January 3, 1982

The Riemann-Hilbert problem: Let X be a Riemann surface, S a finite subset of X, and E a holom. vector bundle over $X-S$ equipped with a holomorphic connection. Since $\dim(X)=1$ the connection is integrable and so if we choose a basepoint outside of S, we get a representation of $\pi_1(X-S)$ on the fibre over the basepoint. In this way we get an equivalence of categories

$$(*)\quad \left\{\begin{matrix}\text{Reps of}\\ \pi_1(X-S)\end{matrix}\right\} \quad \xrightarrow{\sim} \quad \left\{\begin{matrix}\text{holom. v.b. over } X-S\\ \text{with connection}\end{matrix}\right\}$$

Supposing $X-S$ not complete, then I believe Stein theory implies that any holomorphic vector bundle over $X-S$ is trivial. In any case taking a holom. vector bundle E over $X-S$ with a trivialization, then a connection on E can be described by a DE

$$Du = Au$$

where A is a matrix of 1-forms over $X-S$. A different trivialization of E is given by a holomorphic map $g: X-S \longrightarrow GL_n$. If one makes the change of coordinates $u = gv$, then

$$dg\,v + g\,Dv = Ag\,v$$

$$Dv = (g^{-1}Ag - g^{-1}dg)\,v$$

so the connection form changes

$$A \longmapsto g^{-1}Ag - g^{-1}dg$$

Given a connection on the trivial bundle we can integrate to obtain the parallel translation matrix $Y(z \xleftarrow{\gamma} z_0)$

where γ is a homotopy class of paths from the basept z_0 to z. We have

$$d\, Y(z \xleftarrow{\gamma} z_0) = A(z)\, Y(z \xleftarrow{\gamma} z_0)$$

$$Y(z \xleftarrow{\beta\alpha} z_0) = Y(z \xleftarrow{\beta} z_0)\, Y(z_0 \xleftarrow{\alpha} z_0)$$

So $Y(z \xleftarrow{\gamma} z_0)$ is a multi-valued invertible matrix fn. of z. If all the monodromy transformations $Y(z_0 \xleftarrow{\alpha} z_0)$ are trivial, then we can take $g(z) = Y(z \xleftarrow{\gamma} z_0)$ which is independent of γ. Then $A = dg\, g^{-1}$ and so g transform A into the trivial connection. So we see that a bundle with connection having trivial monodromy is isomorphic to the trivial bundle with trivial connection, which is a special case of ⊛.

The next point is to make a more algebraic description of bundles with connection. We know that any holom. vector bundle of $X-S$ can be extended to X. If X is complete, then any holom. v.b. over X is algebraic, and we can ask the connection, which is given over $X-S$, be algebraic. The connection is a map of vector bundles $J_1(E) \to E \otimes \Omega$, so algebraic here means that on the extension to X this map is meromorphic.

Let's analyze these ideas carefully in the case where the rank is 1. So we are given a holomorphic line bundle L over $X-S$. To simplify suppose X is the disk and $S = \{0\}$, so that $X-S = D^*$, and suppose L is the trivial bundle over $X-S$. Let $\tilde{L}$ be an extension of L to X. Then $\tilde{L}$ is trivial so it has a non-vanishing section s which over $X-S$ is given by a holom. map

$$X - S \xrightarrow{s} \mathbb{C}^*$$

s is unique up to multiplying by $f: X \to \mathbb{C}^*$.

The extension $\tilde{L}$ of L to $X-S$ is determined by those sections of L which extend to holomorphic sections of $\tilde{L}$. Thus a function on $X-S$ is a holom section of $\tilde{L}$ when it is of the form fs with $f: X \to \mathbb{C}$ holomorphic. Similarly it is a meromorphic section with pole at O when it is of the form fs with $f: X - S \to \mathbb{C}$ meromorphic.

To see what's happening look at the field K of Laurent series converging in a punctured disk around O. Or one can let K = holomorphic maps $X-S \to \mathbb{C}$. Then any element of K^* has a degree and ones of degree zero have logarithms in K. Let R = convergent power series around O, or R = holom. maps $X \to \mathbb{C}$. Then given $f \in K^*$ of degree O, we have

$$\log f = \sum c_n z^n$$

and we can see that

$$K^*/R^* = \underbrace{\mathbb{Z}}_{\{z^n\}} \times \exp(\underbrace{K_-}_{\text{all } \sum_{n<0} c_n z^n}).$$

So there are an incredible number of different extensions of L to an $\tilde{L}$, and they are going to be different even if meromorphic sections are allowed.

Next consider the connection

$$\frac{du}{dz} = A(z)u$$

given on L over $X-S = D^*$. Suppose we have an extension given by $s \in K^*$, $s: X-S \to \mathbb{C}^*$

whose holom. sections are fs with $f \in R$. Then putting $u = fs$ as a change of variable, we find

$$f's + fs' = Afs \quad \text{or} \quad \frac{df}{dz} = \left(A - \frac{1}{s}\frac{ds}{dz}\right) f$$

How much can A be changed? The good case from the algebraic viewpoint is when $A - \frac{1}{s}\frac{ds}{dz}$ is as non-singular as possible. The point is that for an $s \in K^*$, $\frac{1}{s}\frac{ds}{dz}$ can be rigged to be anything having residue $\in \mathbb{Z}$. So s can be chosen to kill off the terms in A except $\frac{a_0}{z}$, where a_0 can be changed by an integer.

