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July 11, 1986

Let's return to earlier work on linking superconnections with the Grassmannian graph idea. One starts with vector bundles E^0, E^1 with connections D^i and a map $T: E^0 \longrightarrow E^1$. We assume E^0, E^1 come with inner products preserved by the connections. The graph of T, is a subbundle $\Gamma_T \subset E^0 \oplus E^1$, so it inherits a connection by orthogonal projection. To compute the curvature it is convenient to work also with the complement $(\Gamma_T)^\perp = \operatorname{Im}\left\{\begin{pmatrix} -T^* \\ 1 \end{pmatrix} : E^1 \longrightarrow E^0 \oplus E^1\right\}$.

We use the isomorphism

$$1 + L = \begin{pmatrix} 1 & -T^* \\ T & 1 \end{pmatrix} : E^0 \oplus E^1 \xrightarrow{\sim} \Gamma_T \oplus \Gamma_T^\perp = E^0 \oplus E^1$$

to do the calculation. The connection $D = D^0 \oplus D^1$ is to be projected on Γ_T and on $\Gamma_T^\perp$. We can do this by using $1+L$ and then projecting on E^0 and on E^1. Thus we want the connection

$$D + A = \text{diagonal part of } \frac{1}{1+L} D (1+L)$$

$$= D + \frac{1}{1+L}[D, L] = D + \frac{1}{1-L^2}(1-L)[D,L].$$

We conclude

$$\boxed{D + A = D - \frac{1}{1-L^2} L[D,L]}$$

and now compute the curvature

$$(D+A)^2 = D^2 + \frac{1}{1-L^2}\left(-[D,L]L - \cancel{L[D,L]}\right)\frac{1}{1-L^2}L[D,L] - \frac{1}{1-L^2}[D,L]^2$$
$$- \frac{1}{1-L^2}L[D^2, L] + \cancel{\frac{1}{1-L^2}L[D,L]\frac{1}{1-L^2}L[D,L]}$$

$$= \underbrace{D^2 - \frac{1}{1-L^2} L[D^2, L]}_{\substack{\text{diagonal part of } \frac{1}{1+L} D^2 (1+L) \\ \| \\ \frac{1}{1-L^2}\left(D^2 - LD^2L\right)}} - \left(\frac{1}{1-L^2}[D, L]\right)^2$$

(i.e. D^2 projected onto Γ_T and $\Gamma_T^\perp$)

This formula was obtained in June 84 page 39

July 18, 1986

Problem: Link superconnections with Grassmannian graph.

Suppose we have a map $T: E^0 \longrightarrow E^1$ between vector bundles; we know it determines a K-class supported in the singular set of T, and this K-class has a character with the same support. We want representatives for this character cohomology class as diff'l forms or currents. Grassmannian graph produces cycles representing the character in some sense.

The first step should be to assume E^0, E^1 are trivial, in which case we have found some sort of link between the superconnection character and the Grassmannian graph construction.

In this case one has a family $y \longmapsto T_y : V^0 \longrightarrow V^1$ of matrices over Y, and we have a cohomology class supported in the singular set. We suppose V^0 and V^1 are of the same dimension n.

The universal situation is where $Y = \mathrm{Hom}(V^0, V^1)$ and the singular set is the 'determinant locus', i.e. a hypersurface. The graph construction gives an embedding of $\mathrm{Hom}(V^0, V^1)$ as an open cell in $Gr_n(V^0 \oplus V^1)$ and the closure of the "determinant locus" is the complement of the open cell consisting of subspaces complementary to V^0. Let $Z \subset Gr_n(V)$ be the hypersurface of subspaces intersecting V^0 non-trivially. It's clear from the fact that the Grassmannian has only even dim'l cohomology that any coh. class (≠ degree 0) on $Gr_n(V)$ can be uniquely obtained from a class supported in Z.

So it would seem that what we have to do take the character forms on the Grassmannian and deform them into currents supported in Z, or maybe just forms concentrated near Z. What we have available is an action of the multiplicative group $\mathbb{C}^\times$ on the Grassmannian coming from the splitting $V = V^0 \oplus V^1$. This is the rescaling transformation $T \mapsto tT$. It seems we have to use this if we want the construction to be equivariant with respect to $U(V^0) \times U(V^1)$.

It might be useful to bring in the nice desingularization of Z which I learned from Bott, namely, the space $\tilde{Z}$ of pairs (L, W), where L is a line in V^0, and W is an n-diml. subspace containing L. This is a Grassmannian bundle over $\mathbb{P}(V^0)$ and so is non-singular; on the other hand the map $\tilde{Z} \longrightarrow Z$, $(L, W) \mapsto W$ is bijective over the open set of W such that $W \cap V^0$ is 1-diml.

The next step might be to run thru Grass graph, which means that you want the cycles MacPherson + co. attach which are supported in Z.

July 22, 1986

Let $V^0, V^1 \cong \mathbb{C}^n$, $H = \mathrm{Hom}(V^0, V^1)$, $Gr = Gr_n(V^0 \oplus V^1)$; we have the graph map

$$\begin{array}{ccc} H & \longrightarrow & H \times Gr \\ T & \longmapsto & (T, \Gamma_{tT}) \end{array}$$

for each $t \in \mathbb{C}$. The image is a cycle in $H \times Gr$ and we want the limit of this cycle as $t \to \infty$. We therefore want to form the closure of

$$\begin{array}{ccc} \mathbb{C} \times H & \hookrightarrow & \mathbb{P}^1 \times H \times Gr \\ (t, T) & \longmapsto & (t, T, \Gamma_{tT}) \end{array}$$

and to take the fibre of this closure over $t = \infty$. Put another we want to find all pairs (T, W) with $T \in H$, $W \in Gr$ such that $\exists$ sequences t_n, T_n with $t_n \to \infty$, $T_n \to T$, $\Gamma_{t_n T_n} \to W$. Call the set of these pairs Q.

Note that

$$\Gamma_{t_n T_n} = \mathrm{Im}\begin{pmatrix} 1 \\ t_n T_n \end{pmatrix} = \mathrm{Im}\begin{pmatrix} \frac{1}{t_n} \\ T_n \end{pmatrix}$$

and
$$\begin{pmatrix} \frac{1}{t_n} \\ T_n \end{pmatrix} \longrightarrow \begin{pmatrix} 0 \\ T \end{pmatrix}$$

hence we see that for any $(T, W) \in Q$ one has

(1) $$\mathrm{Im}\begin{pmatrix} 0 \\ T \end{pmatrix} \subset W$$

Similarly
$$\Gamma_{t_n T_n}^{\perp} = \mathrm{Im}\begin{pmatrix} -\bar{t}_n T_n^* \\ 1 \end{pmatrix} = \mathrm{Im}\begin{pmatrix} -T_n^* \\ \frac{1}{\bar{t}_n} \end{pmatrix}$$

and so we also have for any pair in Q

② $\quad \mathrm{Im}\begin{pmatrix} T^* \\ 0 \end{pmatrix} \subset W^\perp$

Now let's see if these two conditions characterize elements of Q.

Recall that any $W \in Gr$ has "eigenvalues" relative to the splitting $V^0 \oplus V^1$.

$$\begin{array}{lcccccc}
V^1 & : & (W^\perp \cap V^0) & \oplus & (\qquad) & \oplus & (W \cap V^1) \\
 & & & & \uparrow S & & \\
V^0 & : & \underbrace{(W \cap V^0)} & \oplus & \underbrace{(\cdots\cdots)} & \oplus & \underbrace{(W^\perp \cap V^0)} \\
\lambda & : & 0 & & 0<\lambda<\infty & & \infty
\end{array}$$

The conditions ① and ② say

$$\mathrm{Im}\, T \subset W \cap V^1 \quad , \quad \mathrm{Im}\, T^* \subset W^\perp \cap V^0$$

i.e. the non-zero part of T is entirely contained in $\lambda = \infty$ 'eigenspace'. This means we can split

$$\begin{aligned} V^0 &= \mathrm{Ker}\, T \oplus \mathrm{Im}\, T^* \\ V^1 &= \mathrm{Ker}\, T^* \oplus \mathrm{Im}\, T \end{aligned} \qquad (\text{i.e.} \quad V = \mathrm{Ker}\, L \oplus \mathrm{Im}\, L)$$

and the $V = W \oplus W^\perp$ splitting is compatible. This reduces the situation to the direct sum of the cases where T is zero and where T is invertible. These cases are easy.

We can reformulate ② as

$$W \subset \mathrm{Ker}\,(T \ \ 0) : V^0 \oplus V^1 \longrightarrow V^1$$

i.e. $\quad W \longrightarrow V^0 \xrightarrow{T} V^1 \quad$ is zero

This is clearly satisfied for $(T, W) \in Q$ because if $\quad v_n \oplus t_n T_n v_n \longrightarrow v' \oplus v'' \in W$

then $\quad v_n \longrightarrow v' \, , \qquad t_n T_n v_n \longrightarrow v''$

$$\Downarrow \qquad\qquad\qquad \Downarrow$$

$$T_n \to T \Rightarrow T_n v_n \to T v' , \qquad T_n v_n \longrightarrow 0$$

so $Tv' = 0$, for $v' \oplus v'' \in W$. Thus we can prove the following.

<u>Proposition:</u> a pair $(T, W) \in \mathrm{Hom}(V^0, V') \times Gr_n(V^0 \oplus V')$ is the limit of a sequence $(T_n, \Gamma_{t_n T_n})$ with $t_n \to \infty$ iff [redacted]

$$0 \oplus \mathrm{Im}\, T \subset W \subset \mathrm{Ker}(T) \oplus V'.$$

[redacted] This result implies that the fibre of Q over a T of rank r is a Grassmannian $\simeq G_{n-r}(\mathbb{C}^{2(n-r)})$. Need the dimension of the variety of matrices of rank r: it fibres [redacted] over the Grassmannian of s dimensional subspaces $(s = n-r)$, and the fibre is the space of embeddings of $V^0/\mathrm{Ker}\, T$ into V' which is open in $\mathrm{Hom}(V^0/\mathrm{Ker}\, T, V')$. This gives the dimension

$$s(n-s) + (n-s)n = n^2 - s^2.$$

Thus the matrices of corank s form a variety of codimension s^2.

Now consider pairs (T, W) in Q with T of corank s. This subvariety has dimension

$$n^2 - s^2 + s^2 = n^2$$

Is it possible that we get different irreducible components of Q for different s?

Let's try to resolve the subvariety of Q consisting of (T, W) with corank$(T) \geq s$. We consider partial flags

$$0 \overset{r}{\subset} \overbrace{W_1 \subset W}^{s} \overbrace{\subset W_2}^{s} \overbrace{\subset V}^{r}$$

such that $W_1 \overset{s}{\subset} V^0$, $V^1 \overset{s}{\subset} W_2$ [redacted] together with $T : V/W_2 \longrightarrow W_1$.

Better: We consider T of rank $\leq r$ (or corank $\geq s$) where $T : V^0 \to V^1$. Consider triples (T, W^0, W^1) where $W^0 \in Gr_s(V^0)$, $W^1 \in Gr_r(V^1)$ and $W^0 \subset \underbrace{\operatorname{Ker} T}_{\dim \geq s}$, $\underbrace{\operatorname{Im} T}_{\dim \leq r} \subset W^1$.

The space of the triples fibres over the product of these Grassmannians and the fibre over W^0, W^1 is $\operatorname{Hom}(V^0/W^0, W^1)$, so the dimension is

$$2s(n-s) + (n-s)(n-s) = (n+s)(n-s) = n^2 - s^2.$$

This variety of triples desingularizes the variety of matrices of rank $\leq r$. [redacted]

To obtain the corresponding non-singular variety up in Q we consider quadruples (T, W^0, W^1, W) where $W \in Gr_n(V)$

$$0 \oplus W^1 \subset W \subset W^0 \oplus V^1$$

Conclusion: [redacted] The variety Q of limits of $(T_n, \Gamma_{t_n T_n})$ with $t_n \to \infty$ is of dimension n^2 and is a union of irreducible subvarieties of dim n^2.

(Irreducibility comes from the fact a connected non singular variety is irreducible.)

Question: Is the variety Q a complete intersection?

