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September 15, 1986

Problem: The Connes-Moscovici form. Let

$$\omega_{2k} = tr_s\left(\frac{1}{1-X^2}dX\right)^{2k} \qquad X = \begin{pmatrix} 0 & -T^* \\ T & 0 \end{pmatrix}$$

so that ω_{2k} is up to the constant $\frac{\cancel{(-1)^k}\,2^{2k+1}}{k!}$ the Chern character form represented ch_{2k} on the Grassmannian. Here use superconnection algebra so that dX is even.

We have the map $\varphi_t : X \longrightarrow tX$ on the Grassmannian which becomes a flow in the variable $s = \log t$. One has

$$\partial_s \varphi_t^* = t\partial_t \varphi_t^* = \varphi_t^* \mathcal{L}_v = \mathcal{L}_v \varphi_t^*$$

where v is the 'radial' vector field. Thus for example, as $d\omega_{2k} = 0$

$$t\partial_t \varphi_t^*(\omega_{2k}) = \mathcal{L}_v \varphi_t^*(\omega_{2k}) = \varphi_t^*(d\iota_v \omega_{2k})$$

$$= d\,\varphi_t^*(\iota_v \omega_{2k}).$$

Now $\iota_v dX = X$ so

$$\iota_v(\omega_{2k}) = \iota_v\, tr_s\left(\frac{1}{1-X^2}dX\right)^{2k}$$

$$= 2k\, tr_s\left(\frac{1}{1-X^2}X\left(\frac{1}{1-X^2}dX\right)^{2k-1}\right)$$

where we have used that $\frac{1}{1-X^2}dX$ is even, so no signs are produced as we apply i_v or move the factors cyclically in the supertrace.

Next restrict to the set where X is invertible on which we know $\varphi_t^*(\omega_{2k}) \longrightarrow 0$ as $t \to 0, \infty$. This gives

$$\varphi_t^*(\omega_{2k})\Big|_0^\infty = 0$$

$$\int_0^\infty \frac{dt}{t}\, t\partial_t\, \varphi_t^*(\omega_{2k}) = d\underbrace{\int_0^\infty \frac{dt}{t}\, 2k\, tr_s\left(\frac{t}{1-t^2X^2}X\left(\frac{t}{1-t^2X^2}dX\right)^{2k-1}\right)}$$

(1) $$= \int_0^\infty \frac{dt}{t}\, 2k\, t^{2k}\, tr_s\left(\frac{1}{t^2-X^2}X\left(\frac{1}{t^2-X^2}dX\right)^{2k-1}\right)$$

which is the formula obtained by Connes-Moscovici for the transgressed character.

On the other hand they start from the character form defined via superconnections which should lead to a transgression form like

(2) $$\int_0^\infty dt\;\; tr_s\left(X e^{u(t^2X^2+tdX)}\right)_{(2k-1)}.$$

I would like to understand clearly how they pass between (1) and (2).

First let's examine the setup from a general viewpoint. I've seen that the natural framework to study both the superconn. character forms and the usual character forms involves working with the **family of** forms

$$\alpha = \int_0^\infty \frac{dt}{t} f(t)\, \varphi_t^*(\omega) \qquad\qquad \omega = \omega_{2k}$$

for different $f(t)$. Let's fix such a form α and run through the transgression process. We consider $\varphi_u^*(\alpha)$ for $0<u<\infty$. One has

$$u\,\partial_u \varphi_u^*(\alpha) = \varphi_u^*(L_v\alpha) = d\;\varphi_u^*(\iota_v\alpha)$$

so

$$\varphi_\infty^*(\alpha) - \varphi_0^*(\alpha) = d \int_0^\infty \frac{du}{u} \varphi_u^*(\iota_v \alpha)$$

On the invertible set the left side will be zero so the integral is closed; it's the transgression form obtained from α. We have

$$\int_0^\infty \frac{du}{u} \varphi_u^*(\iota_v \alpha) = \int_0^\infty \frac{du}{u} \int_0^\infty \frac{dt}{t} f(t) \underbrace{\varphi_u^*(\iota_v \varphi_t^*(\omega))}_{\varphi_{ut}^*(\iota_v(\omega))}$$

$$= \int_0^\infty \frac{du}{u} \int_0^\infty \frac{dt}{t} f(tu^{-1}) \, \varphi_t^*(\iota_v(\omega)).$$

$$= \cancel{\int \frac{du}{u} f(tu^{-1})} \int_0^\infty \frac{dt}{t} \underbrace{\left[\int_0^\infty \frac{du}{u} f(tu^{-1}) \right]}_{\text{independent of } t} \varphi_t^*(\iota_v \omega)$$

Thus we see that the transgression form obtained from α is just the appropriate multiple of the transgression form obtained from ω.

Simplest way to see this. The transgression process takes ω on M, pulls it back to $\mathbb{R}_{>0} \times M$ then integrates over the fibres of $\mathbb{R}_{>0} \times M \to M$. The transgression process has the same effect on ω and $\varphi_t^*(\omega)$, so its effect on $\int \frac{dt}{t} f(t) \varphi_t^*(\omega)$ will be $\int \frac{dt}{t} f(t)$ times its effect on ω.

<u>September 17, 1986</u>

An apparent difficulty with using the Connes-Moscovici forms for defining the cyclic cocycle attached to a Dirac operator is that the form is invariant under rescaling the operator and hence it appears impossible to take the classical limit.

This suggests that I go back to the program of the spring in which I seemed to reach some understanding of Connes' doubling process. It would be nice to have a real understanding of his letter to me.

Above all I would like to see what kind of left-invariant forms arise on a group of gauge transformations from the analysis. Let's begin with a review.

Let E be a vector bundle with inner product over a Riemannian spin manifold M, let $\mathcal{G}$ = its group of gauge transformations acting on $\mathcal{H} = L^2(M, S\otimes E)$, where S is the bundle of spinors. We have seen that there is a canonical involution η modulo compacts, in fact modulo some Schatten ideal, obtained by taking ~~the Dirac operator D assoc. to some connection on E and then forming~~ any ψDO with the symbol $c(\xi)/|\xi|$ where $c(\xi)$ is Clifford multiplication. We then have a canonical group hom.

$$\mathcal{G} \longrightarrow U_{res}(\mathcal{H}, \eta)$$

which defines a K-class on $\mathcal{G}$ as the latter space represents K-theory. The above works when M is either even or odd diml., but for simplicity lets suppose now that $\dim(M)$ is odd.

In order to define left-invariant forms on $\mathcal{G}$ we make $\mathcal{G}$ act on a Grassmannian, choose a point of the Grass to obtain a $\mathcal{G}$-map $\mathcal{G} \longrightarrow$ Grass and pull back the invariant forms on the Grassmannian.

Concretely lets choose a Dirac operator $\not{D}$, or really choose a connection D on E and let $\not{D}$ be the associated Dirac operator. Then form

$$A = \frac{\not{D}}{\sqrt{m^2 + \not{D}^2}}$$

which lies over η and is a self-adjoint contraction. Then expand (dilate) A to an involution

$$F = \begin{pmatrix} A & B \\ B & -A \end{pmatrix} \qquad B = \sqrt{1 - A^2} = \frac{m}{\sqrt{m^2 + \not{D}^2}}.$$

Since A doesn't have the eigenvalues ± 1, F is the minimal dilation of A. What we now have is an embedding

$$j = \begin{pmatrix} 1 \\ 0 \end{pmatrix} : \mathcal{H} \longrightarrow \mathcal{H}^{\oplus 2}$$

and the involution on $\mathcal{H}^{\oplus 2}$

$$F = \begin{pmatrix} \not{D} & m \\ m & -\not{D} \end{pmatrix} \frac{1}{\sqrt{m^2 + \not{D}^2}}$$

such that $j^* F j = A$. We can let $\mathcal{G}$ act on $\mathcal{H}^{\oplus 2}$ by $g \longmapsto \begin{pmatrix} g & 0 \\ 0 & 1 \end{pmatrix} = \tilde{g}$ and consider the induced action on the restricted Grassmannian associated to $\mathcal{H}^{\oplus 2}$ and the involution mod $\mathcal{K}$

$$F \equiv \begin{pmatrix} \eta & 0 \\ 0 & -\eta \end{pmatrix}$$

F provides the basepoint so we get left-invariant forms on G. The problem now is to understand what happens as $m \to \infty$. One hopes that these left-invariant forms have limits as $m \to \infty$.

Recall that the k^{th} character form on the Grassmannian is a constant times

$$(*) \qquad \operatorname{tr} F(dF)^{2k}$$

(Note this is closed as dF anti-commutes with F hence $\operatorname{tr}(dF)^{odd} = 0$.)

We have the map $g \longmapsto gF_0g^{-1} = F$ where F_0 is our basepoint, and for this

$$\begin{aligned} dF &= dg\, F_0 g^{-1} + g F_0(-g^{-1}dg\, g^{-1}) \\ &= g\,[g^{-1}dg, F_0]\, g^{-1} \end{aligned}$$

and so the pull back of $(*)$ under this map is

$$\operatorname{tr}\left(F_0 [F_0, \tilde{\theta}]^{2k}\right) \qquad \tilde{\theta} = \tilde{g}^{-1} d\tilde{g}$$

Let's now write F instead of F_0. Recall that

$$F = \begin{pmatrix} A & B \\ B & -A \end{pmatrix} = \begin{pmatrix} D & m \\ m & -D \end{pmatrix} \frac{1}{\sqrt{m^2 + D^2}}$$

$$A = \frac{hD}{\sqrt{1 + h^2D^2}} \qquad B = \frac{1}{\sqrt{1 + h^2 D^2}} \qquad h = \frac{1}{m}$$

and that we are going to apply this to the case where $\tilde{\theta}$ is of the form $\tilde{\theta} = \begin{pmatrix} \theta & 0 \\ 0 & 0 \end{pmatrix}$. Then

$$[F, \tilde{\theta}] = \begin{pmatrix} [A, \theta] & -\theta B \\ B\theta & 0 \end{pmatrix}$$

Now the question is whether

$$\operatorname{tr}\left(F_{\bullet}[F,\tilde{\theta}]^{2k}\right) \quad \text{has a limit as } h \to 0$$

I want to apply Getzler's calculus which allows one to evaluate traces of operators built up out of hD_μ, $h\gamma^\mu$, and endos of E. Thus the operator $h\not{D} = h\gamma^\mu D_\mu$ is bad, but $h^2\not{D}^2 = h^2\left(D_\mu^2 + \frac{1}{2}\gamma^\mu\gamma^\nu F_{\mu\nu}\right)$ is good. Here I work over a torus say to keep things simple.