So what I seem to be getting is the following.

<u>Prop</u>: Let L be a holomorphic line bundle with connection over $X-S$. Then there is an extension of L to a holom. line bundle $\tilde{L}$ over X such that connection has a regular singularity at $z_0 \in S$. (This means that if I take a section s of $\tilde{L}$ near z_0 and non-vanishing at z_0, then $s^{-1}(\nabla s)$ has a simple pole at z_0.) Any two extensions are related by a power of a uniformizing parameter around z_0.

<u>Proof</u>: All this takes place around $z_0 = 0$ in the disk. We know L is given by a representation of the fundamental group, hence can be realized in the punctured disk D^* by the trivial line bundle with connection

$$\frac{du}{dz} = \frac{a}{z} u$$

The existence of $\tilde{L}$ is now clear.

Any extension will have holom. sections around $z=0$ given by fs where $s \in K^*$ is fixed and f ranges over R. The new connection form is $\frac{a}{z} - \frac{1}{s}\frac{ds}{dz}$ and if this has only a simple pole at $z=0$, then one can see easily that $s \in z^n R^*$. So the possible $\tilde{L}$ are of the form $z^n \tilde{L}$ for different n

January 4, 1982

Deligne's thm. says for a smooth variety the category of representations of π_1 is equivalent to the category of algebraic vector bundles with a connection which is regular at ∞.

In the case of $X-S$ where X is a complete smooth curve it says that given a representation of $\pi_1(X-S)$, we can take the associated flat holomorphic vector bundle E over $X-S$ and find an extension $\tilde{E}$ to X such that the connection $\nabla : E \longrightarrow E \otimes \Omega$ extends to

$$\tilde{\nabla} : \tilde{E} \longrightarrow \tilde{E} \otimes \Omega \otimes \mathcal{O}(S)$$

so that the connection has simple poles along S. Now $\tilde{E}$ is not unique, but because X is complete, $\tilde{E}$ has an algebraic structure, and then the induced algebraic structure on E is unique.

January 4, 1982

Let's review the soliton solutions of KdV.

I start from a Schroedinger equation on the line

$$(-\partial_x^2 + q)u = k^2 u$$

where q decays fast as $|x| \to \infty$. This guarantees existence and uniqueness of solutions with asymp. behav:

$$f_k(x) \sim e^{ikx} \qquad x \to +\infty$$

$$\phi_k(x) \sim e^{-ikx} \qquad x \to -\infty$$

for $\operatorname{Im} k \geq 0$. Then I get transfer coefficients $A(k)$, $B(k)$, $k \in \mathbb{R}$, $k \neq 0$ defined by

$$\phi_k = A f_{-k} + B f_k$$

It is known that $2ikA$ is analytic in the UHP and has an asymptotic expansion

$$A(k) = 1 - \frac{1}{2ik}\int_{-\infty}^{\infty} q + \cdots \qquad \text{as } |k| \to \infty \text{ in the UHP.}$$

Also $\quad A(k)A(-k) - B(k)B(-k) = 1 \quad$ for k real.

At points iK in the UHP where $A(iK) = 0$, there are non-zero numbers $B(iK)$ such that

$$\phi_{iK} = B(iK) f_{iK}.$$

One knows $A(k)$ has finitely many zeroes in the UHP and that they are simple.

Suppose $B = 0$ for $k \in \mathbb{R}$. Then

$$A(-k) = \frac{1}{A(k)}$$

gives a meromorphic extension of A to the LHP having simple poles at the points $-iK$. Since $A \to 1$ as $k \to \infty$ it must be rational:

$$A(k) = \prod_K \frac{k - iK}{k + iK}$$

Next we have

$$f_{-k} = \frac{1}{A(k)}\phi_k$$

which gives for each x an extension of f_k to a meromorphic function in the LHP with simple poles at the points $-iK$.

We have

$$\underset{k=-iK}{\mathrm{Res}}\, f_k(x) = \lim_{k\to -iK} (k+iK)\, f_k(x) = \lim_{k\to +iK} (-k+iK)\, \underbrace{f_{-k}(x)}_{\frac{1}{A(k)}\phi_k(x)}$$

* $$= -\frac{B(iK)}{A'(iK)}\, f_{iK}(x)$$

Let us form from the extended k-plane a singular curve with ordinary double points by identifying iK and $-iK$. Over this singular curve we take the line bundle whose sections are meromorphic functions $f(k)$ of k having simple poles at the points $-iK$ satisfying

* $$\underset{k=-iK}{\mathrm{Res}}\, f(k) = -\frac{B(iK)}{A'(iK)}\, f(iK)$$

(Note that if we have N points $-iK$, then the space of functions with at most simple poles at these points is of dim $1+N$, and the residue conditions ~~are~~ are N-conditions, so in general we have a unique $f(k)$ with value 1 at $k=\infty$.)