July 23, 1986

Review of Bott-Chern: $s : \widetilde{\mathbb{C}}^n \longrightarrow E$ holom. frame

$N = s^* s \qquad (N_{ij} = \langle s_i | s_j \rangle)$

The good connection is

$$D = \underbrace{N^{-1} . d' . N}_{D'} + \underbrace{d''}_{D''} = d + \underbrace{N^{-1} d' N}_{\theta}$$

with curvature

$$D^2 = [D'', D'] = d''\theta$$

If N varies wrt t, then

$$\dot{D} = -N^{-1}\dot{N}N^{-1}.d'.N + N^{-1}.d.NN^{-1}\dot{N}$$
$$= [D', J] \qquad \text{where} \quad J = N^{-1}\dot{N}.$$

Thus

$$\partial_t \operatorname{tr}(e^{D^2}) = \operatorname{tr}(e^{D^2}[D, \dot{D}]) = d \operatorname{tr}(e^{D^2}\dot{D})$$
$$= d \operatorname{tr}(e^{D^2}[D', J]) = d d' \operatorname{tr}(e^{D^2} J)$$

since $\quad 0 = [D, D^2] = \underbrace{[D', D^2]}_{2,1} + \underbrace{[D'', D^2]}_{1,2} \Rightarrow [D', D^2] = [D'', D^2] = 0.$

Rmk: From the transgression viewpoint we need

$$\partial_t \operatorname{tr}(e^{D^2}) = d\eta$$

for some η. There are two possibilities in the above setup

$$\eta = \operatorname{tr}(e^{D^2}\dot{D}) = \operatorname{tr}(e^{D^2}[D', J]) = d' \operatorname{tr}(e^{D^2} J)$$

and $\quad \eta = -d'' \operatorname{tr}(e^{D^2} J) = -\operatorname{tr}(e^{D^2} d'' J)$

Next let us look at the holomorphic vector bundle over $H = \mathrm{Hom}(V^0, V^1)$ whose fibre at T is $\Gamma_T = \mathrm{Im}\begin{pmatrix}1\\T\end{pmatrix}$. Then

$$s = \begin{pmatrix}1\\T\end{pmatrix} \qquad s^* = \begin{pmatrix}T^* & 1\end{pmatrix}$$

$$N = 1 + T^*T$$

$$\theta = N^{-1}d'N = \frac{1}{1+T^*T}\,T^*dT$$

$$D^2 = \Omega = d''\theta = -\frac{1}{1+T^*T}\,dT^*T\,\frac{1}{1+T^*T}\,T^*dT + \frac{1}{1+T^*T}\,dT^*dT$$

$$= \frac{1}{1+T^*T}\,dT^*\,\frac{1}{1+TT^*}\,dT$$

(It's not immediate from this formula that $d''\Omega = 0$, however, one can rewrite it

$$\frac{1}{1+T^*T}\,dT^*T\,\frac{1}{1+T^*T}\,T^{-1}dT = -d''\left(\frac{1}{1+T^*T}\right)T^{-1}dT \quad)$$

Next put in the parameter t; this means we have

$$s = \begin{pmatrix}1\\tT\end{pmatrix} \qquad s^* = \begin{pmatrix}tT^* & 1\end{pmatrix} \qquad t \text{ real}$$

$$N = 1 + t^2T^*T$$

Since the rest of the formulas depend on N, it means we get the same forms from changing the metric on $V^0 \oplus V^1$.

$$J = N^{-1}\dot{N} = \frac{1}{1+t^2T^*T}\,2t\,T^*T$$

$$= \frac{1}{1+t^2T^*T}\,2t\left(T^*T + \frac{1}{t^2}\right) - \frac{1}{1+t^2T^*T}\,\frac{2}{t}$$

$$= \frac{2}{t}\left(1 - \frac{1}{1+t^2T^*T}\right)$$

Thus

$$d''J = \frac{2}{t}\,\frac{1}{1+t^2T^*T}\,t^2\,dT^*T\,\frac{1}{1+t^2T^*T}$$

$$= 2t\,\frac{1}{1+t^2T^*T}\,dT^*\,T\,\frac{1}{1+t^2T^*T}$$

$$= 2t\,\frac{1}{1+t^2T^*T}\,dT^*\,\frac{1}{1+t^2TT^*}\,T$$

On the other hand

$$\theta = \frac{1}{1+t^2T^*T}\,t^2T^*dT$$

$$\dot\theta = -\frac{1}{1+t^2T^*T}\,2t\,T^*T\,\frac{1}{1+t^2T^*T}\,t^2T^*dT + \frac{1}{1+t^2T^*T}\,2t\,T^*dT$$

$$= \frac{1}{(1+t^2T^*T)^2}\,2t\,T^*dT$$

These are roughly the same.

Let's now go to the superconnection approach

$$\partial_t\,tr_s\left\{e^{u(L^2+dL)}\right\} = d\,tr_s\left(e^{u(L^2+dL)}\,\dot L\,u\right)$$

Taking the transform

$$\int_0^\infty tr_s\left(e^{u(L^2+dL)} - e^{u(L^2)}\right)e^{-\lambda u}\,\frac{du}{u}$$

$$= -\,tr_s\log\left(1-\frac{1}{\lambda-L^2}\,dL\right)$$

$$= \sum_{k=1}^{\infty}\frac{1}{k}\,tr_s\left(\frac{1}{\lambda-L^2}\,dL\right)^k$$

It seems then that

$$\boxed{\partial_t\left\{\frac{1}{k}\,tr_s\left(\frac{1}{\lambda-L^2}\,dL\right)^k\right\} = d\,tr_s\left\{\left(\frac{1}{\lambda-L^2}\,dL\right)^{k-1}\dot L\right\}}$$

Sign problem: $L = \begin{pmatrix} 0 & -T^* \\ T & 0 \end{pmatrix}$ so that

$D^2 = -\left(\frac{1}{1-L^2}\, dL\right)^2$ for the sub and quotient bundle on the Grassmannian identified with $\tilde{V}^0, \tilde{V}^1$ via $1+L$.

Cohomologically

$$\mathrm{tr}_s\, e^{u(L^2+dL)} \sim \sum{}' u^k\, ch_k$$

so that

$$\int_0^\infty \mathrm{tr}_s\left(e^{u(L^2+dL)}\right) e^{-\lambda u}\frac{du}{u} \sim \sum{}'\left(\int_0^\infty e^{-\lambda u} u^k \frac{du}{u}\right) ch_k$$

$$= \sum \frac{(k-1)!}{\lambda^k}\, ch_k$$

Thus one expects

$$\frac{1}{2k}\,\mathrm{tr}_s\left(\frac{1}{\lambda-L^2}\, dL\right)^{2k} \sim \frac{(k-1)!}{\lambda^k}\, ch_k$$

which is off by a sign $(-1)^k$. These calculations are heuristic at best since the space of L is contractible.

Let's describe what's happening with Connes-Moscovici transgression. Although they work with superconnections I think the whole business is cleaner with the graph forms.

July 24, 1986

Formulas in Connes-Moscovici transgression paper. We have a natural map

$$\tilde{\Gamma} : \text{Susp } GL_n(\mathbb{C}) \longrightarrow Gr_n(\mathbb{C}^{2n})$$

which assigns to $0 \le t \le \infty$ and $T \in GL_n(\mathbb{C})$ the graph $\Gamma_{tT} = \text{Im}\begin{pmatrix} 1 \\ tT \end{pmatrix}$. Pulling back the character form ch_k on the Grassmannian and integrating over the suspension parameter t gives a closed $2k-1$ form on $GL_n(\mathbb{C})$. We now do this explicitly.

We know that

$$\Gamma^* ch_k = \frac{(-1)^k}{2(k!)} \, tr_s \left(\frac{1}{1-L^2} dL \right)^{2k} \qquad L = \begin{pmatrix} 0 & -T^* \\ T & 0 \end{pmatrix}$$

$$= \frac{1}{2(k!)} \, tr_s \left(\frac{1}{1+\phi^2} d\phi \right)^{2k} \qquad \phi = \frac{1}{i} L$$

so

$$\tilde{\Gamma}^* ch_k = \frac{1}{2(k!)} \, tr_s \left(\frac{1}{1+t^2\phi^2} (dt\,\phi + t\,d\phi) \right)^{2k}$$

Now

$$tr_s (X+Y)^{2k} = tr_s(X^{2k}) + tr_s\left(YX^{2k-1} + XYX^{2k-2} + \cdots\right) + O(Y^2)$$

and notice that $tr_s(XZ) = tr_s(ZX)$ if X, Z are odd both as forms and endomorphisms. (There is a Pandora's box of signs here, which I have encountered earlier, due to the different [redacted] algebra structures on $\Omega(M, \text{End}\,E)$, when E is $\mathbb{Z}_2$-graded. I think this explains the sign problem on p. 12. For example assuming $L^2 = 1$, we get in the superformalism

$$tr_s (dL)^2 = tr_s \left\{ \left(dT \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} - dT^* \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \right) \cdot \left(dT \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} - dT^* \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \right) \right\}$$

$$= \mathrm{tr}_s \left\{ dT^* dT \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + dT dT^* \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$$

(note sign $dT \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} (dT^*) \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = - dT dT^* \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$)

$$= \mathrm{tr}(dT^* dT) - \mathrm{tr}(dT\, dT^*) = 2\,\mathrm{tr}(dT^* dT).\;)$$

In any case let's persist with the calculation in matrix forms where it is true that

$$\mathrm{tr}_s(XZ) = \mathrm{tr}_s(ZX)$$

when X, Z are odd both as forms and endomorphisms. Then we have

$$\mathrm{tr}_s (X+Y)^{2k} = \mathrm{tr}_s(X^{2k}) + 2k\, \mathrm{tr}(YX^{2k-1}) + O(Y^2)$$

when X, Y are odd both as forms and endos.

So we have

$$\tilde{\Gamma}^*(ch_k) = \frac{1}{2(k!)} \mathrm{tr}_s \left(\frac{1}{1+t^2 \not{D}^2} t\, d\not{D} \right)^{2k}$$

$$+ dt \frac{1}{(k-1)!} \mathrm{tr}_s \left(\not{D} \left(\frac{1}{1+t^2\not{D}^2} t\, d\not{D} \right)^{2k-1} \frac{1}{1+t^2\not{D}^2} \right)$$

Now integrate over $0 < t < \infty$ to obtain

$$\frac{1}{(k-1)!} \int_0^\infty \mathrm{tr}_s \left\{ \not{D} \left(\frac{1}{1+t^2\not{D}^2} t\, d\not{D} \right)^{2k-1} \frac{t}{1+t^2\not{D}^2} \right\} \frac{dt}{t}$$

or putting $t = \frac{1}{m}$ we get

$$\boxed{\frac{1}{(k-1)!} \int_0^\infty \mathrm{tr}_s \left\{ \not{D} \left(\frac{1}{m^2+\not{D}^2} d\not{D} \right)^{2k-1} \frac{1}{m^2+\not{D}^2} \right\} m^{2k} \frac{dm}{m}}$$

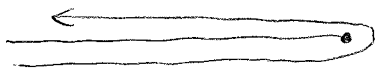
Questions + Problems. Go back to

$$\int_0^\infty tr_s\left\{e^{u(L^2+dL)}\right\}_{(2k)} e^{-\lambda u}\frac{du}{u} = \frac{1}{2k}\, tr_s\left(\frac{1}{\lambda - L^2}\, dL\right)^{2k}$$

and note that the right side is defined not just on $H = \mathrm{Hom}(V^0, V^1)$ but the whole Grass. A natural question is whether the supercharacter forms are defined on the whole Grassmannian. A possible approach would be to use Laplace transform inversion:

$$\frac{1}{u}\, tr_s\left\{e^{u(L^2+dL)}\right\}_{(2k)} = \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} e^{\lambda u}\, \frac{1}{2k}\, tr_s\left(\frac{1}{\lambda - L^2}\, dL\right)^{2k} d\lambda$$

Actually since $L^2 \leq 0$ the contour can be shifted to go around the negative real axis:



And note that it is reasonable to expect the Chern character form

$$ch_k^\lambda = \frac{1}{2\cdot k!}\, tr_s\left(\frac{\sqrt{\lambda}}{\lambda - L^2}\, dL\right)^{2k} = \frac{\lambda^k}{(k-1)!}\,\frac{1}{2k}\, tr_s\left(\frac{1}{\lambda - L^2}\, dL\right)^{2k}$$

to have at most algebraic singularities as $\lambda \to \infty$. If this is the case then the integral on the right will converge for $u > 0$.

July 25, 1986

Notation: $V^0, V^1 \simeq \mathbb{C}^n$, $Gr = Gr_n(V^0 \oplus V^1)$, φ_t is the automorphism on Gr induced by $\begin{pmatrix} 1 & 0 \\ 0 & t \end{pmatrix}$ on $V^0 \oplus V^1$, ch_k = character form of degree $2k$ on Gr. We are interested in the behavior of the form $\varphi_t^*(ch_k)$ as $t \longrightarrow \infty$. We'd like to prove the limit of this form exists as a closed current, and to identify this current with something associated to Grassmannian graph theory.

