2t}}}{(2\pi t)^{n/2}} \varphi(r)$$

We can transform this to a Laplace transform

$$\int_0^\infty e^{-u/t} \varphi(u) \, du$$

where $t \downarrow 0$ corresponds to $s = \frac{1}{t} \longrightarrow +\infty$. So what kind of conditions on $\varphi(u)$, $u \geq 0$ [redacted] guarantee an asymptotic expansion as $\mathrm{Re}(s) \to +\infty$ for

$$\int_0^\infty e^{-su} \varphi(u) \, du$$

Example: $$\int_0^\infty e^{-su} u^{\alpha-1} du = \frac{\Gamma(\alpha)}{s^\alpha}.$$

Standard notation

$$F(s) = \int_0^\infty e^{-st} f(t) \, dt \qquad f(t) = \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} F(s) e^{st} ds$$

~~Assume $e^{-st} f(t)$~~ Now we have the standard question of what to do about $f(t)$ which are not integrable as $t \downarrow 0$.

Basic problem: Consider a distribution $K(y)$ on $\mathbb{R}^n$ which is smooth outside of the point $y=0$. Describe precisely what is meant by K having an asymptotic expansion in terms of homogeneous functions as $y \to 0$. Explain the relation with the asymptotic expansion of the Fourier transform as $k \to \infty$.

Idea: Consider an elliptic operator $D_0 : E \to F$ over a compact manifold M. To it we can attach a line with a metric defined by analytic torsion. Let's now consider operators of the form $D_0 + B$ where B has order < the order of D. Then we get a holomorphic line bundle over the complex vector space of these B. This line bundle $\mathcal{L}$ has the analytic torsion metric, hence it has a connection. Now consider gauge transformations of the form $D \mapsto e^{-itf} D e^{itf}$ where f is a real function on M. This gives an action of the [redacted] group of these functions on the space of B and we know the action lifts to the line bundle preserving the metric and hence the connection. Then we know the action on $\mathcal{L}$ is of the form $\nabla_{\bar{f}} + \varphi_f$ where $\bar{f}$ denotes the vector field on $\{B\}$ which assigns to [redacted] D, the tangent vector $\frac{1}{i}[f, D$[redacted]$]$. Now $\varphi_{\bar{f}}(D)$ is a linear function on the Lie algebra of f, hence should be given by integration against a density, or at least a distribution. What is this distribution?

Take the [redacted] case where D_0 is invertible, whence we get a canonical section s of $\mathcal{L}$ over the space of $D_0 + B = D$. We know that $\nabla s = s \cdot \Theta$ where

$$i(B)\,\Theta = \mathrm{Tr}^{reg}(D^{-1}B) = \text{constant term in the asymp. [redacted] exp. of } \mathrm{Tr}(e^{-t\Delta} D^{-1} B)$$

September 27, 1982

Idea: Consider an elliptic operator $D_0 : E \to F$ over a compact manifold M. We can then consider the family of operators $D = D_0 + B$ where B is of lower order. Over the space $\mathcal{B}$ of these operators we get a determinant line bundle $\mathcal{L}$ which is holomorphic and which has a metric defined by analytic torsion. [redacted] Scalar gauge transformations $D \mapsto e^{-itf} D e^{itf}$ give an action of the additive group of real fns. on M on $(\mathcal{B}, \mathcal{L})$ preserving the metric and hence the connection. Hence to an f we get a function φ_f on $\mathcal{B}$ which gives the difference between the action [redacted] on $\mathcal{L}$ and the lift via the connection of the action on $\mathcal{B}$, where ~~here~~ f is to be thought of as a infinitesimal gauge transf. φ_f is linear in f, hence $\varphi_f(D)$ should be given by a density on M determined by the operator D and the metrics + volume needed to define D^*, etc.

Therefore we should have in great generality [redacted] a density on M whose integral gives the index of D. This last statement follows [redacted] by taking f to be constant whence it leaves D fixed, and [redacted] hence the action of f on $\mathcal{L}$ will be $\varphi_f(D)$ on one hand and also $\text{index}(D) f$ on the other.

I want now to see if the density is [redacted] related in a simple way to the [redacted] local quantities giving the index which are obtained from the heat equation approach to the index thm. Let's first review the heat equation method. One forms the heat kernel trace [redacted] of D^*D and DD^*:

[redacted]

$$\operatorname{Tr}\left(e^{-tD^*D}\right) = h^0 + \sum_{\lambda > 0} e^{-t\lambda}$$

$$\mathrm{Tr}\left(e^{-tDD^*}\right) = h' + \sum_{\lambda>0}' e^{-t\lambda}$$

where λ runs over the non-zero eigenvalues of D^*D, DD^*, and we know these operators have the same non-zero eigenvalues. Thus for any t

$$\mathrm{Ind}(D) = \mathrm{Tr}\left(e^{-tD^*D}\right) - \mathrm{Tr}\left(e^{-tDD^*}\right).$$

On the other hand we know that as $t \downarrow 0$, $\exists$ asymp. exp.

$$\mathrm{Tr}\left(e^{-tD^*D}\right) \sim \sum a_k t^k$$

where the a_k's are local integral expressions calculated from the coefficients of D, D^*. Hence it follows that

$$a_k(D^*D) = a_k(DD^*) \qquad k \neq 0$$

$$\mathrm{Ind}(D) = a_0(D^*D) - a_0(DD^*).$$

My goal should now be to formulate a local index theorem, and the key will be to use infinitesimal gauge transformations coming from a real function f. You know there should be a function $\varphi_f(D)$ which has to come out of the heat kernel formalism. Also it is attached to the perturbation $\frac{1}{i}[f,D]$ of D. This is all very algebraic and maybe should connect up with Connes work.

Let's begin with the case where D is invertible, where $\mathcal{L}$ has a canonical section s with $|s|^2 = e^{-\zeta'(0)}$. Then I know that $\nabla s = s\Theta$ where

$$i(B)\Theta\big|_D = \mathrm{Tr}^{\mathrm{reg}}(D^{-1}B)$$

$=$ the constant term in the asymptotic expansion for $\mathrm{Tr}\left(e^{-tD^*D}D^{-1}B\right)$

Then if $\bar{f}$ denotes the vector $B = \frac{1}{i}[f,D]$ at D we have

$$0 = \left(\nabla_{\bar{f}} + \varphi_f\right)(s) = s\left(i(\bar{f})\Theta + \varphi_f\right)$$

Hence
$$\varphi_f(D) = i(\bar{f})\theta\big|_D = \operatorname{Tr}^{\mathrm{reg}}\left(D^{-1}\left[\tfrac{1}{i}f, D\right]\right)$$

We expect that $\operatorname{Tr}^{\mathrm{reg}}(D^{-1}B) = \int \operatorname{tr}(JB)$, where J is some sort of finite part for D^{-1} along the diagonal.

Let's look in general at the process of extracting a finite part using algebraic methods that yield a determinant.

~~Ideal: Suppose we have a connection in L~~

September 28, 1982

Before one works on determinants of $\bar\partial$ operators on a Riemann surface, one really ought to understand the case of operators on the line or circle. So let's consider operators of the form

$$\frac{d}{dx} + A(x)$$

say over the circle $\mathbb{R}/L\mathbb{Z}$. As A varies we get a family of elliptic operators which are invertible generically. Hence over $\mathcal{A} = \{A\}$ we have the determinant line bundle with the canonical section. We also have the action of the gauge group of invertible functions on the circle.

September 30, 1982

[redacted]

Let's work out the theory of determinants of operators over $S^1 = \mathbb{R}/L\mathbb{Z}$ of the form

$$D = \frac{d}{dx} + \alpha(x)$$

where α is a complex matrix function, and the operators ~~operate~~ on a trivial vector bundle over S^1. The space of D is a big complex vector space $\mathcal{A}^c$ over which we get the determinant line bundle which is holomorphic and which has a canonical section s, generically $\neq 0$ because D is generically invertible. We also have a complex gauge group $\mathcal{G}^c$ of functions $\varphi : S^1 \longrightarrow GL_n$ acting on $(\mathcal{A}, \mathcal{L})$ preserving s.

