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April 21, 1986

Dear Mathai,

I figured out the excision process you need to extend the Thom class in K-theory for the normal bundle to the ambient manifold. This is probably contained in the Atiyah, Bott, Shapiro paper "Clifford Modules".

To fix the notation, let Z be a closed subset of X and let U be an open subset of X containing Z. Excision says the K-theories of U and X with supports in Z are the same:

$$K(X, X-Z) \xrightarrow{\sim} K(U, U-Z)$$

Suppose given an element of the latter which is represented by two vector bundles E^0, E^1 on U and a vector bundle morphism $\varphi : E^0 \longrightarrow E^1$ which is an isomorphism over $U-Z$. We wish to construct an extension of (E^0, E^1, φ) to X. By taking the direct sum of φ with $id : F \rightarrow F$, where F is a vector bundle over U such that $E^0 \oplus F$ is trivial, we can suppose the bundle E^0 extends to a bundle $\tilde{E}^0$ on X. Then we obtain an extension $\tilde{E}^1$ of E^1 to X by gluing E^1 over U and $\tilde{E}^0$ over $X-Z$ together with the "clutching function" given by the isomorphism φ over $U-Z$. There is also a natural extension $\tilde{\varphi} : \tilde{E}^0 \longrightarrow \tilde{E}^1$ given by φ over U and the identity of $\tilde{E}^0$ over $X-Z$. Thus one has an extension $(\tilde{E}^0, \tilde{E}^1, \tilde{\varphi})$ which lifts the class in $K(U, U-Z)$ to $K(X, X-Z)$.

The above construction is not convenient for analytical purposes where one has inner products given on E^0, E^1 and where one wants to work with the hermitian endomorphism $L = \begin{pmatrix} 0 & \varphi^* \\ \varphi & 0 \end{pmatrix}$. In this case one applies the polar decomposition:

$$|L| = \sqrt{L^2} = \begin{pmatrix} \sqrt{\varphi^*\varphi} & 0 \\ 0 & \sqrt{\varphi\varphi^*} \end{pmatrix}$$

$$\frac{L}{|L|} = \begin{pmatrix} 0 & \varphi^*(\varphi\varphi^*)^{-1/2} \\ \varphi(\varphi^*\varphi)^{-1/2} & 0 \end{pmatrix}$$

to obtain a unitary isomorphism $u = \varphi(\varphi^*\varphi)^{-1/2} = (\varphi\varphi^*)^{-1/2}\varphi$ of E^0 with E^1 over $U - Z$. Assuming E^0 extends to $\tilde{E}^0$ we construct $\tilde{E}^1$ as before using this unitary u as clutching function. Then u extends to a unitary isomorphism $\tilde{u}$ of $\tilde{E}^0$ with $\tilde{E}^1$ over $X - Z$. Moreover $u\sqrt{\varphi^*\varphi} = \varphi$ extends over U. Hence if $\{\rho, 1-\rho\}$ is a partition of unity subordinate to the covering $U, X-Z$, we see that

$$\tilde{\varphi} = \begin{cases} \varphi & \text{near } Z \\ u(\rho\sqrt{\varphi^*\varphi} + 1 - \rho) & \text{on } X - Z \end{cases}$$

is a map $\tilde{\varphi} : \tilde{E}^0 \longrightarrow \tilde{E}^1$ which extends φ near Z and which is an isomorphism on $X - Z$.

Problem: Index over $B\mathcal{G}'$.

$B\mathcal{G}'$ classifies pairs (E^0, E^1) of Hilbert bundles together with an index zero isomorphism of E^0 with E^1 modulo compacts.

First construction: Use Kuiper to trivialize E^0, E^1. Then the isomorphism modulo compacts gives a map from the parameter space to $\mathcal{U}(\mathcal{Q})$. Here we are using the fibration

$$\mathcal{G}' \longrightarrow \mathcal{U}(H) \times \mathcal{U}(H) \longrightarrow \mathcal{U}(\mathcal{Q})$$

in the same way we used

$$\mathcal{G} \longrightarrow \mathcal{U}(H) \longrightarrow \mathcal{I}(\mathcal{Q}).$$

Second construction: Use Kuiper to trivialize E^0 so that it becomes the trivial bundle with fibre H. Then E^1 is a Hilbert bundle with a trivialization mod compacts i.e. a map $Y \times H \longrightarrow E^1$ which is an isom. mod $\mathcal{K}$. Now such a bundle can be embedded in the trivial bundle with fibre $H+H$ so that

$$E^1_y \subset H + H \xrightarrow{pr_1} H$$

is consistent with the given isom mod $\mathcal{K}$ of H and E^1_y. This is because the space of such embeddings

$$\left\{ \begin{pmatrix} \alpha \\ \beta \end{pmatrix} : H \longrightarrow H+H \;\middle|\; \begin{matrix} \alpha^*\alpha + \beta^*\beta = 1 \\ \alpha \cong 1 \end{matrix} \right\}$$

is contractible.

What this means is that E^0, E^1 have been embedded in the trivial bundle with fibre $H+H$ so that $E^0 =$ first summand and $E^1 \equiv E^0$ mod $\mathcal{K}$.

Thus we have a map from our parameter space to the ~~restricted~~ Grassmannian.

Next let's consider the general case. Given a pair of Hilbert bundles E^0, E^1 together with an isomorphism mod $\mathcal{K}$ between them, we embed E^0 in a trivial Hilbert bundle: $E^0 \oplus F = \tilde{H}$. Then adding $id : F \leftrightarrows F$ to $E^0 \rightleftarrows E'$ we can suppose E^0 is trivial. Then we are in the above situation, so that in principle at least we have a map to the restricted Grassmannian.