Thus the operator A is bad but B is good.

Let's look now at $[A,\theta]$:

$$[A,\theta] = \left[\frac{h\not{D}}{\sqrt{1+h^2\not{D}^2}},\theta\right] = [h\not{D},\theta]\frac{1}{\sqrt{1+h^2\not{D}^2}} + h\not{D}\left[\frac{1}{\sqrt{1+h^2\not{D}^2}},\theta\right]$$

Now $h^2\not{D}^2 \longrightarrow p^2 + F$, F being the curvature of the bundle E. So unless θ commutes with F the last term will be bad because of the $h\not{D}$.

In general one has

$$\left[\frac{1}{\sqrt{1+h^2\not{D}^2}},\theta\right] = \frac{1}{2\pi i}\oint \frac{1}{\sqrt{1+\lambda}}\underbrace{\left[\frac{1}{\lambda-h^2\not{D}^2},\theta\right]}_{\frac{1}{\lambda-h^2\not{D}^2}[h^2\not{D}^2,\theta]\frac{1}{\lambda-h^2\not{D}^2}} d\lambda$$

Even if $[\overset{\text{curv}}{F},\theta]=0$ one has $[h^2\not{D}^2,\theta] = [h^2D_\mu^2,\theta] = O(h)$ and so the second term in $[A,\theta]$ has a non-zero limit. This suggests we are looking at the wrong sort of thing.

Note that the first term is good and it approaches $[D,\theta]\dfrac{1}{\sqrt{1+p^2+F}}$.

September 18, 1986

It occurs to me that the compatibility of the superconnection forms with embeddings might shed some light on the transgression problem [redacted] studied this spring (see p.325 April 86). Let's review the setup.

$$\mathcal{H} = L^2(M, S \otimes E), \quad \mathcal{G} = \mathrm{Aut}(E)$$

Now the rough idea is that if we lift the canonical involution mod $\mathcal{K}$ given by Clifford mult. to an involution F on $\mathcal{H}$, then we know how to map $\mathcal{G}$ to a Grassmannian and hence obtain left-invariant even forms on $\mathcal{G}$ (assume dim M odd). We want corresponding odd forms on $B\mathcal{G}$. The model for $B\mathcal{G}$ consists of subbundles of the trivial Hilbert bundle over M which are isomorphic to E.

Let's start again. Suppose we have a compact odd-diml Riem. spin manifold M and a family of vector bundles on M param. by Y, i.e. a v.b. E over $Y \times M$. We suppose $E \hookrightarrow \tilde{V}$ whence $E_y = \mathrm{Im}(e_y)$, where e_y is a smooth family param. by Y of projectors over $C^\infty(M)$. Take the Dirac operator ~~over~~ M tensor with V to obtain the Dirac operator $\partial\!\!\!/$ on $L^2(M, S \otimes \tilde{V})$, then reduce it w.r.t. e_y so as to obtain a family $\partial\!\!\!/_y$ acting on $\mathcal{H}_y = L^2(M, S \otimes E_y)$ param. by Y.

What do we mean by the index of this family? The index is an odd K-class hence is represented by a map from $Y \longrightarrow U$. To construct such a map take the Cayley transform

$$g_y = \frac{1+\not{D}_y}{1-\not{D}_y} \qquad (\not{D} \text{ supposed skew-adjoint})$$

to obtain a unitary operator on [redacted] the bundle $\mathcal{H} = \{\mathcal{H}_y\}$. Since $\mathcal{H}$ is embedded in the trivial bundle $\tilde{H}$ over Y with fibre $H = L^2(M, S \otimes \tilde{V})$, we ~~can~~ extend g to $\tilde{g}$ on $\tilde{H}$ by letting $\tilde{g} \equiv -1$ on $\mathcal{H}^{\perp}$ in $\tilde{H}$. Then $\tilde{g}$ is a map from [redacted] Y to unitaries $\equiv -1 \bmod \mathcal{K}$, and it's the desired map.

Next let's consider the odd character forms on the unitary group. We know ~~that~~ restricting the odd character forms [redacted] via $g \mapsto \tilde{g}$ leads to the superconnection character forms [redacted] associated to the induced connection on $\mathcal{H}$ and the skew-adjoint operator $\not{D}$.

So it appears that we prove directly in this way that the superconnection character forms on Y are in fact the pull back ~~of~~ the superconnection character forms on [redacted] $- U(\mathcal{K})$ by the index map.

September 19, 1986

Let's start with a Dirac operator on $L^2(M,S)$, call it X_0, and a family e_y, $y \in Y$, of idempotents in $C^\infty(M) \otimes \mathrm{End}(V)$. Then we obtain a Hilbert bundle $\mathcal{H}$ over Y with an embedding $i : \mathcal{H} \hookrightarrow \tilde{H}_Y$, namely $\mathcal{H}_y = e_y H$. On this Hilbert bundle we have the family of Dirac operators $X_y = \iota_y^* X_0 \iota_y$. $H = L^2(M, S \otimes \tilde{V}_M)$.

This Hilbert bundle $\mathcal{H}$ with the family $X = \{X_y\}$ has an index in K-theory. Yesterday I learned that this index can be described nicely as follows. Using the Cayley transform $\frac{1+X_y}{1-X_y}$ one obtains a family of unitary operators on the fibres of the Hilbert bundle $\mathcal{H}$, and these unitaries are congruent to -1 modulo some Schatten ideal. The unitary $\frac{1+X_y}{1-X_y}$ on $\mathcal{H}_y$ may be extended to H by letting it be -1 on the orthogonal complement to $\mathcal{H}_y$ in H. Thus we obtain a map from Y to the unitaries on H congruent to -1 mod the Schatten ideal. This map represents an element of the K-theory of Y which is the index of the family.

The reason this is a nice description of the index is that on the space $-\mathcal{U}(\mathcal{L}^p)$ are natural differential forms of degrees $> p$, and we know they pull-back to the superconnection character forms on Y associated to the family X on $\mathcal{H}$ and the connection on $\mathcal{H}$ associated to the embedding of $\mathcal{H}$ in $\tilde{H}_Y$. (Problem: The superconnection forms are defined in all degrees yet the character forms on the unitary group require conditions for the traces to exist. It should be the usual game that the heat kernel always is defined but the resolvent has to be regularized.)

?

Let us go on to the question of how to anti-transgress this K-class. Let's suppose all the vector bundles $E_y = \mathrm{Im}\, e_y$ on $\tilde{V}_M$ are isomorphic, e.g. if Y is connected, to a fixed v.b. E_0/M. Let $\mathcal{G} = \mathrm{Aut}(E_0)$. We then obtain over Y a principal $\mathcal{G}$-bundle $\mathcal{P}$ such that $\mathcal{P}_y$ = space of isoms. of $E_0 \xrightarrow{\sim} E_y$. Let $\pi : \mathcal{P} \to Y$ be the canon. map.

When we pull E = the v.b. over $Y \times M$ with fibres E_y over Y up to $\mathcal{P} \times M$ we get a bundle canonically isom. to $pr_2^*(E_0)$:

$$(\pi \times id_M)^*(E) \xleftarrow{\sim} pr_2^*(E_0)$$

Similarly we have a canonical isomorphism

$$\pi^*(\mathcal{H}) \simeq (\tilde{H}_0)_{\mathcal{P}} \qquad H_0 = L^2(M, S \otimes E_0)$$

which means that over $\mathcal{P}$ we have a canon embedding

$$\tilde{H}_0 \hookrightarrow \tilde{H}$$

which at the point of $\mathcal{P}$ represented by

$$E_0 \xrightarrow{\sim} E_y = e_y(\tilde{V}_M) \subset \tilde{V}_M$$

is the effect on $L^2(M, S \otimes ?)$ of this map.

So for each $\xi \in \mathcal{P}$ we have an embedding

$$\iota_\xi : E_0 \hookrightarrow \tilde{V}_M \qquad \text{or}$$

$$\iota_\xi : H_0 \hookrightarrow H$$

whose image is the image of e_y where $y = \pi(\xi)$.

Thus over $\mathcal{P}$ the Hilbert bundle $\mathcal{H}$ becomes canonically trivial, and our family of operators $\{X_y\}$ becomes the family $X_\xi = \iota_\xi^* X_0 \iota_\xi$ on H_0. The index of the family is given by pushing the unitary $\frac{1+X_\xi}{1-X_\xi}$ forward via ι_ξ and extending by -1. This gives a map from $\mathcal{P}$ to $-\mathcal{U}(\mathcal{H})$ which is the composition with π of the previous index map.

Next we want to show this index map is contractible. $\mathcal{P}$ maps to the space of embeddings of E_0 in $\tilde{V}_M$ and the index map is defined on this space of embeddings. Letting V increase gives contractibility - to be precise the space of embedding in $\tilde{V}_M$ contracts to a point within the space of embeddings in $\widetilde{V \oplus V}_M$.

What is going on here is that we have a behavior analogous to connections. The way we show contractibility of the space of connections is to pick a basepoint A_0 and then use the convexity $(1-t)A_0 + tA_1$. Similarly to prove contractibility of embeddings we fix ι_0 and then use the deformation $\sqrt{1-t}\,\iota_0 \oplus \sqrt{t}\,\iota_1$.

Now what I want is an analogue of the ~~map~~ deformation between $\tilde{A}$, $\bar{A}$ in my letter to Singer. $\tilde{A}$ is the tautological connection on $pr_2^*(E_0)$ over $\mathcal{A} \times M$ and it leads to the family $A \mapsto D\!\!\!/_A$ on $L^2(M, S \otimes E_0) =$ the fixed Hilbert space H_0. The analogue of $\tilde{A}$ seems to be the family of Diracs $X_\xi = \iota_\xi^* X_0 \iota_\xi$ on H_0 where $\xi \in \mathcal{P}$. The index map sends ξ to $\frac{1+X_\xi}{1-X_\xi}$ on H_0. The index map which descends modulo $\mathcal{G}$ sends ξ to the unitary $\iota_\xi \frac{1+X_\xi}{1-X_\xi} \iota_\xi^{-1} \oplus -1$ on $H_\xi \oplus H_\xi^\perp = H$.