Consider φ_t as a correspondence. Let $Z_t = \text{graph}(\varphi_t)$
$= (id, \varphi_t)(Gr)$

$$\begin{tikzcd} Z_t \arrow[r, hook, "i"] & Gr \times Gr \arrow[r, "pr_2"] \arrow[d, "pr_1"] & Gr \\ & Gr \arrow[ul, "\Gamma", "\sim"'] & \end{tikzcd}$$

In cohomology one has for any map φ

$$\varphi^*(\alpha) = \Gamma^* pr_2^*(\alpha) \qquad \Gamma = (id, \varphi)$$

$$= pr_{1*} \Gamma_* \Gamma^* pr_2^*(\alpha)$$

$$= pr_{1*}\left(\Gamma_* 1 \cdot pr_2^*(\alpha)\right)$$

Now the point I want to make is that this same formula holds on the level of differential forms, however $\Gamma_* 1$ is a current. In general a cycle in the sense of alg. geometry determines a closed current, because the non-singular subset is oriented naturally and the singular subset has real codimension ≤ 2. In any case [redacted] currents push forward.

So

$$\varphi_t^*(\alpha) = pr_{1*}\left([Z_t]\cdot pr_2^*(\alpha)\right)$$

where $[Z_t]$ denotes the δ fn. current on the subvariety Z_t.

Now the idea will be that since Z_t approaches the cycle Z_∞ as $t \longrightarrow \infty$ we will have by continuity of pr_{1*} and $\cdot pr_2^*(\alpha)$ a ~~[illegible]~~ currents that

$$\lim_{t\to\infty} \varphi_t^*(\alpha) = pr_{1*}\left([Z_\infty]\cdot pr_2^*(\alpha)\right)$$

where now the right side is a current, since Z_∞ is no longer "transverse" to the fibres of pr_1.

But we have an explicit description of Z_∞ as a union of subvarieties with explicit resolutions one for each ~~[illegible]~~ rank. So therefore it might be possible to see exactly what the limiting currents are.

July 26, 1986

Recall the nature of $Z_\infty \subset Gr \times Gr$. It consists of pairs (Γ, W) such that

$$\mathrm{Im}(\Gamma) \subset W \cap V^1 \qquad \text{where } \mathrm{Im}(\Gamma) = \mathrm{Im}(\Gamma \to V^1)$$

$$\mathrm{Im}(W \to V^0) \subset \mathrm{Ker}(\Gamma) \qquad \mathrm{Ker}(\Gamma) = \Gamma \cap V^0$$

i.e.

$$\Gamma + V^0 \subset V^0 \oplus W \cap V^1$$

$$W + V^1 \subset \Gamma \cap V^0 + V^1$$

Alternatively

$$\mathrm{Im}\,\Gamma \subset W \subset \mathrm{Ker}\,\Gamma \oplus V^1$$

I guess it is important to note that the conditions guaranteeing that $(\Gamma, W) \in Z_\infty$ depends only on

$$\varphi_\infty(\Gamma) = \lim_{t\to\infty} \varphi_t(\Gamma) = \underbrace{\mathrm{Ker}\,\Gamma}_{\Gamma \cap V^0} \oplus \underbrace{\mathrm{Im}\,\Gamma}_{(\Gamma + V^0)\cap V^1}$$

and

$$\varphi_0(W) = \lim_{t\to 0} \varphi_t(W) = \underbrace{\mathrm{Im}\,W^t}_{(W+V^1)\cap V^0} \oplus \underbrace{\mathrm{Ker}\,W^t}_{W \cap V^1}$$

and says

$$\begin{cases} \mathrm{Im}\,\Gamma \subset \mathrm{Ker}\,W^t \\ \mathrm{Im}\,W^t \subset \mathrm{Ker}\,\Gamma \end{cases}$$

(sort of like: $\Gamma \cdot W^t = 0$, $W^t \cdot \Gamma = 0$)

Next let's recall how we can resolve Z_∞. The idea is to introduce the space of (K, I, Γ, W) where $K \subset V^0$, $I \subset V^1$ are of complementary dimensions and Γ, W satisfy

$$\mathrm{Im}\,W^t \subset K \subset \mathrm{Ker}\,\Gamma, \qquad \mathrm{Im}\,\Gamma \subset I \subset \mathrm{Ker}\,W^t$$

The point is that $\{(K,I)\}$ is a product of Grassmannians and that $\{(K,I,\Gamma)\}$ is a Grassmannian bundle over the former, the fibre over (K,I) being $Gr_r(V^0/K \oplus I)$ where

$$r = \dim(V^0/K) = \dim(I)$$

$$\{\Gamma \mid K \subset \operatorname{Ker}\Gamma, \ \operatorname{Im}\Gamma \subset I\} = Gr_r(V^0/K \oplus I)$$

$$\Updownarrow$$

$$K \subset \Gamma \subset V^0 \oplus I$$

similarly

$$\{W \mid I \subset \operatorname{Ker} W^t, \ \operatorname{Im} W^t \subset K\} = Gr_s(K \oplus V^1/I)$$

$$\Updownarrow \qquad s = n - r$$

$$I \subset W \subset K \oplus V^1$$

The total dimension of the stratum with $r = \dim(I) = \operatorname{codim} K$ is

$$2r(n-r) + r^2 + (n-r)^2 = n^2.$$

We can think of $\{(K,I)\} = \coprod_{s+r=n} Gr_s(V^0) \times Gr_r(V^1)$ as being the fixpoint set for the action of φ_t.

Let's review what we are doing. Z_t = the cycle in $Gr \times Gr$ which is the graph of φ_t, and $Z_\infty = \lim_{t \to \infty} Z_t$. Think of Z_∞ as a correspondence from Gr to itself. Then we have expressed this correspondence as a sum of simpler correspondences, one for each rank r, $0 \le r \le n$.

We now look at the r-th correspondence more closely. Let us consider (K, I, Γ, W) where $K \in Gr_s(V^0)$, $I \in Gr_r(V')$, $\Gamma, W \in Gr_n(V)$ and

$$K \subset \Gamma \subset V^0 \oplus I$$

$$I \subset W \subset K \oplus V'$$

Then these quadruples form a non-singular projective variety which I will denote $\{(K, I, \Gamma, W)\}$. This variety contains the open dense set where $\Gamma/K \subset V^0/K \oplus I$ is the graph of invertible transf. $V^0/K \xrightarrow{\sim} I$, and where $W/I \subset K \oplus V'/I$ is the graph of an isom. $V'/I \simeq K$.

What I should be doing is to identify the non-singular part of Z_∞ and show it is a disjoint union of irreducible components one for each r. What is the condition that $(\Gamma, W) \in Z_\infty$ comes from a unique (K, I)?

$$(W+V') \cap V^0 \subset K \subset \Gamma \cap V^0 \quad ; \quad (\Gamma + V^0) \cap V' \subset I \subset W \cap V'$$

Thus the condition is that

$$\operatorname{Im}(W^t) = \operatorname{Ker}(\Gamma) \quad , \quad \operatorname{Im} \Gamma = \operatorname{Ker}(W^t).$$

The point is that under specialization the Ker jumps up and the Im jumps down. Another point is that

$$\dim \operatorname{Ker} + \dim \operatorname{Im} = n$$

so that $\boxed{\operatorname{Im}(W^t) = \operatorname{Ker}(\Gamma) \iff \operatorname{Im}\Gamma = \operatorname{Ker}(W^t)}$

So we conclude that the non-singular ("good" at least) part of Z_∞ consists of (Γ, W) satisfying the above boxed condition, and this set splits into irreducible components, one for each $0 \le r \le n$. Moreover we have a natural desingularization of the closure of each component given by $\{(K, I, \Gamma, W)\}$.

We write

$$Z_\infty = \bigcup Z_{\infty,r}$$

where $Z_{\infty,r}$ consists of (Γ, W) with

$$\dim \operatorname{Im} \Gamma \leq r \quad \text{and} \quad \dim \operatorname{Im} W^t \leq s.$$

Maybe better to say

$$\dim(\operatorname{Im} \Gamma) \leq r \leq \dim(\operatorname{Ker} W^t)$$

$$\Updownarrow$$

$$\dim(\operatorname{Im} W^t) \leq s \leq \dim(\operatorname{Ker} \Gamma)$$

Now I should look carefully at the cohomological picture. We know that the cycle Z_∞ in $Gr \times Gr$, when viewed as a correspondence represents the identity map on cohomology. We therefore have decomposed the identity map into sum of parts one for each rank r, $0 \leq r \leq n$.

Next let us look carefully at $Z_{\infty,r}$ which we have seen is resolved by the non-singular variety of (K, I, Γ, W) with $(K, I) \in Gr_s(V^0) \times Gr_r(V^1)$.

Consider the diagram

$$\begin{tikzcd}
\{(K,I,\Gamma,W)\} \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{cart}"] & \{(K,I,W)\} \arrow[r] \arrow[d, "\text{rel. dim } s^2"] & \{W\} = Gr \\
\{(K,I,\Gamma)\} \arrow[r] \arrow[d, "\substack{\text{Desing. of cycle} \\ \text{of codim } s^2}"] & \{(K,I)\} & \\
Gr = \{\Gamma\} & &
\end{tikzcd}$$

This decomposes the correspondence $Z_{s,r}$ into a composition of two correspondences. In fact as the composition of a correspondence

$$\begin{array}{ccc} \{(K, I, \Gamma)\} & \longrightarrow & Gr_s(V^0) \times Gr_r(V^1) \\ \downarrow & & \\ Gr_r(V) & & \end{array}$$

and its transpose. Maybe it would be better to write this differently

$$\begin{array}{ccc} \{(K, I, W)\} & \xrightarrow{\text{rel. codim } r^2} & Gr_r(V) \\ \Big\downarrow{\scriptstyle \text{rel dim } s^2} & & \\ Gr_s(V^0) \times Gr_r(V^1) & & \end{array}$$

Notice that the horizontal arrow is a resolution of the cycle [redacted] of W with $\dim(\mathrm{Ker}\, W^{\dagger}) \geq k$. In other words, we have something like the map which associates to [redacted] a matrix of rank r its kernel and cokernel.

A natural question is what this map does on cohomology. Is it possible to look at this question in the stable range? Seems so, because the Gysin maps involved have degrees $\pm s^2$ and $s = \dim$ [redacted] $K = \dim V^1/I$ stay finite as $n \to \infty$.

So it appears that we have interesting maps

$$H^*(BU_s) \otimes H^*(BU_s) \underset{\Phi_*}{\overset{\Phi^*}{\rightleftarrows}} H^*(BU)$$

with Φ^* of degree $-s^2$, Φ_* of degree $+s^2$.

July 29, 1986

In order to deal with the problem of studying the interesting maps on the previous page, it seems desirable to review how one computes the cohomology class in Gr_n corresponding to the cycle of subspace Γ such that $\dim(\Gamma \cap V^0) \geq s$. For this I seem to need f_* for a flag bundle.

Let's work universally. The full flag bundle over BU_n is

$$f : (BU_1)^n \longrightarrow BU_n$$

where f is given by Whitney sum. (More precisely: $PU_n \times^{U_n} (U_n/U_1^n) \sim B(U_1^n) = (BU_1)^n$.) One knows from the iterated projective bundle construction that

$$H^*((BU_1)^n) = k[u_1, \ldots, u_n] \qquad u_i = pr_i^*(c_1)$$

is a free module over

$$H^*(BU_n) = k[c_1, \ldots, c_n] \qquad c_i = c_i(\text{can. bdl})$$

where $f^*\left(\sum_{i=0}^{n} c_i t^i\right) = \prod_1^n (1 + tu_i)$. A basis is given by the monomials

$$u_1^{a_1} \cdots u_n^{a_n} \qquad \text{with} \qquad 0 \leq a_i \leq n - i$$

and moreover f_* on this basis is given by

$$f_*(u_1^{a_1} \cdots u_n^{a_n}) = \begin{cases} 1 & \text{if all } a_i = n-i \\ 0 & \text{otherwise} \end{cases}$$

Prop: One has

$$f_*(\alpha) = \frac{\sum \operatorname{sgn}(\sigma)\, \sigma(\alpha)}{\prod_{i<j} (u_i - u_j)} \qquad \sigma \in \Sigma_n$$

Proof: Both sides being linear over $H^*(BU_n)$ it suffices to check the formula on the basis elements. But if $\alpha = u_1^{a_1} \cdots u_n^{a_n}$, then

$$\sum \mathrm{sgn}(\sigma)\, \sigma(\alpha) = \det \begin{pmatrix} u_1^{a_1} & u_2^{a_1} & \cdots & u_n^{a_1} \\ u_1^{a_2} & u_2^{a_2} & & u_n^{a_2} \\ \vdots & & \ddots & \\ u_1^{a_n} & & & u_n^{a_n} \end{pmatrix}$$

and this is zero if two of the a_i coincide which is the case unless $a_i = n-i$. In this last case the determinant is the denominator $\prod_{i<j}(u_i - u_j)$ by the theorem of VanderMonde.

