The Riemann-Hilbert problem: Let $X$ be a Riemann surface, $S$ a finite subset of $X$, and $E$ a holomorphic vector bundle over $X-S$ equipped with a holomorphic connection. Since $\dim(X) = 1$, the connection is integrable and so if we choose a basepoint outside of $S$, we get a representation of $\pi_1(X-S)$ on the fibre over the basepoint. In this way we get an equivalence of categories

$$\mathfrak{K} \left\{ \text{Rep}_k \pi_1(X-S) \right\} \sim \left\{ \text{holom. vec. b. over } X-S \right\}_{\text{with connection}}$$

Supposing $X-S$ not complete, then I believe Stein theory implies that any holomorphic vector bundle over $X-S$ is trivial. In any case taking a holomorphic vector bundle $E$ over $X-S$ with a trivialization, then a connection on $E$ can be described by a DE

$$Du = Au$$

where $A$ is a matrix of 1-forms over $X-S$. A different trivialization of $E$ is given by a holomorphic map $g: X-S \to GL_n$. If one makes the change of coordinates $u = g \nu$, then

$$dg \nu + g \nu Du = Ag \nu$$

$$D \nu = (g^{-1}Ag - g^{-1}dg) \nu$$

so the connection form changes

$$A \mapsto g^{-1}Ag - g^{-1}dg$$

Given a connection on the trivial bundle we can integrate to obtain the parallel translation matrix $Y(z \in \mathbb{C})$. 
where $X$ is a homotopy class of paths from the base $z_0$ to $z$. We have

$$d\gamma(z \overset{p}{\leftarrow} z_0) = A(z)\gamma(z \overset{p}{\leftarrow} z_0)$$

$$\gamma(z \overset{q}{\leftarrow} z_0) = \gamma(z \overset{p}{\leftarrow} z_0)\gamma(z_0 \overset{q}{\leftarrow} z_0)$$

So $\gamma(z_0 \overset{q}{\leftarrow} z_0)$ is a multi-valued matrix function of $z$. If all the monodromy transformations $\gamma(z_0 \overset{q}{\leftarrow} z_0)$ are invertible, then we can take $g(z) = \gamma(z \overset{p}{\leftarrow} z_0)$ which is independent of $X$. Then $A = \partial dg g^{-1}$ and so $g$ transforms $A$ into the trivial connection. So we see that a bundle with connection having trivial monodromy is isomorphic to the trivial bundle with trivial connection, which is a special case of $\otimes$.

The next point is to make a more algebraic description of bundles with connection. We know that any vector bundle of $X$ can be extended to $X$. If $X$ is complete, then any holomorphic vector bundle over $X$ is algebraic, and we can ask the connection, which is given over $X$ over $X$, be algebraic. The connection is a map of vector bundles $\nabla (E) \rightarrow E \otimes \Omega$, so algebraic here means that in the extension to $X$ this map is meromorphic.

Let's analyze these ideas carefully in the case where the rank is 1. So we are given a holomorphic line bundle $L$ over $X$. To simplify, suppose $X$ is the disk and $S = \{0 \}$, so that $X$ is $D^*$, and suppose $L$ is the trivial bundle over $X$. Let $\tilde{L}$ be an extension of $L$ to $X$. Then $\tilde{L}$ is trivial so it has a non-vanishing
section $s$ which over $X-S$ is given by a holom. map

$$X \rightarrow S \xrightarrow{s} \mathbb{C}^*$$

$s$ is unique up to multiplying by $f: X \rightarrow \mathbb{C}^*$.

The extension $\tilde{L}$ of $L$ to $X-S$ is determined by those sections of $L$ which extend to holomorphic sections of $\tilde{L}$. Thus a function on $X-S$ is a section of $\tilde{L}$ when it is of the form $fs$ with $f: X \rightarrow \mathbb{C}$ holomorphic.

Similarly it is a meromorphic section with pole at $0$ when it is of the form $fs$ with $f: X \rightarrow S \rightarrow \mathbb{C}$ meromorphic.

To see what's happening look at the field of Laurent series converging in a punctured disk around $0$. Or one can let $K^*$ holomorphic maps $X-S \rightarrow \mathbb{C}$. Then any element of $K^*$ has a degree and ones of degree zero have logarithms. Let $R = \text{convergent power series around } 0$, or $R = \text{holom. maps } X \rightarrow \mathbb{C}$. Then given $f \in K^*$ of degree $0$, we have

$$\log f = \sum c_n z^n$$

and we can see that $\{z^n\}_{n < 0}$ gives all $\sum c_n z^n$

$$K^*/R^* = \mathbb{Z} \times \exp(K_+).$$

So there are an incredible number of different extensions of $L$ to an $\tilde{L}$, and they are going to be different even if meromorphic sections are allowed.