April 22, 1986

Here is the index map over $B\mathcal{G}'$. Given two Hilbert bundles E^0, E' with an isomorphism mod compacts between them, we first add the identity map of a suitable bundle in order to assume E^0 is trivial, say $\tilde{H}$ with fibre H. We lift the isomorphism mod compacts to a essentially unitary contraction operator

$$T : \tilde{H} \longrightarrow E'.$$

Then we obtain an embedding

$$\begin{pmatrix} T^* \\ j\sqrt{1-TT^*} \end{pmatrix} : E' \hookrightarrow \tilde{H} \oplus \tilde{H}$$

where j is any embedding of E' into H. Then at each point y of the parameter space we get a subspace E'_y of $H \oplus H$ which is congruent to $H \oplus 0$ mod $\mathcal{K}$. This gives a map from the parameter space to the restricted Grassmannian.

April 22, 1986 (cont.)

Lets try to understand this construction a bit better when we take for BG' the space of pairs (e,e') of infinite rank + nullity projectors on H such that $e \equiv e'$ mod compacts. Then we have the Hilbert bundles $(e,e') \longmapsto eH, e'H$ and the essentially unitary contractions

$$\begin{tikzcd} eH \arrow[r, shift left, "e'e"] & e'H \arrow[l, shift left, "ee'"] \end{tikzcd}$$

We first add an identity map to make the first bundle trivial:

$$\begin{tikzcd} H = \begin{array}{c} eH \\ \oplus \\ (1-e)H \end{array} \arrow[r, "\begin{array}{c} e'e \\ \oplus \\ (1-e) \end{array}"] & \underbrace{\begin{array}{c} e'H \\ \oplus \\ (1-e)H \end{array}}_{E^1} \end{tikzcd}$$

Call this map T. We use $T^* : E^1 \longrightarrow H$ plus some other map from E^1 to H to make an embedding of E^1 into $H \oplus H$.

$$\begin{tikzcd}[row sep=small] & \overbrace{H}^{} & \oplus & H \\ E^1 = \quad e'H \arrow[r, "ee'"] \arrow[rrr, dashed, bend left] & eH & \oplus & (1-e)H \\ \oplus & \oplus & & \\ (1-e)H \arrow[r, "1-e"'] & (1-e)H & & \end{tikzcd}$$

dotted arrow is $(1-e)e'$

Thus we seem to obtain the following map to the restricted Grassmannian. Given e, e' we have an isomorphism

$$H \oplus H = \begin{array}{c} eH \\ \oplus \\ (1-e)H \end{array} \oplus \begin{array}{c} (1-e)H \\ \oplus \\ eH \end{array} \simeq \begin{array}{c} H \\ \oplus \\ H \end{array}$$

depending on e. We then take the subspace $\begin{array}{c} e'H \\ \oplus \\ (1-e)H \end{array}$ in $\begin{array}{c} H \\ \oplus \\ H \end{array}$ and transform back by this isom.

April 23, 1986

I discovered two days ago that there is an index map on the space of pairs of projectors (e,e') congruent mod compacts to the restricted Grassmannian. It seems worthwhile to understand this very well.

Let's consider the case where e, e' project onto lines L, L' in $H = \mathbb{C}^2$. Thus the space of pairs (e,e') is $\mathbb{P}^1 \times \mathbb{P}^1$, and we are trying to map it to $Gr_2(\mathbb{C}^4)$. This map is determined by a 2 diml vector bundle over $\mathbb{P}^1 \times \mathbb{P}^1$. It is the index bundle, formally the difference of $pr_1^* \mathcal{O}(-1)$ and $pr_2^* \mathcal{O}(-1)$, which we realize concretely as the bundle

$$pr_1^* \mathcal{O}(-1) \oplus pr_2^* \mathcal{O}(-1)^{\perp}$$

whose fibre at (L', L) is $\quad L' \oplus L^{\perp} = e'H + (1-e)H.$

This vector bundle is canonically trivial when restricted to $\Delta(\mathbb{P}^1) \subset \mathbb{P}^1 \times \mathbb{P}^1$. We want to embed it in the trivial bundle with fibre $H \oplus H$ so that it gives the first factor H over $\Delta(\mathbb{P}^1)$. This means that we have the embedding $L' \oplus L^{\perp} \subset H \oplus H$ given when $L' = L$ as the isom. $L \oplus L^{\perp} = H \xhookrightarrow{i_1} H \oplus H$ and then it must be extended to the rest of $\mathbb{P}^1 \times \mathbb{P}^1$.

What I have done is to fix L in $H \oplus 0$, and then let L' vary in $L \oplus L^{\perp} \subset H \oplus H$ using the natural identification of $L \oplus L^{\perp}$ with H. I would like to find something a bit more symmetric with respect to L and L'. The above has the property that if $E = \operatorname{Im}\{L' \oplus L^{\perp} \hookrightarrow H \oplus H\}$, then $L = E \cap (H \oplus 0)$

I don't know if it is important to find a symmetric formula, but I would like a simple formula so that I can see the character forms.

The normal way to proceed would be to work with the normal bundle to $\Delta\mathbb{P}^1$ and to see how to extend the embedding to it.

$\mathbb{P}^1$ is the 2 sphere, hence there is a map of $\mathbb{P}^1 \times \mathbb{P}^1$ to the unit interval given by the distance. This interval is the quotient by the action of $PU(2)$. Over the ends of the interval we have $\mathbb{P}^1$ embedded diagonally and as the graph of the antipodal map $L \longmapsto L^\perp$.

Given $(L, L') \in \mathbb{P}^1 \times \mathbb{P}^1$ we can think of L' being on the half great circle joining L to $L^\perp$. There's an open covering of $\mathbb{P}^1 \times \mathbb{P}^1$ by two open sets: one consists of (L, L') with $L' \neq L^\perp$ and it deform. retracts to $\Delta\mathbb{P}^1$; the other consists of (L, L') with $L' \neq L$ and it deform. retracts to the graph of $L \longmapsto L^\perp$. The intersection deform. retract to the set of (L, L') where L' is 45° w.r.t. L. This set is $\simeq U(2)/$scalars.