An easier way to get at the same result is via fixpoint formalism. The map

$$f_* : H^*((BU_1)^n) \longrightarrow H^*(BU_n)$$

is the same as the Gysin map in equivariant cohomology $H^*_{U_n}$ for the map $U_n/T_n \longrightarrow pt$. If we restrict to the subgroup $T_n = U_1^n$, then we can localize. This means we are computing

$$\begin{tikzcd} H^*_{T_n}(U_n/T_n)[\delta^{-1}] \arrow[d, "f_*"] & H^*_{T_n}(U_n/T_n) \arrow[l] \arrow[d, "f_*"] & H^*_{U_n}(U_n/T_n) \arrow[l] \arrow[d, "f_*"] \\ H^*_{T_n}(pt)[\delta^{-1}] & H^*_{T_n}(pt) \arrow[l] & H^*_{U_n}(pt) \arrow[l] \end{tikzcd}$$

where $\delta = \prod_{i<j}(u_i - u_j)$. The localization formula gives the answer for $f_*(\alpha)$ as a sum over fixpts, i.e. over $\sigma \in \Sigma_n$.

Let's consider a vector bundle E over X of rank n and a subspace V^0 of sections, say generic in some sense to be made precise. Consider the cycle in X consisting of those x such that $ev_x : V^0 \longrightarrow E_x$ has a non-zero kernel. We want the cohomology class of this cycle.

Let's consider in the product $\mathbb{P}V^0 \times X$ the subset Z of pairs (L, x) such that $L \subset \operatorname{Ker}(ev_x)$. Over $\mathbb{P}V^0 \times X$ ~~the vector bundle~~ there is a canonical v.b. map $pr_1^* \mathcal{O}(-1) \longrightarrow pr_2^* E$ whose value at (L, x) is the composition

$$L \subset V^0 \xrightarrow{ev_x} E_x$$

Thus we have a canonical section of $pr_1^*\mathcal{O}(1) \otimes pr_2^* E$ which vanishes on the subset Z in question. In good cases the section will be transversal to zero, and so $Z \subset \mathbb{P}V^0 \times X$ will be a submanifold whose coh. class is

$$c_n(pr_1^*\mathcal{O}(1) \otimes pr_2^* E) = u^n + u^{n-1}c_1(E) + \cdots + c_n(E)$$

where $n = \operatorname{rank} E$ and $u = c_1 \mathcal{O}(1)$. The cycle in X where ev_x has a non-zero kernel is the image of Z under pr_2. Thus the cycle of x in X, where V^0 fails to inject into E_x has the cohomology class

$$(pr_2)_* \left(u^n + u^{n-1}c_1(E) + \cdots + c_n(E)\right) = c_{n-d+1}(E)$$

where $d = \dim V^0$. (Note that integrating over $\mathbb{P}V^0$ picks up the coefficient of u^{d-1}.) For example if $d = 1$, i.e. we have a generic section of E, then its zero set has class $c_n(E)$, $n = \operatorname{rank} E$.

July 30, 1986

Review the problem: $V^0 \subset V$, $\dim V^0 = d$, $\dim V = N$. In $Gr_n(V)$ we have the cycle of Γ such that $\dim(\Gamma \cap V^0) \geq s$, and the problem is to find the coh. class. I recall that the answer is supposed to be a Hankel determinant of size $s \times s$, at least in certain cases.

Here's a general approach to the problem. We consider within $Gr_s(V^0) \times Gr_n(V)$ the subvariety $\tilde{Z}$ of (K, Γ) such that $K \subset \Gamma$. This is a non-singular variety of dim

$$\dim Gr_s(V^0) + \dim Gr_{n-s}(V/K)$$

$$= \quad s(d-s) \quad + (n-s)(N-n)$$

Generically on this subvariety one has $K = \Gamma \cap V^0$ (assuming $d-s+n-s \leq N-s$ or $N-n-d+s \geq 0$), and so this subvariety resolves the cycle Z of Γ with $\dim(\Gamma \cap V^0) \geq s$. The codim. of this cycle Z is

$$n(N-n) - s(d-s) - (n-s)(N-n) = s(N-n-d+s)$$

so we are seeking in $H^*(Gr_n V)$ a coh. class of dimension $s(N-n-d+s)$.

Now $\tilde{Z}$ is the locus on $Gr_s(V^0) \times Gr_n(V)$ where the canonical hom. $pr_1^*(S) \subset \tilde{V}^0 \subset \tilde{V} \longrightarrow pr_2^*(Q)$ is zero. Thus

$$[\tilde{Z}] = c_{sm}\left(pr_1^* \check{S} \otimes pr_2^* Q\right) \qquad m = N-n$$

All we have to do is to apply $(pr_2)_*$ to get

[2] in $H^*(Gr_n V)$.

So one is lead to the following problem in algebra. One expresses $c_{sm}(pr_1^* S \otimes pr_2^* Q)$ in terms of $pr_1^* c_i(S)$ and $pr_2^* c_j(Q)$ and looks for the part picked out by the top cycle of $Gr_s(V^\circ)$.

Consider the case $s = 1$ in which case we have

$$c_m(pr_1^* \mathcal{O}(1) \otimes pr_2^* Q) = u^m + u^{m-1} c_1^Q + \cdots + c_m^Q$$

which is to be integrated over $\mathbb{P}V$, which picks out the coefficient of u^{d-1}, which is $c_{m-d+1}(Q)$. However suppose instead of the quotient bundle Q one were to use the subbundle S over $Gr_n(V)$. One has

$$\begin{aligned} & u^m + u^{m-1} c_1(Q) + \cdots + c_m(Q) \\ = \; & u^m \left(1 + u^{-1} c_1(Q) + \cdots + u^{-m} c_m(Q)\right) \\ = \; & u^m \frac{1}{1 + u^{-1} c_1(S) + \cdots + u^{-n} c_n(S)} \end{aligned}$$

But if one thinks of the operator of multiplying by $1 + Tc_1 + \cdots + T^n c_n$ on $k[T]/(T^N)$, the matrix is

$$\begin{pmatrix} 1 & c_1 & c_2 & c_3 & \\ & 1 & c_1 & c_2 & \\ & & 1 & c_1 & \\ & 0 & & & \ddots \end{pmatrix} \qquad \begin{pmatrix} 1 & b_1 & b_2 \\ & 1 & b_1 \\ & & 1 \\ & & & \ddots \end{pmatrix}$$

and the inverse matrix has the same form, say $\uparrow$ where

$$(-1)^{i} b_i = \det \begin{pmatrix} c_1 & c_2 & c_i & c_{i+1} \\ 1 & c_1 & c_2 & \\ & 1 & c_1 & \\ & & & c_1 \\ & & & 1 \end{pmatrix} = \begin{vmatrix} c_1 & c_2 & \cdots & c_i \\ 1 & c_1 & & \vdots \\ & 1 & \ddots & \\ & & 1 & c_1 \end{vmatrix}$$

$i+1$ st column

Thus we conclude that

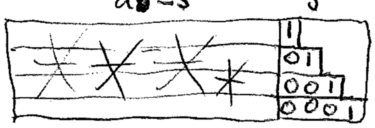
$$\frac{1}{1+Tc_1+T^2c_2+\cdots} = 1 - Tc_1 + T^2 \begin{vmatrix} c_1 & c_2 \\ 1 & c_1 \end{vmatrix} - T^3 \begin{vmatrix} c_1 & c_2 & c_3 \\ 1 & c_1 & c_2 \\ 0 & 1 & c_1 \end{vmatrix} + \cdots$$

so that
$$c_i(Q) = (-1)^i \begin{vmatrix} c_1 & & & c_i \\ 1 & c_1 & & \\ 0 & 1 & \ddots & \\ & 0 & & c_1 \end{vmatrix} \quad (S)$$

In particular for $s=1$ we get the cohomology class
$$(-1)^{m-d+1} \begin{vmatrix} c_1 & c_2 & \cdots & c_{m-d+1} \\ 1 & c_1 & & \vdots \\ & 1 & \ddots & \\ & & 1 & c_1 \end{vmatrix}$$

Next let's try $s=2$. The idea I have is to use a resolution of Z that is close to the Schubert cell picture. Let us choose a hyperplane $V' \subset V^0$ and consider flags $0 \subset K' \subset K$ such that with $K' \subset V'$, and $K \subset V^0$. This is a blowup of the Grassmannian associated to the fat Schubert cell:

$d-s$ $\quad s$



The above approach of trying to write things in terms of the subbundle is a red herring. One really should work with the quotient bundle Q

since that is connected with sections.

The real problem involved in the above is how to integrate over $Gr_s(V^\circ)$ in analogy with integrating over $\mathbb{P}(V^\circ)$. Let's do this carefully for $s = 2$. Put $V = V^\circ$

The idea is to use the flag manifold $Dr_{1,2}(V)$. This is the projective bundle of $\tilde{V}/\mathcal{O}(-1)$ over $\mathbb{P}V$. What we want to do is to figure out integration over $Dr_{1,2}(V)$, that is, the map $b_* a^*$, where a, b are the maps

$$\begin{array}{ccc} Dr_{1,2}(V) & \xrightarrow{a} & (BU_1)^2 \\ \downarrow{\scriptstyle b} & & \\ pt. & & \end{array}$$

Now
$$H^*(\mathbb{P}V) = k[u_1]/u_1^d$$

$$H^*(D_{12}) = k[u_1, u_2]/\left(u_1^d,\ u_2^{d-1} + c_1(\tilde{V}/\mathcal{O}(-1))u_2^{d-2} + \cdots\right)$$

Now
$$c_t(\tilde{V}/\mathcal{O}(-1)) = \frac{c_t(\tilde{V})}{c_t(\mathcal{O}(-1))} = \frac{1}{1 - tu_1}, \text{ so}$$

that
$$c_i(\tilde{V}/\mathcal{O}(-1)) = u_1^i. \quad \text{Thus}$$

$$H^*(D_{12}) = k[u_1, u_2]\Big/\Big(u_1^d,\ \underbrace{u_1^{d-1} + u_1^{d-2}u_2 + \cdots + u_2^{d-1}}_{\frac{u_1^d - u_2^d}{u_1 - u_2}}\Big)$$

We want the linear functional on this which is homogeneous of degree $(d-1) + (d-2)$ and has the value 1 on $u_1^{d-1}u_2^{d-2}$.

Let $b_i \in H_i(BU_1)$ be dual to u^i, i.e. have inner product 1 with u^i. Then the linear functional we are after has to be a linear comb. of $b_i \otimes b_j$, $i+j = (d-1)+(d-2)$ and as it vanishes on monomials involving either u_1^d or u_2^d it can only involve $b_{d-1} \otimes b_{d-2}$, $b_{d-2} \otimes b_{d-1}$. As it must vanish on

$$u_1^{d-2}\left(u_1^{d-1} + \cdots + u_2^{d-1}\right) \equiv u_1^{d-2} u_2^{d-2} + u_1^{d-2} u_2^{d-1}$$

we conclude that

$$[D_{12}(V)] = b_{d-1} \otimes b_{d-2} - b_{d-2} \otimes b_{d-1}$$

So we then consider $Gr_2(V)$:

$$\begin{tikzcd} D_{12} \arrow[r, hook] \arrow[d, "g"'] & (BU_1)^2 \arrow[d] \\ Gr_2 \arrow[r, hook] \arrow[d, "f"'] & BU_2 \\ pt & \end{tikzcd}$$

We have $\qquad f_*(\alpha) = f_* g_*(u_2 g^*(\alpha)) \qquad$ as $g_*(u_2) = 1$

so it's clear that integrating a symmetric poly p in u_1, u_2 over $Gr_2(V)$ is given by

$$p(u_1, u_2)\,[Gr_2(V)] = \text{coeff of } u_1^{d-1} u_2^{d-2} - \text{coeff of } u_1^{d-2} u_2^{d-1} \text{ in } u_2\, p(u_1, u_2)$$

So now we know how to integrate

$$c_{2n}\left(pr_1^*(\check{S}) \otimes pr_2^*(E)\right) \quad \text{on } Gr_2(V^0) \times X$$

over the Grassmannian. We see this Chern class is $\prod_{i=1}^{2}\left(u_i^n + u_i^{n-1} c_1 + \cdots + c_n\right)$, so multiply by u_2 and look for the appropriate coefficient. We clearly get

$$c_{n-d+1}\, c_{n-d+3} - c_{n-d+2}^2$$

At this point I think the general case can be treated by similar methods, and I should now go back to understand the earlier questions about Grass. graph. I recall wanting to compute a certain decomposition of the identity map on the cohomology of a Grassmannian. In fact its [redacted] stable, so there's [redacted] a correspondence

$$BU_s \times BU_s \dashrightarrow BU$$

whose effect on cohomology is to be computed.