Next consider the connection

$$\frac{du}{dz} = A(z)u$$

given an $L$ over $X-S = D^*$. Suppose we have an extension given by $s \in K^*$, $s: X-S \rightarrow \mathbb{C}^*$.
whose holom. sections are $fs$ with $f \in R$. Then putting $u = fs$ as a change of variable, we find

$$f's + fs' = Af$$
$$a \frac{df}{dz} = \left( A(0 - \frac{1}{s} \frac{ds}{dz}) \right) f$$

How much can $A$ be changed? The good case from the algebraic viewpoint is when $A - \frac{1}{s} \frac{ds}{dz}$ is as non-singular as possible. The point is that for an $s \in K^*$, $\frac{1}{s} \frac{ds}{dz}$ can be rigged to be anything having residue $c \in Z$. So $s$ is chosen to kill off the term, in $A$ except $\frac{a}{z}$, where $a$ can be changed by an integer.

So what I seem to be getting is the following.

Prop: Let $L$ be a holomorphic line bundle with connection over $x_S$. Then there is an extension of $L$ to a line bundle over $\mathbb{P}^\times$ such that connection has a regular singularity at $z_0 \in S$. (This means that if I take a section of $L$ near $z_0$ and non-vanishing at $z_0$, then $s'(\varphi S)$ has a simple pole at $z_0$.) Any two extensions are related by a power of a uniformizing parameter around $z_0$.

Proof: All this takes place around $z_0 = 0$ in the disk. We know $L$ is given by a representation of the fundamental group, hence can be realized in the punctured disk $D^\times$ by the trivial line bundle with connection

$$\frac{du}{dz} = \frac{a}{z} u$$

The existence of $\tilde{L}$ is now clear.

Any extension will have holom. sections around $z_0$ given by $fs$ where $s \in K^*$ is fixed and $f$ varies over $R$. The new connection form is $\frac{a}{z} - \frac{1}{2} \frac{ds}{dz}$ and
if this has only a simple pole at \( z=0 \), then one can see easily that \( s \in \mathbb{R}^n \). So the possible \( \mathcal{E} \) are of the form \( \mathbb{C}^n \mathcal{E} \) for different \( n \).

January 4, 1982

Deligne's thm. says for a smooth variety the category of representations of \( \pi_1 \) is equivalent to the category of algebraic vector bundles with a connection which is regular at \( \infty \).

In the case of \( X-S \) where \( X \) is a complete smooth curve it says that given a representation of \( \pi_1(X-S) \), we can take the associated flat holomorphic vector bundle \( E \) over \( X-S \) and find an extension \( \tilde{E} \) to \( X \) such that the connection \( \nabla: E \to E \otimes \Omega \) extends to \( \tilde{E} \to \tilde{E} \otimes \Omega \otimes \Omega(S) \) so that the connection has simple poles along \( S \). Now \( \tilde{E} \) is not unique, but because \( X \) is complete, \( \tilde{E} \) has an algebraic structure, and then the induced algebraic structure on \( E \) is unique.
January 4, 1982

Let's review the soliton solutions of KdV.

I start from a Schrödinger equation on the line

\[ (-\partial_x^2 + g) u = k^2 u \]

where \( g \) decays fast as \( |x| \to \infty \). This guarantees existence and uniqueness of solutions with asymp. behav.

\[ f_k(x) \sim e^{ik x} \quad x \to +\infty \]

\[ f_k(x) \sim e^{-ik x} \quad x \to -\infty \]

for \( \text{Im} k > 0 \). Then I get transfer coefficients \( A(k), B(k), \ k \in \mathbb{R} \) defined by

\[ \phi_k = A f_k + B f_{-k} \]

It is known that \( 2ikA \) is analytic in the UHP and has an asymptotic expansion

\[ A(k) = 1 - \frac{i}{2ik} \int_{-\infty}^{\infty} g + \ldots \quad \text{as } |k| \to \infty \]

in the UHP.

Also \( A(k)A(-k) - B(k)B(-k) = 1 \) for \( k \) real.

At points \( ik \) in the UHP where \( A(ik) = 0 \), there are non-zero numbers \( B(ik) \) such that

\[ \phi_{ik} = B(ik) \phi_{-ik} \]

Suppose \( B = 0 \) for \( k \in \mathbb{R} \). Then

\[ A(-k) = \frac{1}{A(k)} \]

gives a meromorphic extension of \( A \) to the LHP.

Since \( A \to 1 \) as \( k \to \infty \)

it must be rational:

\[ A(k) = \prod_k \frac{k-iK}{k+iK} \]

Next we have

\[ f_{-k} = \frac{1}{A(k)} \phi_k \]

which gives for each \( x \) an extension of \( f_k \) to a meromorphic function in the LHP with simple poles at the points \(-ik\).
We have
\[
\text{Res } f_k(x) = \lim_{k \to -iK} f_k(x) = \lim_{k \to iK} (-k+iK)f_k(x) = \frac{1}{A(k) f_k(x)}
\]

\[
= -\frac{B(iK)}{A'(iK)} f_iK(x)
\]

Let us form from the extended $k$-plane a singular curve with ordinary double points by identifying $iK$ and $-iK$. Over this singular curve we take the line bundle whose sections are meromorphic functions $f_k$ having simple poles at the points $-iK$ satisfying
\[
\text{Res } f(k) = -\frac{B(iK)}{A'(iK)} f(iK)
\]

(Note that if we have $N$ points $-iK$, then the space of functions with simple poles at these points is of dim $1+N$, and the residue conditions are $N$ conditions, so in general we have a unique $f(k)$ with value $1$ at $k = \infty$.)
Review solutions again:
\[ (-\partial^2 + q) \psi = k^2 \psi \]

\[ \psi_k = A \phi_{-iK} + B \phi_{iK} \quad \text{on } \Re \{ z \} \]

\[ \psi_{iK} = B (iK) f_{iK} \]

at points \( iK \) in UHP where \( A \phi_{iK} = 0 \).