<u>January 6, 1982</u>

Review solitons again:

$$(-\partial^2+q)u = k^2 u$$

$$\phi_k = A f_{-k} + B f_k \qquad \text{on } \mathbb{R}-\{0\}$$

$$\phi_{iK} = B(iK) f_{iK} \qquad \text{at points } iK \text{ in UHP where } A(iK)=0.$$

Suppose $B(k)=0$ on $\mathbb{R}$. Then we know

$$A(k) = \prod \frac{k-iK}{k+iK}$$

and for any x, $f_k(x)$ is a meromorphic function of k with only simple poles at the points $-iK$ satisfying

$$\circledast \qquad \operatorname*{res}_{k=-iK} f_k(x) = -\frac{B(iK)}{A'(iK)} f_{iK}(x)$$

Also as $k\to\infty$ one [redacted] has

$$f_k(x) \sim e^{-ikx}\left(1+\frac{a_1(x)}{k}+\frac{a_2(x)}{k^2}+\cdots\right).$$

So now I want to rewrite this in [redacted] the following way. Draw a circle S^1 containing the points $\pm iK$, and work with analytic functions on this circle. Let H_- be the space [redacted] of functions on the circle which extend to meromorphic functions inside with only simple poles at the points iK satisfying the residue condition $\circledast$. Let H_+ be the functions on the circle given by convergent power series $a_0 + a_1/k + a_2/k^2 + \cdots$ Then $H_+ \cap H_-$ is the sections of a line bundle over the singular curve obtained by identifying iK and $-iK$. So we know it is one-dimensional. So we know for sufficiently small x that $f_k(x) \in e^{-ikx} H_+ \cap H_-$ will exist

So now what I should do see what conditions on H_- lead to a Schroedinger equation. Notice that H_-

contains all analytic functions inside the circle which vanish at the points iK. Hence H_- is commensurable with the standard H_-^{st}, consisting of convergent series $a_0 + a_1 k + \cdots\cdots$ on the circle.

It seems that the algebraic functions on this singular curve are polynomials in k which carry H_- into itself, because the residue condition forces the values of the polynomial to be the same at the points $\pm iK$. So for example H_- is stable under multiplication by k^2.

But now

$$f_k(x) = e^{ikx}\left(1 + \frac{a_1(x)}{k} + \cdots\right) \in H_-$$

for all x so we can differentiate

$$f_k''(x) = (ik)^2 f_k + 2ik\, e^{ikx}\left(\frac{a_1'(x)}{k} + \cdots\cdots\right) \in H_-$$
$$+ e^{ikx}\left(\frac{a_1''(x)}{k} + \cdots\cdots\right)$$

so

$$f_k'' + k^2 f_k = e^{ikx}\left(2i\, a_1'(x) + \frac{2i\, a_2' + a_1''}{k} + \cdots\cdots\right) \in H_-$$

because $k^2 H_- \subset H_-$. Thus we have

$$f_k'' + k^2 f_k = q f_k \qquad \text{where} \quad q = 2i\, a_1'.$$

Let's derive KdV equation. To simplify the notation put $z = ik$ and then write

$$f(x,t) = e^{xz + tz^3}\left(1 + \frac{a_1}{z} + \frac{a_2}{z^2} + \cdots\right) \in H_-$$

This formal gadget is called the Baker-Akhiezer fn.

$$\dot{f} = e^{xz+tz^3}\left[z^3 + a_1 z^2 + a_2 z + a_3 + \cdots \quad + \dot{a}_1 z^{-1} + \dot{a}_2 z^{-2} + \cdots\right]$$

It might be simpler to put

$$f(x,t) = e^{xz+tz^3} h(x,t) \qquad\qquad h = 1 + a_1 z^{-1} + a_2 z^{-2} + \cdots$$

Then $\quad f'' = e^{xz+tz^3}\left[z^2h + 2zh' + h''\right] \quad \in H_-$

$$f'' = \underbrace{z^2 f}_{\in H_-} + e^{xz+tz^3}\left[2a_1' + 2a_2'/z + a_1''/z\right] \in H_-$$

$\Rightarrow \quad f'' = z^2 f + 2a_1' f \qquad\qquad 2a_1'a_1 = 2a_2' + a_1''$

$\Rightarrow \quad a_2' = a_1 a_1' - \dfrac{a_1''}{2}$

$a_2 = \frac{1}{2}(a_1^2 - a_1') + \text{const.} \qquad$ const $= 0$ since $a_n = 0$ at $x = 0$

$\dot f = e^{xz+tz^3}\left[z^3 h + \dot h\right] \qquad\qquad f' = e^{xz+tz^3}\left[zh + h'\right]$

$z \qquad a_1'/z$

a_1

a_2/z

$f''' = \text{——}\left[z^3h + 3z^2h' + 3zh'' + h'''\right]$

$3z\,a_1' \qquad 3a_1'' \qquad + a_1'''/z$

$+ \qquad$

$3a_2' \qquad +3a_2''/z$

$+$

$3a_3'/z$

$$\dot f - f''' + 3a_1'f' = e^{\sum x_n z^n}\left[\dot a_1/z + (-3a_2' - 3a_1'' + 3a_1'a_1) + (-3a_3' - 3a_2'' - a_1''' + 3a_1'a_2)/z\right]$$

$$3\left[a_1'a_1 - a_1'' - a_2'\right] = 3\left[a_1'a_1 - a_1'' - a_1a_1' + a_1''/2\right] = -\frac{3}{2}a_1''$$