Let's recapitulate: One has a space $\mathcal{P}$ of embeddings $\iota_\xi : E_0 \hookrightarrow \tilde{V}_M$ stable under $\mathcal{G} = \mathrm{Aut}(E_0)$. Hence over $\mathcal{P}$ one has embeddings

$$\iota_\xi : H_0 = L^2(M, S\otimes E_0) \longrightarrow H = L^2(M, S\otimes \tilde{V}_M)$$

On the latter we have given a Dirac operator $\not{D}$ and then one obtains the family of Diracs

$$X_\xi = \iota_\xi^* \not{D} \iota_\xi \quad \text{on } H^0$$

We can now construct two index maps from $\mathcal{P}$ to spaces representing K-theory. The first takes the family $\frac{1+X_\xi}{1-X_\xi}$ of unitaries on H_0. The second takes the unitary $\frac{1+X_\xi}{1-X_\xi}$ pushes it forward under ι_ξ (thereby obtaining $\frac{1+e_\xi \not{D} e_\xi}{1- \quad ''}$ $\quad e_\xi = \iota_\xi \iota_\xi^*$) to obtain a unitary on $\mathrm{Im}\, i_\xi$ which then one extends by -1 on the orthogonal complement.

The way to construct a homotopy between these index maps is to choose an embedding $\iota_0 : H^0 \hookrightarrow H$ whose image is $\perp$ to all the ι_ξ and then use the deformation

$$\sqrt{1-t}\, \iota_0 + \sqrt{t}\, \iota_\xi$$

to deform the family $\{\iota_\xi\}$ to the constant family with value ι_0. ? Why isn't this the ~~[illegible]~~ deformation to a basepoint of $\mathcal{P}$?

In order to decipher this, consider $pr_2^*(E_0) = \mathcal{A}\times E_0$ over $\mathcal{A}\times M$. This is ~~a~~ $\mathcal{G}$-bundle with $\mathcal{G}$ acting both on $\mathcal{A}$ and on E_0. The tautological connection

$\tilde{A}$ is flat in the $\mathcal{A}$-direction. Recall the $\mathcal{A}$-direction part of the connection gives rise to the connection in the Hilbert bundle $\mathcal{A}\times H_0$ over $\mathcal{A}$. Thus we are dealing with the trivial $\mathcal{G}$-bundle over $\mathcal{A}$ with fibre the $\mathcal{G}$-representation H_0. Such a $\mathcal{G}$-bundle has the invariant connection d. In fact a $\mathcal{G}$-bundle over a point has a unique connection, necessarily invariant and the inverse image of this connection is then invariant.

Next we notice that since any $\mathcal{G}$ bundle embeds in the trivial $\mathcal{G}$ bundle belonging to a representation, it is reasonable to work with equivariant embeddings in representations instead of invariant connections.

Now let's go back to our setup where over $\mathcal{P}$ we have the trivial $\mathcal{G}$-bundle with fibre the representation $H_0 = L^2(M, S\otimes E_0)$. We have the embedding $\tilde{H}_0 \hookrightarrow \tilde{H}$ which over ξ is the embedding $\iota_\xi : H_0 \hookrightarrow H$ induced by $\iota_\xi : E_0 \hookrightarrow \tilde{V}_M$. This embedding $\tilde{H}_0 \hookrightarrow \tilde{H}$ is equivariant provided $\mathcal{G}$ acts as it does on H_0 and trivially on H.

Now consider the family of embeddings

$$\sqrt{1-t}\; id_{H_0} + \sqrt{t}\,\iota_\xi \;:\; H_0 \longrightarrow H_0 \oplus H$$

As ξ varies it defines a 1-parameter family of embeddings of $\tilde{H}_0$ into $\tilde{H}_0 \oplus \tilde{H}$ over $\mathcal{P}$. For each t this embedding is equivariant provided $\mathcal{G}$ acts as it should on H_0 and trivially on H. Thus the induced connection on $(\tilde{H}_0)_{\mathcal{P}}$ is $\mathcal{G}$-invariant for each t.

The family of operators $X_\xi = \iota_\xi^* \not{D}\, \iota_\xi$ doesn't change during the deformation, so we are not taking the deformation which contracts the whole index map.

At any rate the conclusion is that the family of embeddings is $\mathcal{G}$ invariant, so it leads to a $\mathcal{G}$-invariant connection on $\tilde{H}_0$ over P, and this combined with the equivariant family X_ξ will produce $\mathcal{G}$-invariant forms on P.

Perhaps one should add that this deformation deforms the index map in a $\mathcal{G}$-invariant way from a map to a unitary group on H (trivial $\mathcal{G}$-action) to a unitary group on H_0 (non trivial $\mathcal{G}$-action). ?

Finally let us look at the case where P consists of a single $\mathcal{G}$-orbit. Fix a basepoint $O \in P$ so that the other points are $\xi = O \cdot g^{-1}$ with $g \in \mathcal{G}$; to $\xi = O g^{-1}$ we have $\iota_\xi = \iota_0 g^{-1} : E_0 \hookrightarrow \tilde{V}$. Also

$$X_\xi = \iota_\xi^* \emptyset \iota_\xi = g \underbrace{\iota_0^* \emptyset \iota_0}_{X_0} g^{-1}$$

and the path of connections is the linear path joining d (= the connection induced by the constant embedding $\tilde{H}_0 \subset \tilde{H}_0 \oplus \tilde{H}$) and the connection induced by the embedding $\tilde{H}_0 \subset \tilde{H} \underset{\substack{\uparrow \\ \text{constant}}}{\subset} \tilde{H}_0 \oplus \tilde{H}$. The first embedding at g is the map $\iota_0 g^{-1} : H_0 \longrightarrow H$, so we have $H_0 \xrightarrow{g^{-1}} H_0 \xrightarrow{\iota_0} H \xrightarrow{in_2} H_0 \oplus H$ where the latter two are constant over $\mathcal{G} = P$. Thus the connection appears to be

$$g \cdot d \cdot g^{-1} \qquad \text{at} \quad t = 1$$

and $(1-t)d + t\, g \cdot d \cdot g^{-1}$ in general. Thus we obtain on the trivial bundle $\tilde{H}_0$ the 1-parameter family of superconnections

$$(1-t)\, d + t\, g \cdot d \cdot g^{-1} + g X_0 g^{-1}$$

Applying the gauge transformation g and changing t to $1-t$ leads to

$$d + t\, g^{-1}dg + X_0$$

Actually I should write δ instead of d to conform to previous notation.

I think I have looked at the superconnection forms associated to this family:

$$\int_0^1 dt \; tr_s\left(e^{X_0^2 + t[X_0,\theta] + (t^2-t)\theta^2}\theta\right)$$

The difficulty is that this form is not closed rather applying δ gives

$$tr_s\left(e^{X_0^2 + [X_0,\theta]}\right) - tr_s\left(e^{X_0^2}\right)$$

and the first term isn't zero. On the other hand Connes-Moscovici explain how this term is a coboundary using the deformation $X_0 \to tX_0$ with $t \downarrow 0$.

September 20, 1986

Suppose $\not{D}$ a Dirac operator in $H = L^2(M, S\otimes E)$, $\mathcal{G} = \mathrm{Aut}(E)$. Over $\mathcal{G}$ we consider the trivial bundle with fibre H; on this bundle we have the 1-parameter family of superconnections

$$\delta + t\theta + \not{D}$$

where $\theta = g^{-1}\delta g$ is the Maurer-Cartan form on $\mathcal{G}$. Yesterday I saw that this family is what one gets by restricting the linear path between the connections $\tilde{A}$ $\bar{A}$ to the $\mathcal{G}$-orbit of $\not{D}$. Provided I can find a suitable version of the Bott theorem this will lead to forms describing the transgression of the character of the index.

The transgression forms should be obtained by taking the closed form over $\mathbb{R}\times\mathcal{G}$:

$$\mathrm{tr}_s\left\{e^{(dt\,\partial_t + \delta + t\theta + \not{D})^2}\right\}_{(2k)}$$

and integrating over $0 \le t \le 1$. This form won't be closed because there will be boundary terms at $t=0,1$. At $t=1$ we get

$$\mathrm{tr}_s\left(e^{\not{D}^2 + [\not{D},\theta]}\right)_{(2k)}$$

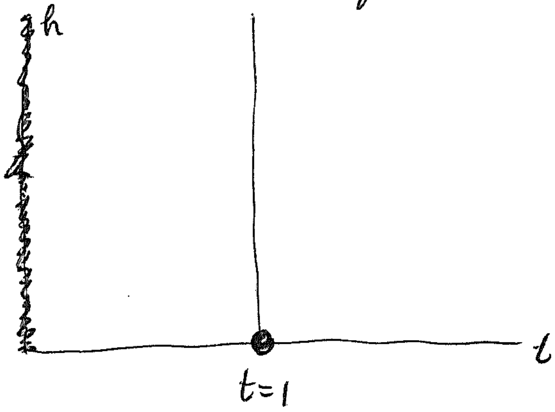
which is apparently $\neq 0$, since Connes + Moscovici define their transgression forms using the family $\delta + \theta + h\not{D}$ $\quad h\in(0,\infty)$. The endpt. limits $h\to 0,\infty$ vanish.

This suggests looking at the closed form on $\mathbb{R}\times\mathbb{R}_{>0}\times\mathcal{G}$ given by

$$\mathrm{tr}_s\left(e^{(dt\,\partial_t + dh\,\partial_h + \delta + t\theta + h\not{D})^2}\right)_{(2k)}$$

In order to obtain a closed $(2k-1)$ form, we might look for a curve in $[0,1]\times\mathbb{R}_{>0}$ where the endpoints

vanish and integrate over this curve.
The places where this form vanishes are



The Only thing to try ~~seems~~ to be $t = 1-h$, but it doesn't lead to anything simple.