Suppose \( B(k) = 0 \) on \( \Re \). Then we know

\[ A(k) = \prod \frac{k - iK}{k + iK} \]

and for any \( x \), \( f_k(x) \) is a meromorphic function of \( k \) with only simple poles at the points \(-iK\) satisfying

\[ \text{Res}_{k=-iK} f_k(x) = -\frac{B(iK)}{A'(iK)} f_{iK}(x) \]

Also as \( k \to \infty \) one has

\[ f_k(x) \sim e^{iKx} \left( 1 + \frac{a_1(x)}{k} + \frac{a_2(x)}{k^2} + \ldots \right) \]

So now I want to rewrite this in the following way. Draw a circle \( S \) containing the points \( \pm iK \) and work with analytic functions on this circle. Let \( H_- \) be the space of functions on the circle which extend to meromorphic functions inside with simple poles at the points \( iK \) satisfying the residue condition \( \circ \). Let \( H_+ \) be the functions on the circle given by convergent power series \( a_0 + a_1/k + a_2/k^2 + \ldots \). Then \( H_+ \cap H_- \) is the sections of a line bundle over the singular curve obtained by identifying \( iK \) and \(-iK\). So it is one-dimensional. So we know for sufficiently small \( x \) that \( f_k(x) \in e^{iKx} H_+ \cap H_- \) will exist.

So now what I should do see what conditions on \( H_- \) lead to a Schrödinger equation. Notice that \( H_- \)
contains all analytic functions inside the circle which vanish at the points $iK$. Hence $H_-$ is commensurable with the standard $H_{st}^i$, consisting of convergent series $a_0 + a_1 x + ...$ on the circle.

It seems that the algebraic function on this singular curve are polynomials in $iK$ which carry $H_-$ into itself, because the residue condition forces the values of the polynomial to be the same at the points $\pm iK$. So for example $H_-$ is stable under multiplication by $k^2$.

But now
$$f_k(x) = e^{ikx}(1 + \frac{a_1(x)}{k} + ... \in H_-$$
for all $x$ so we can differentiate
$$f_k''(x) = (ik)^2 f_k + 2ik e^{ikx}(\frac{a_1(x)}{k} + ... \in H_-$$
$$+ e^{ikx}(\frac{a_2''(x)}{k} + ...$$
so
$$f_k'' + k^2 f_k = e^{ikx}(2i a_1(x) + \frac{2i a_1'}{k} + ... \in H_-$$
because $k^2 H_- \in H_-$. Thus we have
$$f_k'' + k^2 f_k = \varrho f_k$$
where $\varrho = 2i a_1'$.

Let's derive KdV equation. To simplify the notation put $\varepsilon = ik$ and then write
$$f_k(x, t) = e^{x \varepsilon + t \varepsilon^3}(1 + \frac{a_1}{\varepsilon} + \frac{a_2}{\varepsilon^2} + ... \in H_-$$
This formal gadget is called the Baker--Akhiezer fn.
$$f = e^{x \varepsilon + t \varepsilon^3}[\varepsilon^3 + a_1 \varepsilon^2 + a_2 \varepsilon + a_3 + ... + a_1 \varepsilon^{-1} + a_2 \varepsilon^{-2} + ...]$$
It might be simpler to put
$$f(x, t) = e^{x \varepsilon + t \varepsilon^3} h(x, t) \quad h = 1 + a_1 \varepsilon^{-1} + a_2 \varepsilon^{-2} + ...$$
\[ f'' = e^{xz+\tau z^3} \left[ z^2 h + 2zh' + h'' \right] \in H_r \]
\[ f'' = z^2 f' + e^{xz+\tau z^3} \left[ 2a_1' + 2a_2'/z + a_1''/z \right] \in H_r \]
\[ \Rightarrow f'' = z^2 f' + 2a_1' f \]
\[ 2a_1' a_1 = 2a_2' + a_1'' \]
\[ a_2' = a_1 a_1' - a_1'' \]
\[ a_2 = \frac{1}{2} (a_1^2 - a_1') + \text{const} \]
\[ \begin{align*}
\dot{f} &= e^{xz+\tau z^3} \left[ z^3 h + h'' \right] \\
\dot{f}'' &= e^{xz+\tau z^3} \left[ z^3 h + 3z^2 h' + 3z h'' + h''' \right] \\
&= \frac{3a_1' + a_1''}{3a_1' + 3a_1''/z} + \frac{3a_1' + a_1''}{3a_1' + a_1''/z} + \frac{3a_1' + a_1''}{3a_1' + a_1''/z} \\
\end{align*} \]
\[ \dot{f} - f'' + 3a_1' f' = \sum_{k=0}^{\infty} x^k \left[ a_1/z + (-3a_2' - 3a_1' + 3a_1 q_1) + (-3a_3' - 3a_2' - a_1''/z) \right] \]
\[ 3[a_1' a_1 - a_1''/z] = 3[a_1' a_1 - a_1''/z] + a_1''/z = -\frac{3}{2} a_1'' \]
\[ \text{so} \quad \dot{f} = f'' - 3a_1' f' - \frac{3}{2} a_1'' f \quad \text{But} \quad \delta = 2a_1' \]
\[ \dot{f}'' - \frac{3}{2} \delta f' - \frac{3}{4} \delta f \]
Recall that if \( L = -\partial_x^2 + \delta \), then KdV is
\[ L^* = [M, L] \quad M = \partial_x^3 - \frac{3}{2} \delta \partial_x - \frac{3}{4} \delta' \]
So therefore the \( f \) we have constructed satisfies
\[ f = M f \]
Inside $V=L^2(S')^d$ I want to understand the family of all subspaces $W$ complementary to a given incoming subspace $W_-$. Suppose we fix $W_+ = (W_-)^\perp$. Then subspaces $W$ complementary to $W_-$ are given by the graph of a linear map $T: W_+ \rightarrow W_-$.