~~Now the vector bundle $(L, L') \longmapsto$~~ [redacted] $L' \oplus L^\perp$ ~~which we want to embed in $H \oplus H$ looks nice at the ends.~~ [redacted] ~~Where $L \neq L'$ we have a canonical identification~~

~~$$L \oplus L^\perp = H = H \oplus 0 \hookrightarrow H \oplus H$$~~

~~and where~~

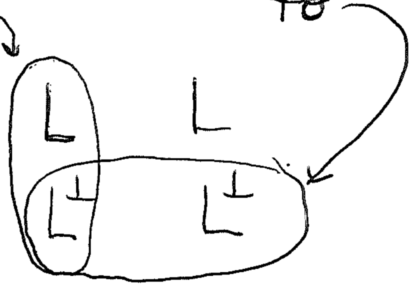
The vector bundle over $\mathbb{P}^1 \times \mathbb{P}^1$ given by $(L, L') \longmapsto L^\perp \oplus L'$ which we want to embed in $H \oplus H$ looks nice at the ends. Where $L = L'$ we have ~~[illegible]~~ a canonical identification $L^\perp \oplus L' = H$ which we can embed into $H \oplus H$ as the first factor. Where $L^\perp = L'$ we have $L' \oplus L'$ which we can embed in an obvious way into $H \oplus H$. We would like now to extend these embeddings in a natural way.

Think of a general pair (L, L') as a point on the half great circle joining L to $L^\perp$. At the ends we have

$$(L, L) \longmapsto L^\perp \oplus L = H = H \oplus 0 \hookrightarrow H \oplus H$$

$$(L, L^\perp) \longmapsto L^\perp \oplus L^\perp \subset H \oplus H.$$

So we want therefore a path in the Grassmannian going from to



depending on a ~~[illegible]~~ line ~~[illegible]~~ L' 45° relative to L. More generally as L' goes from L to $L^\perp$ we want the 2 plane to go from $\begin{smallmatrix} L \\ \oplus \\ L^\perp \end{smallmatrix}$ to $L^\perp \oplus L^\perp$.

The only way I can see how to do this is to embed L' in $L \oplus L^\perp$ as I did above.

This construction treats (L, L') asymmetrically because the 2 plane E attached to this pair has

$$E(L,L') \cap H = L^{\perp}.$$

Perhaps the asymmetry is inevitable because of what the family looks like at the ends.

April 24, 1986

Transgression: Suppose $E \xrightarrow{\pi} B$ is a principal G-bundle, with E connected. Let η be a transgression form on E: $d\eta = \pi^*(\omega)$. Let $\iota_e : G \to E$ be the map $g \longmapsto e.g$. Why are the forms

$$\iota_e^*(\eta) \quad \text{on } G$$

for different e cohomologous? Note that η is not closed, so the answer is not the obvious homotopy argument, at least on the surface.

However suppose we choose a path in E joining e to e'. The we get

$$\begin{tikzcd} I \times G \arrow[r, "h"] \arrow[d] & E \arrow[d] \\ I \arrow[r, "\bar{h}"] & B \end{tikzcd}$$

and $d\, h^*(\eta) = h^*(d\eta) = \bar{h}^*(\omega)$. If degree$(\omega) \geq 2$ this implies $h^*(\eta)$ is closed, hence $\iota_e^*(\eta)$ and $\iota_{e'}^*(\eta)$ are cohomologous.

This argument breaks down when deg$(\omega) = 1$. For example take $\mathbb{Z} \to \mathbb{R} \to S^1$, $\eta =$ function x. Then the restriction of η to the different fibres gives different locally constant functions on $\mathbb{Z}$ and these are are different in $H^0(\mathbb{Z},\mathbb{C})$.

This suggests transgression is linked to filtration ideas.

Even though I have constructed a map from $B\mathcal{G}'$ to the restricted Grassmannian, there appear to be difficulties using it for the purposes of transgression. What I want is a map of fibre spaces

$$\begin{tikzcd} \mathcal{G}' \arrow[d] & U(\mathcal{X}) \arrow[d] \\ P\mathcal{G}' \arrow[r] \arrow[d] & P_{res} \arrow[d] \\ B\mathcal{G}' \arrow[r] & \mathcal{I}_{res,(0)} \end{tikzcd}$$

consistent with the map

$$\mathcal{G}' \longrightarrow U(\mathcal{X})$$
$$(g,g') \longmapsto g'g^{-1}.$$

Then I can take the transgression form on P_{res} and produce one on $P\mathcal{G}'$.

Unfortunately the map $\mathcal{G}' \longrightarrow U(\mathcal{X})$ is not a homomorphism.

Another way to see the difficulty is to try to work with the vector bundle

$$(e,e') \longmapsto (\operatorname{Im} e)^{\perp} \oplus (\operatorname{Im} e')$$

on $B\mathcal{G}'$ which the map to the Grassmannian classifies. If we left to $P\mathcal{G}'$, then we have frames in $(\operatorname{Im} e)$ and $(\operatorname{Im} e')$, but that does not give us a frame in $(\operatorname{Im} e)^{\perp}$.

Let us consider another question. Put $G = U_{res}$. We have seen there is a canonical element of $K^0(G)$. It is represented by either of the maps

$$U(2) \xleftarrow{\text{Toep.}} G \longrightarrow \mathcal{I}_{res}$$

but actually we should take the former as it is completely canonical (independent of a choice of F over ε.)