Let's see if we can understand why the Connes-Moscovici form

$$\int_0^\infty \frac{dm}{m} \; tr_s\left(\frac{m}{m^2-X^2} X \left(\frac{m}{m^2-X^2} dX\right)^{2k-1}\right)$$

is well-defined, when X is an n-dimensional Dirac operator and $2k$ is sufficiently large. The trace is a sum over the eigenvalues and dX will be a zeroth order operator. It is perhaps simpler to understand the behavior of the form

$$(*) \qquad tr_s\left(\frac{m}{m^2-X^2} dX\right)^{2k}$$

as m goes to 0 or ∞. As $m \to 0$ assuming X is invertible this becomes

$$\sim m^{2k} \, tr_s\left(\frac{-1}{X^2} dX\right)^{2k}$$

X is of order 1, so we need the trace of an operator of order $-4k$. Thus the form $(*)$ is actually defined for $2(2k) > n = \dim(M)$.

Let's now look at the $m \to \infty$ end.

Let's assume first that dX commutes with X, then the trace can be evaluated as a sum over eigenvalues of X, and these are like lattice points, and the sum over lattice points converges when the analogous integral converges. So we want the behavior of

$$\int d^n x \left(\frac{m}{m^2+x^2}\right)^{2k} = m^{n-2k} \int d^n x \left(\frac{1}{1+x^2}\right)^{2k}$$

as $m \to \infty$ and this goes to zero provided $2k > n$.

We still need to understand what happens when dX doesn't commute with X, however this should be the leading term in the general case. You are somehow involved with the behavior of the operator $\frac{1}{\lambda - X^2}$ for large λ, which is roughly equivalent to the behavior of e^{tX^2} for small t.

Now however for Dirac operators one has more precise statements. For example if $dX = [X, \theta]$, then in fact we know by Getzler's method that

$$\lim_{m\to\infty} tr_s \left(\frac{m}{m^2 - X^2}[X,\theta]\right)^{2k} = \lim_{h \to 0} tr_s \left(\frac{1}{1-h^2X^2}[hX,\theta]\right)^{2k}$$

has a limit for all $2k$. The limit is zero once $2k > \dim(M) = n$ by the skew symmetry properties of differential forms. The vanishing result is the same. It seems this vanishing argument, which depends only on the growth of the eigenvalues is the appropriate analogue of the Bott theorem.

Program: Let $\not{D}$ be a Dirac op on $H = L^2(M, S\otimes\tilde{V})$, let $\{e_y, y\in Y\}$ be a family of projectors in $C^\infty(M)\otimes \mathrm{End}(V)$, whence we get a family of Dirac operators $X_y = e_y \not{D} e_y$ on the fibres of the Hilbert bundle $\mathcal{H}/Y$ with $\mathcal{H}_y = e_y H = L^2(M, S\otimes E_y)$, where $E_y = \mathrm{Im}(e_y \text{ on } \tilde{V}_M)$. We have described the index of this family as a map from Y to $-\mathcal{U}_{res}(H)$. We have also discussed anti-transgressing this index to a K-class of the opposite parity over $\mathcal{G} = \mathrm{Aut}(E_0)$. What we really obtain is some sort of K-class of the same parity over the suspension of $\mathcal{G}$, i.e. a map from the suspension of $\mathcal{G}$ to unitaries. What remains is then to bring in a version of periodicity which would tell me that $\Omega\mathcal{U} \sim \mathbb{Z}\times B\mathcal{U}$ and somehow might suggest a way to map $\mathcal{G}$ to the Grassmannian.

Already the case $M = S^1$ is probably very interesting, and it probably connects up with the work done this spring on different periodicity maps, in particular problems with the different kinds of loops in $\mathcal{U}$.

When we do anti-transgression from $B\mathcal{G}$ to $\mathcal{G}$ we use the canonical map $\Sigma\mathcal{G} \longrightarrow B\mathcal{G}$ which results from the principal bundle $\mathcal{G}*\mathcal{G} \longrightarrow \Sigma\mathcal{G}$. In the context of Hilbert spaces where $P\mathcal{G}$ ~~is to be thought of as consisting of embeddings of H_0 into~~ can be viewed as certain embeddings of H_0 into H, we see $\mathcal{G}*\mathcal{G}$ as embeddings of the form

$$\sqrt{1-t}\,\iota_0 g_0 + \sqrt{t}\,\iota_1 g_1$$

where ι_0, ι_1 are $\perp$ embeddings of H_0 into H. So we suppose $H = H_0 \oplus H_0$ with $\iota_0 = \begin{pmatrix}1\\0\end{pmatrix}$ $\iota_1 = \begin{pmatrix}0\\1\end{pmatrix}$.

Then $\Sigma \mathcal{G}$ appears as the space of subspaces $\Gamma_u = \operatorname{Im}\begin{pmatrix} 1 \\ ug \end{pmatrix}$, with $0 \leq u \leq \infty$, $g \in \mathcal{G}$, of $H_0 \oplus H_0 = H$. We have on H the Dirac operator $\not{D}$, say $\not{D}_0 \oplus \not{D}_0$, where $\not{D}_0$ is a given Dirac operator on H_0. Then for each $\Gamma_{ug} \in \Sigma \mathcal{G}$ we reduce $\not{D}$ to Γ_{ug}, take the Cayley transform and extend by -1, so as to obtain an element of $-U(H)$. This describes the index map from $\Sigma \mathcal{G}$ to $-U(H)$, and my problem is to understand the corresponding map $\mathcal{G} \longrightarrow \mathbb{Z} \times BU$.

Note that I have a map

$$\mathcal{G} \longrightarrow \Omega(-U(H))$$

and I want therefore some kind of map

$$\Omega U \longrightarrow \mathbb{Z} \times BU$$

which is a map in the periodicity thm. This suggests bringing in the Dirac operator on the circle ~~or~~ $[0,1]$ in some way, because this is the way such periodicity maps are produced.

September 22, 1986

Let $H_0 = L^2(M, S\otimes E_0)$, $\mathcal{G} = \mathrm{Aut}(E_0)$, and let $\not{D}$ be a Dirac operator on H_0. Yesterday we constructed a family of Dirac operators on M with parameter space $\Sigma(\mathcal{G})$ and an index map $\Sigma(\mathcal{G}) \longrightarrow -\mathcal{U}(H)$ for this family. The problem is now to use periodicity to convert this to a map $\mathcal{G} \longrightarrow \mathbb{Z}\times BU$. This involves integrating over the suspension coordinate, and I feel that this means one wants to bring in the Dirac operator on the circle. (Possibly one might want to do things over the torus or 2-sphere instead).

Let's review the index map defined on $\Sigma(\mathcal{G})$. We set $H = H^{\oplus 2}$. To $(g,t)\in \mathcal{G}\times[0,1]$ we associate the embedding

$$\sqrt{1-t}\,\iota_0 + \sqrt{t}\,\iota_1 g \qquad \text{where } \iota_0, \iota_1 \text{ are } \binom{1}{0}, \binom{0}{1}$$

in some order; this is an embedding of H_0 into $\tilde{H} = H_0\oplus H_0$ and defines over $\Sigma(\mathcal{G})$ a subbundle $\mathcal{H}$ of $\tilde{H}$. The fibre of $\mathcal{H}$ at (g,t) is the graph Γ_{ug} with $u = \sqrt{\frac{t}{1-t}}$. The index map is defined by contracting $\not{D}\oplus\not{D}$ to Γ_{ug}, taking Cayley transform, and extending by -1 to obtain an elt. of $-\mathcal{U}(H)$.

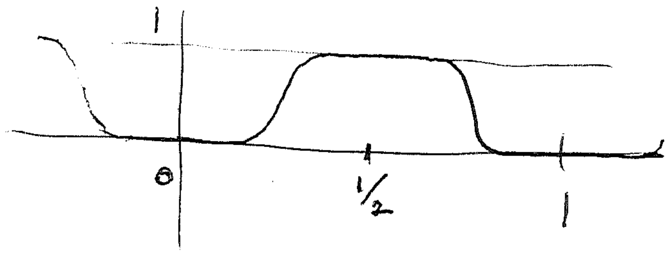
Now $\Sigma(\mathcal{G})$ is not a manifold and I recall from the work this spring that there are two ways of remedying this. Let's return to the finite-diml setup considered then where one looked at the Bott map

$$\Sigma(U_n) \longrightarrow Gr_n(\mathbb{C}^{2n})$$

instead of a gauge transf. group $\mathcal{G}$. So now I am looking instead of $\mathcal{G}, H_0$ at $U(n)$, $\mathbb{C}^n$.

We have over $\Sigma(U_n)$ a canonical n plane bundle embedded in the trivial bundle $\widetilde{\mathbb{C}^{2n}} = \widetilde{\mathbb{C}^n \oplus \mathbb{C}^n}$, whose fibre at (g,t) is Γ_{ug}.

Now over $U(n) \times S^1$ is a canonical vector bundle E equipped with partial connection in the S^1-direction whose sections over $\{g\} \times S^1$ are $f: \mathbb{R} \to \mathbb{C}^n$ such that $f(x+1) = g f(x)$. In other words this bundle results by taking the trivial bundle $\widetilde{\mathbb{C}^n}$ over $U(n) \times [0,1]$ and using the identity map of $U(n)$ as a clutching function. We use the partition of unity over S^1 given by $1-\rho(x)$, $\rho(x)$ where ρ:



$$\rho(x+1) = \rho(x)$$

to embed this canonical bundle E in $\widetilde{\mathbb{C}^{2n}}$ as follows. Consider the embedding of $\widetilde{\mathbb{C}^n}$ into $\widetilde{\mathbb{C}^{2n}}$ over $\mathcal{G} \times [0,1]$

$$\sqrt{1-\rho(x)}\, \iota_0 + \sqrt{\rho(x)}\, \iota_1 g \qquad 0 \le x \le \tfrac{1}{2}$$

$$\sqrt{1-\rho(x)}\, \iota_0 g + \sqrt{\rho(x)}\, \iota_1 g \qquad \tfrac{1}{2} \le x \le 1$$

This is smooth and is compatible with the identification at the ends $x = 0, 1$ via g. What this embedding means is that we follow the path Γ_{ug}, $0 \le u \le \infty$ for the first half of the circle and then we come back using Γ_{u1}, $1 = 1_{U(n)}$. Thus we have a smooth map

$$U(n) \times S^1 \longrightarrow Gr_n(\mathbb{C}^{2n})$$

which factors thru $\Sigma(U(n))$.