So $\quad \dot f = f''' - 3a_1'f' - \frac{3}{2}a_1''f \qquad$ But $q = 2a'$

$\quad = f''' - \frac{3}{2}qf' - \frac{3}{4}q'f$

Recall that if $\quad L = -\partial^2 + q$, then KdV is

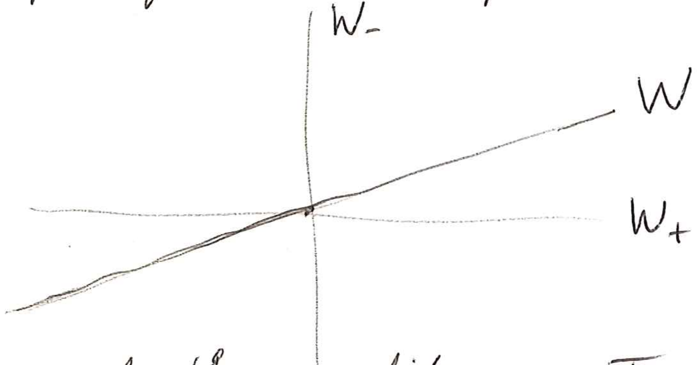
$\dot L = [M, L] \qquad M = \partial^3 - \frac{3}{2}q\partial - \frac{3}{4}q'$

So therefore the f we have constructed satisfies

$\dot f = Mf$

January 7, 1982

Inside $V = L^2(S^1)^d$ I want to understand the family of all outgoing subspaces W complementary to a given incoming subspace W_-. Suppose we fix $W_+ = (W_-)^\perp$. Then subspaces W complementary to W_- are given by the graph of a linear map $T: W_+ \to W_-$



I want the condition on T corresponding to $zW \subset W$.

Take $d=1$, $W_+ = H_+ = \text{span of } 1, z, z^2, \dots$ and $W_- = \text{span of } z^{-1}, z^{-2}, \dots$. I know that $W = fW_+$, where $f: S^1 \to S^1$ has degree 0. So the matrix of f relative to the decomposition $V = W_- \oplus W_+$ is

$$f = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

then

$$W = \left\{ \begin{pmatrix} Bx \\ Dx \end{pmatrix} \,\middle|\, x \in W_+ \right\} = \left\{ \begin{pmatrix} BD^{-1}x \\ x \end{pmatrix} \,\middle|\, x \in W_+ \right\}.$$

This shows $T = BD^{-1}$, and it is not clear that there is an easy way to formulate the condition $zW \subset W$ in terms of T.

I would like to understand in $L^2(S^1)$ the possible subspaces W such that $z^2 W \subset W$ and which are complementary to $W_- = z^{-1}H_- = \text{span of } z^{-1}, z^{-2}, z^{3}, \dots$. The beautiful thing is that $L^2(S^1)$ with the operator z^2 is equivalent to $L^2(S^1)^2$ with the operator z. So what we are looking at is simply all outgoing subspaces in $L^2(S^1)^2$

which are complementary to a given (incoming) W_-, and these are classified by certain maps $S^1 \to U_2$. They should be of degree 0, and [redacted] the pair (W_-, W) defines a rank 2 vector bundle E over $\mathbb{P}^1$ with $h^0 = h^1 = 0$, and hence $E \simeq \mathcal{O}(-1) \oplus \mathcal{O}(-1)$.

Restate formally: Suppose one gives a subspace W inside the space of formal Laurent series $\sum_{n \le N} c_n z^n$ which is complementary to [redacted] $z^{-1}\mathbb{C}[[z^{-1}]]$, and is such that $z^2 W \subset W$. Is there then a curve which is a double covering of the projective line with variable z^2 such that W is the space of sections of a line bundle over the complement of ∞.

What seems to be clear is W is a free module of rank 2 over $\mathbb{C}[z^2]$. You get a basis because

$$W_n = \dim\left(z^n \mathbb{C}[[z^{-1}]] \cap W\right) = n+1 \qquad n \ge 0.$$

[redacted], so you can pick generators in this space for $n = 0, 1$. Now you need another endom. of W which commutes with z^2, and it should be a uniformizer on the curve at ∞, so should be a series $z + c_0 + c_1 z^{-1} + \cdots$. NO.

Example: W = space of rational functions $f(z)$ in z having at most a simple pole[redacted] at $z = -q$ satisfying the condition

$$\operatorname*{res}_{z=-q} f(z) = c f(q)$$

Here the ring R belonging to the affine curve is the subring of $\mathbb{C}[z]$ consisting of f such that $f(-q) = f(q)$. Thus

$$R = \mathbb{C}[z^2] + \mathbb{C}[z^2]\, z(z^2 - q^2)$$

and R is generated by $X = z^2$ and $Y = z(z^2 - g^2)$ with the relation

$$Y^2 = z^2(z^4 - 2g^2z^2 + g^4)$$

$$Y^2 = X^3 - 2g^2X^2 + g^4X$$

In general if I take [redacted] several distinct [redacted] $g_1^2, \dots,$ [redacted] g_n^2 then the ring becomes

$$R = \underbrace{\mathbb{C}[z^2]}_{X} + \mathbb{C}[z^2]\,\overbrace{z\prod_j (z^2 - g_j^2)}^{Y}$$

with the relation

$$Y^2 = X\left(\prod (X - g_j^2)\right)^2$$

Notice that this is in the form $Y^2 = X f(X)^2$ where $f(X)$ has distinct roots. The distinct roots part is related to the residue condition which defines the line bundle.