On the other hand the fibration

$$G \longrightarrow U(H) \xrightarrow{\cdot\varepsilon} \mathcal{I}(2)$$

provides a homotopy equivalence

$$\Omega\mathcal{I}(2) \sim G$$

and periodicity provides a h. eq.

$$U(2) \rightsquigarrow \Omega\mathcal{I}(2)$$

The question is whether

$$\begin{tikzcd} G \arrow[r, phantom, "\sim"] \arrow[dr] & \Omega\mathcal{I}(2) \\ & U(2) \arrow[u, "\sim"'] \end{tikzcd}$$

commutes. This can be proved as follows: First from the AS proof we have that

$$\begin{tikzcd} \mathcal{F}_0 \arrow[rr] \arrow[d, "\sim"] & & \mathcal{I}_{res} \\ U(2) \arrow[r, "Bott"'] & \Omega(\mathcal{I}(2)) \arrow[ur] & \end{tikzcd}$$

path lifting + endpt for

$$\begin{tikzcd} \mathcal{I}^0_{res} \arrow[d] \\ \mathcal{I}(H) \arrow[d] \\ \mathcal{I}(2) \end{tikzcd}$$

Commutes. Next we have the map of fibrations

$$\begin{tikzcd} G \arrow[r] \arrow[d] & U(H) \arrow[r] \arrow[d, "\sim"] & \mathcal{I}(2) \arrow[d, "1"] \\ \mathcal{I}_{res} \arrow[r] & \mathcal{I}(H) \arrow[r] & \mathcal{I}(2) \end{tikzcd}$$

which gives a comm. triangle

$$\begin{tikzcd}
 & \mathcal{G} \arrow[d] \\
\Omega\mathcal{I}(2) \arrow[ur] \arrow[r] & \mathcal{I}_{res}
\end{tikzcd}$$

So we get a big comm. diagram

$$\begin{tikzcd}
 & \mathcal{G} \arrow[dl] \arrow[dr] & \\
\mathcal{F}_0 \arrow[ur, dotted] \arrow[rr] \arrow[d, "\sim"] & & \mathcal{I}_{res} \\
U(2) \arrow[r] & \Omega\mathcal{I}(2) \arrow[uu] \arrow[ur] &
\end{tikzcd}$$

Graded version of the above: There is a canonical element of $K^1(\mathcal{G}')$ which is represented by the map

$$U(\mathcal{K}^+) \hookrightarrow \mathcal{G}'$$

which is a homotopy equivalence. This is because $\mathcal{G}'$ is the semi-direct product of $U(H^+)$ acting on $U(\mathcal{K}^+)$. Apparently there is no natural map from $\mathcal{G}'$ to $\mathcal{I}(2)$.

We would like to check compatibility of

$$\mathcal{I}(2^+) \longrightarrow \Omega U(2^+)$$

with the maps

$$\begin{tikzcd}
 & U(\mathcal{K}^+) \\
\Omega U(2^+) \arrow[ur] \arrow[dr] & \\
 & \mathcal{G}'
\end{tikzcd}$$

defined by the fibrations

$$\begin{tikzcd}
U(\mathcal{K}^+) \arrow[r] \arrow[d] & U(H^+) \arrow[r] \arrow[d] & U(2^+) \arrow[d, equal] \\
\mathcal{G}' \arrow[r] & U(H^+)\times U(H^-) \arrow[r] & U(2^+)
\end{tikzcd}$$

This is all obvious and not very interesting.

April 25, 1986

Yesterday I learned that to use the canonical map $\mathcal{G}=U_{res}\longrightarrow U(\mathcal{Q}^+)$ is not advisable, since there doesn't seem to be a corresponding map $\mathcal{G}'\longrightarrow \mathcal{I}(\mathcal{Q})$. If we adopt this viewpoint, then the maps $\mathcal{G}\longrightarrow \mathcal{I}_{res}$, $\mathcal{G}'\longrightarrow U(\mathcal{K})$, which require the choice of an involution for their definition, are the best we can do for representing the canonical classes in $K^0(\mathcal{G})$, $K^1(\mathcal{G}')$ respectively.

Now on the classifying space level we do have canonical maps. Thus the fibrations

$$\mathcal{G}=U_{res}\longrightarrow U(H)\longrightarrow \mathcal{I}(\mathcal{Q})$$

$$\mathcal{G}'\longrightarrow U(H^+)\times U(H^-)\longrightarrow U(\mathcal{Q})$$

determine canonical homotopy equivalences

$$B\mathcal{G}\simeq\mathcal{I}(\mathcal{Q}) \quad , \quad B\mathcal{G}'\simeq U(\mathcal{Q})$$

which are consistent with the arrows

$$\mathcal{G}\longrightarrow \mathcal{I}_{res}$$

$$\Omega\mathcal{I}(\mathcal{Q})\longrightarrow \mathcal{I}_{res}$$

the latter being defined by

$$\mathcal{I}_{res}\longrightarrow \mathcal{I}(H)\longrightarrow \mathcal{I}(\mathcal{Q}).$$

Now we want to use different classifying spaces. $B\mathcal{G}$ is the Grassmannian of involutions F on $H=H^+\oplus H^-$ which commute with the grading ε mod $\mathcal{K}$ and which induce non-trivial splittings of H^+, H^- mod $\mathcal{K}$. The principal bundle $P\mathcal{G}$ over $B\mathcal{G}$ consists of all embeddings $V\overset{i}{\hookrightarrow}H$

such that [redacted] $\mathrm{Im}(j)$ is in $B\mathcal{G}$ and such that the splitting mod $\mathcal{K}$ of V induced by j, ε **coincides** with the given $\eta \in \mathcal{I}(2)$.

If we want to link the principal bundles $U(V) \longrightarrow \mathcal{I}(2)$ and $P\mathcal{G} \to B\mathcal{G}$, we look at the mixed space

$$\mathcal{I}(2) \longleftarrow (U(V) \times P\mathcal{G})_{\mathcal{G}} \longrightarrow B\mathcal{G}$$

which consists of all embedding $V \hookrightarrow H$ whose image belongs to the Grassmannian $B\mathcal{G}$. The fibre over a point of $B\mathcal{G}$ is the isoms of V with this subspace; it's just $U(V) \sim pt$.