(The other method for smoothing is to consider smooth loops $g(t): \mathbb{R} \to U(2n)$ such that $g(t+1) = g(t)$ and $\varepsilon g(t)\varepsilon = g(-t)$ and $g(0) = 1$, $g(\frac{1}{2}) = -1$, or something like this.)

<u>September 23, 1986</u>

It has become clear that I have to understand the case where $M = S^1$. Specifically consider $E \hookrightarrow \tilde{V}$ defined by a projector $e \in C^\infty(S^1) \otimes \mathrm{End}(V)$. Take the Dirac operator $X = \partial_x$ on $H = L^2(S^1, \tilde{V})$ and contract it to $eH = L^2(S^1, E)$. As e varies we get a family of Dirac operators over S^1 with parameter space $Y = \mathcal{L}\, Gr(V)$. We have the index map $Y \longrightarrow -U(H)$ which gives rise to the superconnection forms on Y. But we also have the monodromy map which associates to a Dirac operator over S^1 the parallel transport on the fibre over the basepoint. Thus if we restrict to $Y = \Omega\, Gr(V)$ we get a monodromy map

$$Y \longrightarrow U(V)$$

which is to be compared with the preceding index map.

Relative to $\tilde{V} = E \oplus E^\perp$ we have

$$X = \begin{pmatrix} X' & i^* X j \\ j^* X i & X'' \end{pmatrix}$$

where X', X'' are the induced Dirac operators on $E, E^\perp$ respectively and the off diagonal parts are 0-th order. The index is the unitary

$$\begin{pmatrix} \frac{1+X'}{1-X'} & 0 \\ 0 & -1 \end{pmatrix} \quad \text{on } H.$$

Let's recall what we learned last spring about a Dirac on S^1, say $X = \partial_x + A$ on $S^1 = \mathbb{R}/2\pi\mathbb{Z}$. This operator has a monodromy

$$\tau = T\left\{ e^{-\int_0^{2\pi} A\,dx} \right\}$$

which is a unitary transformation on the fibre over the basepoint $0 \in S^1$. The eigenvalues of τ lie on the unit circle and the eigenvalues of X are simply the numbers $i\lambda \in i\mathbb{R}$ such that $e^{2\pi i\lambda}$ is an eigenvalue of τ. More specifically a basis of eigenvectors for τ leads to a basis of eigenvectors of X stable under multiplication by $z = e^{ix}$.

We saw that if we form from X the unitary operator $f(\frac{1}{i}X)$, where $f(\lambda)$ is

$$f(\lambda) = \begin{cases} -1 & \lambda \leq -1 \\ e^{\pi i\lambda} & -1 \leq \lambda \leq 1 \\ -1 & 1 \leq \lambda \end{cases}$$

then $f(\frac{1}{i}X)$ is essentially the monodromy operator extend by -1 in the following sense. The space spanned by the eigenvectors of $\frac{1}{i}X$ with eigenvalues in $(-1,1)$ is embedded isometrically into the fibre over the basepoint by the evaluation map.

In any case it seems clear that there is a nice homotopy between $f(\frac{1}{i}X)$ and $\frac{1+X}{1-X}$.

Suppose we apply the rescaling transformation $X \longrightarrow tX$ with $t \longrightarrow +\infty$.

First I should review a bit about the Kasparov C^*-module viewpoint. Refer back to April 1986 p. 311-12.

The important thing for the index are the zero modes and one has the basic fact evaluation at basepoint gives

$$\mathrm{Ker}(\not D_y) \xrightarrow{\sim} \mathrm{Ker}(\tau_y - 1)$$

Unfortunately these subspaces jump. Nevertheless we have two bundles over Y each equipped with a unitary transformation. The first is the Hilbert space bundle with unitary $\left\{\frac{1+\not D_y}{1-\not D_y}\right\}$ and the second is the bundle over Y obtained by taking the fibre over the basept of S^1 equipped with the monodromy autom. $\{\tau_y\}$. The above isomorphism saying that the zero modes are the same should be enough in the Kasparov theory to show that these index maps are equivalent.

It may be necessary to include not just the zero modes of $\frac{1}{i}\not D$ but also eigenspaces with eigenvalues in an interval $(-\varepsilon, \varepsilon)$.

At the moment I am having difficulty in finding the next step. One idea is to use the fact that the infinite unitary group is a classifying space for the Q-category; one has two classifying maps for a Q-structure over Y, hence they have to be homotopic. This idea perhaps is linked to the procedure used in defining the metric on the determinant bundle where you have an open covering U_a, where a is not an eigenvalues. This raises the issue of the differential forms - say for example you have a family (V_y, X_y) making a sort of Kasparov family (sections of $V = \{V_y\}$ should make a Hilbert module over $C(Y)$, then how can you define forms representing its character.

In order to bring things a bit into focus, let's return to the case where I have the complete data required to define the differential forms.

This means we have a family $\{e_y\}$ of projectors over $C^\infty(S^1) \otimes \mathrm{End}(V)$, and hence the Hilbert bundle $\mathcal{H}$ over Y with
$$\mathcal{H}_y = e_y \underbrace{L^2(S^1, V)}_{H} = L^2(S^1, \underbrace{e_y \tilde{V}}_{E_y})$$
comes with an embedding into the trivial bundle $\tilde{H}$. For each y we have $\tilde{V} = E_y \oplus E_y^\perp$ and the Dirac operator $X = \partial_x$ on $L^2(S^1, V)$ decomposes
$$X = \begin{pmatrix} X_y' & i^* X j \\ j^* X i & X_y'' \end{pmatrix}. \qquad i = i_y,\ j = j_y$$
The index is then the map
$$y \longmapsto \begin{pmatrix} \frac{1+X_y'}{1-X_y'} & 0 \\ 0 & -1 \end{pmatrix} = \tilde{g}_y$$
from Y to $-\mathcal{U}(H)$.

Similarly we can look over the basepoint ξ and we have $(E_y)_\xi \oplus (E^\perp)_\xi = V$ and we can extend the monodromy automorphism τ_y on $(E_y)_\xi$ by -1 to a unitary $\tilde{\tau}_y$ on V, thus obtaining a map $Y \longrightarrow \mathcal{U}(V)$. Now our problem is to prove homotopy ~~commutativity~~ of

$$\begin{tikzcd} Y \arrow[r] \arrow[dr] & \mathcal{U}(V) \arrow[d] \\ & \mathcal{U}(H) \end{tikzcd}$$

$\leftarrow$ induced by any embedding $V \hookrightarrow H$

As mentioned above it's probably more fruitful to think of classifying maps rather than explicit homotopies. This means constructing something big which maps both to Y and $\mathcal{U}(V)$.

One idea is to pick an interval $(-\varepsilon, \varepsilon)$

on S^1 and consider unitaries not having $\varepsilon, -\varepsilon$ as eigenvalues. This is an open set $\mathcal{U}_{(\varepsilon,\varepsilon)}$ which deforms to the Grassmannian. If I move all eigenvalues outside $(-\varepsilon, \varepsilon)$ to -1, then the character forms move to superconnection forms.

Is it possible to understand stratification of the unitary group by multiplicity of the eigenvalue $\underline{1}$ using the covering by the open sets $\mathcal{U}_{(-\varepsilon,\varepsilon)}$?

September 24, 1986

$\mathcal{G} = \mathrm{Aut}(E_0)$ acts on $H_0 = L^2(M, S\otimes E_0)$, and $\not{D}$ is given on H_0. We make a family of ops. with parameter space $\Sigma(\mathcal{G})$. The index of the family is a map

$$\Sigma(\mathcal{G}) \longrightarrow \mathcal{U}$$

assuming $\dim(M)$ odd. By periodicity this yields a map

$$\mathcal{G} \longrightarrow \Omega\mathcal{U} = \mathbb{Z}\times B\mathcal{U}.$$

The problem is to set things up so that the last map is realized with a Hilbert space on which $\mathcal{G}$ acts and a point of a restricted Grassmannian of this Hilbert space.

Here's how the family with parameter space $\Sigma(\mathcal{G})$ is obtained. For each $(g,t)\in\Sigma(\mathcal{G})$ let

$$E_{g,t} = \mathrm{Im}\left\{\begin{pmatrix}\sqrt{1-t}\\ \sqrt{t}\,g\end{pmatrix} : E_0 \longrightarrow E_0\oplus E_0\right\}$$

$$= \text{graph of } ug \qquad u = \sqrt{\tfrac{t}{1-t}}$$

$$H_{g,t} = L^2(M, S\otimes E_{g,t}) = \text{graph of } ug \text{ on } H_0.$$

The Dirac operator $\not{D}^{\oplus 2}$ on $H_0^{\oplus 2}$ compresses to a Dirac operator $\not{D}_{gt}$ on $H_{g,t}$. If we use the isom. $H_0 \longrightarrow H_{g,t}$ given by $\begin{pmatrix}\sqrt{1-t}\\ \sqrt{t}\,g\end{pmatrix}$, then

$$\not{D}_{gt} = (1-t)\not{D} + t\,g^{-1}\not{D}g = \not{D} + t\,g^{-1}[\not{D},g]$$

is a linear path.

Now we have to take the index of the family $\not{D}_{gt}$ on H_{gt} for $(g,t)\in\Sigma(\mathcal{G})$. The scheme I've been using is to take the Cayley transform of $\not{D}_{gt}$ extend to the complement of $H_{gt}\subset H_0^{\oplus 2} = H$ by -1, thus obtaining a map $\Sigma(\mathcal{G}) \longrightarrow -\mathcal{U}(H)$.

I want to return to the above to see if this index map

$$\Sigma(\mathcal{G}) \longrightarrow -U(H)$$

is $\mathcal{G}$-equivariant. But now let's record some of Graeme's ideas.

We want to discuss spectral flow. Suppose we have a family of odd dim'l Dirac operators parametrized by the circle. One follows the eigenvalues of the operators around the circle and a certain number change sign; the net sign change is the spectral flow. The theorem says the spectral flow equals the index of the total Dirac operator.