What are the orbits of $\mathcal{G}^c$ on $\mathcal{A}^c$? They should be described by the monodromy which is a conjugacy class in GL_n. To see this think of D as a connection in the trivial bundle of dim. n over S^1. Since any v.b. over S^1 is trivial, the $\mathcal{G}^c$ orbits on $\mathcal{A}^c$ are the isom. classes of v.b. over S^1 with connection, i.e. the isom. classes of repns. of

Z of dim. n etc.

If α is constant, then the monodromy transf. is $e^{-L\alpha}$. Hence because $\exp : \mathfrak{gl}_n \to Gl_n$ is onto each $\mathcal{G}^c$-orbit contains D with α constant.

The line bundle $\mathcal{L}$ has the analytic torsion metric given where D is invertible by

$$|s|^2 = e^{-\zeta_\Delta'(0)} \qquad \Delta = D^* D.$$

Hence there is a curvature to be computed.

First look at $\zeta_\Delta(s) = \frac{1}{\Gamma(s)} \int_0^\infty \mathrm{Tr}(e^{-t\Delta}) t^{s-1} dt$.

Because we are in 1-dimension, the heat kernel has

$$\mathrm{Tr}(e^{-t\Delta}) \sim \frac{1}{\sqrt{t}} (a_0 + a_1 t + \cdots) \qquad \text{as } t \downarrow 0.$$

Consequently the integral above can have at most simple poles at $s = \frac{1}{2}, -\frac{1}{2}, \ldots$. Hence

$$\zeta_\Delta(0) = 0$$

and

$$\zeta_\Delta'(0) = \left[\frac{1}{s\Gamma(s)} \int_0^\infty \mathrm{Tr}(e^{-t\Delta}) t^s \frac{dt}{t} \right]_{s \to 0}$$

$$\boxed{\zeta_\Delta'(0) = \int_0^\infty \mathrm{Tr}(e^{-t\Delta}) \frac{dt}{t}}$$

This has to be properly interpreted because the integral isn't convergent; but it is something Graeme mentioned.

We need to go over the heat kernel $e^{-t\Delta}$:

$$-\Delta = (-D^*)(D) = \underbrace{(\partial_x - \alpha^*)}_{\tilde{D}} \underbrace{(\partial_x + \alpha)}_{D}$$

Put $\phi = \dfrac{e^{-\frac{u}{t}}}{\sqrt{t}}$. Then

$$\phi^{-1}(\partial_t + \Delta)\phi = \partial_t + \frac{u}{t^2} - \frac{1}{2t} - \left(\tilde{D} - \frac{1}{t}\partial_x u\right)\left(D - \frac{1}{t}\partial_x u\right)$$

$$= \frac{1}{t^2}\left[u - (\partial_x u)^2\right] + \frac{1}{t}\left[-\frac{1}{2} + \partial_x u \cdot D + \partial_x u\,\tilde{D} + \partial_x^2 u\right] + \partial_t - \tilde{D}D$$

So $(\partial_x u)^2 = u$ which is satisfied by $u = \frac{x^2}{4}$ whence $\partial_x^2 u - \frac{1}{2} = 0$. Thus

$$\langle x | e^{-t\Delta} | x' \rangle \sim \frac{e^{-\frac{(x-x')^2}{4t}}}{\sqrt{4\pi t}}\left(A_0(x,x') + t A_1(x,x') + \dots\right)$$

where $$(\partial_x u)(D + \tilde{D}) A_0 = 0, \qquad A_0(x',x') = I$$

and the rest of the A_n can be ground out recursively.

Now $$D + \tilde{D} = \partial_x + \alpha + \partial_x - \alpha^* = 2\partial_x + (\alpha - \alpha^*)$$

hence from $$\left(\partial_x + \frac{\alpha - \alpha^*}{2}\right) A_0 = 0$$ we get

$$A_0(x,x') = I - \left.\frac{\alpha - \alpha^*}{2}\right|_{x'}(x - x') + O((x-x')^2)$$

Return to the general program. Using the canonical section s one computes the connection form as

$$\theta = \partial \log |s|^2$$

which I know to be

$$i(\delta\alpha)\theta = \mathrm{Tr}\left(e^{-t\Delta} D^{-1} \delta\alpha\right) \quad (\text{const coeff. of})$$

Because we are in dimension 1 the Green's function $G(x,x')$ for D has a jump of 1 as x passes x'.

$$\mathrm{Tr}\left(e^{-t\Delta} D^{-1}\delta\alpha\right) = \int dx' \left[\int dx \left(\frac{e^{-\frac{(x-x')^2}{4t}}}{\sqrt{4\pi t}} A_t(x',x)\right) G(x,x')\right] \delta\alpha(x')$$

To see what's happening at $t \to 0$, let's set $x' = 0$, and look at

$$\underbrace{\int dx \; \frac{e^{-\frac{x^2}{4t}}}{\sqrt{4\pi t}}}_{\substack{\text{approaching } \delta(0)\\ \text{symmetrically}}} \Big(\underbrace{A_0(0,x) + t A_1(0,x) + \cdots}_{\substack{\text{smooth}\\ = I \text{ at}\\ x=0}}\Big) \underbrace{G(x,0)}_{\text{jump discont.}}$$

It's clear that we get the average $\frac{1}{2}[G(0^+,0) + G(0^-,0)]$ as $t \downarrow 0$.

Conclude: $\qquad \mathrm{Tr}^{(reg)}(D^{-1}\delta\alpha) = \int J\,\delta\alpha\, dx$

where $\quad J(x) = \frac{1}{2}\left[G(x^+,x) + G(x^-,x)\right]$. This regularized trace is obviously analytic in D, because D^{-1} is. Hence the curvature is zero. Consequently there is an [redacted] analytic function $D \longmapsto \det(D)$ over $\mathcal{A}$, unique up to a constant, such that

$$e^{-\zeta'_\Delta(0)} = |\det(D)|^2.$$

One [redacted] has $\qquad \delta \log\det(D) = \mathrm{Tr}^{(reg)}(D^{-1}\delta D).$

Example: Consider $\partial_x - i\omega$ where ω is constant on $\mathbb{R}/L\mathbb{Z}$. The Green's function is easily found to be

$$G(x,x') = \begin{cases} \dfrac{e^{i\omega x}}{e^{i\omega x'} - e^{i\omega(L+x')}} & 0 \le x' < x \le L \\[2ex] \dfrac{e^{i\omega(x+L)}}{e^{i\omega x'} - e^{i\omega(L+x')}} & 0 \le x < x' \le L \end{cases}$$

hence

$$J(x) = \frac{1}{2}\,\frac{e^{i\omega x} + e^{i\omega(x+L)}}{e^{i\omega x} - e^{+i\omega(x+L)}} = \frac{1}{2}\,\frac{1+e^{i\omega L}}{1-e^{+i\omega L}}$$

$$= \frac{-1}{2i}\,\frac{\cos\frac{\omega L}{2}}{\sin\frac{\omega L}{2}}$$

Thus
$$\delta \log \det(\partial_x - i\omega) = \int J(x)(-i\,\delta\omega)\,dx$$
$$= L \cdot \frac{1}{2} \cot\left(\omega \frac{L}{2}\right) \delta\omega$$
which integrates to
$$\det(\partial_x - i\omega) = \text{const} \cdot \sin\left(\omega \frac{L}{2}\right)$$

Consider a gauge transformation
$$e^{ikx}(\partial_x - i\omega)e^{-ikx} = \partial_x - i(\omega + k)$$
where $k \in \frac{2\pi}{L}\mathbb{Z}$ in order to be defined on $\mathbb{R}/L\mathbb{Z}$.
Now
$$\sin\left(\left(\omega + \frac{2\pi}{L}n\right)\frac{L}{2}\right) = \sin\left(\omega\frac{L}{2} + \pi n\right)$$
$$= (-1)^n \sin\left(\omega \frac{L}{2}\right)$$
hence the determinant will not be completely gauge invariant, although I can hope it will be invariant under [redacted] infinitesimal gauge transformations.