[redacted] Whatever model we choose for $B\mathcal{G}$, it classifies Hilbert bundles with a non trivial splitting mod $\mathcal{K}$. Over $\mathcal{I}(2)$ one has the [redacted] trivial Hilbert bundle with fibre V and the different splittings of V mod $\mathcal{K}$. Over the model for $B\mathcal{G}$ chosen we have the Hilbert bundle which is the subbundle, as for a Grassmannian: a point of $B\mathcal{G}$ is a subspace $W \subset H^+ \oplus H^-$ such that it commutes with ε modulo $\mathcal{K}$, and the induced splittings mod $\mathcal{K}$ of H^+ and H^- are non-trivial.

Let's say it better: $\mathcal{I}(2)$ is the space of non-trivial splittings mod $\mathcal{K}$ of V. $B\mathcal{G}$ is space of subspaces $W \subset H^+ \oplus H^-$ as above. $(U(V) \times P\mathcal{G})_{\mathcal{G}}$ is the space of embeddings $V \overset{j}{\hookrightarrow} H$ whose image is in $B\mathcal{G}$. Such an embedding induces a splitting mod $\mathcal{K}$ of V, whence we have the map to $\mathcal{I}(2)$. The fibre over $\tilde{\eta}$ is the space of embeddings $V \hookrightarrow H^+ \oplus H^-$ with the right kind of image and such that ε induces $\tilde{\eta}$. This is equivalent to $P\mathcal{G}$ and so is contractible.

What does it mean to lift a map $Y \longrightarrow \mathcal{I}(2)$ up to $(\mathcal{U}(V) \times P\mathcal{G})_{\varepsilon g}$? It means that we go from the trivial bundle over Y with fibre V and family of mod $\mathcal{K}$-splittings to a ~~[illegible]~~ fixed Hilbert space with splitting $H = H^+ \oplus H^-$ and a family of embeddings $j : V \longrightarrow H$.

Consider a fixed pair (V, A) with $A \in \mathcal{F}_1(V)$, and (H, F) with $F \in \mathcal{I}(H)$. We consider the space of embeddings $j : V \hookrightarrow H$ such that $j^* F j = A$. Let $\iota : V \longrightarrow W$, F' be a minimal expansion of A. Then for each j there is a unique embedding of W in H compatible with F' and F. So the space of embeddings j with F contracting to A is the same as the space of embeddings of the $F' = 1$ eigenspace on W into the $F = 1$ eigenspace on V product with the same for -1. Thus the space of embeddings is a product of two Stiefel manifolds. In infinite dimensions this space will be contractible.

The principal bundle $P\mathcal{G}$ consists of embeddings $j : V \longrightarrow H$ such that ε induces non-trivial involutions mod $\mathcal{K}$ on V and $V^\perp$ and such that the ~~involutory on~~ induced $\mathcal{K}$-splitting on V is the given ~~one~~ η. We see this space sits over $\mathcal{F}_1(V)_\eta$ with contractible fibres, so it should be possible to prove $P\mathcal{G}$ is contractible in this way.

Summary: I have two classifying spaces for $\mathcal{G} = U_{res}$. The first is $\mathcal{I}(2)$ and it describes K-splittings on the trivial Hilbert bundle. The second, which I denoted $B\mathcal{G}$, is a suitable Grassmannian of subspaces in a fixed Hilbert space $H = H^+ \oplus H^-$ with grading ε. ~~On the former picture the~~ It describes Hilbert bundles with K-splitting which are embedded in a trivial situation with fibre $H = H^+ \oplus H^-$.

We would like to go from one description to the other by a constructive procedure. Replace $\mathcal{I}(2)$ by $\mathcal{F}_1 = \mathcal{F}_1(V)$. Given a family of A in $\mathcal{F}_1(V)$ we can expand it to the family of involutions on $V \oplus V$

$$F = \begin{pmatrix} A & B \\ B & -A \end{pmatrix}$$

and embed further into $\underbrace{(V \oplus V)}_{H^+} \oplus \underbrace{(V \oplus V)}_{H^-} = H$ by splitting $V \oplus V$ into the $F = 1$ and $F = -1$ eigenspaces and then putting these into the two factors of H. This gives a family of embeddings of V into H such that ε contracts to the family of A's.

Formula: The embedding of V in $V^{\oplus 4}$ is given by

$$j = \begin{pmatrix} \frac{A+1}{2} \\ \frac{B}{2} \\ \frac{A-1}{2} \\ \frac{B}{2} \end{pmatrix} \qquad j^* j = \left(\frac{A+1}{2}\right)^2 + \left(\frac{B}{2}\right)^2 + \left(\frac{A-1}{2}\right)^2 + \left(\frac{B}{2}\right)^2$$

$$= \frac{2A^2 + 2 + 2B^2}{4} = 1$$

Then $j^* \varepsilon j$ is

$$\begin{pmatrix} \frac{A+1}{2} & \frac{B}{2} & \frac{A-1}{2} & \frac{B}{2} \end{pmatrix} \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix} \begin{pmatrix} \frac{A+1}{2} \\ \frac{B}{2} \\ \frac{A-1}{2} \\ \frac{B}{2} \end{pmatrix} = \left(\frac{A+1}{2}\right)^2 + \cancel{\left(\frac{B}{2}\right)^2} - \left(\frac{A-1}{2}\right)^2 - \cancel{\left(\frac{B}{2}\right)^2}$$

$$= A$$

Notice that we have constructed a section:

$$\begin{tikzcd}
\mathcal{F}_1(V) \arrow[d, "\sim"'] \arrow[r, dashed, bend left] & (U(V)\times P\mathcal{G})/\mathcal{G} \arrow[l] \arrow[r] \arrow[dl] & B\mathcal{G} \\
\mathcal{J}(Q(V)) & &
\end{tikzcd}$$

using $H = V^{\oplus 4}$, $\varepsilon = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}$.