To be more specific let's consider an odd dim'l Riem. spin manifold M and a vector bundle with comp. connection and inner product E over $S^1 \times M$. Then we can form the Dirac op on $S^1 \times M$ with coeffs in E, and the family $\not{D}$ of Dirac operators $\not{D}_y$ on M with coefficients in E_y for each $y \in S^1$. The theorem equates the index of the former with the spectral flow of the latter.

Using the transverse part of the connection given on E we can trivialize the pull back of E to $\mathbb{R} \times M$ in the transverse direction. What this means is that E is obtained by taking the quotient of $\mathbb{R} \times E_0$ over $\mathbb{R} \times M$ by the $\mathbb{Z}$ action defined using an a gauge transf. g of E_0. The family $\not{D}_y$ then becomes a family of Dirac operators $\not{D}_t$ on E_0 over M satisfying

$$\not{D}_{t+1} = g^{-1} \not{D}_t \, g$$

and the total Dirac operator lifts to the pair

$$\partial_t + \frac{1}{i} \not{D}_t \quad , \quad +\partial_t - \frac{1}{i} \not{D}_t$$

Set $A(t) = \frac{1}{i}\not{D}_t$, so we have the elliptic operator

$$\begin{pmatrix} 0 & \partial_t - A(t) \\ \partial_t + A(t) & 0 \end{pmatrix}$$

acting on functions on $\mathbb{R}$ with values in $L^2(M, S\otimes E_0)^{\oplus 2}$ satisfying

$$f(t+1) = g^{-1} f(t) .$$

A key idea in understanding the spectral flow theorem is the adiabatic approximation. This means looking at the operator

$$\varepsilon \partial_t + A(t)$$

in the limit as $\varepsilon \downarrow 0$.

There is some reason to believe that if we let g vary over $\mathcal{G}$, then the index of the family of total Dirac operators $\varepsilon\partial_t + A(t)$ is particular simple.

Let's try to recall Graeme's theorem. He considers something like $\Omega(\mathcal{A}/\mathcal{G})$, and replaces it by the space of pairs $(A(t), g)$ where $A(t)$ is a 1-parameter family of connections such that $g^{-1}A(0)\,g = A(1)$. Maybe one requires $A(t)$ to be defined and smooth for all t and to be constant for $t \notin (0,1)$. This space of pairs call it $\mathcal{X}$ maps to Fredholm operators by forming $\varepsilon\partial_t + A(t)$. One gets equivalent operators by letting V be the solutions of

$$\big(\varepsilon\partial_t + A(t)\big)\psi(t) = 0$$

over $[0,1]$ and considering

$$V \oplus H \xrightarrow{\text{endpts} + \Gamma_g} H \oplus H$$

September 26, 1986

Program: $\mathcal{G} = \mathrm{Aut}(E_0)$, $H_0 = L^2(M, S\otimes E_0)$, $\not{D}$ given on H_0. There is a family of Dirac operators over M parametrized by $\Sigma_1(\mathcal{G})$, namely $(1-t)\not{D} + t g^{-1}\not{D} g$. The idea is to couple this with the Dirac ∂_t on the time coordinate so as to get a family param. by $\mathcal{G}$. This is to be done $\mathcal{G}$ equivariantly with $\mathcal{G}$ acting ~~on~~ the Hilbert space.

There are two ideas I want to try:

1) adiabatic approximation

2) loop group $(\mathcal{L}U_{2n})^\sigma$ attached to the loop space of the Grassmannian $Gr_n(\mathbb{C}^{2n})$.

Let's begin with the second. Change notation $\mathcal{G} = \mathcal{L}(U_{2n})$, $\mathcal{A} = \mathcal{L}(\mathfrak{u}_{2n})$, $\mathcal{G}' = \Omega(U_{2n}; 1, 1)$ (smooth periodic). Recall that one has a (right) $\mathcal{G}$-action on $\mathcal{A}$:

$$g^{-1}(\partial_t + A) g = \partial_t + (g^{-1}g' + g^{-1}Ag)$$

To $A \in \mathcal{A}$ one associates $h: \mathbb{R} \to U_{2n}$ $\ni$

$$h' = hA$$
$$h(0) = 1,$$

and in this way elts of $\mathcal{A}$ can be identified with smooth $h: \mathbb{R} \to U_{2n}$ $\ni$ $h(t+1) = h(1)h(t)$, $h(0) = 1$.

and

$$(h * g)(t) = g(0)^{-1} h(t) g(t)$$

We have the principal bundle

$$\mathcal{G}' \longrightarrow \mathcal{A} \xrightarrow{\text{monodromy}} U_{2n}$$
$$h \longmapsto h(1).$$

and $\mathcal{G}$ acts on it; the action on $\mathcal{U}_{2n}$ being by evaluation $g \mapsto g(0) \in \mathcal{U}_{2n}$ and then conjugation.

We also have a $\mathcal{G}$-equivariant map which associates to $A \in \mathcal{A}$ the Dirac operator $\frac{1}{2\pi i}(\partial_t + A)$ on $L^2(S^1, \mathbb{C}^{2n}) = H$. One of the most important $\mathcal{G}$-orbits on $\mathcal{A}$ is the space of connections with monodromy -1, which is a smooth version of $\Omega(\mathcal{U}_{2n}; 1, -1)$. The corresponding Dirac operators are invertible, so they yield involutions via polar decomposition. This gives a map ($\mathcal{G}$-equivariant)

$$\mathcal{A}_{\tau=-1} \longrightarrow Gr_{res}(H)$$

It has the property that on composing with the Bott map

$$\begin{array}{ccccc} Gr(\mathbb{C}^{2n}) & \longrightarrow & \mathcal{A}_{\tau=-1} & \longrightarrow & Gr_{res}(H) \\ \cup\!\!\!\!\!\!\;\;\, F & \longmapsto & \frac{1}{2\pi i}\partial_t + \frac{1}{2}F & \longmapsto & 1 \otimes (F=1) \oplus z H^2 \otimes \mathbb{C}^{2n} \end{array}$$

one gets an embedding of a small Grassmannian into a large one.

All the above holds with $2n$ replaced by n. Now let $\varepsilon = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ on $\mathbb{C}^{2n}$ and define σ on $H = L^2(S^1, \mathbb{C}^{2n})$ by

$$(\sigma f)(t) = \varepsilon f(-t)$$

Then σ induces actions on $\mathcal{G}$, $\mathcal{A}$.

All this stuff is contained in p410-411, May 1986.

I think it is more important to concentrate on the $\mathcal{G}$-equivariance of the resulting family of operators being sought. Let's start again with $\mathcal{G}$, $\not{D}$ acting on H_0. We have already found the family of operators over $\Sigma(\mathcal{G})$: For each $(g,t) \in \Sigma(\mathcal{G})$

we have the subspace

$$H_{g,t} = \operatorname{Im}\left\{ \begin{pmatrix} \sqrt{1-t} \\ \sqrt{t}\, g \end{pmatrix} : H_0 \to H_0 \oplus H_0 \right\}$$

and we contract $\not{D} \oplus \not{D}$ to it to obtain $\not{D}_{g,t}$. Relative to the isom. $\begin{pmatrix} \sqrt{1-t} \\ \sqrt{t}\, g \end{pmatrix} : H_0 \xrightarrow{\sim} H_{g,t}$, this operator becomes

$$\not{D}_{g,t} = (1-t)\not{D} + t\, g^{-1} \not{D} g .$$

Now we propose somehow to [redacted] couple this family with the [redacted] Dirac on the t axis. This means we introduce some sort of Hilbert space $\mathcal{H}_g$ of functions $f(t)$ of t such that $f(t) \in H_{g,t}$. For example, lets [redacted] use the identification $H_0 \simeq H_{g,t}$; then we can take $\mathcal{H}_g$ [redacted] to be the space of [redacted] functions $f(t)$ with values in H_0 such that $f(1) = g^{-1} f(0)$, and on $\mathcal{H}_g$ we consider the operator

$$\mathcal{D}_g = \partial_t + (1-t)\not{D} + t\, g^{-1} \not{D} g$$

(Perhaps one wants to use a different metric on the t axis, so that in effect one is working over $\mathbb{R}$.)

At any rate the important thing the question to be answered is whether any of these choices for the t axis structure will lead to a G-equivariant situation.

What we have at the moment is a [redacted] family of Hilbert spaces $\mathcal{H}_g$ with operators [redacted] $\mathcal{D}_g$ parametrized by G. Thus we have a Hilbert bundle over G equipped with a mod $\mathcal{K}$ splitting. What we want eventually is a [redacted] homomorphism from G to $U_{res}(H_\eta)$, for some H, η.

I have reached the following natural question. Suppose we have a family over G given by a Hilbert bundle together with a family of Fredholms in the fibres. This represents an elt of $K(G)$. I would like to know when it is equivalent to the K-class defined by a group homomorphism from G to $U_{res}(H,\eta)$.

This is a K-theoretic analogue of asking when a cohomology class on the group comes from the Lie algebra.

Let's try thinking of the analogue of a left-invariant closed form on G as a representation of G on a Hilbert space H together with an involution F on H which G preserves modulo compacts. Let's think of the analogue of a cocycle on G as a graded Hilbert bundle over G with odd self adjoint contraction which is an involution mod $\mathcal{K}$. The natural question is how to go from the former to the latter.

If we are given H, F with G acting on H, then clearly we get a graded Hilbert bundle by taking (H^+, gH^+) and the orthogonal projections furnish the Fredholm operators.

It would be nice if this family had a nice index map which would go from G to unitaries congruent to -1. This index map shouldn't be a group homomorphism, but I'd like it to be nice when it comes to Dirac operators.