Let's take the case $s=1$. We have in general

$$\begin{array}{ccc} Y = \{(K,I,W)\} & \xrightarrow{p_3} & Gr_n(V^0 \oplus V^1) \\ \downarrow{\scriptstyle p_{12}} & & \\ Gr_s(V^0) \times Gr_{n-s}(V^1) & & \end{array}$$

where $\dim(V^0) = \dim(V^1) = n$ (large), [redacted] $\dim K = s$

$$I < W < K \oplus V^1$$

Thus (Y, p_{12}) is a Grassmannian bundle of s planes in the vector bundle whose fibre at (K,I) is $K \oplus V^1/I$.

Notation: S^s = subbundle over $Gr_s(V^0)$ (fibre K)
Q^s = quotient bundle over $Gr_{n-s}(V^1)$ (fibre V^1/I)
S^{n-s} = sub " " " (fibre I)

$\mathcal{S}^s$ = canon subbundle over $Y \subset p_1^*(S^s) \oplus p_2^*(Q^s)$ (fibre W/I)

Then we have the exact sequence

$$0 \longrightarrow p_2^*(S^{n-s}) \longrightarrow p_3^*(S^n) \longrightarrow \mathcal{S}^s \longrightarrow 0$$

Let φ be a multiplicative characteristic class. Then

$$p_3^*(\varphi(S^n)) = p_2^*\varphi(S^{n-s})\ \varphi(\mathcal{S}^s)$$

and so

$$(p_{12})_*\, p_3^*(\varphi(S^n)) = \underbrace{pr_1^*\,\varphi(S^{n-s})}_{pr_1^*\frac{1}{\varphi(Q^s)}}\ (p_{12})_*\{\varphi(\mathcal{S}^s)\}$$

July 31, 1986:

I am trying to compute the map on cohomology associated to the correspondence

$$\begin{tikzcd} Y = \{(K, I, W)\} \arrow[r] \arrow[d, "f"] & Gr_n(V) \\ \mathbb{P}(V^0) \times \mathbb{P}(\check{V}^1) & \end{tikzcd}$$

where Y is the space of triples such that

$$I \subset W \subset K \oplus V^1$$

Thus Y is the projective line bundle of the 2 plane bundle whose fibre over (K, I) is $K \oplus V^1/I$. Let $\mathcal{O}_Y(-1)$ denote the canonical sub line bundle over Y. Then one has exact sequences over Y

$$0 \longrightarrow \mathcal{J} \longrightarrow \mathcal{W} \longrightarrow \mathcal{O}_Y(-1) \longrightarrow 0$$

$$0 \longrightarrow \mathcal{J} \longrightarrow \tilde{V}^1 \longrightarrow \mathcal{O}_{\mathbb{P}(\check{V}^1)}(1) \longrightarrow 0.$$

We will need a formula for f_* where $f : \mathbb{P}\mathcal{E} \longrightarrow X$, which is

$$f_*\{p(\xi)\} = \operatorname{Res}\left\{\frac{p(T)\, dT}{T^n + c_1(\mathcal{E})T^{n-1} + \cdots + c_n(\mathcal{E})}\right\}$$

where $\xi = c_1(\mathcal{O}(1))$. One can check this by assuming $\mathcal{E} = L_1 \oplus \cdots \oplus L_n$. If $s_i : X \to \mathbb{P}\mathcal{E}$ is the section where $\mathcal{O}(-1) = L_i$, then

$$(s_i)_* 1 = c_1(\mathcal{E}/L_i \otimes \mathcal{O}(1)) = \prod_{j \neq i} (\xi + c_1(L_j))$$

and the formula checks for something in the image of $(s_i)_*$. These span after localization.

In the present case

$$\mathcal{E} = \mathcal{O}_{\mathbb{P}V^0}(-1) + \mathcal{O}_{\mathbb{P}\check{V}^1}(1)$$

Let $u_1 = c_1(\mathcal{O}_{\mathbb{P}V^0}(1))$, $u_2 = c_1(\mathcal{O}_{\mathbb{P}\check{V}^1}(1))$. Then

$$T^2 + c_1(\mathcal{E})T + c_2(\mathcal{E}) = (T - u_1)(T + u_2)$$

and

$$f_*(\varphi(\xi)) = \operatorname{Res} \frac{\varphi(T)\,dT}{(T-u_1)(T+u_2)}$$

$$= \frac{1}{u_1+u_2}\big(\varphi(u_1) - \varphi(-u_2)\big)$$

Now let's compute $f_*\{ch(\mathcal{W})\}$. We have

$$ch(\mathcal{W}) = ch(\mathcal{I}) + ch(\mathcal{O}_Y(-1))$$

$$= ch(\mathcal{I}) + e^{-\xi}.$$

Since $\mathcal{I}$ comes from the base $f_* ch(\mathcal{I}) = 0$. Thus

$$f_*\{ch(\mathcal{W})\} = \frac{1}{u_1+u_2}\left(e^{-u_1} - e^{u_2}\right)$$

Next we want to ~~[illegible]~~ compute what happens to this ~~[illegible]~~ under the homom. assoc. to the correspondence

$$\begin{array}{ccc} Y' = \{(K, I, \Gamma)\} & \longrightarrow & \mathbb{P}(V^0) \times \mathbb{P}(\check{V}^1) \\ \big\downarrow & & \\ Gr_n(V) & & \end{array}$$

where (K, I, Γ) are subject to

$$K \subset \Gamma \subset V^0 \oplus I$$

In this case we determine the class of Y' in the product $\mathbb{P}V^0 \times \mathbb{P}(\check{V}^1) \times Gr_n(V)$ and then we integrate over the product of projective spaces. The condition $K \subset \Gamma$ is where a section of $\mathcal{Q} \otimes \mathcal{O}_{\mathbb{P}V^0}(1)$ vanishes, so ~~[illegible]~~ the corresponding subvariety has coh. class

$$c_n(\mathcal{Q} \otimes \mathcal{O}_{\mathbb{P}V^0}(1)) = u_1^n + c_1(\mathcal{Q})\,u_1^{n-1} + \cdots + c_n(\mathcal{Q})$$

Once we are on this subvariety the condition $\Gamma \subset V^0 \oplus I$ is equivalent to $\Gamma/K \subset V^0/K \oplus I$, which is where a map $\mathcal{S}/\mathcal{O}_{\mathbb{P}V^0}(-1) \longrightarrow \mathcal{O}_{\mathbb{P}(\check{V}^1)}(1)$ vanishes. So the class is

$$c_{n-1}\left(\left(\mathcal{S}/\mathcal{O}_{\mathbb{P}V^0}(-1)\right)^\vee \otimes \mathcal{O}_{\mathbb{P}(\check{V}^1)}(1)\right).$$

Now

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}V^0}(-1) \longrightarrow \mathcal{S} \longrightarrow \mathcal{S}/\mathcal{O}_{\mathbb{P}V^0}(-1) \longrightarrow 0$$

$$0 \longleftarrow \mathcal{O}_{\mathbb{P}V^0}(1) \longleftarrow \mathcal{S}^\vee \longleftarrow \left(\mathcal{S}/\mathcal{O}_{\mathbb{P}V^0}(-1)\right)^\vee \longleftarrow 0$$

so formally at least

$$c_{n-1}\left(\left(\mathcal{S}/\mathcal{O}_{\mathbb{P}V^0}(-1)\right)^\vee \otimes \mathcal{O}_{\mathbb{P}(\check{V}^1)}(1)\right) = \frac{c_n(\mathcal{S}^\vee \otimes \mathcal{O}_{\mathbb{P}\check{V}^1}(1))}{c_1(\mathcal{O}_{\mathbb{P}V^0}(1) \otimes \mathcal{O}_{\mathbb{P}\check{V}^1}(1))}$$

$$= \frac{u_2^n + c_1(\mathcal{S}^\vee)\,u_2^{n-1} + \cdots + c_n(\mathcal{S}^\vee)}{u_1 + u_2}$$

What probably is true is that

$$c_n(\mathcal{Q} \otimes \mathcal{O}_{\mathbb{P}V^0}(1)) \cdot c_{n-1}(\cdots\cdots) = \frac{(u_1^n + c_1(\mathcal{Q})u_1^{n-1} + \cdots)(u_2^n + c_1(\mathcal{S}^\vee)u_2^{n-1} + \cdots)}{u_1 + u_2}$$

since we ought to be able to see that the denominator divides the numerator.

Let's review the problem. We consider

$$V = V^0 \oplus V^1 \qquad \dim V^0 = m, \ \dim V^1 = n$$

and we consider the cycle in $Gr_n(V)$ consisting of Γ such that

$$\dim(\Gamma \cap V^0) \geq 1$$

We have a desingularization of this cycle consisting of (K, I, Γ) with $K \in \mathbb{P}(V^0)$, $I \in \check{\mathbb{P}}(V^1)$ and

$$K \subset \Gamma \subset V^0 \oplus I$$

In effect if we call the variety of these triples Y, then Y is a Grassmannian bundle over $\mathbb{P}(V^0) \times \check{\mathbb{P}}(V^1)$. It has dimension

$$m-1 \ + \ n-1 \ + \ (n-1)(m-1) \ = \ nm - 1.$$

Given (K, I), a generic Γ over (K, I) is such that $\Gamma/K \subset V^0/K \oplus I$ is the graph of a map from I to V^0/K. This shows that generically Γ determines (K, I), which means that the map $Y \longrightarrow Gr_n(V)$ is (1-1) over the set of $\Gamma \ni \dim(\Gamma \cap V^0) = 1$.

Now the problem is to compute the map in cohomology associated to the correspondence

$$\begin{array}{ccc} Y & \longrightarrow & \mathbb{P}(V^0) \times \check{\mathbb{P}}(V^1) \\ \downarrow & & \\ Gr_n(V) & & \end{array}$$

We get a nice basis for the cohomology of $\mathbb{P}(V^0) \times \check{\mathbb{P}}(V^1)$ by taking the classes of the subvarieties

$$\mathbb{P}(W^0) \times \check{\mathbb{P}}(V^1/W^1)$$

where $W^0 \subset V^0$ and $W^1 \subset V^1$. Thus the problem is now to find the cohomology class of the subvar. of the Grassmannian consisting of Γ such that

$$\left[\begin{array}{l}\Gamma \cap V^0 \text{ is a line in } W^0 \\ \Gamma + V^0 \text{ is a hyperplane in } V \text{ containing } V^0 \oplus W^1\end{array}\right.$$

What is the dimension of this cycle? Clearly it is

$$(\dim W^0) - 1 + \dim (V^1/W^1) - 1 + (n-1)(m-1)$$

since we have pointed out that a Γ with $\Gamma \cap V^0$ = a line K in V^0 and $\Gamma + V^0$ = a hyperplane $V^0 \oplus I$ in V is the same thing as a subspace Γ/K of $V^0/K \oplus I$ complementary to V^0/K, i.e. the graph of a map from I to V^0/K.

It appears that the natural way to resolve this cycle is to consider the partial flag manifold

$$Dr_{1,n,n+m-1}(V)$$

consisting of $K \subset \Gamma \subset H$, and the map to

$$Dr_{1,n+m-1}(V)$$

In the latter we consider the subvariety of $K \subset H$ with $K \subset W^0$ and $V^0 \oplus W^1 \subset H$. It's here that I have to calculate the codimension carefully. The dimensions are

$$\dim Dr_{1,n+m-1} = (n+m-1) + (n+m-2) \qquad -$$

$$\dim \left\{ (K \subset H) \,\middle|\, \begin{array}{l} K \subset W^0 \\ V^0 + W^1 \subset H \end{array} \right\} = \dim W^0 - 1 + \dim V^1/W^1 - 1 \qquad +$$

$$\dim Dr_{1,n,n+m-1} = nm + n - 1 + m - 1 \qquad +$$

$$\dim \text{resolution} = nm - n - m + 1 + \dim W^0 - 1 + \dim V^1/W^1 - 1$$

$$= (\dim W^0) - 1 + (\dim V^1/W^1) - 1 + (n-1)(m-1)$$

which works.