So let us see if there is an anomaly connected with the above regularization process. Locally we can choose φ such that
$$e^{\varphi}\partial_x e^{-\varphi} = \partial_x + \alpha$$
Actually this way of writing things assumes that we have an abelian situation; it's better to use an invertible φ with
$$\varphi\,\partial_x \varphi^{-1} = \partial_x + \alpha$$
i.e.
$$-\varphi'\varphi^{-1} = \alpha$$
Then
$$(\partial_x + \alpha)\,G(x, x') = \delta(x - x')$$
$$\overset{\shortparallel}{\varphi\,\partial_x(\varphi^{-1}G)}$$

Let us see if there is an anomaly connected with the above regularization procedure. We have

$$J(x) = \frac{1}{2} + G(x^-, x)$$

and if $D = \partial_x + \alpha$, then

$$[D, J] = \partial_x J + \alpha J - J\alpha$$
$$= \partial_1 G(x^-, x) + \partial_2 G(x^-, x) + \alpha(x)\left(\tfrac{1}{2} + G(x^-, x)\right)$$
$$- \left(\tfrac{1}{2} + G(x^-, x)\right)\alpha(x)$$

Now $\partial_1 G(x^-, x) + \alpha(x^-) G(x^-, x) = 0$, and since $\alpha(x^-) = \alpha(x)$ we see immediately that

$$[D, J] = 0.$$

Another way to check this is to note that if $D = \varphi D_0 \varphi^{-1}$, then

$$G(x, x') = \varphi(x)\, G_0(x, x')\, \varphi(x')^{-1}$$

and hence we will have

$$G(x^\pm, x) = \varphi(x)\, G_0(x^\pm, x)\, \varphi(x)^{-1}$$

and so

$$J(x) = \varphi(x)\, J_0(x)\, \varphi(x)^{-1}.$$

Hence

$$[D, J] = \varphi [D_0, J_0] \varphi^{-1}.$$

So we can reach the case where $D_0 = \partial_x + \alpha$ and α is constant. However it's more or less clear that the formula

$$J_0 = \frac{i}{2}\cot\left(\frac{wL}{2}\right) \qquad \text{for } D_0 = \partial_x - iw$$

should hold for w a matrix, hence $[D_0, J_0] = 0$.

So I conclude that the anomaly is trivial, so $\det(D)$ is invariant under infinitesimal gauge transformations.

October 1, 1982

I am trying to calculate $\det(\partial_x + \alpha)$ where x runs over $S^1 = \mathbb{R}/L\mathbb{Z}$, or $\mathbb{R}$ and α has compact support. The idea I have is that this determinant is a simple function of the monodromy

$$M = T\left\{e^{-\int_0^L \alpha\, dx}\right\}.$$

In fact over S^1 we know $\partial_x + \alpha$ is invertible $\iff$ $\operatorname{Ker}(\partial_x + \alpha) \neq 0$ and also

$$\operatorname{Ker}(\partial_x + \alpha) = \operatorname{Ker}(I - M)$$

hence we expect a formula for $\det(\partial_x + \alpha)$ in terms of $\det(I - M)$.

Consider the example $\partial_x - i\omega$ ω constant where using the symmetric regularization of the Green's function we got

$$\det(\partial_x - i\omega) = \text{const} \cdot \sin\left(\frac{\omega L}{2}\right)$$

Hence

$$M = e^{+\int_0^L i\omega\, dx} = e^{i\omega L}.$$

and

$$\sin\left(\frac{\omega L}{2}\right) = \frac{e^{-i\frac{\omega L}{2}}}{2i}\left(e^{i\omega L} - 1\right).$$

If we define the determinant using the regularization $G(x^-, x) = J(x)$ we get something else, which I want now to compute.

First from the viewpoint of working over $\mathbb{R}$ with α of compact support, it seems natural to use

$$\det(\partial_x + \alpha) = \det(1 + \partial_x^{-1}\alpha)$$

and hence to use "the" Green's function G_0 of ∂_x. This is $H(x - x') + \text{constant}$, and there is no apparent way to pick the constant. Picking the constant is essentially the same as a regularization process, e.g. if $G_0 = H(x-x') - \frac{1}{2}$, then

$$G(x,x') - G_0(x,x')$$

is continuous and its value when $x' = x$ is

$$G(x^-,x) + \frac{1}{2} = \frac{1}{2}\left[G(x^-,x) + G(x^+,x)\right].$$

So let's compute $\det(\partial_x - i\omega)$ with the regularization $J(x) = G(x^-,x)$. From p. 170

$$G(x^-,x) = \frac{e^{i\omega(x+L)}}{e^{i\omega x} - e^{i\omega(x+L)}} = \frac{1}{e^{-i\omega L} - 1}$$

so

$$\delta \log \det(\partial_x - i\omega) = \int_0^L \frac{1}{e^{-i\omega L}-1}(-i\delta\omega)\,dx = \frac{-iL\delta\omega}{e^{-i\omega L}-1}$$

$$= \frac{e^{i\omega L}\, iL\delta\omega}{e^{i\omega L}-1}$$

so

$$\det(\partial_x - i\omega) = \text{const.}\left(e^{+i\omega L} - 1\right)$$

$$= \text{const.}\,(M - I)$$

Notice that this determinant is a function of the monodromy and hence is gauge-invariant

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Actually the above argument for $\partial_x - i\omega$ is completely general. Let $\varphi(x) = U(x,0)$ for the operator $\partial_x + \alpha$. Then

$$(\partial_x + \alpha)\varphi = 0 \qquad \text{or} \qquad \alpha = -\varphi'\varphi^{-1}.$$

$$M = \varphi(L).$$

So

$$G(x,x') = \begin{cases} \varphi(x)A & x > x' \\ \varphi(x)B & x < x' \end{cases}$$

$$\varphi(L)A = B \qquad\qquad MA = B$$

$$1 = \varphi(x')A - \varphi(x')B = \varphi(x')[A - MA]$$

so
$$A = \left[\varphi(x')(1-M)\right]^{-1} = (1-M)^{-1}\varphi(x')^{-1}$$
$$B = M(1-M)^{-1}\varphi(x')^{-1}$$

and so
$$J(x) = G(x^-, x) = \varphi(x)\, M(1-M)^{-1}\varphi(x)^{-1}$$

Also
$$\delta M = -\int_0^L dx\ \underbrace{U(L,x)}_{M\varphi(x)^{-1}}\ \delta\alpha(x)\ \underbrace{U(x,0)}_{\varphi(x)}$$

and
$$\delta \log\det(\partial_x + \alpha) = \int dx\ \mathrm{tr}\big(G(x^-,x)\,\delta\alpha(x)\big)$$
$$= \int dx\ \mathrm{tr}\left\{\varphi(x)\, M(1-M)^{-1}\varphi(x)^{-1}\delta\alpha(x)\right\}$$

So
$$(M-1)^{-1}\delta M = \int_0^L dx\ (1-M)^{-1} M\, \varphi(x)^{-1}\delta\alpha(x)\,\varphi(x)$$

hence
$$\delta\log\det(\partial_x+\alpha) = \mathrm{tr}\ (M-1)^{-1}\delta M$$
$$= \delta\log\det(M-1).$$

<u>Conclusion</u>: If $\det(\partial_x + \alpha)$ is defined using the regularization $G(x^-, x)$, then one has
$$\det(\partial_x + \alpha) = \text{const}\ \det(1-M)$$

If one uses $J(x) = G(x^-, x) + \frac{1}{2} = \frac{1}{2}\left[G(x^+,x) + G(x^-,x)\right]$
then one gets
$$\text{const}\cdot \det\left(M^{1/2} - M^{-1/2}\right)$$

and finally $J(x) = G(x^+, x)$ yields $\det(1-M^{-1})$.

Next project: Can you work in the [redacted] determinants occurring as $\langle 0|S|0\rangle$?

One
$$M = T\{e^{-\int \alpha(x)dx}\}$$
so
$$\det(M) = e^{-\int tr(\alpha)dx}$$ and
$$\delta \log\det(M) = -\int tr(\delta\alpha)dx$$

[redacted] Since
$$\delta \log\det(\partial_x + \alpha) = \int tr\left(G(x,x)\,\delta\alpha(x)\right)dx$$
it follows that changing the regularization process by a constant c alters the determinant by $(\det M)^{-c}$.