September 27, 1986

Let $\mathcal{G} = \mathrm{Aut}(E_0)$ act on $H_0 = L^2(M, S \otimes E_0)$ and let η be the canonical involution mod K associated to the "phase" of any Dirac operator. Over $B\mathcal{G}$ we have the Hilbert bundle $P\mathcal{G} \times^{\mathcal{G}} H_0$ with involution mod K, and this family has an index which I can construct as a map

$$B\mathcal{G} \longrightarrow \mathcal{U} \quad \Big(= \underset{\text{on same } H}{\text{unitaries} \equiv 1 \bmod K} \Big)$$

I propose to restrict this index to the suspension of $\mathcal{G}$, or better, take loops

$$\mathcal{G} = \Omega B\mathcal{G} \longrightarrow \Omega\mathcal{U}$$

and then apply Bott periodicity to get a map

(1) $$\mathcal{G} \longrightarrow \mathbb{Z} \times B\mathcal{U}.$$

What I would like to end up with is something equivalent to the map

(2) $$\mathcal{G} \longrightarrow U_{res}(H_0, \eta) \xrightarrow{\text{given by choice of } F \text{ over } \eta} \mathcal{J}_{res}(H_0, \eta)$$

~~The viewpoint I have is that the map (1) which should be realized as a family of operators on $\mathcal{G}$, is the analogue of a cohomology class on $\mathcal{G}$, whereas the analogue of a~~

The viewpoint is as follows. The map (1) can be [redacted] realized as a family of operators parametrized by $\mathcal{G}$, and in fact as a more or less explicit map from $\mathcal{G}$ to a restricted Grass. Such a map is analogous to a cochain on $\mathcal{G}$, whereas the map (2) is the analogue of a left-invariant form on $\mathcal{G}$.

(Hopefully, by "more or less explicit index map" one means something obtained using constructions fitting into the Kasparov cup product theory.)

Now the real point here has to do with periodicity and can be discussed with $\mathcal{G} = U_{res}$. From the viewpoint of explicit index maps I think the natural spaces are U and the restricted Grassmannian $\mathcal{I}_{res}$, because these spaces carry the nice differential forms. On the other hand one has the groups $\mathcal{G} = U_{res}$ and its graded analogue $\mathcal{G}'$ pairs of unitaries preserving a Fredholm operator modulo compacts. Actually there is one of these groups for each $k \in \mathbb{Z}$.

Note that U_{res} acts on $\mathcal{I}_{res}$ and that $\mathcal{G}'$ acts on U; more precisely when the Fredholm from H^+ to H^- has index zero U is identified with the isomorphisms of H^+ with H^- congruent mod $\mathcal{K}$ to the Fredholm operator.

My problem is this: Over $B\mathcal{G}$ one has an explicit family of operators and one first needs an index map

$$B\mathcal{G} \longrightarrow U$$

Next one applies Ω to get

$$\mathcal{G} \longrightarrow \Omega(U)$$

Next one needs a periodicity equivalence

$$\Omega(\mathcal{U}) \sim \mathcal{I}_{res}$$

and this should be compatible in the sense that

$$\mathcal{G} \longrightarrow \Omega(\mathcal{U}) \sim \mathcal{I}_{res}$$

is homotopic to $\mathcal{G} \longrightarrow \mathcal{U}_{res} \longrightarrow \mathcal{U}_{res}/\mathcal{U}^F = \mathcal{I}_{res}$. It is not easy to pin down what is required.

September 28, 1986

It seems that we are dealing with a problem in K-theory which would be easy if we knew more about Kasparov theory. Let $\mathcal{G} = U_{res}$. From

$$U_{res} \longrightarrow U(H) \longrightarrow \mathcal{I}(2)$$

and Kuiper we know $BU_{res} \sim \mathcal{I}(2)$, so we have a canonical ~~map~~ equivalence

$$(*) \qquad \mathcal{G} \sim \Omega B\mathcal{G} \sim \Omega \mathcal{I}(2).$$

On the other hand by periodicity

$$\Omega \mathcal{I}(2) \sim \mathcal{I}_{res}$$

so we have a canonical equivalence $\mathcal{G} \sim \mathcal{I}_{res}$ and our problem is to see that it is the same as the map $\mathcal{G} = U_{res} \longrightarrow \mathcal{I}_{res}$ defined by a basepoint of $\mathcal{I}_{res}$.

Now certain loops in $\mathcal{I}(2)$ we know how to handle from the Atiyah-Singer proof

$$\begin{array}{ccccc} \mathcal{F}_0 & \longrightarrow & \Omega(U(H); \varepsilon, -\mathcal{I}_{res}) & \longrightarrow & -\mathcal{I}_{res} \\ \wr\downarrow & & \downarrow & & \\ U(2) & \longrightarrow & \Omega(\mathcal{I}(2); \varepsilon, -\varepsilon) & & \end{array}$$

so it would be nice if the map $(*)$ factored through a ~~map~~ $\mathcal{G} \longrightarrow U(2)$. Such a map is provided by the Toeplitz construction, which I dislike because of its asymmetry and also because it doesn't seem to have a graded analogue.

However we should first see what kind of map (*) is.

As usual, over $\Sigma\mathcal{G}$ we considered the family of subspaces $\Gamma_{ug} \subset H_0 \oplus H_0 = H$, which gives a Hilbert subbundle $\Gamma \subset \tilde{H}$. The involution η mod $\mathcal{K}$ on H_0 gives $\eta^{\oplus 2}$ on H and induces an η on the bundle Γ. In order to obtain the map

$$\Sigma\mathcal{G} \longrightarrow \mathcal{I}(\mathcal{Q}) \qquad \mathcal{Q} = \mathcal{B}(H)/\mathcal{K}(H)$$

we need to extend this η on Γ trivially to to $\Gamma^\perp$, say by ± 1.

(I thought infinite repetition would prove a canonical extension, but this doesn't seem to work. Recall that this method replaces the embedding $\Gamma \subset H$ by

$$\Gamma = 1 \otimes \Gamma \subset H^2(S^1) \otimes H = \hat{H}.$$

The orthogonal complement of Γ in $\hat{H}$ is canonically isomorphic to $\hat{H}$ via the loop

$$ze + (1-e) : \hat{H} \hookrightarrow \hat{H}$$

where e = projector on H with image Γ. Thus we now have an embedding $\Gamma \subset \tilde{H}$ whose complement is canonically trivialized. However there is still no natural η on $\hat{H} = H^2(S^1) \otimes H$. One lifts η to an F on H and then gets $1 \otimes F$ on $\hat{H}$ however $1 \otimes \mathcal{K}(H) \not\subset \mathcal{K}(\hat{H})$ so choices of F matter.)

So I conclude that it should not matter whether the η on Γ is extended by $+1$ or -1; the important point is that the involution mod $\mathcal{K}$ on $\Gamma^\perp$ be trivial <u>from the viewpoint</u> of K-theory.

Approach: Over $\Sigma_1(\mathcal{G})$ we have a canonical Hilbert bundle Γ with mod $\mathcal{K}$ involution. This is a standard representative for Kasparov theory. On the t axis we have the operator ∂_t, which upon being changed into the Hilbert operator gives a standard Kasparov gadget. According to KK-theory there is a cup-product. How cleanly can the construction of the cup-product in this case be done?

What is the general idea?

In thinking through the $\mathcal{A}^\sigma$ business the following points emerged.

1) Recall σ acts on $L^2(S^1, H)$, $H = H_0 \oplus H_0$ by $(\sigma f)(t) = \varepsilon f(-t)$, $\varepsilon = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. The two eigenspaces of σ both identifiable with $L^2([0,1], H)$ so that a Dirac operator on $L^2(S^1, H)$ reversed by σ is like a graded Dirac operator.

2) It seems that the operator

$$(*) \qquad \partial_t + \pi \begin{pmatrix} 0 & -g^* \\ g & 0 \end{pmatrix} \qquad \text{on } L^2(S^1, H)$$

is the analogue of the Dirac operator over $[0,1]$ with coefficients in the bundle of $\Gamma_{ug} \subset H_0 \oplus H_0$. Notice that the action of $\mathcal{G} \times \mathcal{G}$ on $H_0 \oplus H_0$ preserves the subbundle as well as the operators $(*)$.

3) If $\not{D}$ is a Dirac operator on H_0, then we can couple it to $(*)$ via

$$(**) \qquad \partial_t + \pi \begin{pmatrix} 0 & -g^* \\ g & 0 \end{pmatrix} + \begin{pmatrix} 0 & -\not{D} \\ \not{D} & 0 \end{pmatrix}$$

This seems to be the analogue of the family induced by $\not{D} \oplus \not{D}$ on $\Gamma \subset H_0 \oplus H_0$, which recall with the isom $\begin{pmatrix}\sqrt{1-t}\\ \sqrt{t}\,g\end{pmatrix} : H_0 \longrightarrow \Gamma_{ug}$ is $(1-t)\not{D} + t\, g^{-1} \not{D} g$. Neither this family nor $(**)$ has any invariance properties w.r.t. $\mathcal{G}$, except that both simplify noticeably for $g = 1$.

What this perhaps means is that I am still quite far from my goal of getting the family over $\Sigma \mathcal{G}$ linked via periodicity to the map $\mathcal{G} \longrightarrow U_{res}$.

September 29, 1986

The problem remains one in K-theory - how to relate the family over $\Sigma(\mathcal{G})$ to the map $\mathcal{G} \longrightarrow \mathcal{G}_{res}$ via periodicity.