January 10, 1982

Notes on the Krichever theory (Mumford's paper).

We start with a complete curve X, a smooth point P on X, a torsion-free rank 1 sheaf $\mathcal{F}$ on X such that $h^0(\mathcal{F}) = h^1(\mathcal{F}) = 0$. Let z^{-1} be a uniformizing parameter on X at P and U a small disk around P on which z is a coordinate.

We construct a deformation $\mathcal{F}_t$ of $\mathcal{F}$ by requiring $\mathcal{F}_t = \mathcal{F}$ over $X - P$, but a holom. section of $\mathcal{F}_t$ over U is of the form $e^{tz} f$ where f is a holomorphic section of $\mathcal{F}$ over U. Thus if

$$V = \Gamma_{hol}(U-P, \mathcal{F}) \quad , \quad W = \Gamma_{hol}(X-P, \mathcal{F}) \quad , \quad W_- = \Gamma_{hol}(U, \mathcal{F})$$

then $\mathcal{F}_t$ is the holomorphic sheaf with

$$V = \Gamma_{hol}(U-P, \mathcal{F}_t) \quad , \quad W = \Gamma_{hol}(X-P, \mathcal{F}) \quad , \quad e^{tz} W_- = \Gamma_{hol}(U, \mathcal{F}_t)$$

For small t it should be true that $h^0(\mathcal{F}_t) = h^1(\mathcal{F}_t) = 0$, or equivalently, because of the Čech computation of the cohomology by the covering $X = (X-P) \cup U$, that

$$(*) \qquad V = e^{tz} W_- \oplus W.$$

Now put $H_- = z W_-$, so that when $(*)$ holds, we have that $e^{tz} H_- \cap W$ is 1-dimensional. If we choose a non-vanishing section s of $\mathcal{F}$ over U, then I can identify W_- with convergent series $\sum_{n \geq 0} c_n z^{-n}$ in U. So for each t we get a unique section

$$s_0(t) = e^{tz}\left(1 + \frac{a_1}{z} + \frac{a_2}{z^2} + \dots\right) \in e^{tz} H_- \cap W$$

of $\mathcal{F}_t(P)$ called the Baker-Akhieser function. If we successively differentiate

$$s_n(t) = \left(\frac{\partial}{\partial t}\right)^n s_0(t)$$

then $s_0(t), \dots, s_n(t)$ forms a basis for $\Gamma(X, \mathcal{F}_t((n+1)P))$. So now let $R = \Gamma_{alg}(X-P, \mathcal{O}_X)$. Then if $a \in R$ has order

n, so that $\quad a = \alpha z^n + \sum_{-\infty < k < n} c_k z^k \quad$ we have

$$a\, s_0(t) \in \Gamma(X, \mathcal{F}_t((n+1)P))$$

so

$$a s_0 = \alpha s_n + \sum_{i=0}^{n-1} a_i(t) s_i$$

$$= \left[\alpha \left(\frac{\partial}{\partial t}\right)^n + \sum_{i=0}^{n-1} a_i(t) \left(\frac{\partial}{\partial t}\right)^i \right] s_0$$

In this way we associate to each element of R a differential operator in $\mathbb{C}[[t]][\frac{\partial}{\partial t}]$, whose leading coefficient is constant. The order of the operator is minus the order of the element of R at the point P.

So what one obtains in this way is a commutative subring of $\mathbb{C}[[t]][\frac{\partial}{\partial t}]$ containing two operators

$$A = a_m(t)\partial^m + \cdots\cdots + a_0(t)$$
$$B = b_n(t)\partial^n + \cdots\cdots + b_0(t)$$

where the leading coefficients a_m, b_n are constants $\neq 0$, and where $(m, n) = 1$. In Mumford's paper, he doesn't want to normalize the leading coefficient of s_0, and takes any

$$s_0(t) = e^{tz}\left(a_0 + \frac{a_1}{z} + \cdots\right) \in e^{tz} H_- \cap W$$

where $a_0(t) \neq 0$. Then the subring of differential operators is determined up to conjugacy by a $u(t) \in \mathbb{C}[[t]]$, $u(0) \neq 0$. Then the conjugacy class of the operator ring doesn't depend very much on the choice of z. In effect if

$$z' = z + c_0 + c_1/z + \cdots$$

is another coordinate at P, then

$$e^{tz'} H_- = e^{tz} e^{tc_0 + tc_1/z + \cdots} H_- = e^{tz} H_-$$

so the Baker-Akhieser function is independent of the choice of z, provided one fixes its image in m_P / m_P^2.