\[ W_- \xrightarrow{T} W \]

\[ W_+ \]

I want the condition on $T$ corresponding to $z W \subset W$.

Take $d=1$, \[ W_+ = H_+ = \text{span of } 1, z, z^2, \ldots \] and \[ W_- = \text{span of } z^{-1}, z^{-2}, \ldots. \]

I know that $W = f W_+$, where $f: S' \rightarrow S'$ has degree $0$. So the matrix of $f$ relative to the decomposition $V = W_+ \oplus W_+$ is

\[
\begin{pmatrix}
A & B \\
C & D
\end{pmatrix}
\]

then \[ W = \{(Bx) \leftarrow (Dx) \mid x \in W_+\} = \{(BD^{-1}x) \leftarrow x \mid x \in W_+\}. \]

This shows $T = BD^{-1}$, and it is not clear that there is an easy way to formulate the condition $z W \subset W$ in terms of $T$.

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I would like to understand in $L^2(S')$ the possible subspaces $W$ such that $z^2 W \subset W$ and which are complementary to $W_- = z^{-1} H_- = \text{span of } z^{-1}, z^2, z^3, \ldots$. The beautiful thing is that $L^2(S')$ with the operator $z^2$ is equivalent to $L^2(S')^2$ with the operator $z$. So what we are looking at is simply all outgoing subspaces in $L^2(S')^2$. 
which are complementary to a given $W_-$, and these are classified by a maps $s' \to H_2$. They should be of degree 0, and the pair $(W_-, W)$ defines a rank 2 vector bundle over $P^1$ with $h^0 = h^1 = 0$, and hence $E \cong O(-1) \oplus O(-1)$.

Restate formally: Suppose one gives a subspace $W$ inside the space of formal Laurent series $\sum_{n \leq N} c_n z^n$ which is complementary to $z^{-1} \mathbb{C}[z^{-1}]$, and is such that $z^2 W \subset W$. Is there then a curve which is a double covering of the projective line with variable $z^2$ such that $W$ is the space of sections of a line bundle over the complement of $\infty$.

What seems to be clear is $W$ is a free module of rank 2 over $\mathbb{C}[z^2]$. You get a basis because

$$W_n = \dim \left( z^n \mathbb{C}[z^{-1}] \cap W \right) = n + 1 \quad n \geq 0.$$ 

so you can pick generators in this space for $n = 0, 1$. Now you need another endomorphism $W$ which commutes with $z^2$, and it should be a uniformizer on the curve at $\infty$, so should be a series $z + c_0 + c_1 z^{-1} + \ldots$. No.

Example: $W =$ space of rational functions in $z$ having at most a simple pole at $z = -\beta$ satisfying the condition

$$\operatorname{res} f(z) = c f(\beta).$$

Here the ring $R$ belonging to the affine curve is the subring of $\mathbb{C}[z^2]$ consisting of $f$ such that $f(-\beta) = f(\beta)$. Thus

$$R = \mathbb{C}[z^2] + \mathbb{C}[z^2] z (z^2 - \beta^2).$$
and \( R \) is generated by \( X = z^2 \) and \( Y = z(z^2 - g^2) \) with the relation
\[
Y^2 = z^2(z^4 - 2g^2z^2 + g^4)
\]
\[
Y^2 = X^3 - 2g^2X^2 + g^4X
\]

In general if I take several \( g_i \), then the ring becomes

\[
R = \mathbb{C}[z^2] + \mathbb{C}[z^2] \cdot \prod_i (z^2 - g_i^2)
\]

with the relation
\[
Y^2 = X\left(\prod_i (X - g_i^2)\right)^2
\]

Notice that this is in the form \( Y^2 = Xf(X)^2 \) where \( f(X) \) has distinct roots. The distinct roots are related to the residue condition which defines the line bundle.
January 10, 1982

Notes on the Krichever theory (Mumford's paper).

We start with a complete curve $X$, a smooth point $P$ on $X$, a torsion-free rank 1 sheaf $\mathcal{F}$ on $X$ such that $h^0(\mathcal{F}) = h^1(\mathcal{F}) = 0$. Let $\epsilon$ be a uniformizing parameter on $X$ at $P$ and $U$ a disk around $P$ on which $\epsilon$ is a coordinate.