Now determinants of the form $\langle 0|S|0\rangle$ result from different [redacted] boundary conditions for the differential operator $\partial_x + \alpha$. Let α operator on the vector space V. Then the propagator $U(L,0) = M : V \longrightarrow V$ tells us the boundary values of solutions of $(\partial_x+\alpha)\psi = 0$. A set of boundary conditions for the operator should therefore be a subspace W of $V\times V$ which is complementary (at least generically in α) for the graph of M. Notice that when W is complementary to Γ_M [redacted] one can construct a Green's function. Namely you choose any Green's fn. $G_0(x,x')$, i.e. solution of $(\partial_x+\alpha)G_0(x,x') = \delta(x-x')$, and then look at its boundary values. [redacted] Specifically given $v \in V$, look at $G_0(x,x')v$ at $x = 0, L$. [redacted] This gives an element of $V\times V$, which can, by assumption, be expressed as the sum of the boundary values of a solution [redacted] in $\operatorname{Ker}(\partial_x+\alpha)$ and something in W. Thus changing $G_0(x,x')v$ by a solution in $\operatorname{Ker}(\partial_x+\alpha)$ we get a solution $G(x,x')v$ of $(\partial_x+\alpha)\psi = \delta(x-x')v$ which is unique. Etc.

Supposing $W \subset V\times V$ given we want to calculate the determinant. We still have

$$\delta M = -\int_0^L dx \ M\varphi(x)^{-1}\delta\alpha(x)\varphi(x)$$

$$\delta \log\det(\partial_x+\alpha) = \int tr\left(G(x,x)\delta\alpha(x)\right) dx$$

$$G(x,x') = \begin{cases} \varphi(x) A \ \varphi(x')^{-1} & x > x' \\ \varphi(x) B \varphi(x')^{-1} & x < x' \end{cases}$$

where $A - B = 1$. The boundary condition says that

$$W \supset \{ (G(0,x')v, G(L,x')v \mid v \in V\}$$
$$= \{ (Bv, MAv) \mid v \in V\}.$$

For example this says

$$(Bv, MAv) = (-v, 0) + (Av, MAv) \in W$$

and since $W \oplus \Gamma_M = V \times V$, this determines Av. So A, B are some kind of functions of M, W. Let's use

$$J(x) = G(x^-, x) = \varphi(x) B \varphi(x)^{-1}$$

Then

$$\delta \log\det(\partial_x+\alpha) = \int tr\left(B\varphi(x)^{-1}\delta\alpha(x)\varphi(x)\right)dx$$

$$= tr \ BM^{-1} \underbrace{\int M\varphi(x)^{-1}\delta\alpha(x)\varphi(x)\,dx}_{-\delta M}$$

$$= tr\{(-B)M^{-1}\delta M\}$$

The obvious candidate for this determinant is the determinant of the map

$$\bar{\Gamma}_M : V \longrightarrow V\times V/W.$$

One has

$$\delta \log\det(\bar{\Gamma}_M) = Tr\left(\bar{\Gamma}_M^{-1}\delta\bar{\Gamma}_M\right)$$

Now $\Gamma_M v = (v, Mv)$, so $\delta \Gamma_M (v) = (0, \delta M v)$.

By defn. of B we have

$$(Bv, M(B+1)v) \in W$$

$$(0, Mv) + \Gamma_M (Bv) \equiv 0 \quad (\mathrm{mod}\ W)$$

$$(0, v) \equiv -\Gamma_M (BM^{-1} v) \quad \text{''}$$

$$(0, \delta M v) \equiv -\Gamma_M (BM^{-1} \delta M v) \quad \text{''}$$

$$\delta \bar{\Gamma}_M (v) = \bar{\Gamma}_M (-BM^{-1} \delta M v)$$

$$\therefore \quad \bar{\Gamma}_M^{-1} \delta \bar{\Gamma}_M = -BM^{-1} \delta M$$

Conclusion: Using the Green's fn. for $\partial_x + \alpha$ with boundary conditions W and the regularization $J(x) = G(x^-, x)$ we have

$$\boxed{\det(\partial_x + \alpha) = \text{const.} \det(\bar{\Gamma}_M : V \to V \times V/W)}$$

October 2, 1982

Consider the differential operator $\partial_x + \alpha$ over the interval $[0, L]$, where α is a matrix function of x, i.e. $\alpha(x)$ operates on a vector space V. Let $\varphi(x)$ be the propagator from 0 to x for solutions of

(1) $$(\partial_x + \alpha)\psi = 0.$$

Thus $\varphi(x) \in \mathrm{Aut}(V)$ satisfies

$$\begin{cases} (\partial_x + \alpha)\varphi = 0 \\ \varphi(0) = \mathrm{Id}. \end{cases}$$

The boundary values $(\psi(0), \psi(L))$ of the solutions of (1) form a subspace of $V \times V$ which is the image of the graph map

$$\Gamma_M : V \longrightarrow V \times V$$

where $M = \varphi(L)$ is the monodromy of (1).

A subspace $W \subset V \times V$ of the same dim. as V gives us boundary condition for (1). For generic α the graph of M is complementary to W, and so one can solve the inhomogeneous equation

(2) $$(\partial_x + \alpha)\psi = f$$

uniquely for a ψ satisfying the boundary conditions: $(\psi(0), \psi(L)) \in W$. (To see this, choose a solution of (2) and add a suitable solution of (1).) Thus the operator with the boundary conditions is invertible $\iff$ the map

$$\bar{\Gamma}_M : V \longrightarrow V \times V / W$$

induced by Γ_M is an isomorphism.

What I want to do now is to justify the formula

(3) $$\boxed{\det(\partial_x + \alpha) = \text{const.} \det(\bar{\Gamma}_M)}$$

where $\det(\Gamma_M)$ is defined by choosing volumes in V and $V\times V/W$. The determinant of a diff'l operator D with boundary conditions is defined up to a constant factor by the variational formula

$$\delta \log \det(D) = \mathrm{Tr}^{(reg)}(D^{-1}\delta D).$$

Therefore to justify (3), I need only specify the regularization process, and show that both sides of (3) have the same $\delta \log$.

The Green's function $G(x,x')$ or kernel for D^{-1} is given by

$$G(x,x') = \begin{cases} \varphi(x)\,A\,\varphi(x')^{-1} & x > x' \\ \varphi(x)\,B\,\varphi(x')^{-1} & x < x' \end{cases}$$

where $A - B = I$ so that G jumps by I as x crosses x', and where the boundary conditions are satisfied:

$$\big(G(0,x')v,\ G(L,x')v\big) \in W \qquad \text{for any } v.$$

$$\parallel \ \big(B\varphi(x')^{-1}v,\ MA\,\varphi(x')^{-1}v\big)$$

$$\Longleftrightarrow \ \underbrace{\big(Bv,\ M(1+B)v\big)}_{\Gamma_M(Bv)\,+\,(0,\,Mv)} \in W \qquad \text{for any } v$$

Since $\Gamma_M(V) \oplus W = V\times V$, the condition

(4) $$\Gamma_M(Bv) + (0, Mv) \in W$$

defines Bv uniquely.

Now

$$\mathrm{Tr}^{(reg)}(D^{-1}\delta D) = \int \mathrm{tr}\big(J(x)\,\delta\alpha(x)\big)\,dx$$

where $J(x)$ is a way of making sense of $G(x,x')$ when $x = x'$. We will take the regularization:

(5) $$J(x) = G(x^-, x) = \varphi(x)\,B\,\varphi(x)^{-1}$$

whence

(6) $$\delta \log \det(\partial_x + \alpha) = \int \operatorname{tr}\left(\varphi(x) B \varphi(x)^{-1} \delta\alpha(x)\right) dx.$$

On the other hand we have

$$\delta \log \det(\bar{\Gamma}_M) = \operatorname{tr}(\bar{\Gamma}_M^{-1} \delta \bar{\Gamma}_M)$$

and $\Gamma_M(v) = (v, Mv)$ so $\delta\Gamma_M v = (0, \delta M v)$.