In this theory there is an interesting lemma which says that if we take an operator $A = a_m(t)\partial^m + a_{m-1}(t)\partial + \cdots + a_m(t)$

in $\mathbb{C}[[t]][\partial]$ with $a_m(0) \neq 0$, and look at its commutant in the ring $\Psi D\{t\}$ of formal pseudo-diffl. operators $\sum_{-\infty < n \leq N} c_n(t)\partial^n$, then the commutant is generated by $A^{1/m}$, in fact, every commuting pseudo-diffl. ops. is of the form

$$\sum_{-\infty < n \leq N}{}' c_n (A^{1/m})^n$$

I want to consider the following problem. Let's start with V = analytic function in a punctured disk $U - P$, and H_- = analytic functions in U, and suppose I give a subspace W of V which is complementary to $z^{-1}H_-$:

$$V = z^{-1}H_- \oplus W$$

Then I should be able to define the Baker-Akhieser function

$$s_0(t) = e^{tz}\left(1 + \frac{a_1}{z} + \frac{a_2}{z^2} + \cdots\right) \in e^{tz}H_- \cap W$$

at least for small t. Next I suppose that

$$z^2 W \subset W$$

which implies that if W came from an X, F as above, then $z^2 \in R$. Thus z^2 is a merom. function on the curve with only a double pole at P, so we have a hyper-elliptic curve which might be degenerate.

I have seen that then to z^2 belongs a differential operator of the second order. Namely

$$\left(\frac{\partial}{\partial t}\right)^2 s_0 = z^2 s_0 + q(t)s_0 \qquad q(t) = 2a_1'(t).$$

Thus in the corresponding ring of operators, assuming W comes from an (X, F), we have the operator

$$A = \left(\frac{\partial}{\partial t}\right)^2 - q(t).$$

Thus if we have a curve, we must be able to find an operator B commuting with A of odd degree. By the lemma on the commuting operators

B will be a linear combination of $(A^{1/2})^n$, and so we will find a B of the form $p(A)A^{1/2}$ for some polynomial $p(A)$.) Then the curve will have the equation

NO $$B^2 = A\,p(A)^2 .$$

(Why should p be a polyn.?)

We see that any W gives an operator $\partial^2 - q(t)$. Conversely given this operator, we can construct a Baker-Akhieser function

$$s_0(t) = e^{tz}\left(1 + \frac{a_1}{z} + \frac{a_2}{z^2} + \cdots\right)$$

as the formal solution of the equation $(\partial^2 - z^2)s_0 = q(t)s_0$. Then W should be spanned by the functions $s_0(t)(z)$ for different t, in some sense. Does this have a sense? We can always consider $s_0(0), s_0'(0), s_0''(0), \dots$ which will give a family of formal series $1 + O(\frac{1}{z})$, $z + \cdots$, $z^2 + \cdots$, and should give a free module of rank 2 over $\mathbb{C}[z^2]$, which is complementary to $z^{-1}H_-$. Let's go over this carefully.

Let W be a subspace of $\mathbb{C}[[z^{-1}]][z] = V$ which is complementary to $z^{-1}\underbrace{\mathbb{C}[[z^{-1}]]}_{H_-}$ and such that $z^2 W \subset W$. Then we know that

$$W_n = z^n H_- \cap W$$

is $(n+1)$-dimensional, and that $\bigcup W_n = W$. There is a unique element in W_n modulo W_{n-1} of the form z^n + lower degree terms. I claim now that there are unique series $a_n(t) \in \mathbb{C}[[t]]$ for $n \geq 1$ such that

$$s(t,z) = e^{tz}\left(1 + \frac{a_1(t)}{z} + \frac{a_2(t)}{z^2} + \cdots\right) \in W[[t]]$$

Think of e^{tz} as being in $V[[t]]$, $V =$ field $\mathbb{C}[[z^{-1}]][z]$. Let's get the coefficients by differentiating: The condition

$$1 + \frac{a_1(0)}{z} + \frac{a_2(0)}{z} + \cdots \in H_- \cap W$$

determines $a_1(0), a_2(0), \ldots$. Then we want

$$z(1 + a_1(0)/z + a_2(0)/z + \cdots) + \frac{a_1'(0)}{z} + \frac{a_2'(0)}{z^2} + \cdots \in zH_- \cap W$$

We know that $zH_- \cap W$ contains an elt of the form $z +$ lower terms unique modulo $H_- \cap W$, and we can use elts of $H_- \cap W$ to get a unique element of $zH_- \cap W$ of the form $z + c +$ lower terms with a given c. Thus one sees $a_1'(0), a_2'(0), \ldots$ exist and are unique. Next want

$$z^2\left(1 + \frac{a_1(0)}{z} + \frac{a_2(0)}{z^2} + \cdots\right) + 2z\left(\frac{a_1'(0)}{z} + \frac{a_2'(0)}{z^2} + \cdots\right) + \left(\frac{a_1''(0)}{z} + \frac{a_2''(0)}{z^2} + \cdots\right) \in z^2H_- \cap W$$

and so it is clear that $a_1''(0), a_2''(0), \ldots$ exist and are unique.

Finally we use the fact that $z^2W \subset W$ and one sees as before that

$$\left(\frac{\partial}{\partial t}\right)^2 s(t,z) - z^2 s(t,z) = q(t)\, s(t,z) \qquad q(t) = 2a_1'(t).$$

On the other hand starting from $q(t) \in \mathbb{C}[[t]]$, we can solve this DE uniquely by a formal series

$$s(t,z) = e^{tz}\left(1 + \frac{a_1(t)}{z} + \frac{a_2(t)}{z^2} + \cdots\right)$$

and then we obtain W by taking the derivatives $\left(\frac{\partial}{\partial t}\right)^n s$ and evaluating at $t=0$. This gives a basis for W. So we seem to have established:

Proposition: There is a 1-1 correspondence between $q(t) \in \mathbb{C}[[t]]$ and between subspaces W of $\mathbb{C}[[z^{-1}]][z] = V$ complementary to $z^{-1}\mathbb{C}[[z^{-1}]]$ such that $z^2 W \subset W$, and such that $1 \in W$.