We construct a deformation $\mathcal{F}_t$ of $\mathcal{F}$ by requiring $\mathcal{F}_t = \mathcal{F}$ over $X - P$, but a section of $\mathcal{F}_t$ over $U$ is of the form $e^{tz}f$ where $f$ is a holomorphic section of $\mathcal{F}$ over $U$. Thus if

$$V = \Gamma_{\text{hol}}(U - P, \mathcal{F}), \quad W = \Gamma_{\text{hol}}(X - P, \mathcal{F}), \quad e^{tz}W = \Gamma_{\text{hol}}(U, \mathcal{F})$$

then $\mathcal{F}_t$ is the holomorphic sheaf with

$$V = \Gamma_{\text{hol}}(U - P, \mathcal{F}_t), \quad W = \Gamma_{\text{hol}}(X - P, \mathcal{F}_t), \quad e^{tz}W = \Gamma_{\text{hol}}(U, \mathcal{F}_t)$$

For small $t$ it should be true that $h^0(\mathcal{F}_t) = h^1(\mathcal{F}_t) = 0$, or equivalently, because of the Cech computation of the cohomology by the covering $X = (X - P) \cup U$, that

$$V = e^{tz}W \oplus W.$$

Now put $H_\epsilon = e^{zW}$, so that when $\otimes$ holds, we have that $e^{tz}H_\epsilon \cap W$ is 1-dimensional. If we choose a non-vanishing section $s_0$ of $\mathcal{F}$ over $U$, then I can identify $H_\epsilon$ with convergent series $\sum c_n z^{-n}$ in $U$.

So for each $t$, we get a unique section

$$s_0(t) = e^{tz} \left(1 + \frac{a_1}{z} + \frac{a_2}{z^2} + \cdots\right) e^{tZ}H_\epsilon \cap W$$

of $\mathcal{F}_t(P)$ called the Baker-Akhiezer function. If we successively differentiate

$$s_n(t) = \left(\frac{\partial}{\partial t}\right)^n s_0(t)$$

then $s_0^{(t)}, s_1^{(t)}, \ldots, s_n^{(t)}$ forms a basis for $\Gamma(X, \mathcal{F}_t((n+1)P))$. So now let $R = \Gamma(X, \mathcal{F}_t(\mathcal{O}_X))$. Then $\otimes$ if $a \in R$ has order
In this way we associate to each element of $R$ a differential operator in $\mathcal{O}(t)[\frac{\partial}{\partial t}]$, whose leading coefficient is constant. The order of the operator is the order of the element of $R$ at the point $P$.

So what one obtains in this way is a commutative subring of $\mathcal{O}(t)[\frac{\partial}{\partial t}]$ containing two operators

$$A = a_n\frac{\partial}{\partial t}^n + \cdots + a_0(t)$$

$$B = b_n(t)\frac{\partial}{\partial t}^n + \cdots + b_0(t)$$

where the leading coefficients $a_n, b_n$ are constants $\neq 0$, and where $(n, n) = 1$. In Mumford's paper, he doesn't want to normalize the leading coefficient of $S_0$, and takes any

$$S_0(t) = e^{t\frac{\partial}{\partial t}}(a_0 + \frac{a_1}{t} + \cdots) \in e^{t\frac{\partial}{\partial t}} H_+ \cap W$$

where $a_0(t) \neq 0$. Then the subring of differential operators is determined up to conjugacy by a $\psi(t) \in \mathcal{O}(t)[\frac{\partial}{\partial t}], \psi(0) \neq 0$. Then the conjugacy class of the operator ring doesn't depend very much on the choice of $z$. In effect if

$$z' = z + c_0 t^2 + c_1/t + \cdots$$

is another coordinate at $P$, then

$$e^{t\frac{\partial}{\partial t}} = e^{t\frac{\partial}{\partial t}} e^{2c_0 t^2 + tc_1/t + \cdots} \in e^{t\frac{\partial}{\partial t}} H_+$$

so the Baker-Akhieser function is independent of the choice of $z$, provided one fixes its image in $m_P/m_P^2$.

In this theory there is an interesting lemma which says that if we take an operator $A = a_n\frac{\partial}{\partial t}^n + a_{n-1}(t)\frac{\partial}{\partial t}^{n-1} + \cdots + a_0(t)$
in $C[[z]]$ with $a_m(t) \neq 0$. Then look at its commutant in the ring $R[t]$ of (formal) pseudo-diff. operators $\sum c_n(t) \partial^n$, then the commutant is generated by $A^m$, in fact, every commuting pseudo-diff. op is of the form

$$\sum c_n(A^m)^n$$

$-\infty < n < \infty$.

I want to consider the following problem. Let's start with $V$ = analytic function in a punctured disk $U \setminus \{P\}$, and $H_-$ = analytic functions in $U$, and suppose I give a subspace $W$ of $V$ which is complementary to $z^{-1}H_-$:

$$V = z^{-1}H_- \oplus W$$

Then I should be able to define the Baker-Akhiezer function

$$s_0(t) = e^{t^2} \left(1 + \frac{a_1}{2} t + \frac{a_2}{3} t^2 + \cdots\right) e^{-t^2} H_- \cap W$$

at least for small $t$. Next I suppose that

$$z^2 W \subset W$$

which implies that if $W$ came from an $X$, $F$ as above, then $z^2 \in R$. Thus $z^2$ is a meromorphic function on the curve with only a double pole at $P$, so we have a hyper-elliptic curve which might be degenerate.