From (4) we have

$$(0, Mv) \equiv -\Gamma_M(Bv) \quad (\text{mod } W)$$

$$\delta\Gamma_M v = (0, \delta M v) = (0, MM^{-1}\delta M v)$$
$$\equiv -\Gamma_M(BM^{-1}\delta M v).$$

Hence
$$\bar{\Gamma}_M^{-1} \delta\bar{\Gamma}_M = -BM^{-1}\delta M.$$

Finally because M = propagator $U(L,0)$ for $\partial_x + \alpha$ we have

$$\delta M = -\int_0^L dx \; \underbrace{U(L,x)}_{M\varphi(x)^{-1}} \delta\alpha(x) \underbrace{U(x,0)}_{\varphi(x)}.$$

So we get

$$\delta \log \det(\bar{\Gamma}_M) = \operatorname{tr}(-BM^{-1}\delta M) = \int dx \operatorname{tr}(B\varphi^{-1}\delta\alpha\,\varphi)$$
$$= \delta \log \det(\partial_x + \alpha),$$

thereby justifying (3).

Notice that because

$$\delta \log \det M = \operatorname{tr}(M^{-1}\delta M) = -\int \operatorname{tr}(\delta\alpha)\, dx$$

changing the regularization to $J(x) = G(x^-; x) + cI$, with c a constant (e.g. $G(x^+, x) = G(x^-; x) + I$), has the effect of changing $\det(\partial_x + \alpha)$ by the factor $(\det M)^{-c}$.

October 3, 1982

The case of $\partial_x + \alpha$ on $\mathbb{R}$ with α of compact support goes as follows. Let $\varphi(x) = U(x, -\infty)$ and $M = U(\infty, -\infty)$. If we work on a large interval $[-L, L]$ containing the support of α, and use periodic boundary conditions, then the determinant of the operator is $\det(M - I)$. This is independent of L and is the obvious candidate for the determinant of $\partial_x + \alpha$ over $\mathbb{R}$. The condition $\det(M - I) \neq 0$ is also what is needed to define the Green's function. In effect

$$G(x, x') = \begin{cases} \varphi(x) A \varphi(x')^{-1} & x > x' \\ \varphi(x) B \varphi(x')^{-1} & x < x' \end{cases}$$

$A - B = I.$

$$G(x, x') = MA\varphi(x')^{-1} \qquad x \gg 0$$
$$G(x, x') = B\varphi(x')^{-1} \qquad x \ll 0$$

If we require these two values at $\pm\infty$ to coincide, which is the periodic boundary condition, then we get

$$MA = B, \qquad B = M + MB, \quad B = (1-M)^{-1}M$$

so

$$\delta \log \det(\partial_x + \alpha) = \int \operatorname{tr}\Big(\underbrace{G(x^-, x)\,\delta\alpha(x)}_{\varphi(x)B\varphi(x)^{-1}\delta\alpha(x)}\Big)dx$$

$$\delta \log \det(M - I) = \operatorname{Tr}\big((M-I)^{-1}\delta M\big) = \operatorname{Tr}(1-M)^{-1}\int \underbrace{U(\infty, x)}_{M\varphi(x)^{-1}}\delta\alpha(x)\underbrace{U(x, -\infty)}_{\varphi(x)}$$

This example is interesting because the Green's function is not an L^2-inverse as you are used to. Rather one is inverting the operator, better, solving the inhomogeneous equation

$$(\partial_x + \alpha)\psi = f$$

with $f \in C_0^\infty(\mathbb{R})$, with ψ satisfying $\psi(-\infty) = \psi(+\infty)$.

October 5, 1982

Consider an elliptic operator $D: E \longrightarrow F$ and let $D: W \longrightarrow V$ be the effect on global sections. Suppose metrics etc. chosen so that [redacted] W and V decompose into eigenspaces for the Laplaceans. Denote these eigenspaces by W_λ, V_λ $\lambda \geq 0$ and to fix the ideas suppose that these at are most 1-diml. Then as we vary the operator D slightly these eigenlines will vary in a smooth fashion, so we get a bunch of line bundles over our family of operators. Moreover it seems that these line bundles have natural connections because they sit inside a Hilbert space.

October 6, 1982

I'd like to see if it is possible to generalize the way you define a metric on $\mathcal{L}$ with analytic torsion to a way of defining $ch\, f_!$ as a differential form. The approach would be to do things over a space of Fredholm operators $D = D_0 + B$: $W \longrightarrow V$. The answer will be in $\Lambda \mathcal{B}^*$ where $\mathcal{B}$ = space of B.

As B varies over $\mathcal{B}$, the subspace of W where $D^*D < \lambda$ forms a vector bundle in a nbd. of $B = 0$ when λ is not an eigenvalue of D^*D. Call this bundle $\mathcal{W}_{<\lambda}$, and define $\mathcal{V}_{<\lambda}$ similarly. These bundles have connections, so curvature forms, so we can form

$$ch(\mathcal{W}_{<\lambda}) - ch(\mathcal{V}_{<\lambda})$$

and the problem is to see how this changes as λ is [redacted] varied, then [redacted] to compensate.

Suppose to simplify that all the eigenspaces of $D_0^*D_0$ are 1-diml. Then the vector bundles $\mathcal{W}_{<\lambda}$, $\mathcal{V}_{<\lambda}$ have a natural splitting into orthogonal lines, i.e. a reduction of the structural group to the torus. Hence it might be true that the connections respect [redacted] the splittings, but this point should be examined carefully.

So let's review standard perturbation theory, which gives [redacted] the connections. Let λ be a simple eigenvalue of a self-adjoint operator [redacted] H and let $H\psi = \lambda\psi$ with $\|\psi\| = 1$. Consider a variation $H + \delta H$ and let's calculate the corresponding changes $\delta\lambda$, $\delta\psi$ to first order in δH.

$$\big(H + \delta H - (\lambda + \delta\lambda)\big)(\psi + \delta\psi) = 0$$

$$(*) \qquad (H - \lambda)\,\delta\psi + (\delta H - \delta\lambda)\psi = 0$$

Take $(\psi|$ and you get

$$(\psi | \delta H \psi) = \delta\lambda .$$

Now $\delta\psi$ is uniquely determined by the eqns. $(*)$ provided one also requires $(\psi|\delta\psi) = 0$. And this is how the connection in the line bundle is defined. Recall that the tangent space to a line L in $\mathbb{P}(V)$ is $\mathrm{Hom}(L, V/L)$ canonically and that given a tangent vector $A : L \to V/L$ and a [redacted] $\psi \in L$, the corresponding tangent vector at ψ given by the connection is $A\psi \in V/L = L^\perp$.

So

$$\delta\psi = (\lambda - H)^{-1}(\delta H - \delta\lambda)\psi \qquad \text{in } (\mathbb{C}\psi)^\perp .$$

Over the Grassmannian of p-planes in n-space is the subbundle S. Calculate its curvature at a point represented by a p-plane $W \subset V$. The curvature will be a skew-symmetric form on the tangent space $\mathrm{Hom}(W, V/W)$ with values in $\mathrm{End}(W)$.

Planes near to W can be described as the graph of maps $T : W \longrightarrow W^\perp$. If w_i is an orthonormal base for W, then $s_i : T \longrightarrow w_i + Tw_i$ gives ~~th~~.

a holomorphic frame for $\mathcal{S}$ near W.

$$(s_i \mid s_j) = h_{ij}$$

$$d(s_i \mid s_j) = (\underbrace{Ds_i}_{s_k\theta_{ki}} \mid s_j) + (s_i \mid \underbrace{Ds_j}_{s_k\theta_{kj}})$$

$$= \text{c.c.} \qquad + (s_i \mid s_k)\,\theta_{kj}$$

$$\therefore\ \theta = h^{-1}\partial h$$

$$K = d\theta + \theta\theta$$

$$= h^{-1}\bar\partial\partial h + \underbrace{dh^{-1}}_{-h^{-1}dh\,h^{-1}}\partial h + h^{-1}\partial h\, h^{-1}\partial h$$

$$K = h^{-1}\bar\partial\partial h - h^{-1}\bar\partial h\, h^{-1}\partial h$$

Now $$h_{ij}(T) = (w_i + Tw_i \mid w_j + Tw_j) = \delta_{ij} + (Tw_i \mid Tw_j)$$

$$= \delta_{ij} + \sum_k \bar t_{ki}\, t_{kj}$$

at $T=0$ one has

$$K_{ij} = \bar\partial\partial \sum_k \bar t_{ki} t_{kj} = \sum_k d\bar t_{ki}\, dt_{kj}$$

The next thing I want to calculate is this: ~~[illegible] the partial flag manifold [illegible]~~ I used the holomorphic structure on the Grassmannian in the above curvature calculation. However in general there is something called the Grassmannian connection which goes as follows: Suppose we have a bundle E which is a direct summand of a vector bundle F equipped with a connection. Then if $E \xrightarrow{i} F \xrightarrow{p} E$ are the injection and projection we get an induced connection on E by

$$E \xrightarrow{i} F \xrightarrow{D} F\otimes T^* \xrightarrow{p\otimes \mathrm{id}} E\otimes T^*.$$

I want to compare this with the above connection.