Next we want to go the other way. Thus we start with a family of subspaces $W \subset H$ such that ε induces non trivial K-splittings on W and $W^\perp$. If F is the involution determined by W, we can contract F to H^+ obtaining an element of $\mathcal{F}_1(H^+)$. Thus we have a map

$$B\mathcal{G} \longrightarrow \mathcal{F}_1(H^+).$$

Let's compute the composition with the previous map. The involution determined by j is

$$F = 2jj^* - 1$$

and if $i : H^+ \hookrightarrow H$, the contraction of F to H^+ is

$$i^*Fi = 2(i^*j\,j^*i) - 1$$

$$i = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \qquad j = \begin{pmatrix} \frac{A+1}{2} \\ B/2 \\ \frac{A-1}{2} \\ B/2 \end{pmatrix}$$

$$i^*j = \begin{pmatrix} \frac{A+1}{2} \\ B \end{pmatrix}$$

$$2i^*jj^*i - 1 = \begin{pmatrix} 2\left(\frac{A+1}{2}\right)^2 - 1 & \frac{(A+1)B}{2} \\ B(A+1)/2 & \frac{B^2}{2} - 1 \end{pmatrix} = \begin{pmatrix} A - \frac{B^2}{2} & \frac{B(A+1)}{2} \\ \frac{B(A+1)}{2} & B^2/2 - 1 \end{pmatrix}$$

Mod K this is $\begin{pmatrix} A & 0 \\ 0 & -1 \end{pmatrix}$ which is just the family A on V extended by -1 on the complementary copy of V in $H^+ = V \oplus V$.

Idea for tomorrow: The involution

$$\gamma = \begin{pmatrix} A & B \\ B & -A \end{pmatrix} \qquad B = \sqrt{1-A^2}$$

is obtained by applying a 2×2 matrix function to A. So if I want to study the commutator $[F, \gamma]$ or $[e, \gamma]$, I might be able to express the result in terms of the resolvent of A, and use the derivation property of the inverse

$$\left[e, \frac{1}{\lambda - A}\right] = \frac{1}{\lambda - A} [e, -A] \frac{1}{\lambda - A}$$

Graded case of the two classifying spaces: $B\mathcal{G}'$ versus $U(2)$.

April 26, 1986

The problem under discussion is to construct a 1-form on the model for $B\mathcal{G}$, which, I recall, consists of projectors e on $H^+ \oplus H^-$ commuting mod $\mathcal{K}$ with the grading ε.

This 1-form should be constructed using the operators e, ε and the operator 1-form de. To get scalar valued forms one ~~[illegible]~~ has to use the trace. The only candidates then are of the form

$$tr\left(\Phi(\varepsilon, e)\, de\right),$$

since no matter where de occurs in the trace of a monomial constructed with e, ε, it can be moved to the right. (Actually, Φ is apt to be a non-polynomial function of ε, e).

I have discussed the index map

$$(*) \qquad B\mathcal{G} \longrightarrow \mathcal{F}_1(H^+)$$

which takes e into the contraction of $F = 2e-1$ on H^+. Recall the formulas

$$F = \begin{pmatrix} \alpha & \beta^* \\ \beta & \gamma \end{pmatrix} \qquad \frac{F\varepsilon + \varepsilon F}{2} = \frac{g+g^{-1}}{2} = \begin{pmatrix} \alpha & 0 \\ 0 & -\gamma \end{pmatrix}$$

$$\left|\frac{g-g^{-1}}{2}\right| = \begin{pmatrix} \sqrt{\beta^*\beta} & 0 \\ 0 & \sqrt{\beta\beta^*} \end{pmatrix}$$

Digression: Yesterday I constructed a map $\mathcal{F}_1(V) \longrightarrow B\mathcal{G}$ with $H = V^{\oplus 4}$. However there is an obvious map $\mathcal{F}_1(V) \longrightarrow B\mathcal{G}$ with $H = V^{\oplus 2}$ namely sending A to the $+1$ eigenspace for $F_A = \begin{pmatrix} A & B \\ B & -A \end{pmatrix}$

This map is a section of the index map (*).

Summary: $H = V \oplus V$

$$\begin{tikzcd}
F_A & A \arrow[l, mapsto] \arrow[dl, dashed, bend right] \\
B\mathcal{G} \arrow[r] & \mathcal{F}_1(H^+) \\
F \arrow[r, mapsto] & \text{contraction of } F \text{ to } H^+
\end{tikzcd}$$

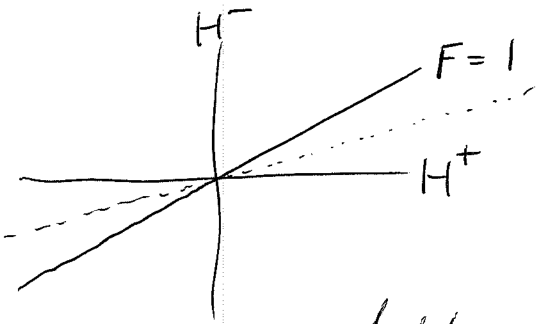
$$\begin{pmatrix} A & B \\ B & -A \end{pmatrix} \longmapsfrom A$$

$$\begin{pmatrix} \alpha & \beta^* \\ \beta & \gamma \end{pmatrix} \longmapsto \alpha .$$

But though we have maps, we need also maps of the principal $\mathcal{G}$-bundles over these spaces.

For example consider $A \mapsto F_A$. On $\mathcal{F}_1(V)$ we have the constant Hilbert bundle with fibre V and the family of K-splittings defined by the various A's in $\mathcal{F}_1(V)$. Pulling back via $A \mapsto F_A$ we have also the Hilbert bundle whose fibre at A is the $F_A = 1$ eigenspace in H equipped with the K-splitting induced by ε in H. Do we have a canonical isomorphism of H^+ with $\{F_A = 1\}$ compatible with the K-splittings?