I have seen that then to $z^2$ belongs a differential operator of the second order. Namely

$$\left(\frac{\partial}{\partial t}\right)^2 s_0 = z^2 s_0 + q(t) s_0 \quad q(t) = 2 a_1'(t).$$

Thus in the corresponding ring of operators, assuming $W$ comes from an $M(X, F)$, we have the operator

$$A = \left(\frac{\partial}{\partial t}\right)^2 - q(t).$$

Thus if we have a curve, we must be able to find an operator $B$ commuting with $A$ of odd degree. By the lemma on the commuting operators...
B will be a linear combination of \((A^{1/2})^n\), and so we will find a \(B\) of the form \(p(A)A^{1/2}\) for some polynomial \(p(A)\). Then the curve will have the equation

\[
B^2 = A p(A)^2.
\]

Why should \(p\) be a polynomial?

We see that any \(W\) gives an operator \(\partial^2 - \partial_t\).

Conversely, given this operator we can construct a Baker-Akhieser function

\[
s_0(t) = e^{tz}(1 + \frac{a_1(t)}{z} + \frac{a_2(t)}{z^2} + \ldots)
\]

as the formal solution of the equation \((\partial^2 - z^2)s_0 = \partial_t s_0\).

Then \(W\) should be spanned by the functions \(s_0(t)(z)\) for different \(t\), in some sense. Does this have a sense?

We can always consider \(s_0(0), s_0'(0), s_0''(0), \ldots\) which will give a family of formal series \(1 + O(z), z, z^2, \ldots\) and should give a free module of rank 2 over \(\mathbb{C}[z^2]\), which is complementary to \(z^{-1}H_\cdot\). Let's go over this carefully.

Let \(W\) be a subspace of \(\mathbb{C}[[z^{-1}]][z] = V\) which is complementary to \(z^{-1}C[[z^{-1}]]\) and such that \(z^2 W \subset W\).

Then we know that

\[
W_n = z^n H_\cdot \cap W
\]

is \((n+1)\)-dimensional, and that \(U W_n = W\). There is a unique element in \(W_n\) modulo \(W_{n-1}\) of the form \(z^n + \text{lower degree terms}\). I claim now that there are unique series \(a_n(t) \in \mathbb{C}[[t]]\) for \(n > 1\) such that

\[
\tilde{s}(t, z) = e^{tz}(1 + \frac{a_1(t)}{z} + \frac{a_2(t)}{z^2} + \ldots) \in W[[t]]
\]
Think of $e^z$ as being in $V[[t]]$, $V = \text{field } \mathbb{C}[z]$. Let's get the coefficients by differentiating: The condition

$$1 + \frac{a'_1(z)}{2} + \frac{a'_2(z)}{2^2} + \cdots \in H_{-n} W$$

determines $a_1(z), a_2(z), \ldots$. Then we want

$$z \left( 1 + \frac{a_1(z)}{2} + \frac{a_2(z)}{2^2} + \cdots \right) + \frac{a'_1(z)}{2} + \frac{a'_2(z)}{2^2} + \cdots \in H_{-n} W$$

We know that $zH_{-n} W$ contains an elt of the form $z + \text{lower terms unique modulo } H_{-n} W$, and we can use elts of $H_{-n} W$ to get a unique element of $zH_{-n} W$ of the form $z + c + \text{lower terms with a given } c$. Thus one sees $a'_1(z), a'_2(z), \ldots$ exist and are unique. Next we want

$$z^2 \left( 1 + \frac{a'_1(z)}{2} + \frac{a'_2(z)}{2^2} + \cdots \right) + 2z \left( \frac{a''_1(z)}{2} + \frac{a''_2(z)}{2^2} + \cdots \right) + \left( \frac{a'''(z)}{2} + \frac{a'''(z)}{2^2} + \cdots \right) \in z^2 H_{-n} W$$

and so it is clear that $a''_1(z), a''_2(z), \ldots$ exist and are unique.

Finally, we use the fact that $z^2 W \subset W$ and one sees $\exp z$ as before that

$$\left( \frac{d}{dt} \right)^2 s(t, z) - z^2 s(t, z) = g(t) s(t, z) \quad g(t) = 2a'_1(t).$$

On the other hand starting from $g(t) \in C[[t]]$, we can solve this DE uniquely by a formal series

$$s(t, z) = e^{t z} \left( 1 + \frac{a_1(t)}{2} + \frac{a_2(t)}{2^2} + \cdots \right)$$

and then we obtain $W$ by taking the derivatives $\left( \frac{d}{dt} \right)^n$ of $s$ and evaluating at $t=0$. This gives a basis for $W$, so we seem to have established:

**Proposition:** There is a 1-1 correspondence between $g(t) \in C[[t]]$ and between subspaces $W$ of $C[z^{-1}][z] = V$ complementary to $z^{-1} C[z^{-1}]$ such that $z^2 W \subset W$, and such that $1 \in W$.