First do for $\mathbb{P}^1$. $\quad s(z) = \begin{pmatrix} 1 \\ z \end{pmatrix} \in \mathbb{C}^2$.

The orthogonal projection on the line generated by $s(z)$ is

$$p_z \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 1 \\ z \end{pmatrix} \frac{a + b\bar{z}}{1+|z|^2}$$

for it kills $\begin{pmatrix} -\bar{z} \\ 1 \end{pmatrix}$ which is $\perp$ to $s(z)$, and reproduces $s(z)$. Then

$$Ds = p_z\, ds = p_z \begin{pmatrix} 0 \\ dz \end{pmatrix} = s \cdot \frac{\bar{z}\,dz}{1+|z|^2}$$

so the connection form relative to s is $\frac{\bar{z}\,dz}{1+|z|^2}$ which is the same as $\partial \log |s|^2 = \partial \log(1+|z|^2)$.

In general the orthogonal complement of $\Gamma_T = \operatorname{Im}\begin{pmatrix} I \\ T \end{pmatrix}$ is $\operatorname{Im}\begin{pmatrix} -T^* \\ I \end{pmatrix}$, so the orthogonal projection on Γ_T is

$$p_T = \begin{pmatrix} I \\ T \end{pmatrix} (1+T^*T)^{-1} \begin{pmatrix} I & T^* \end{pmatrix}$$

hence

$$\begin{aligned} Ds_i &= p_T\, d\begin{pmatrix} I \\ T \end{pmatrix} e_i \\ &= \begin{pmatrix} I \\ T \end{pmatrix} (1+T^*T)^{-1} \begin{pmatrix} I & T^* \end{pmatrix} \begin{pmatrix} 0 \\ dT \end{pmatrix} e_i \\ &= \begin{pmatrix} I \\ T \end{pmatrix} \underbrace{(1+T^*T)^{-1}\, T^*\, dT}_{h^{-1}\partial h} e_i \qquad h = 1+T^*T \end{aligned}$$

so it works. So I conclude that the connection on the subbundle obtained from metric + holom. structure on the Grassmannian coincides with the Grassmannian connection.

Next consider the manifold $U(n)/U(1)^2 \times U(n-2)$ consisting of pairs of orthogonal lines (L_1, L_2) in V. Then over this manifold I have two line bundles $\mathcal{L}_1, \mathcal{L}_2$ and the 2-plane bundle $\mathcal{L}_1 \oplus \mathcal{L}_2$ all equipped with embeddings into the trivial bundle with fibre V. Hence we can ask whether the connection on $\mathcal{L}_1 \oplus \mathcal{L}_2$ coincides with the direct sum of the connections on the $\mathcal{L}_i$. This seems unlikely because if I parallel translate an element of $\mathcal{L}_1$ relative to the connection on $\mathcal{L}_1 \oplus \mathcal{L}_2$, then it seems that we should get a component in $\mathcal{L}_2$. One can see this clearer when $n=2$. Then $\mathcal{L}_1 \oplus \mathcal{L}_2$ is trivial so its Grassmannian connection has zero curvature, whereas the curvature forms for $\mathcal{L}_1$ and $\mathcal{L}_2$ are non-zero.

<u>Question</u>: The manifold $U(n)/U(1)^2 \times U(n-2)$ thought of as the partial flag manifold of flags $0 \subset L_1 \subset W$ in V is a complex manifold, so the bundle $\mathcal{L}_2$ has in addition to its Grassmannian connection a connection coming from the holomorphic structure and metric. Do these connections coincide?

I think the way to handle this question is to use the fact that the basic skew-forms live over the space of projection operators, and that the metric enables you to lift from flags to projection operators. Thus we replace $U(n)/U(1)^2 \times U(n-2)$ by $GL(n)/GL(1)^2 \times GL(n-2)$ which is the space of splittings $V = L_1 \oplus L_2 \oplus A$. Then this space should have canonical Grassmannian connections on the various bundles. Unfortunately this doesn't seem to help us with the Question.

Let's consider $n=2$ where $U(2)/U(1)^2 = \mathbb{P}^1$. We have to calculate the Grassmannian connection on the quotient bundle $\mathcal{O}(1)$. The projection operator on the subbundle is

$$p_z = \begin{pmatrix} 1 \\ z \end{pmatrix} \frac{1}{1+|z|^2} \begin{pmatrix} 1 & \bar{z} \end{pmatrix} = \frac{1}{1+|z|^2} \begin{pmatrix} 1 & \bar{z} \\ z & |z|^2 \end{pmatrix}$$

and so the projection operator on the quotient bundle is

$$1-p_z = \frac{1}{1+|z|^2}\begin{pmatrix}|z|^2 & -\bar z\\ -z & 1\end{pmatrix} = \begin{pmatrix}-\bar z\\ 1\end{pmatrix}\frac{1}{1+|z|^2}$$ [redacted] $$(-z \;\; 1)$$

a holomorphic section of $O(1)$ is given by the image of $\begin{pmatrix}0\\1\end{pmatrix}$, which when lifted to $L^\perp$ is

$$s = (1-p_z)\begin{pmatrix}0\\1\end{pmatrix} = \frac{1}{1+|z|^2}\begin{pmatrix}-\bar z\\ 1\end{pmatrix}$$

One has $|s|^2 = \frac{1}{1+|z|^2}$ whence the connection obtained from holom. structure + metric is

$$Ds = s\,\theta \qquad \theta = \partial \log |s|^2 = -\frac{\bar z\, dz}{1+|z|^2}$$

The Grassmannian connection is given by [redacted]
$D^G s = (1-p)\,ds = s(-z \;\; 1)\,ds$, so

$$\theta^G = (-z \;\; 1)\, d\, \frac{1}{1+|z|^2}\begin{pmatrix}-\bar z\\ 1\end{pmatrix}$$

$$= d\,1 -$$ [redacted] $$[d(-z \;\; 1)]\frac{1}{1+|z|^2}\begin{pmatrix}-\bar z\\ 1\end{pmatrix}$$

$$= -\frac{\bar z\, dz}{1+|z|^2}$$

and the two connections coincide.

<u>Proposition:</u>

[redacted] Let $0 \to E' \xrightarrow{i} E \xrightarrow{p} E'' \to 0$ be an exact sequence of holomorphic vector bundles, let E be given a metric and E', E'' the induced metrics. Then splitting the above sequence via the metrics allows us to induce from the canon. connection on E connections on E' and E''. Claim these induced connections are canonical for the induced metrics.

Proof for E': Take a holom. section s of E'. [redacted] The induced connection on E' is

$$E' \xrightarrow{i} E \xrightarrow{D} E\otimes T^* \xrightarrow{i^*\otimes id} E'\otimes T^*;$$

call this $D' = i^* D i$. I have to see that $D's$ is of type (1,0). But $i(s)$ is a holom. section of E, so $Di(s) \in E \otimes T^{1,0}$ so $i^*Di(s) \in E' \otimes T^{1,0}$. I also have to see that D' preserves the metric.

$$d(s_1|s_2) = d(is_1|is_2) = (Dis_1|is_2) + (is_1|Dis_2)$$
$$= (i^*Dis_1|s_2) + (s_1|i^*Dis_2)$$

so it works.

Proof for E'': This time $D'' = pDp^*$ is the induced connection. Clearly it preserves metric by the same argument. So let s'' be a holom. section of E'' and lift it: $s'' = ps$ locally, where s is a holom. section of E. Fix a point at which we want to test that $D''s''$ is of type (1,0). I can assume that $i^*s = 0$ at this point. (In effect pick a holom. s' of E' with $s' = i^*s$ at the point, then replace s by $s - is'$.)