August 2, 1986

Consider $D_{1,2}(V)$ as the space of (L_1, L_2) where L_1, L_2 are two perpendicular lines. Then we have an embedding

$$D_{1,2}(V) \xhookrightarrow{i} \mathbb{P}V \times \mathbb{P}V$$

The image is the place where the orthogonal projection homomorphism

$$\mathcal{L}_2 \subset \tilde{V} \twoheadrightarrow \mathcal{L}_1$$

vanishes, hence ignoring orientation questions, one has

$$\iota_* 1 = c_1(\mathcal{L}_1 \otimes \check{\mathcal{L}}_2) = u_2 - u_1$$

where $u_i = c_1(\mathcal{L}_i^\vee)$. Similarly for

$$D_{1,2,\ldots,n}(V) \xhookrightarrow{i} (\mathbb{P}V)^n$$

one can note that

$$D_{1,\ldots,s}(V) \hookrightarrow D_{1,\ldots,s-1}(V) \times \mathbb{P}V$$

is the submanifold, where $L_s \perp L_1 \oplus \cdots \oplus L_{s-1}$, which has the class

$$c_{s-1}\left(\mathcal{L}_s^\vee \otimes (\mathcal{L}_1 \oplus \cdots \oplus \mathcal{L}_{s-1})\right) = \prod_{i=1}^{s-1} (u_s - u_i).$$

Thus up to sign

$$\iota_* 1 = \prod_{i<j} (u_j - u_i)$$

Is there a sensible way to treat the orientation questions. The idea might be to fix L_1 and look at the fibres over L_1. The fibre of $D_{1,2}(V)$ over L_1 is $\mathbb{P}(L_1^\perp)$, which is naturally embedded into $\mathbb{P}(V)$. The normal bundle at L_2

for this embedding is the cokernel of

$$\mathrm{Hom}(L_2, L_1^\perp / L_2) \longrightarrow \mathrm{Hom}(L_2, V/L_2)$$

which is $\mathrm{Hom}(L_2, V/L_1^\perp) = \mathrm{Hom}(L_2, L_1)$. Since this all involves complex vector spases and maps, which have natural orientations, it seems that

$$i_* 1 = c_1(\mathcal{L}_2^\vee \otimes \mathcal{L}_1) = u_2 - u_1$$

As a check take $V = \mathbb{C}^2$, whence i is the graph of the anti-podal map $L \to L^\perp$ on $\mathbb{P}(\mathbb{C}^2) = \mathbb{C}P^1$. This leads me to suspect $i_* 1 = u_1 - u_2$. So it's more or less clear that the above orientation arguments are wrong.

Let us now consider inside $D_{1,2}(V)$, the subvariety M of (L_1, L_2) such that $L_1 \subset V^0$, $L_2 \subset V^1$, where as before $V = V^0 \oplus V^1$. We then have

$$\mathbb{P}V^0 \times \mathbb{P}V^1 = M \subset D_{1,2}(V) \overset{i}{\subset} \mathbb{P}V \times \mathbb{P}V$$

Now I know that there are exact sequences

$$0 \longrightarrow H^*(D_{1,2}) \xrightarrow{i_*} H^*(\mathbb{P}V \times \mathbb{P}V) \xrightarrow{\Delta} H^*(\mathbb{P}V) \longrightarrow 0$$

so that I will know $[M]$ from $i_*[M] = u_1^n u_2^m$. We have

$$(i^* i_* 1) \cdot [M] = i^*(u_1^n u_2^m)$$

To simplify write $u_j = i^* u_j$. Thus

$$(u_1 - u_2)[M] = u_1^n u_2^m$$

and because the solution is unique we have

$$[M] = u_1^{n-1} u_2^m + u_1^{n-2} u_2^{m+1} + \cdots + u_2^{n+m-1}$$

$$= \frac{u_1^n - u_2^n}{u_1 - u_2} . u_2^m$$

Here one uses that $u_2^{n+m} = 0$, which holds more generally when $V^0 \perp V^1$.

Having done this calculation, let us now return to the task of computing the effect of the correspondence

$$\begin{array}{ccc} Y = \{K, I, \Gamma)\} & \longrightarrow & \mathbb{P}V^0 \times \check{\mathbb{P}}V^1 \\ \downarrow & & \\ Gr_n(\Gamma) & & \end{array}$$

on cohomology.

It appears that maybe we can do this thing directly always using the idea of embedding into a trivial situation. Let's recall that Y consists of triples (K, I, Γ) with $K \in \mathbb{P}(V^0)$, $I \in \check{\mathbb{P}}(V^1)$, $\Gamma \in Gr_n(V)$ with $K \subset \Gamma \subset V^0 \oplus I$. We therefore have a cartesian square at the top of:

$$\begin{tikzcd}
Y \arrow[r, "f'"] \arrow[d, hook, "i'"] \arrow[dr, phantom, "\text{tr. cart}"] & \mathbb{P}V^0 \times \check{\mathbb{P}}V^1 \arrow[d, hook, "i"] \\
D_{1,n,N-1}(V) \arrow[r, "f"] \arrow[d, hook, "k"] \arrow[dddd, bend right, "q"'] & D_{1,N-1}(V) \arrow[ddd, hook, "j"] \\
Gr_n \times D_{1,N-1} \arrow[ur, "p'"] \arrow[d, hook, "j'"] \arrow[r, phantom, "\text{tr. cart.}"] & \\
Gr_n \times \mathbb{P}V \times \check{\mathbb{P}}V \arrow[d, "p_1"] \arrow[r, "p"] & \mathbb{P}V \times \check{\mathbb{P}}V \\
Gr_n &
\end{tikzcd}$$

We want to compute

$$(g i')_* (f'^*)(\alpha) \qquad \alpha \in H^*(\mathbb{P}V^0 \times \mathbb{P}V')$$

and the first point is that α comes from $\beta \in H^*(\mathbb{P}V \times \check{\mathbb{P}}V)$, so that what we are after is

$$(p_1)_* \Big(\underbrace{j'_* k_* i'_* 1}_{[Y]} \cdot p^* \beta \Big)$$

So the problem is to calculate

$$[Y] = j'_* k_* i'_* 1$$

Now.

$$i'_* 1 = f^* i_* 1$$

$$= k^* (p')^* i_* 1$$

$$\therefore \quad k_* i'_* 1 = (k_* 1) \cdot (p')^* i_* 1$$

Now we know $i_* 1 = j^*(\gamma) \qquad \gamma \in H^*(\mathbb{P}V \times \check{\mathbb{P}}V)$

so $\qquad p'^* i_* 1 = (j')^* p^* \gamma$

so

$$[Y] = j'_* \{ k_* 1 \cdot (p')^* i_* 1 \}$$

$$= j'_* \{ k_* 1 \cdot (j')^* p^* \gamma \}$$

$$= j'_* k_* 1 \cdot p^* \gamma$$

So I need $j'_* k_* 1 = [O_{1,n,N-1}]$ in $Gr_n \times \mathbb{P}V \times \check{\mathbb{P}}V$

and

$$\gamma = (-1)^m \frac{u_1^n - u_2^n}{u_1 - u_2} u_2^m \qquad \text{from the above}$$

The next point is that

$$D_{1,n,N-1}(V) \qquad \dim = (m+n)-1 + (n-1)m + m-1$$
$$= nm+n+m-2$$
$$\cap$$
$$Gr_n \times \mathbb{P}V \times \check{\mathbb{P}}V \qquad \dim = nm + (n+m-1) + (n+m-1)$$

has $\operatorname{codim} = n+m$ which means that the conditions

$$K \subset \Gamma \qquad \operatorname{codim}\ m$$
$$\Gamma \subset H \qquad \operatorname{codim}\ n$$

are independent. These define respectively the classes

$$c_m\left(\mathcal{Q}_{Gr} \otimes \mathcal{O}_{\mathbb{P}V}(1)\right) = u_1^m + c_1(\mathcal{Q})u_1^{m-1} + \dots + c_m(\mathcal{Q})$$

$$c_n\left(\check{\mathcal{S}}_{Gr} \otimes \mathcal{O}_{\hat{\mathbb{P}}V}(1)\right) = (-u_2)^n + c_1(\check{\mathcal{S}})(-u_2)^{n-1} + \dots$$
$$= (-1)^n\left\{u_2^n + c_1(\mathcal{S})u_2^{n-1} + \dots + c_n(\mathcal{S})\right\}$$

Thus

$$[D_{1,n,N-1}] = (-1)^n\left(u_1^m + c_1(\mathcal{Q})u_1^{m-1} + \dots + c_m(\mathcal{Q})\right)\left(u_2^n + c_1(\mathcal{S})u_2^{n-1} + \dots\right)$$

and so the integration map appears to send $u_1^k u_2^l \in H^*(\mathbb{P}V^0 \times \check{\mathbb{P}}V^1)$ to

$$\text{coeff of } u_1^{N-1}u_2^{N-1} \text{ in } \left\{ \left(u_1^m + c_1(\mathcal{Q})u_1^{m-1} + \dots\right)\left(u_2^n + c_1(\mathcal{S})u_2^{n-1} + \dots\right) \times \underbrace{\frac{u_1^n - u_2^n}{u_1 - u_2}}_{= u_1^{n-1} + u_1^{n-2}u_2 + \dots + u_2^{n-1}} u_2^m \cdot u_1^k u_2^l \right\}$$

Check: $k = l = 0$. As the only way to get u_1^{m+n-1} is to use the u_1^m. Get

$$\text{coeff of } u_2^{N-1} \text{ in } \left(u_2^n + c_1(\mathcal{S})u_2^{n-1} + \dots\right)u_2^m = c_1(\mathcal{S})$$

August 3, 1986

Today I checked that the correspondence

$$\begin{array}{ccc} \{(K,I,\Gamma,W)\} & \longrightarrow & Gr_n(V) \\ \downarrow & & \\ Gr_n(V) & & \end{array}$$

where $(K,I) \in \mathbb{P}(V^0) \times \check{\mathbb{P}}(V^1)$ reproduced the classes ch_d for $d = 1, 2, 3$. I don't really see much point in further computations, although at the next level $d=4$ should appear the codim 4 stratum of Γ where $\Gamma \cap V^0$ has dim 2.

It appears I can do the $s=1$ calculation in general:

$$\left(u_1^m + c_1(2)u_1^{m-1} + \cdots + c_m(2)\right)\left(u_1^{n-1} + u_1^{n-2}u_2 + \cdots + u_2^{n-1}\right) u_1^k$$

$$\Big\downarrow \text{coeff of } u_1^{N-1}$$

$$\left(u_2^n + c_1(S)u_2^{n-1} + \cdots\right)\left(c_k(2) + c_{k-1}(2)u_2 + \cdots + c_0(2)u_2^k\right) u_2^{m+l}$$

$$\Big\downarrow \text{coeff of } u_2^{N-1}$$

$$\boxed{c_{l+1}(S)\, c_k(2) + \cdots\cdots + c_{k+l+1}(S)\, c_0(2)}$$

The above is the formula for the image of $u_1^k u_2^l \in H^*(PV^0 \times \check{P}V^1)$ under the correspondence

$$\begin{array}{ccc} Y = \{(K,I,\Gamma)\} & \longrightarrow & \mathbb{P}(V^0) \times \check{\mathbb{P}}(V^1) \\ \downarrow & & \\ Gr_n(V) & & \end{array}$$

Now apply this to the image of $d! \, ch_d(S)$ under the correspondence

$$\begin{array}{ccc} \{(K, I, W)\} & \longrightarrow & Gr_n(V) \\ \downarrow & & \\ \mathbb{P}(V^0) \times \check{\mathbb{P}}(V^1) & & \end{array}$$

which I think I found to be

$$\frac{u_1^d - u_2^d}{u_1 - u_2} = \sum_{k+l=d-1} u_1^k u_2^l \qquad \left(\text{sign } (-1)^{d} \, ?\right)$$

and we find

$$\sum_{k+l=d-1} c_{l+1}(S) c_k(Q) + \cdots + c_{k+l+1}(S) c_0(Q)$$

$$= \begin{cases} c_1(S) c_{d-1}(Q) + \cdots + c_d(S) c_0(Q) \\ \qquad c_2(S) c_{d-2}(Q) + \cdots + c_d(S) c_0(Q) \\ \\ \end{cases}$$

$$= d \, c_d(S) c_0(Q) + (d-1) c_{d-1}(S) c_1(Q) + \cdots + c_1(S) c_{d-1}(Q)$$

But note

$$+ t \frac{d}{dt} \log c_{+t}(L) = t \frac{d}{dt} \left\{ + \log (1 + t \, c_1(L)) \right\}$$

$$= \frac{t c_1(L)}{1 + t c_1(L)} = \sum_{d \geq 1} (-1)^{d-1} t^d \, c_1(L)^d$$

so

$$\frac{t \partial_t (c_t)}{c_t} = \sum (-1)^{d-1} t^d \, d! \, ch_d$$

since $c_t(Q) = \dfrac{1}{c_t(S)}$ it appears that the correspondence on $Gr_n(V)$ we are looking at reproduces the class $ch_d(S)$.

This is surprising as I expected to need the $s=2$ correspondence in degree 4.

Calculation of dimensions

q	$\dim H^{2q}(BU)$	$\dim H^{2(q-1)}(BU_1^2)$	$\dim H^{2(q-4)}(BU_2^2)$
0	1		
1	1	1	
2	2	2	
3	3	3	
4	5	4	1
5	7	5	2
6	11	6	5

This suggests that we might have a decomposition of $H^*(BU)$ into the cohomology of the strata:

$$H^*(BU) \simeq \bigoplus_{s \geq 0} H^*(BU_s^2)[s^2]$$

If true, then on the Poincaré series level we have an identity for the partition fn.

$$\prod_{n=1}^{\infty} \frac{1}{1-t^n} \overset{?}{\doteq} 1 + \frac{t}{(1-t)^2} + \frac{t^4}{(1-t)^2(1-t^2)^2} + \frac{t^9}{(1-t)^2(1-t^2)^2(1-t^3)^2} + \cdots$$

An obvious way to try to prove this is to show that the different correspondences are orthogonal idempotents. Another method might be to use Frances' Morse theory applied to a circle action on the Grassmannian. The subbundle over the Grassmannian is equivariant, hence all the cohomology can be lifted to equivariant classes.