This proposition is probably not too interesting as it amounts to a funny parameterization of the fat cell in a suitable Bruhat setup. The interesting case seems to be when in addition to $A = z^2$ there is another endomorphism B of V such that $BW \subset W$. I am assuming that B commutes with z, hence $B = f(z)$ where $f \in \mathbb{C}[[z^{-1}]][z]$. Now B is an endo of W which commutes with $z^2 = A$, and hence satisfies a quadratic equ.

$$B^2 + f(A)B + g(A) = 0$$

According to Mumford's picture just as $A = z^2$ became transformed into the operator $\partial^2 - q$, the operator B will be transformed into a differential operator. This went as follows. To get the operator from $A = z^2$, we form $As_x = z^2 s_x$ which we can write

$$As_x = [\partial^2 - q(x)]\, s_x$$

Note: CHANGE t above to x

because the elts $s_x, \partial s_x, \partial^2 s_x$ form a basis for W_2. Thus if $B = z^m +$ lower terms we will have

$$Bs_x = [\partial^m + b_1(x)\partial^{m-1} + \cdots + b_m(x)]\, s_x$$

One should read this equation as saying the operator $[\partial^m + \cdots + b_m(x)]$ has the eigenfunction s_x with the eigenvalue $B(z)$. Hence since the differential operators have $s_x(z)$ as common eigenfunctions, they commute. Precisely one has

$$[\partial^2 - q(x)][\partial^m + \cdots + b_m]\, s_x = [\partial^2 - q(x)]\, B(z) s_x$$
$$= B(z)\, [\partial^2 - q(x)]\, s_x = B(z) A(z)\, s_x$$

and this is the same the other way around.

January 11, 1982

Review yesterday's result. Starting with a subspace W of $V = \mathbb{C}[[z^{-1}]][z]$ which is complementary to $z^{-1}\mathbb{C}[[z^{-1}]]$, one constructs a unique Baker-Akhieser function

$$s_x = e^{xz}\left(1 + \frac{a_1(x)}{z} + \frac{a_2(x)}{z^2} + \cdots\right) \in W[[x]]$$

where the $a_i(x) \in \mathbb{C}[[x]]$. This s_x completely determines W since we get a basis for W by taking the derivatives $(\partial_x^n s_x)(x=0)$. When $z^2 W \subset W$ we have

$$\circledast \qquad (\partial^2 - q(x))\, s_x = z^2 s_x$$

where $q(x) = 2a_1'(x)$. On the other hand starting with a given choice for $s_0 = \left(1 + \frac{a_1(0)}{z} + \frac{a_2(0)}{z^2} + \cdots\right)$ we can construct the series s_x from the Schroedinger equation. Thus we have a parameterization of the W's using pairs of series, namely s_0 and $q(x)$.

A simpler parameterization would be to give v_0 = unique element of W of form $1 + \frac{c_1}{z} + \frac{c_2}{z^2} + \cdots$ and v_1 = unique elt of W of form $z + \frac{b_1}{z} + \frac{b_2}{z^2} + \cdots$, for then W has the basis $\{z^{2n} v_0, z^{2n} v_1\}$. Note $v_0 = s_0$.

January 12, 1982

Formal expansions for the resolvent of $L = -\partial^2 + q$. I want to use the heat operator

(1) $$e^{-tL} = e^{-tL_0} - \int_0^t dt_1\, e^{-(t-t_1)L_0}\, q\, e^{-t_1 L_0} \cdots\cdots ;$$

If I use $\int^\infty e^{t\lambda} e^{-tL}\, dt = -\frac{1}{\lambda - L}$, then (1) goes into the geometric expansion for the resolvent

$$\frac{1}{\lambda - L} = \frac{1}{\lambda - L_0} + \frac{1}{\lambda - L_0}\, q\, \frac{1}{\lambda - L_0} + \cdots$$

The goal is to compute asymptotic expansions for $\langle a | e^{-tL} | a \rangle$ or $\langle a | \frac{1}{\lambda - L} | a \rangle$.

According to Feynman there is a path integral formula for the heat kernel.

$$\langle a | e^{-tL} | a \rangle = \int_{x(0)=x(t)=a} \mathcal{D}x\; e^{-S(x)}$$

where
$$S(x) = \int_0^t \left[\tfrac{1}{4}\dot{x}^2 + q(x)\right] dt'$$

I'm interested in the case where t is small $\to 0$, and so I change ~~variables from~~ $x(t')$, $0 \le t' \le t$ to $\hat{x}(\tau)$, $0 \le \tau \le 1$ by the rule $x(\tau t) = \hat{x}(\tau)$. Better to think of putting $t' = t\tau$ whence $dt' = t\, d\tau$ and $\frac{dx}{dt'} = \frac{1}{t}\frac{dx}{d\tau}$, and so

$$S(x) = \frac{1}{t}\int_0^1 \left[\frac{1}{4}\left(\frac{dx}{d\tau}\right)^2 + t^2 q(x)\right] d\tau$$

where x is a path $\tau \mapsto x(\tau)$ defined for $0 \le \tau \le 1$ with $x(0) = x(1) = a$. Next put

$$x(\tau) = a + \hat{x}(\tau)$$

where $\hat{x}(0) = \hat{x}(1) = 0$, and then the action becomes

$$S(\hat{x}) = \frac{1}{4t}\int_0^1 \left(\frac{d\hat{x}}{d\tau}\right)^2 d\tau + t\sum_{n=0}^{\infty} \frac{g^{(n)}(a)}{n!}\int_0^1 \hat{x}(\tau)^n \, d\tau$$