Now
$$s = p^*\underbrace{ps}_{s''} + i\,i^*s$$

$$\underbrace{pDs}_{\in E''\otimes T^{1,0}} = \underbrace{pDp^*s''}_{D''s''} + pDi\,i^*s$$

However pDi is a 0-th order operator, so at the point of interest $pDi\,i^*s = 0$ because $i^*s = 0$. Thus $D''s'' \in E''\otimes T^{1,0}$ at this point. Q.E.D.

It follows that the question on p. 187 has an affirmative answer.

October 7, 1982

η-invariant. This is defined for a self-adjoint operator as the value of $\eta_A(s) = \sum_{\lambda \neq 0} \operatorname{sgn}(\lambda)|\lambda|^{-s}$ at $s=0$, when the makes sense. For first order operators in odd dimensions it is defined (thm. of Atiyah Patodi Singer). In these cases $\zeta_{A^2}(0) = 0$, hence $\sum_{\lambda \neq 0}' |\lambda|^{-s} = 0$ at $s=0$, so the η invariant is essentially the number of negative eigenvalues. Hence one is getting a phase for the determinant of A.

In finite dimensions we have

$$\det A = \prod \lambda = \prod \operatorname{sgn}(\lambda)|\lambda|$$

$$e^{-\zeta'_{A^2}(0)} = \prod |\lambda^2|$$

$$\log \det A = -\frac{1}{2}\zeta'_{A^2}(0) + i\pi \underbrace{\sum_{\lambda<0}' 1}_{\text{essentially } \eta_A(0)}.$$

So Atiyah's idea is to treat [redacted] the η-invariant as the phase of $\det(A)$.

Let's work out formulas in the case of

$$\frac{1}{i}\partial_x + \alpha \qquad \text{over } S^1$$

where α is a hermitian matrix function. In this case I already have an idea of what the determinant is, so I should begin by computing the η-invariant.

$0 < x$ [redacted]

$$\sum_{n \geq 0}' (x+n)^{-s} = \frac{1}{\Gamma(s)} \int_0^\infty \underbrace{\sum_{n\geq 0} e^{-t(x+n)}}_{\frac{e^{-tx}}{1-e^{-t}}} t^s \frac{dt}{t}$$

$$\frac{1}{1-e^{-t}} = \frac{1}{t} + \frac{1}{2} + \frac{t}{12} \qquad\qquad \frac{e^{-tx}}{1-e^{-t}}\, t^s \frac{dt}{t}.$$

So
$$\sum_{n\geq 0} (x+n)^{-s}\Big|_{s=0} = \text{constant term in } e^{-tx}\frac{1}{1-e^{-t}}$$
$$\left(1-tx+\cdots\right)\left(\frac{1}{t}+\frac{1}{2}+\frac{t}{12}+\cdots\right)$$
$$= -x+\frac{1}{2} \qquad \text{for } x>0$$

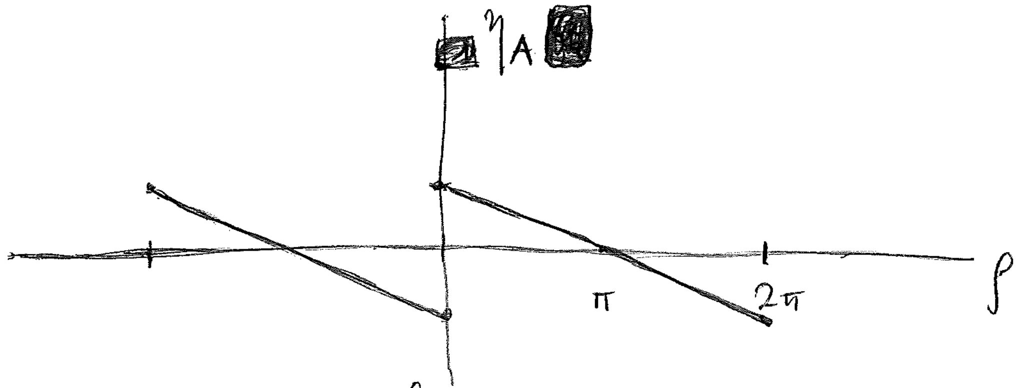
If $0<x<1$, then
$$\sum_{n<0} \underbrace{\operatorname{sgn}(x+n)}_{-1}\,|x+n|^{-s} = \sum_{n\geq 0}(-1)\,|n+1-x|^{-s}$$
$$= -\left\{-(1-x)+\frac{1}{2}\right\} = -x+\frac{1}{2}.$$

Thus
$$\sum_{n\in\mathbb{Z}} \operatorname{sgn}(x+n)\,|x+n|^{-s}\Big|_{s=0} = \begin{cases}\text{periodic fn. of period } 1\\ \text{which} = -2x+1 \quad \text{for } 0<x<1.\end{cases}$$

Consider $A=\frac{1}{i}\partial_x+\alpha$ on $S^1=\mathbb{R}/L\mathbb{Z}$ where α is hermitian. The monodromy for $\frac{1}{i}\partial_x+\alpha-\lambda$ is
$$T\left\{e^{-i\int_0^L(\alpha-\lambda)dx}\right\} = e^{i\lambda L}M$$
where M is the monodromy for $\frac{1}{i}\partial_x+\alpha$. In this case M is unitary and I can suppose it diagonal, so let's assume $M=e^{-i\rho}$. The eigenvalues are λ such that
$$e^{i\lambda L}e^{-i\rho}=1 \qquad \lambda L-\rho\in 2\pi\mathbb{Z}$$
or $\quad \lambda\in +\frac{\rho}{L}+\frac{2\pi}{L}\mathbb{Z}$. Hence
$$\eta_A(s) = \sum_{n\in\mathbb{Z}} \operatorname{sgn}\left(+\frac{\rho}{L}+\frac{2\pi n}{L}\right)\left|+\frac{\rho}{L}+\frac{2\pi n}{L}\right|^{-s}$$
$$= \left(\frac{2\pi}{L}\right)^{-s}\sum_n \operatorname{sgn}\left(+\frac{\rho}{2\pi}+n\right)\left|+\frac{\rho}{2\pi}+n\right|^{-s}$$

Thus
$$\eta_A(0) = -2\frac{\rho}{2\pi}+1 \qquad \text{if } 0<\rho<2\pi$$
$$= -\frac{\rho}{\pi}+1$$

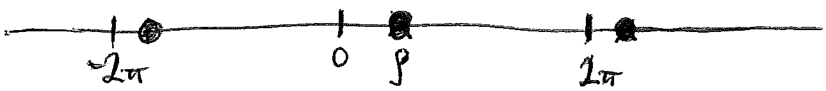
Picture: Think of ρ as being small >0, so that one has a small positive eigenvalue $\lambda = \frac{\rho}{L}$



The point is that in the sum

$$\eta_A(s) = \sum' \operatorname{sgn}(\lambda)\, |\lambda|^{-s}$$

we have the following eigenvalues



as $\rho \downarrow 0$ the sum gives 1 for the other terms cancel. As ρ increases, the term just above 2π is smaller than the term just above -2π, so η decreases.

~~By my formulas~~

I need formulas expressing $\det(A)$ in terms of torsion and η. Do in finite dimensions.

$$\eta_A(0) = \sum \operatorname{sgn}\lambda\ |\lambda|^{-s}\Big|_{s=0}$$

$$\zeta_{A^2}(0) = \sum' \qquad |\lambda|^{-2s}\Big|_{s=0} = \sum' |A|^{-s}\Big|_{s=0}$$

$$\eta_A(0) - \zeta_{A^2}(0) = -2 \sum_{\lambda<0} |\lambda|^{-s}\Big|_{s=0} = -2\, n_-$$

$$\det(A) = \left(\det(A^2)\right)^{1/2} (-1)^{n_-}$$

$$= e^{-\frac{1}{2}\zeta'_{A^2}(0)} \left(e^{\pm i\pi}\right)^{-\frac{1}{2}\eta_A(0)} \qquad \text{if } \zeta_{A^2}(0) = 0$$

$$= e^{-\frac{1}{2}\zeta'_{A^2}(0) \pm i\frac{\pi}{2}\eta_A(0)}$$

In the above example

$$\det A = e^{-i\rho} - 1 \qquad \text{(up to a constant ind. of } \rho)$$

$$= e^{-i\frac{\rho}{2}} \underbrace{\frac{\left(e^{i\frac{\rho}{2}} - e^{-i\frac{\rho}{2}}\right)}{2i}}_{\sin \rho/2 \text{ which is } > 0} (-2i)$$

$$= e^{-i\frac{\rho}{2} - i\frac{\pi}{2}} \left(2 \sin \frac{\rho}{2}\right)$$

$$i\frac{\pi}{2}\eta_A(0) = i\frac{\pi}{2}\left[-\frac{\rho}{\pi} + 1\right] = -i\frac{\rho}{2} + i\frac{\pi}{2}$$