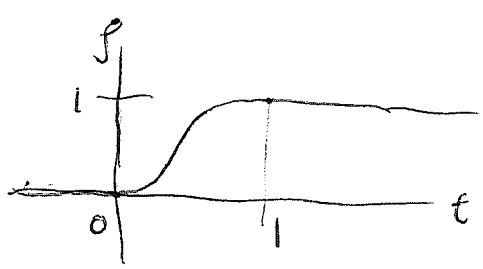
Recall that over $\Sigma(\mathcal{G})$ we have a canonical subbundle Π embedded in the trivial bundle $\widetilde{H_0 \oplus H_0}$ whose fibre at (g,t) is $\operatorname{Im}\begin{pmatrix}\sqrt{1-t}\\ \sqrt{t}\,g\end{pmatrix}$. We use this embedding to trivialize Π over $\mathcal{G}\times[0,1]$. Thus over $\mathcal{G}\times[0,1]$, Π becomes $\tilde{H}_0$ with the connection

$$\begin{pmatrix}\sqrt{1-t}\\ \sqrt{t}\,g\end{pmatrix}^*(dt\,\partial_t+\delta)\begin{pmatrix}\sqrt{1-t}\\ \sqrt{t}\,g\end{pmatrix} = dt\,\partial_t + \delta + t\,g^{-1}\delta g$$

and the family of Dirac operators

$$\begin{pmatrix}\sqrt{1-t}\\ \sqrt{t}\,g\end{pmatrix}^*\begin{pmatrix}\not{D} & 0\\ 0 & \not{D}\end{pmatrix}\begin{pmatrix}\sqrt{1-t}\\ \sqrt{t}\,g\end{pmatrix} = (1-t)\not{D} + t\,g^{-1}\not{D}\,g$$

It might be convenient to use a more refined map $\mathcal{G}\times[0,1] \longrightarrow \Sigma(\mathcal{G})$ using a different parametrization, i.e. $(g,t)\longmapsto(g,\rho(t))$ where $\rho(t)$:



Then we will have the connection & family

$$dt\,\partial_t + \delta + \rho\,g^{-1}\delta g\;,\qquad (1-\rho)\not{D} + \rho\,g^{-1}\not{D}\,g \;=\; \not{D} + \rho\,g^{-1}[\not{D},g]$$

In other words we have taken (up to the reparametrization ρ) the linear path joining the

Note that so far we are constructing essentially the transgressed cohomology class on $\mathcal{G}$, but it's not yet been deformed to a Lie class.

Next we want to form Dirac operators over $[0,1] \times M$ using Dirac on $[0,1]$. This depends on a choice of metric on $[0,1]$, and since we have already allowed for distortion by putting in ρ, we can take a constant metric. This gives the operator

$$h\partial_t + i\left(\not{D} + \rho(t) g^{-1}[\not{D}, g]\right)$$

where h is related to the length of $[0,1]$ for the metric. The above operator is to be combined with its adjoint to get the full Dirac. It also operates on the space of functions $f(t)$ from $[0,1]$ to H_0 satisfying the boundary condition

$$f(1) = g^{-1} f(0)$$

(So we are working in $L^2([0,1], H_0)$ with an unbdd operator whose domain incorporates the boundary condition.)

Now we have our family of Dirac operators, rather unbounded Fredholm operators, parametrized by $\mathcal{G}$. Graeme analyzes the index of this family as follows. What's important for the index are the kernel and cokernel and these can be found using the space V of solutions of

$$\left(h\partial_t + i\left(\not{D} + \rho(t) g^{-1}[\not{D}, g]\right)\right)\psi = 0$$

over $[0,1] \times M$. One then has a map

$$V \xrightarrow{\varepsilon} H_0 \oplus H_0$$ giving the boundary values and one has a Fredholm operator

$$H_0 \oplus V \xrightarrow{\Gamma_g - \varepsilon} H_0 \oplus H_0$$

whose kernel and cokernel can be identified with those of the Dirac operator + boundary conditions.

At this point one wants to bring in the "adiabatic approximation" which means letting h go to zero or to ∞. $h \to 0$ seems to mean letting the operator in the t directions become "classical" hence it should be like working over $\Sigma(\mathcal{G})$ and integrating over t the superconnection forms on $\Sigma(\mathcal{G})$ to get forms on $\mathcal{G}$. These should not be left-invariant. Thus $h \to 0$ seems to be wrong.

So we try $h \to \infty$ which obviously has the limiting equation $\partial_t \psi = 0$. In this case we have the map

$$H_0 \oplus H_0 \xrightarrow{\Gamma_g - \Delta} H_0 \oplus H_0$$

which is equivalent to

$$H_0 \xrightarrow{g-1} H_0 .$$

This is the formal identity

$$\text{Index}\,(\partial_t + \cdots) = \text{Index}\,(g-1).$$

However $g-1$ is not a Fredholm operator* so it is necessary to do the limit more carefully. *at least for $\mathcal{G}$ = group of gauge transf acting on $H_0 = L^2(M, S \otimes E_0)$.

Note that our $h \to \infty$ limit is essentially the same as a classical limit over M, i.e. replacing $\not{D}$ by $\varepsilon \not{D}$ and letting $\varepsilon \downarrow 0$. Recall that the superconnection forms on $\mathcal{G}$ associated to the family of operators on $S^1 \times M$ results from the Dirac operator on $Y \times S^1 \times M$ (here $Y \subset \mathcal{G}$) by letting the Y directions become classical. So the limit (adiabatic) of the superconnection forms we are investigating should be obtained by taking forms on $Y \times M$ associated to a family of operators on S^1 and integrating these forms over M. It would be interesting to see what sort of forms on $\mathcal{G}$ are obtained.

The first question is whether there is a natural family of Dirac operators on S^1 parametryzed by $\mathcal{G} \times M$. This is clear: one has over $\mathcal{G} \times M$ a vector bundle $pr_2^*(E_0)$ equipped with a canonical automorphism. One uses this autom as a clutching function to produce a vector bundle over $S^1 \times \mathcal{G} \times M$ and there is a canonical partial connection in the S^1-direction. In fact we can do all our old constructions working with E_0 and D rather than H_0 and $\not{D}$. So we get a canonical subbundle of $pr_3^*(E_0 \oplus E_0)$ over $S^1 \times \mathcal{G} \times M$; upon lifting to $[0,1] \times \mathcal{G} \times M$ it is isomorphic to $pr_3^*(E_0)$ with the connection

$$dt\,\partial_t + \delta + \rho g^{-1}\delta(g) + (1-\rho)D + \rho g^{-1} D . g$$

which is the linear path essentially between

$$\delta + D \qquad \text{and} \qquad g^{-1} . (\delta + D) . g$$

Thus we [redacted] have the data required to [redacted] construct superconnection forms of odd degree on

$\mathcal{G} \times M$.

In general a family of Diracs on the circle where only the gauge fields vary has a superconn. of the form

$$\nabla + X\sigma$$

where $X = \partial_t$ in a framing flat in the S^1-direction. The curvature is

$$\underbrace{X^2}_{\partial_t^2} + \underbrace{[\nabla, X]\sigma + \nabla^2}_{0\text{th order}}$$

and the superconnection form

$$(*) \qquad tr_\sigma \left(e^{X^2 + [\nabla, X]\sigma + \nabla^2} \right)$$

therefore involves taking the trace of an operator [redacted] of the [redacted] form

$$e^{\partial_t^2 - V(t)}$$

i.e. a heat operator associated to a 1-diml. Schroedinger operator. It doesn't seem [redacted] reasonable to expect there to be a nice formula for the form $(*)$.

September 30, 1986

Adiabatic approximation: Examples are the precessing top and the perturbation effects in celestial mechanics - the way the elliptical orbit of a planet precesses due to the gravitational attraction of other planets. Basically we have a small perturbation of a high frequency periodic motion. In a normal size time interval, we see many cycles of the unperturbed ~~periodic~~ motion, so we see a definite unperturbed periodic orbit at each time. Over a long time we see this periodic orbit varying as influenced by the perturbation.

For example, consider the top. The configuration space is $SO(3)$ and for the unperturbed the trajectories are geodesics, i.e. one sees the top spinning periodically about an axis. Thus the orbits are described by $\vec{\omega} \in \mathbb{R}^3 - (0)$, the angular rotation vector. The effect of gravity is to supply a torque and from the equation

$$\frac{d}{dt}\underbrace{L}_{\text{ang. momentum}} = \underbrace{\tau}_{\text{torque}}$$

one gets the precessing of the top.

I guess one can consider a time dependent perturbation in the adiabatic approximation, but to keep things simple, let's suppose that the perturbation is time independent. I would like to understand things quantum mechanically, where one has a free Hamiltonian H_0 and the perturbation V. It seems part of the picture that when

V is constant in time the effect is to produce "motion" in the ω eigenspace of H_0, where ω is the high frequency of the motion being considered. This is also true if V varies slowly in time.

One possibility is to consider the perturbation εV acting for the time $\frac{1}{\varepsilon}$. This means we want to know if

$$e^{+iTH_0}\, e^{-iT(H_0+\frac{1}{\varepsilon}V)} = e^{iTH_0}\, e^{-i(TH_0+V)}$$

has a limit as $T \to \infty$.

In the case of $\partial_t + A(t)$, there are two limits possible: $\partial_t + \varepsilon A(t)$ as either $\varepsilon \to \infty$ or $\varepsilon \to 0$. According to Graeme $\varepsilon \to \infty$ is the adiabatic limit, and he claims to be able to link the limiting Fredholm operator with the Toeplitz operator associated to g. ~~[illegible]~~

The limit as $\varepsilon \to \infty$ tends to concentrate things near the zero modes of $A(t)$. Evidently he has some feeling for the space V of solutions of $(\partial_t + \varepsilon A(t))\psi = 0$ over $[0,1]\times M$ in the limit as $\varepsilon \to \infty$, or really the boundary value map $V \longrightarrow H_0 \times H_0$.

I somehow favor $\varepsilon \to 0$. For the future we should note that each of these limits is a classical limit. Thus if I take the family of operators parametrized by $\mathcal{G}$ on $S^1\times M$ and take the superconnection forms, since these are already obtained from a total Dirac on $\mathcal{G}\times S^1\times M$ by letting the

G-directions go classical, if I now let the S^1 direction go classical $(\varepsilon \to \infty)$ I should be getting the superconnection forms for the family of operators on M param. by $G \times S^1$, integrated over S^1 to get forms on G. These are not left-invariant since if we take the limit as the M-directions become classical, we get forms on G which are not left-invariant.

Similarly if I take the supercon. forms for the family of ops on $S^1 \times M$ param. by G and take the $\varepsilon \to 0$ limit, this amounts to letting the M directions becoming classical, which should give supercom. forms for the family of ops. on S^1 param. by $G \times M$, integrated over M to yield forms on G. These forms are also not left, since we could let the S^1 become classical ~~obtain~~ the same forms as above, namely the character forms over $G \times S^1 \times M$ integrated over $S^1 \times M$ forms on G.

Conclude: It seems as if the adiabatic limit is not going to provide the means to transgress the index of the family param. by $S^1 \times G$ into the nice equivariant family represented by $G \to I_{res}$.

Needed perhaps is a way to bring in the continuous cohomology of G. After all, the reason that the index classes on BG transgress to left invariant forms on G is that they vanish in the continuous cohomology.