Picture



We want a "rotation" taking H^+ into $F=1$, i.e. a u such that $u \varepsilon u^{-1} = F$. We want a square root of $g = F\varepsilon$ which rotates twice the angle.

Suppose we can find u such that

$$\boxed{u^2 = F\varepsilon \quad , \quad \varepsilon u \varepsilon = u^{-1}}$$

Then $\quad u \varepsilon u^{-1} = u^2 \varepsilon = F \varepsilon \varepsilon = F \quad$ and also

$$F u^{-1} F = F\varepsilon u \varepsilon F = u^2 u (u^2)^{-1} = u .$$

In general we don't expect to be able to extract a square root u of $F\varepsilon$ as above because of the -1 eigenspace. However for F_A we can

$$F_A \varepsilon = \begin{pmatrix} A & -B \\ B & A \end{pmatrix} \qquad \text{think as} \quad \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

$$\cos\left(\frac{\theta}{2}\right) = \sqrt{\frac{1+\cos\theta}{2}} = \sqrt{\frac{1+A}{2}}, \quad \sin\left(\frac{\theta}{2}\right) = \sqrt{\frac{1-\cos\theta}{2}} = \sqrt{\frac{1-A}{2}}$$

Thus we set

$$u = \begin{pmatrix} \sqrt{\frac{1+A}{2}} & -\sqrt{\frac{1-A}{2}} \\ \sqrt{\frac{1-A}{2}} & \sqrt{\frac{1+A}{2}} \end{pmatrix} \qquad u^* = \varepsilon u \varepsilon = u^{-1}$$

Thus we conclude that there is a natural map of principal $\mathcal{G}$-bundles

$$\begin{tikzcd}
\mathcal{F}_1(V) \times_{\mathcal{O}(2)} U(V) \arrow[r] \arrow[d] & \mathcal{P}\mathcal{G} \arrow[d] \\
\mathcal{F}_1(V) \arrow[r] & B\mathcal{G}
\end{tikzcd}$$

April 27, 1986

Let's consider the graded case, where $\mathcal{G}'$ = pairs of unitaries (g_1, g_2) on H which are congruent mod $\mathcal{K}$. This is the same as graded unitaries on $V' = V \oplus V$ commuting with $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ mod $\mathcal{K}$. Our model for $B\mathcal{G}'$ consists of pairs of involutions $F_1, F_2 \in \mathcal{I}(H)$ congruent mod $\mathcal{K}$.

In analogy with the ungraded case we want to identify $B\mathcal{G}'$ and $\mathcal{F}_0(V') = \{ T$ ess unitary contraction on $V \}$ up to homotopy. And actually we want to identify these as different classifying spaces for $\mathcal{G}'$.

Now a $\mathcal{G}'$-torsor is the same thing as a pair of Hilbert bundles E^+, E^- and a mod $\mathcal{K}$ isomorphism between them. A map from Y to $\mathcal{F}_0(V)$ is the same as an essentially unitary map between the trivial Hilbert bundle with fibres V and itself. A map from Y to our model for $B\mathcal{G}'$ is the same as a pair of Hilbert bundles E^+, E^- embedded in the trivial Hilbert bundle with fibre H such that the projection $E^+ \longrightarrow E^-$ is Fredholm.

Here's how to ~~construct a~~ "classify" a pair E^+, E^- equipped with a mod $\mathcal{K}$ isomorphism. First we lift the mod $\mathcal{K}$ isomorphism to an essentially unitary contraction $T : E^+ \longrightarrow E^-$. Then we expand this to a unitary

$$\begin{matrix} E^+ \\ \oplus \\ E^- \end{matrix} \xrightarrow{\begin{pmatrix} \sqrt{1-T^*T} & T^* \\ T & \sqrt{1-TT^*} \end{pmatrix}} \begin{matrix} E^+ \\ \oplus \\ E^- \end{matrix}$$

and finally we embed $E^+ \oplus E^-$ into the trivial Hilbert bundle with fibre H. We then obtain an embedding

$$\begin{tikzcd}
E^+ \arrow[r, "T"] \arrow[dr, phantom, "\bigcap"] & E^- \\
E^+ \oplus E^- \arrow[r, "u"] \arrow[d, "j"] \arrow[dr, phantom, "\cap"] & E^+ \oplus E^- \arrow[d, "ju^{-1}"] \\
H \arrow[r, "id"] & H
\end{tikzcd}$$

such that T is the contraction of the identity.

April 29, 1986

Recall the map $B\mathcal{G}' \longrightarrow \mathcal{F}_0$. Given two Hilbert bundles $T: E^+ \to E^-$ and a mod $\mathcal{K}$-isom., we choose isos. $E^\pm \oplus H \simeq H$ and extend T by the identity on H. In the case of our model for $B\mathcal{G}'$, where $E^\pm$ are already embedded in H, the isomorphisms desired can be obtained via the Eilenberg trick. It turns out that this is nicely linked to the Toeplitz setup over S^1 as follows.

Suppose E embedded in H with orth. comp. $E^\perp$. The Eilenberg isom. is (with $H' = H \oplus H \oplus \cdots\cdot$)

$$\begin{aligned} H' &= (E + E^\perp) + (E + E^\perp) + \cdots\cdots \\ &\wr\| \\ E + H' &= E + (E^\perp + E) + (E^\perp + E) + \cdots\cdots \end{aligned}$$

But I want to think of the embedding of H' into H' with complement E.

$$\begin{tikzcd}[column sep=small] E & \oplus E^\perp \arrow[dr, phantom] & \oplus E & \oplus E^\perp & \oplus \cdots\cdot \\ E & \oplus E^\perp & \oplus E & \oplus E^\perp & \oplus E \arrow[from=1-1, to=2-3, "\|"] \arrow[from=1-3, to=2-5, "\|"] \end{tikzcd}$$

This embedding is given by the matrix

$$\begin{pmatrix} 1-e & & \\ e & 1-e & \\ & e & \\ & & \ddots \end{pmatrix} = ze + (1-e)$$

where we identify: $H' = \bigoplus_{n \geq 0} z^n H$.