This seems to suggest formulas like

$$\det(A) = e^{-\frac{1}{2}\zeta'_{A^2}(0) + i\frac{\pi}{2}\eta_A(0)}$$

$$= 1 - e^{-i\rho} = 1 - M$$

however the other regularization process $G(x^+, x)$ would give

$$\det(A) = e^{-\frac{1}{2}\zeta'_{A^2}(0) - i\frac{\pi}{2}\eta_A(0)}$$

Let's leave the heuristics and try to understand the variational behavior of the η-invariant.

$$\eta_A(s) = \sum (\operatorname{sgn} \lambda)\, |\lambda|\, |\lambda|^{-s-1}$$

$$= \operatorname{Tr}\left(A (A^2)^{-\frac{s+1}{2}}\right)$$

which makes sense because $A^2 > 0$. Put $B = A^2$, and let B undergo an infinitesimal variation δB. Assume only that A commutes with B.

$$\delta \operatorname{Tr}(AB^{-s}) = \frac{1}{\Gamma(s)} \int_0^\infty \delta \operatorname{Tr}(Ae^{-tB})\, t^s \frac{dt}{t}$$

$$\delta e^{-tB} = \int_0^t dt_1\, e^{-(t-t_1)B}(-\delta B)\, e^{-t_1 B}$$

So
$$\mathrm{Tr}(A\delta e^{-tB}) = -\int_0^t dt_1\, \mathrm{Tr}(A e^{-(t-t_1)B}\,\delta B\, e^{-t_1 B})$$
$$= -t\,\mathrm{Tr}(\text{[redacted]}\, A e^{-tB}\,\delta B)$$

$\therefore$
$$\delta\,\mathrm{Tr}(AB^{-s}) = \frac{1}{\Gamma(s)}\int_0^\infty (-1)\,\mathrm{Tr}(Ae^{-tB}\,\delta B)\, t^s\, dt$$
$$= \frac{1}{\Gamma(s)}\ \text{[redacted]}\ \Big\{ \cancel{\big[+\mathrm{Tr}(Ae^{-tB}B^{-1}\delta B)\,t^s\big]_0^\infty} - \int_0^\infty \mathrm{Tr}(Ae^{-tB}B^{-1}\delta B)\, s\, t^{s-1}\,dt \Big\}$$
$$\delta\,\mathrm{Tr}(AB^{-s}) = -s\,\mathrm{Tr}(AB^{-s-1}\,\text{[redacted]}\,\delta B)$$

So
$$\delta\eta_A(s) = \mathrm{Tr}\big(\delta A\, B^{-\frac{s+1}{2}}\big) + \Big(-\frac{s+1}{2}\Big)\mathrm{Tr}\Big(AB^{-\frac{s+1}{2}-1}\underbrace{\delta B}_{A\delta A+\delta A A}\Big)$$
$$= \Big[2\Big(-\frac{s+1}{2}\Big)+1\Big]\,\mathrm{Tr}\big(B^{-\frac{s+1}{2}}\delta A\big)$$
$$\delta\eta_A(s) = -s\,\mathrm{Tr}\big(B^{-(\frac{s+1}{2})}\delta A\big)$$

[redacted] Now B is 2nd order on a manifold of odd dimension m so $\mathrm{Tr}(B^{-\frac{s+1}{2}}\delta A)$ has at most simple poles at $\frac{s+1}{2} = \frac{m}{2}, \frac{m}{2}-1, \ldots$ which are integer values of s. So $\mathrm{Tr}(B^{-\frac{s+1}{2}}\delta A)$ can have at most a simple pole at $s=0$ and so $\delta\eta_A(s)$ is regular at $s=0$ and it is given by a local formula which we get from the heat kernel expansion of $\mathrm{Tr}(e^{-tB}\delta A)$, coeff. of $t^{-1/2}$.

$$\delta\eta_A(s) = -s\,\frac{1}{\Gamma(\frac{s+1}{2})}\int_0^\infty \underbrace{\mathrm{Tr}(e^{-tB}\delta A)}_{c\,t^{-1/2}}\, t^{\frac{s+1}{2}}\,\frac{dt}{t}$$

$$\underset{s\to 0}{\sim}\ -s\,\frac{1}{\Gamma(\frac{1}{2})}\, c \underbrace{\int_0^1 t^{\frac{s}{2}-1}\,dt}_{2/s} = -\frac{2c}{\sqrt{\pi}}$$

This must be the argument that the residue doesn't change, as well as the fact that $\delta\eta(0)$ is given by a local formula.

Fascinating point: Suppose we define

$$\log \det(A) = -\tfrac{1}{2}\zeta'_{A^2}(0) + i\tfrac{\pi}{2}\eta_A(0)$$

and notice that this not subject to the usual ambiguity in the definition of $\det(A)$. Then corresponding to this definition is a regularization process for $\mathrm{Tr}(A^{-1}\delta A)$. It is not the obvious candidate $\mathrm{Tr}(e^{-tB}A^{-1}\delta A)\big|_{t=0}$ because

$$\delta\left(-\tfrac{1}{2}\zeta'_{A^2}(0)\right) = \tfrac{1}{2}\,\mathrm{Tr}\Big(e^{-tB}B^{-1}\underbrace{\delta B}_{A\delta A+\delta A\cdot A}\Big)\Big|_{t=0}$$

$$= \mathrm{Tr}\left(e^{-tB}A^{-1}\delta A\right)\Big|_{t=0}$$

so the obvious heat equation candidate just gives the real part. This corresponds exactly to what we saw for the 1-diml. case.

October 8, 1982

An important point that I learned last night is that heat kernel traces of the form

$$\operatorname{Tr}\left(e^{-t\Delta}K\right)$$

don't always have asymptotic expansions. This arose as follows: Start with

$$\eta(s) = \operatorname{Tr}\left(A\, B^{-\left(\frac{s+1}{2}\right)}\right)$$

$$\delta\eta(s) = -s\operatorname{Tr}\left(B^{-\left(\frac{s+1}{2}\right)}\delta A\right)$$

derived yesterday. Then

$$\delta\eta(s) = -s\operatorname{Tr}\left(B^{-\left(\frac{s+1}{2}\right)}\delta A\right)$$

$$= -s\frac{1}{\Gamma\left(\frac{s+1}{2}\right)}\int_0^\infty \underbrace{\operatorname{Tr}\left(e^{-tB}\delta A\right)}_{\text{has } c\,t^{-1/2} \text{ term}} t^{\frac{s+1}{2}}\frac{dt}{t}$$

$$\underset{s\to 0}{\sim} -s\frac{1}{\Gamma\left(\frac{1}{2}\right)}\, c\int_0^1 t^{\frac{s}{2}}\frac{dt}{t} = -\frac{2}{\sqrt{\pi}}\, c$$

assuming $\operatorname{Tr}(e^{-tB}\delta A)$ has an asymptotic expansion. But also

$$\delta\eta(s) = -s\operatorname{Tr}\left(B^{-\frac{s}{2}}B^{-1/2}\delta A\right)$$

$$= -s\frac{1}{\Gamma\left(\frac{s}{2}\right)}\int_0^\infty \underbrace{\operatorname{Tr}\left(e^{-tB}B^{-1/2}\delta A\right)}_{\text{has } c_1 t^0 \text{ term}} t^{s/2}\frac{dt}{t}$$

$$\underset{s\to 0}{\sim} -s\frac{1}{\Gamma\left(\frac{s}{2}\right)}\, c_1\frac{2}{s} \sim -c_1 s \to 0$$

assuming that $\operatorname{Tr}(e^{-tB}B^{-1/2}\delta A)$ has an asymptotic expansion. So we reach a contradiction.

(resolved on p. 200)