Recall that the s-th correspondence is

$$\begin{array}{ccc}
 & & Y = \{(K, I, W)\} \longrightarrow Gr_n(V) \\
 & & \Big\downarrow \text{dim } s^2 \\
Z = \{(K, I, \Gamma)\} & \longrightarrow & Gr_s(V^0) \times \check{Gr}_s(V^1) \\
\Big\downarrow \text{cod } s^2 & & \\
Gr_n(V) & &
\end{array}$$

where the conditions are

$$\overset{r}{I} \subset \overset{n}{W} \subset \overset{s}{K} \oplus \overset{n}{V^1}$$

$$K \subset \Gamma \subset V^0 \oplus I$$

I would like to prove this correspondence is idempotent and it suffices to show that

$$Y \circ Z = id \quad \text{on} \quad H^*(Gr_s(V^0) \times \check{Gr}_s(V^1)).$$

This is stronger, because it identifies the image of the projector $Z \circ Y$ with the cohomology.

$$\begin{array}{ccc}
 & & \{(K, I, \Gamma)\} \longrightarrow Gr_s(V^0) \times \check{Gr}_s(V^1) \\
 & & \Big\downarrow \\
(K', I', W) & \longrightarrow & Gr_n(V) \\
\Big\downarrow & & \\
Gr_s(V^0) \times \check{Gr}_s(V^1) & &
\end{array}$$

August 4, 1986

Consider the $\mathbb{G}_m$ action on $Gr_n(V^0 \oplus V^1)$, $\dim V^0 = m$, $\dim V^1 = n$, given by $\begin{pmatrix} 1 & 0 \\ 0 & t \end{pmatrix}$. The fixpoint set is

$$\coprod_{s=0}^{n} \underbrace{Gr_s(V^0) \times \check{Gr}_s(V^1)}_{F_s} \qquad \check{Gr}_s(V^1) = Gr_{n-s}(V^1)$$

We now define a correspondence Φ_s which will map cohomology of F_s to cohomology of $Gr_n(V)$.

$$\begin{tikzcd} Y_s = \{(K, I, \Gamma)\} \arrow[r] \arrow[d, "\mathrm{cod}\ s^2"] & F_s \\ Gr_n(V) & \end{tikzcd}$$

where the condition defining Y_s is

$$\overset{s}{K} \subset \overset{n}{\Gamma} \subset \overset{m}{V^0} \oplus \overset{n-s}{I}$$

Note that

$$\begin{aligned} \dim Y_s &= s(m-s) + s(n-s) + (n-s)(m-s) \\ &= \cancel{sm} - s^2 + \cancel{sn} - \cancel{s^2} + nm - \cancel{sm} - \cancel{ns} + \cancel{s^2} \\ &= nm - s^2 \end{aligned}$$

We also define a correspondence Ψ_s which maps $H^*(Gr_n V)$ to $H^*(F_s)$

$$\begin{tikzcd} Y_s' = \{(K, I, W)\} \arrow[r] \arrow[d, "\dim s^2"] & Gr_n(V) \\ F_s & \end{tikzcd}$$

where the defining condition for Y_s' is

$$\overset{n-s}{I} \subset \overset{n}{W} \subset \overset{s}{K} \oplus \overset{n}{V^1}$$

Next we consider $\Phi_t \circ \Phi_s$

$$
\begin{tikzcd}
 & \{(K,I,\Gamma)\} \arrow[r] \arrow[d, "\operatorname{cod} s^2"] & F_s \\
\{(K',I',W)\} \arrow[r] \arrow[d, "\dim t^2"] & Gr_n(V) & \\
F_t & &
\end{tikzcd}
$$

The fibre product consists of (K',I',W), (K,I,Γ) with $W = \Gamma$, i.e.

$$
\begin{array}{ccccc}
\overset{s}{K} & \subset & \Gamma & \subset & V^0 \oplus I \\
 & & \| & & \\
\overset{n-t}{I'} & \subset & W & \subset & K' \oplus V'
\end{array}
$$

Suppose that $t = s$. Then $\Gamma = K \oplus I \subset V^0 \oplus V'$ and $K = K'$, $I = I'$. It follows that

$$\boxed{\Phi_s \circ \Phi_s = \mathrm{id}}$$

which means that Φ_s embeds $H^*(F_s)$ as a direct summand of $H^*(Gr_n(V))$. Similarly one can see that if $t < s$, then the fibre product is empty (for then $\Gamma \supset K \oplus I$ is impossible dimensionally.) So

$$\boxed{\Phi_t \circ \Phi_s = 0 \qquad t < s}$$

It follows that

$$\bigoplus_s H^*(F_s) \xhookrightarrow{\sum \Phi_s} H^*(Gr_n V)$$

is injective. If one has $\sum \Phi_s(\alpha_s) = 0$, look at the smallest s_0 such that $\alpha_{s_0} \neq 0$; then apply Φ_{s_0}.

On the other hand I feel from the whole approach to this theory via Grassmannian graph that one should have

$$\sum_s \Phi_s \cdot \Psi_s = id \quad \text{on } H^*(Gr_n V).$$

This comes from the fact that the identity corresp. on $Gr_n(V)$ is deformed via $\begin{pmatrix} 1 & 0 \\ 0 & t \end{pmatrix}$ as $t \to \infty$ to the disjoint union of the correspondence $\Phi_s \cdot \Psi_s$.

Assuming this to be the case we have projectors $e_s = \Phi_s \Psi_s$ on $H^*(Gr_n V)$ such that

$$\begin{cases} 1 = e_0 + e_1 + \cdots + e_n \\ e_t e_s = 0 \quad \text{for } t < s. \end{cases}$$

Then

$$e_{n-1} = e_{n-1}^2 + e_n e_{n-1} \implies e_n e_{n-1} = 0$$

$$\implies e_{n-1} + e_n \text{ is a projector}$$

and

$$1 = (e_0 + \cdots + e_{n-2}) + (e_{n-1} + e_n)$$

$$e_{n-2} = e_{n-2}^2 + (e_{n-1} + e_n) e_{n-2}$$

$$\implies (e_{n-1} + e_n) e_{n-2} = 0$$

so left multiplying gives $e_{n-1} e_{n-2} = e_n e_{n-2} = 0$.

(Better argument than an induction:

$$e_j = (e_0 + \cdots + e_n) e_j = e_j^2 + e_{j+1} e_j + \cdots + e_n e_j$$

$$\implies e_{j+1} e_j + \cdots + e_n e_j = 0$$

$$\implies e_{j+1}(\text{———}) = e_{j+1} e_j = 0$$

$$\implies e_{j+2}(e_{j+2} e_j + \cdots + e_n e_j) = e_{j+2} e_j = 0 \qquad \text{etc.})$$

Conclude that $\boxed{e_i e_j = 0 \quad i \neq j}$

I want next to work out a Morse theory approach to these results. This means that I want to think of the F_s as critical submanifolds for a ~~nondegenerate~~ Morse function in Bott's sense. Then we should build up by attaching each F_s via the negative part of its normal bundle.

Let's look at the normal bundle to F_s at a point (K, I). The tangent space to $Gr_n(V)$ at $K \oplus I$ is

$$\mathrm{Hom}(K\oplus I,\ V/K\oplus I) = \mathrm{Hom}(K\oplus I,\ V^0/K \oplus V^1/I)$$

The action of $\begin{pmatrix} 1 & 0 \\ 0 & t \end{pmatrix}$ breaks up this tangent space into three parts: First, where the action is trivial

$$\mathrm{Hom}(K, V^0/K) \oplus \mathrm{Hom}(I, V^1/I)$$

which is the tangent space to F_s, second, where the action is via the character $t \mapsto t$

$$\mathrm{Hom}(K, V^1/I)$$

which is the "negative" bundle for the Morse theory, and third, where the action is via $t \mapsto t^{-1}$

$$\mathrm{Hom}(I, V^0/K).$$

(Here I think of the Morse flow given by $\begin{pmatrix} 1 & 0 \\ 0 & e^t \end{pmatrix}$ as $t \to \infty$ so that the minimum is given by the critical point $F_0 = Gr_0(V^0) \times \check{G}r_0(V^1) = pt.$)

If we choose complements for K in V^0 and I in V^1, then the graph construction identifies the tangent space with an open cell in the Grassmannian equivariantly. So the negative submanifold through (K, I) will consist of subspaces W looking like the graph of a map $K\oplus I \longrightarrow K^{\perp}\oplus I^{\perp}$ which is at most nonzero from K to $I^{\perp}$. This means that

$$I \subset W \subset K \oplus V'$$

and that $W/I \subset K \oplus V'/I$ is the graph of a map $K \longrightarrow V'/I$, i.e. complementary to V'/I. Conversely if W satisfies these two conditions then $\varphi_t(W) \longrightarrow K\oplus I$ as $t \longrightarrow -\infty$

Anyway we conclude that

$$Y_s' = \{(K, I, W)\} \longrightarrow Gr_n(V)$$

is a compactification of the negative subbundle for F_s, and similarly

$$Y_s = \{(K, I, \Gamma)\} \longrightarrow Gr_n(V)$$

is the positive subbundle. Note that Y_s' is of relative dim s^2, so the index is s^2.

Now I would like to carry out the Morse decomposition. Put $X = Gr_n(V)$ and introduce the increasing sequence of closed subspaces

$$X_0 \subset X_1 \subset X_2 \subset \cdots$$

$$X_s = \bigcup_{t\le s} \operatorname{Im} Y_t' = \{W \mid \varphi_x(W) \to \coprod_{t\le s} F_t \quad \text{as } x \to 0\}$$

Then X_s consists of W such that $\exists (K,I) \in Gr_s(V^0)\times Gr_{n-s}(V')$ with $I \subset W \subset K\oplus V'$, or better those $W \ni$

$$\dim W \cap V' \geqslant n-s$$

$$X_s - X_{s-1} = \{ W \mid \dim(W \cap V') = n-s \}$$

$$\begin{tikzcd} & V' \arrow[dr, dash] & \\ \overset{n-s}{I} \arrow[ur, dash] \arrow[dr, dash] & & \overset{s+n}{K \oplus V'} \\ & W \arrow[ur, dash] & \end{tikzcd}$$

Clearly $X_s - X_{s-1}$ is the vector bundle over F_s with fibre $\mathrm{Hom}(K, V'/I)$. Thus X_s/X_{s-1} is the Thom space of this vector bundle and we have a Gysin sequence

$$\longrightarrow H^{*-2s^2}(F_s) \xrightarrow{\iota_*} H^*(X_s) \longrightarrow H^*(X_{s-1}) \longrightarrow \cdots$$

The goal is to show that this sequence is split exact, precisely that ι_* is injective. To do this the fastest method seems to be the following. First note that F_s has only even dimensional cohomology. So arguing by induction the above exact sequence shows that $H^{odd}(X_s) = 0$ for all s and that the sequence is exact. (This is just the old fact - if all indices of critical points are even, then the Morse theory is perfect.)

But a slightly better understanding results if we use equivariant cohomology for the circle action. In equivariant cohomology ι_* is injective because $\iota^*\iota_*$ is multiplication by the Euler class which is a non-zero divisor as $H^*_S(F_s)$ is free. Thus inductively $H^*_S(X_s)$ is free for each s because one has short exact sequences

$$0 \longrightarrow H^*_S(F_s) \longrightarrow H^*_S(X_s) \longrightarrow H^*_S(X_{s-1}) \longrightarrow 0.$$

Then one gets the assertion about H^* by using $H^*(X) = H^*_S(X)/u\,H^*_S(X)$ (from usual Gysin)

At this point I have checked the fact that the cohomology of the Grassmannian decomposes into $\bigoplus_s H^{*-2s^2}(F_s)$, but I still want to check the explicit nature of the decomposition.

So far I haven't ~~[illegible]~~ used the ~~[illegible]~~ positive manifolds through the critical submanifolds. Let's fix notation: (slightly different from the above)

$$Y_s = \underset{\text{thru } F_s}{\text{negative subm.}} = \{(K, I, W) \mid I \subset W \subset K \oplus V'\}$$

$$Z_s = \underset{\text{thru } F_s}{\text{pos. subm.}} = \{(K, I, P) \mid K \subset P \subset V^0 \oplus I\}$$

These are resolutions of subvarieties of X, e.g. Y_s resolves $X_s = \{W \mid \dim(W \cap V') \geq n-s\}$.