This proposition is probably not too interesting as it amounts to a funny parameterization of the
fort cell in a suitable Bruhat setup. The interesting case seems to be when in addition to \( z^2 \) there is another endomorphism \( B \) of \( V \) such that \( BW \subset W \). I am assuming that \( B \) commutes with \( z \), hence \( B = f(z) \) where \( f \in C[[z^2]] \). Now \( B \) is an endo of \( W \) which commutes with \( z^2 = A \), and hence satisfies a quadratic eqn.

\[
B^2 + f(A)B + g(A) = 0
\]

According to Mumford's picture just as \( A = z^2 \) became transformed into the operator \( \partial^2 - g \), the operator \( B \) will be transformed into a differential operator. This went as follows. To get the operator from \( A = z^2 \), we form \( A s_x = z^2 s_x \) which we can write

\[
A s_x = [\partial^2 - g(x)] s_x
\]

Note: change \( x \)

because the ects \( s_x, \partial s_x, \partial s_x \) form a basis for \( W \). Thus if \( B = z^m + \) lower terms we will have

\[
B s_x = [\partial^m + b_1(x) \partial^{m-1} + \ldots + b_m(x)] s_x
\]

One should read this equation as saying the operator \( [\partial^m + \ldots + b_m(x)] \) has the eigenfunction \( s_x \) with the eigenvalue \( B(z) \). Hence since the differential operators have \( s_x(z) \) as common eigenfunctions, they commute. Precisely one has

\[
[\partial^2 - g(x)][\partial^m + \ldots + b_m] s_x = [\partial^2 - g(x)] B(z) s_x = B(z) [\partial^2 - g(x)] s_x = B(z) A(z) s_x
\]

and this is the same the other way around.
January 11, 1982

Review yesterday's result. Starting with a subspace \( W \) of \( V = \mathcal{C}[[z^{-1}]][z] \) which is complementary to \( z^{-1}\mathcal{C}[[z^{-1}]] \), one constructs a Baker-Akhiezer function

\[ s_x = e^{\frac{a_1(x)}{z} + \frac{a_2(x)}{z^2} + \ldots} e^{W[[x]]} \]

where the \( a_i(x) \in \mathcal{C}[[x]] \). This \( s_x \) completely determines \( W \) since we get a basis for \( W \) by taking the derivatives \( (\partial_x^n s_x)(x = 0) \). When \( z^2 W \subset W \) we have

\[ (\partial^2 - g(x)) s_x = z^2 s_x \]

where \( g(x) = 2a_1''(x) \). On the other hand starting with a given choice for \( s_0 = (1 + \frac{a_1(0)}{z} + \frac{a_2(0)}{z^2} + \ldots) \) we can construct the series \( s_x \) from the Schroedinger equation. Thus we have a parameterization of \( W \)'s using pairs of series, namely \( s_0 \) and \( g(x) \).

A simpler parameterization would be to give

\( v_0 = \text{unique element of } W \text{ of form } 1 + \frac{a_1(0)}{z} + \frac{a_2(0)}{z^2} + \ldots \)

and \( v_1 = \text{unique elt of } W \text{ of form } z + \frac{b_1(0)}{z} + \frac{b_2(0)}{z^2} + \ldots \),

for then \( W \) has the basis \( \{ z^{2n}v_0, z^{2n}v_1 \} \). Note \( v_0 = s_0 \).
Formal expansions for the resolvent of \( L = -\partial^2 + \varphi \). I want to use the heat operator

\[
e^{-tL} = e^{-t\lambda_0} - \int_0^t e^{-t\lambda_1} \log e^{-t\lambda_0}
\]

If I use \( \int e^{t\lambda} e^{-t\lambda} \, dt = -\frac{1}{\lambda - \lambda_0} \), then (1) goes into

the geometric expansion for the resolvent:

\[
\frac{1}{\lambda - \lambda_0} = \frac{1}{\lambda - \lambda_0} + \frac{1}{\lambda - \lambda_0} \frac{1}{\lambda - \lambda_0} + \ldots
\]

The goal is to compute asymptotic expansions for

\[ \langle a | e^{-tL} | a \rangle \] or \[ \langle a | \frac{1}{\lambda - \lambda} | a \rangle \].

According to Feynman there is a path integral formula for the heat kernel.

\[
\langle a | e^{-tL} | a \rangle = \int \mathcal{D}x \, e^{-S(x)}
\]

\[ x(0) = x(t) = a \]

where

\[
S(x) = \int_0^t \left[ \frac{1}{4} \dot{x}^2 + \varphi(x) \right] \, dt
\]