So when we evaluate the path integral formally by expanding out the exponential of the second term and doing the Gaussian integral, we have a vertex $-tg^{(n)}(a)$ for each $n \geq 0$ of multiplicity n. Also for each τ. Now to do the Gaussian integrals we need the inverse of $-\frac{1}{2t}\left(\frac{d}{d\tau}\right)^2$ on $[0,1]$ with Dirichlet conditions, which is

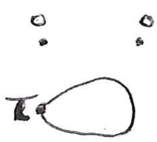
$$G(\tau,\tau') = (-2t)\frac{\tau_<(1-\tau_>)}{(-1)}$$

$$= (2t)\,\tau_<(1-\tau_>)$$

So each vertex and edge contributes a factor of t.

Coefficient of t: Only possibility is a mult. 0 vertex $-tg(a)$.

Coefficient of t^2: Graphs:



gives terms $\dfrac{(-tg(a))^2}{2}$ and $\dfrac{-tg''(a)}{2}\, 2t\int_0^1 \tau(1-\tau)\,d\tau = \dfrac{-t^2 g''(a)}{6}$

Coefficient of t^3: connected graphs are

So it seems that

$$\langle a|e^{-tL}|a\rangle = \frac{1}{\sqrt{4\pi t}}\left[1 - tg(a) + t^2\left[\frac{g(a)^2}{2} - \frac{g''(a)}{6}\right] + \cdots\right]$$

Let's calculate the resolvent $\frac{1}{\lambda - L}$ formally for large λ. Put $-z^2 = \lambda$. The idea will be to compute the Green's function using the formal solution $e^{xz}(1+\frac{a_1}{z}+\cdots)$ and the similar thing for $-z$. Fix the point a and now construct the formal solution

$$f_z(x) = e^{xz}\left[1 + \frac{a_1(x)}{z} + \frac{a_2(x)}{z^2} + \cdots\right]$$

such that the $a_n(x)$ vanish at $x = a$.

$$f_z'(a) = ze^{az} + e^{az}\left[\frac{a_1'(a)}{z} + \frac{a_2'(a)}{z^2} + \cdots\right]$$

$$f_{-z}'(a) = -ze^{-az} + e^{-az}\left[\frac{a_1'(a)}{-z} + \frac{a_2'(a)}{z^2} + \cdots\right]$$

Then $$W(f_{-z}, f_{+z}) = \begin{pmatrix} 1 & 1 \\ -z + \frac{a_1'(a)}{-z} + \cdots & z + \frac{a_1'(a)}{z} + \cdots \end{pmatrix}$$

$$= 2z + \frac{2a_1'(a)}{z} + \frac{2a_3'(a)}{z^3} + \cdots\cdots$$

So we get from

$$G(x,x') = \frac{f_{-z}(x_<)\, f_z(x_>)}{W(f_{-z}, f_z)}$$

So $$G(x,x) = \frac{\left(1 - \frac{a_1(x)}{z} + \frac{a_2(x)}{z^2} - \cdots\right)\left(1 + \frac{a_1(x)}{z} + \frac{a_2(x)}{z^2} \cdots\right)}{2z\left(1 + \frac{a_1'(x)}{z^2} + \frac{a_3'(x)}{z^4} + \cdots\right)}$$

$$= \frac{1}{2z}\left(1 - \frac{a_1(x)^2}{z^2} + \frac{2a_2(x)}{z^2} - \frac{a_1'(x)}{z^2}\right.$$

Recall the recursion relation

$$2a_n' = (-\partial^2 + q)\, a_{n-1} \qquad a_1' = \frac{q}{2} \qquad a_1(x) = \frac{1}{2}\int_a^x q$$

$$a_2' = \frac{1}{2}\left(-\frac{q'}{2} + q\int_a^x \frac{q}{2}\right)$$

$$\therefore\ a_2(x) = -\frac{1}{4}\left[q(x) - q(a)\right] + \frac{1}{4}\left(\int_a^x q\right)^2$$

$$2a_2 - a_1^2 - a_1' = -\frac{1}{2}\left[q(x) - q(a)\right] + \frac{1}{2}\left(\int_a^x q\right)^2 - \left(\frac{1}{2}\int_a^x q\right)^2 - \frac{q(x)}{2}$$

Now $$G(a,a) = \frac{1}{2z\left(1 + \frac{a_1'(a)}{z^2} + \frac{a_3'(a)}{z^4} + \cdots\right)}$$

$$G(a,a) = \frac{1}{2z}\left(1 - \frac{a_1'(a)}{z^2} + \frac{a_1'(a)^2 - a_3'(a)}{z^4} + \cdots\right)$$

Unfortunately this seems complicated, and it's not clear that I shouldn't use an f_z, f_{-z} with bdry conditions independent of the point a I evaluate the Green's function at.