We want to lift the image of $\iota_* : H^*(F_s) \to H^*(X_s)$ into $H^*(X)$, and we do this as follows.

$$\begin{tikzcd} Z_s \arrow[r, "j"] \arrow[d, shift right, "\pi'"'] & X \\ F_s \arrow[u, shift right, hook, "i'"'] \arrow[r, hook, shift left, "i"] & Y_s \arrow[u, "j'"'] \arrow[l, shift left, "\pi"] \end{tikzcd}$$

We've seen that ~~[illegible]~~ the map j, j' are transversal with intersection F_s. Now

$$(j')^* j_* (\pi')^* \alpha = \iota_* (i')^* \pi'^* \alpha = \iota_*(\alpha).$$

Thus we see that (modulo problems with the difference between X_s and its resolution Y_s), that the correspondence
$$\begin{tikzcd} Z_s \arrow[r] \arrow[d] & X \\ F_s \end{tikzcd}$$
gives the desired map of $H^*(F_s)$ to $H^*(X)$.

<u>August 5, 1986</u>

Here is a way to avoid the difficulty with the singularities of X_s:

$$\begin{tikzcd}
H^*(Z_s) \arrow[r, "\partial^*"] \arrow[d, "i'^*"] & H^*(X, X_{s-1}) \arrow[r] \arrow[d] & H^*(X) \arrow[d] \\
H^*(F_s) \arrow[r, "i_*", "\sim"'] & H^*(X_s, X_{s-1}) \arrow[r, hook] \arrow[d, "\wr"] & H^*(X_s) \arrow[d] \\
 & H^*(Y_s, \partial Y_s) \arrow[r] & H^*(Y_s)
\end{tikzcd}$$

The point is to use the transversal intersection

$$\begin{tikzcd}
Z_s \arrow[r, hook] & X - X_{s-1} \\
F_s \arrow[u, hook] \arrow[r, hook] & X_s - X_{s-1} \arrow[u, hook] \arrow[d, equal] \\
 & Y_s - Y_{s-1}
\end{tikzcd}$$

Conversations with Graeme: He gave me two nice proofs of the identity

$$\boxed{\frac{1}{\prod_{i>0}(1-q^i)} = \sum_{s\geq 0} \frac{q^{s^2}}{\left(\prod_{i=1}^{s}(1-q^i)\right)^2}}$$

One ~~[illegible]~~ uses the Jacobi identity

$$\prod_{i\geq 1}\left(1+t^{-1}q^{-\frac{1}{2}+i}\right)\left(1+t\,q^{-\frac{1}{2}+i}\right) = \frac{\sum q^{\frac{n^2}{2}} t^n}{\prod_{i>0}(1-q^i)}$$

Taking the coefficient of t^0 gives

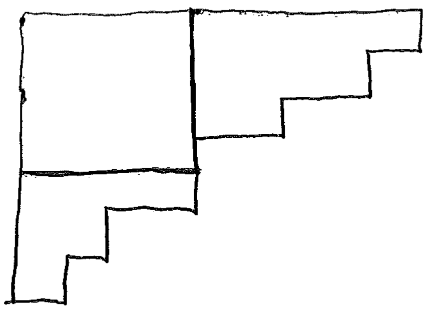
$$\frac{1}{\prod_{i>0}(1-q^i)} = \sum_s \left(q^{-\frac{s}{2}}\right)^2 \left(\sum_{i_1<\dots<i_s} q^{i_1+\dots+i_s}\right)^2$$

Now
$$\sum_{i_1<\dots<i_s} q^{i_1+\dots+i_s} = \sum_{0<i_1} q^{s i_1} \sum_{0<i_2-i_1<i_3-i_2<\dots<i_s-i_1} q^{(i_2-i_1)+\dots+(i_s-i_1)}$$

$$= \frac{q^s}{1-q^s}\cdot\frac{q^{s-1}}{1-q^{s-1}}\cdots\frac{q}{1-q} = \frac{q^{\frac{s(s+1)}{2}}}{\prod_{i=1}^{s}(1-q^i)}$$

so the identity follows.

The other proof comes from Hardy + Wright and is based on the fact that in an Young diagram one can put a ! largest square



and so obtain an s, a partition of s, and a dual partition of s.

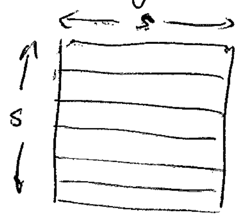
Returning to the first proof, it is obtained from an isomorphism of representations

$$\bigoplus_{s\geq 0} \Lambda^s H_+ \otimes \Lambda^s \overline{H_-} \cong S(H_+)$$

of the circle. In general there are many mysteries connected with the coincidence between the cohomology classes of Schubert varieties and characters. For example the cycle in the Grassmannian $Gr_n(V)$ where $\dim \Gamma \cap V^0 \geq s$ which we know is represented by

$$\begin{vmatrix} c_1 & \cdots & c_s \\ c_2 & \cdots & c_{s+1} \\ & & \\ c_s & \cdots & c_{2s-1} \end{vmatrix}$$

is linked to the representation [redacted] belonging to the Young diagram



whose character is given by a similar determinant.

The only mechanism I know which links cohomology to characters involves Frobenius.

Is there any significance to the isom

$$H^*(\mathbb{Z} \times BU) \simeq \Lambda H^+ \otimes \Lambda \overline{H^-} \ ?$$

I still have to check that the sum of the projectors for different s on $H^*(Gr_n(V))$ is the identity. This we understand partially in terms of a deformation of the [redacted] diagonal cohomology class in $H^*(X \times X)$, $X = Gr_n(V)$, to the sum of the correspondences belonging to each s.

Note that the diagonal $\Delta : X \longrightarrow X \times X$ is where the canonical section of

$$Hom\,(pr_1^*(S), pr_2^*(Q)),$$

which associates to (Γ, W) the map $\Gamma \subset V \longrightarrow V/W$, vanishes. Thus

$$\Delta_* 1 = c_{mn}(pr_1^*(\check{S}) \otimes pr_2^*(Q))$$

Let's call this canonical section s, and [redacted] look at the question of doing the [redacted] deformation for the vector bundle + section. The idea here is that if I find a vector bundle of rank mn $\mathcal{E}$ over $\mathbb{P}^1 \times$ [redacted] $X \times X$ agreeing with $pr_1^*(\check{S}) \otimes pr_2^*(Q)$ at $t = 1$, then $\Delta_* 1$ will be the restriction of $c_{mn}(\mathcal{E})$ for any t, so that if a properly vanishing section of $\mathcal{E}_\infty$ is found, then its zero set represents $\Delta_* 1$.

~~[illegible]~~

Note that $pr_1^*(\check{S}) \otimes pr_2^*(Q)$ over $X \times X$ admits $V^* \otimes V$ as a space of sections, e.g., a homomorphism $\theta : V \to V$ induces $\Gamma \longrightarrow V/W$ for each (Γ, W). So $\varphi_t = \begin{pmatrix} 1 & 0 \\ 0 & t \end{pmatrix}$ (rel. to $V = V^0 \oplus V^1$) gives a family of sections transversal to zero. The problem is now to extend the family $\mathcal{E}_t = pr_1^*(\check{S}) \otimes pr_2^*(Q)$ over $\mathbb{G}_m \times X \times X$ to $\mathbb{P}^1 \times X \times X$ in such a way that a suitable section can be found over $t = \infty$.

One method to construct $\underset{\sim}{\mathcal{E}}$ might be to extend the trivial bundle $V^* \otimes V$ over $\mathbb{G}_m$ to a vector bundle $\mathcal{H}$ over $\mathbb{P}^1$. We have a map

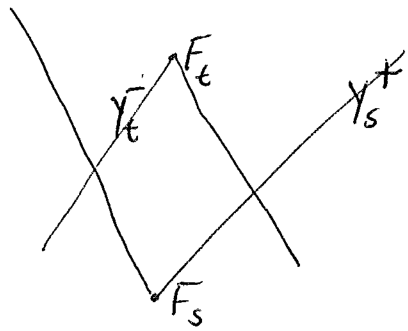
$$\mathbb{G}_m \times X \times X \longrightarrow \check{Gr}_{mn}(\mathcal{H})$$

which might extend to $\mathbb{P}^1 \times X \times X$. For example, I could take $\mathcal{H} = \widetilde{V^* \otimes V}$ over $\mathbb{P}^1$, but then the section φ_t doesn't have a limit.

<u>August 6, 1986</u>

Let's calculate the composition of correspondences

$$\begin{tikzcd} & Y_s^+ = \{(K,I,\Gamma)\} \arrow[r] \arrow[d, "\operatorname{cod} s^2"] & F_s \\ Y_t^- = \{(K',I',W)\} \arrow[r] \arrow[d, "\dim t^2"] & Gr_n(V) & \\ F_t & & \end{tikzcd}$$



carefully for $t > s$.

$$\overset{s}{K} \subset \overset{n}{\Gamma} \subset \overset{m}{V^0} \oplus \overset{n-s}{I}$$

$$\dim(Y_s^+) = s(m-s) + (n-s)s + (n-s)(m-s) = nm - s^2$$

$$\overset{n-t}{I'} \subset \overset{n}{W} \subset \overset{t}{K'} \oplus \overset{n}{V^1}$$

$$\dim(Y_t^-) = t(m-t) + (n-t)t + t^2 = mt + nt - t^2$$

$$nm - (mt + nt - t^2) = (n-t)(m-t)$$

Let's now compute the fibre product of Y_t^- and Y_s^+ over $X = Gr_n(V)$. It consists of $(K', I', K, I, \Gamma) \ni$

$$\begin{array}{ccccc} K & \subset & \Gamma & \subset & V^0 \oplus I \\ & & \| & & \\ I' & \subset & \Gamma & \subset & K' \oplus V^1 \end{array}$$

i.e.
$$\overset{s}{K} \oplus \overset{n-t}{I'} \subset \overset{n}{\Gamma} \subset \overset{t}{K'} \oplus \overset{n-s}{I}.$$

This intersection is non-singular: it is pairs of flags $(\overset{s}{K}, \overset{t}{K'})$ in V^0 and $(\overset{n-t}{I'}, \overset{n-s}{I})$ in V^1 together with Γ^n as above. It's dimension is

$$\left[s(m-s) + (t-s)(m-t)\right] + \left[(n-t)(t) + (t-s)s\right] + (t-s)(t-s)$$

$$\cancel{sm} - s^2 + tm - t^2 - \cancel{sm} + \cancel{st} + nt - \cancel{t^2} + \cancel{ts} - \cancel{s^2} + \cancel{t^2} - 2\cancel{s}t + \cancel{s^2}$$

$$= mt + nt - t^2 - s^2 = (\dim Y_t^-) - s^2$$

This shows that the fibre product has the correct dimension. With a little more work one can probably check the transversality and conclude that the fibre product indeed cohomologically represents the product of the correspondences.

However note that where $t > s$, the map from the fibre product to $F_t \times F_s$ has fibres of dim $(t-s)^2 > 0$. This means that the cohomology class of the fibre product in $H^*(F_t \times F_s)$ is zero for $t > s$.

Thus we have checked that the correspondences Y_s^+, Y_s^- define injections and projections respectively relative to the decomp.

$$H^*(X) = \bigoplus_{s \geq 0} H^{*-2s^2}(F_s).$$

I think that the next project will be to understand exactly what it means for the class ch_k to come from F_1. I know if I start with the character form of degree $2k$ on X and construct the associated form on F_1 by the correspondence

$$\begin{array}{ccc} Y_1 = \{(K, I, W)\} & \longrightarrow & Gr_n(V) \\ \downarrow & & \\ \mathbb{P}V^0 \times \check{\mathbb{P}}V^1 & & \end{array}$$

then everything is invariant under $U(V^0) \times U(V^1)$, so we obtain an invariant $2k-2$ form on $\mathbb{P}V^0 \times \check{\mathbb{P}}V^1$. Since we know the class of the form it is uniquely determined.

Similarly since we know that the class of ch_k vanishes when moved to F_s, we know that the character form on X, when restricted to Y_s^- and integrated over the fibre to F_s, gives zero.

What I would like to do further is to see if there is any significance to the fact that ch_k comes from F_1 and not from F_s for $s > 1$. We have found a closed form on Y_1^+ which when pushed into X represents ch_k. This means we have a current representing ch_k supported on the image of Y_1^+ which is the cycle of Γ with $\dim(\operatorname{Ker}\Gamma) \geq 1$.

Given $f: S \longrightarrow X$ we can ask for the same kind of behavior. Thus we take the character form on X pull back to S; in fact take the whole deformation $f^* \varphi_t^* (ch_k)$. The question is whether these forms converge to a current as $t \longrightarrow \infty$, and if so, can one describe the current as the push-forward of a form on a variety over S.