I'm interested in the case where \( t \) is small \( \to 0 \), and so I change variables from \( x(t') \), \( 0 \leq t' \leq \tau \) to \( x(t) \), \( 0 \leq t \leq 1 \) by the rule \( x(t') = \hat{x}(t) \). Better to think of putting \( t' = t \tau \) whence \( dt' = \tau dt \) and

\[
\frac{dx}{dt'} = \frac{1}{t} \frac{dx}{dt}
\]

and so

\[
S(x) = \frac{1}{\tau} \int_0^1 \left[ \frac{1}{4} \left( \frac{dx}{dt} \right)^2 + \varphi(x) \right] \, dt
\]

where \( x \) is a path \( t \to x(t) \) defined for \( 0 \leq t \leq 1 \) with \( x(0) = x(1) = a \). Next put

\[
x(t) = a + \hat{x}(t)
\]

where \( \hat{x}(0) = \hat{x}(1) = 0 \), and then the action becomes
\[ S(x) = \frac{1}{4t} \int_0^1 \left( \frac{d^2}{dt^2} x(t) \right)^2 dt + t \sum_{n=0}^{\infty} \frac{g^{(n)}(0)}{n!} \int_0^1 x(t)^n dt \]

So when we evaluate the path integral formally by expanding out the exponential of the second term and doing the Gaussian integral, we have a vertex \(-t g^{(n)}(a)\) for each \(n > 0\) of multiplicity \(n\). Also for each \(t\).

Now to do the Gaussian integrals we need the inverse of \(-\frac{1}{2t} \frac{d^2}{dt^2}\) in \([0,1]\) with Dirichlet conditions, which is

\[ G(t, t') = -2t \frac{\tau < (1-t') >}{(-1)^{\tau < (1-t') >}} \]

\[ = (2t) \tau < (1-t') > \]

So each vertex and edge contributes a factor of \(t\).

Coefficient of \(t\): Only possibility is a small, 0 vertex \(-tg(a)\).

Coefficient of \(t^2\): Graphs:

\[ \text{Coefficient of } t^2: \text{ Connected graphs are } \]

\[ \text{Coefficient of } t^3: \text{ Connected graphs are } \]

\[ \text{Coefficient of } t^4: \text{ Connected graphs are } \]

\[ \text{Coefficient of } t^5: \text{ Connected graphs are } \]

So it seems that

\[ \langle a | e^{-t^2} | a \rangle = \frac{1}{\sqrt{4\pi t}} \left[ 1 - tg(a) + t^2 \left( \frac{g(a)^2}{2} - \frac{g''(a)}{6} \right) + \ldots \right] \]

Let's calculate the resolvent \( \frac{1}{1-L} \) formally for large \(t\). Let \(-\varepsilon^2 = 1\). The idea will be to compute the Green's function using the formal solution \( e^{x^2} (1 + \frac{a}{\varepsilon} + \ldots) \) and the similar thing for \(-2\). Fix the point \(a\) and now construct the formal solution.
\[ f_z(x) = e^{\frac{x^2}{2}} \left(1 + \frac{a_1(x)}{z} + \frac{a_2(x)}{z^2} + \ldots\right) \]

such that the \( a_n(x) \) vanish at \( x = a \).

\[ f_z'(a) = z e^{\frac{a^2}{2}} + e^{\frac{a^2}{2}} \left[ \frac{a_1'(a)}{z} + \frac{a_2'(a)}{z^2} + \ldots\right] \]

\[ f_{-z}'(a) = -z e^{-\frac{a^2}{2}} + e^{-\frac{a^2}{2}} \left[ \frac{a_1'(a)}{-z} + \frac{a_2'(a)}{z^2} + \ldots\right] \]

Then

\[
W(f_{-z}, f_z) = \begin{pmatrix}
0 & 1 \\
-z + \frac{a_1'(a)}{-z} + \ldots & z + \frac{a_1'(a)}{z} + \ldots
\end{pmatrix}
\]

\[ = 2z + \frac{2a_1'(a)}{z} + \frac{2a_2'(a)}{z^2} + \ldots \]

So we get from

\[ G(x, x') = \frac{f_{-z}(x < ) f_z(x > )}{W(f_{-z}, f_z)} \]

\[ G(x, x) = \frac{1 - \frac{a_1(x)}{z}}{2z} \left(1 + \frac{a_1(x)}{z} + \frac{a_2(x)}{z^2} + \ldots\right) \left(1 + \frac{a_1(x)}{z} + \frac{a_2(x)}{z^2} + \ldots\right) \]

Recall the recursion relation

\[ 2a_n = -2^2 + 9 \] \( a_{n-1} \)

\[ a_1' = \frac{9}{2} \]

\[ a_2' = -2 \left( -\frac{9}{2} + \int \frac{9}{a} \right) \]

\[ a_2(x) = \frac{1}{4} \left[ g(x) - g(a) \right] + \left( \int g(x) \right)^2 - \left( \int g(a) \right)^2 \]

\[ 2a_2 - a_1 - a_1' = -\frac{1}{2} \left[ g(x) - g(a) \right] + \left( \int g(x) \right)^2 - \left( \frac{1}{2} \int g(a) \right)^2 \]

Now

\[ G(a, a) = \frac{1}{2z \left(1 + \frac{a_1'(a)}{z} + \frac{a_2'(a)}{z^2} + \ldots\right)} \]
\[ G(a, a) = \frac{1}{2\pi i} \left( 1 - \frac{a'(a)}{z^2} + \frac{a''(a)^2 - a'''(a)}{z^4} + \ldots \right) \]

Unfortunately, this seems complicated, and it's not clear that I shouldn't use and \( f_2, f_{-2} \) with very conditions independent of the point at which I evaluate the Green's function at.