Suppose given two projections e_0, e on H. I want to compute the operator on H' given by

$$\begin{tikzcd} H' \arrow[r, phantom, "\simeq"] & e_0H \oplus H' \arrow[d, "\begin{pmatrix} ee_0 & 0 \\ 0 & 1 \end{pmatrix}"] & \longleftarrow \begin{pmatrix} e_0\iota_0^* \\ z^*e+(1-e) \end{pmatrix} \\ H' \arrow[r, phantom, "\simeq"] & eH \oplus H' & \longleftarrow \begin{pmatrix} \iota_0 e & ze+(1-e) \end{pmatrix} \end{tikzcd}$$

where $\iota_0 : H \hookrightarrow H'$ is the first factor.

$$\begin{pmatrix} \iota_0 e & ze+(1-e) \end{pmatrix} \begin{pmatrix} ee_0\iota_0^* \\ z^*e_0+(1-e_0) \end{pmatrix} = \iota_0(ee_0)\iota_0^* + (ze+(1-e))(z^*e_0+(1-e_0))$$

$$= (\iota_0\iota_0^* + zz^*)ee_0 + ze(1-e_0) + z^*(1-e)e_0 + (1-e)(1-e_0)$$

$$= ee_0 + ze(1-e_0) + z^*(1-e)e_0 + (1-e)(1-e_0)$$

(In this calculation one is working in $\mathrm{End}(H) \otimes \mathrm{End}(H^2(S^1))$ so that e, e_0 commute with z, z^*, ι_0, etc.

Notice that this ~~map~~ operator on H' is the contraction to $H^2(S^1)\hat{\otimes} H$ of the operator on $L^2(S^1)\hat{\otimes} H$ given by

$$ee_0 + ze(1-e_0) + z^{-1}(1-e)e_0 + (1-e)(1-e_0)$$

$$= (ze + (1-e))(z^{-1}e_0 + (1-e_0))$$

$$= (ze+(1-e)) \cdot (ze_0 + (1-e_0))^{-1}$$

Thus we have the following description of the map $B\mathcal{U}' \longrightarrow \mathcal{F}_0$.

Change notation slightly. Put V for the original Hilbert space, $H = L^2(S^1) = \underbrace{H_+}_{\langle z^n \mid n \geq 0 \rangle} \oplus H_-$, so that $H' = H_+ \hat{\otimes} V$.

Now the loops $ze + (1-e)$, $ze_0 + (1-e_0)$ will carry $H_+ \hat{\otimes} V$ into subspaces which should be congruent module $\mathcal{K}(H \hat{\otimes} V)$ because $e \equiv e_0 \mod \mathcal{K}(V)$. This is clear because

$$(ze + 1 - e)H' = 1 \otimes (1-e)V + zH_+ \hat{\otimes} V \subset H_+ \otimes V$$

$$(ze_0 + 1 - e_0)H' = 1 \otimes \underset{\nwarrow}{(1-e_0)V} + zH_+ \tilde{\otimes} V \subset H_+ \otimes V$$

and the subspaces ⟶ are congruent mod $\mathcal{K}$

~~Therefore the loop~~

$$\cancel{g = [ze + (1-e)][ze_0 + (1-e_0)]^{-1}}$$

~~will belong to the restricted unitary group of $H \hat{\otimes} V$ relative to the subspace $H_+ \hat{\otimes} V$. We should say more, namely that because~~

What I want to see is that $g(H^+ \otimes V)$ is congruent mod $\mathcal{K}$ to $H^+ \otimes V$, where

$$g = (ze + 1 - e)(z^{-1}e_0 + 1 - e_0)$$

Let's work with

$$zg = (ze + 1 - e)(e_0 + z(1 - e_0)).$$

Now $zH^+ \otimes V \subset (e_0 + z(1-e_0)) \cdot H^+ \otimes V \subset H^+ \otimes V$

so $zg(H^+ \otimes V)$ is sandwiched between

$$(ze + 1 - e)(H^+ \otimes V) \subset H^+ \otimes V$$

and

$$(ze + 1 - e)(zH^+ \otimes V) \supset z^2 H^+ \otimes V.$$

Thus the subspace $zg(H^+ \otimes V)$ is equivalent to a subspace of $1\otimes V \oplus z\otimes V$, say either the intersection of $\underline{\qquad\qquad\qquad}$ with $zg(H^+\otimes V)$ or the orthogonal complement $(H^+\otimes V) \ominus zg(H^+\otimes V)$. Let's call the intersection W and its complement $W^\perp$.

We have some obvious inclusions

$$z\otimes(1-e)V \subset W \quad , \quad W^\perp \supset 1\otimes eV$$

Let's start with

$$zH^+\otimes V \subset \big(e_0+z(1-e_0)\big)(H_+\otimes V) \subset H_+\otimes V$$

and multiply by $(ze+1-e)$ to get

$$z^2H^+\otimes eV + zH^+\otimes(1-e)V \subset zg(H^+\otimes V) \subset zH_+\otimes eV \oplus H_+\otimes(1-e)V$$

$$z\otimes(1-e)V \subset W \subset z\otimes V + 1\otimes(1-e)V$$

$$\begin{array}{r|c|c|}
 & \{{1 \atop z}\}\times eV & \{{1 \atop z}\}\otimes(1-e)V \\
\hline
1\otimes V: & & \\
\hline
z\otimes V, & & z\otimes(1-e)V \\
\hline
\end{array}$$

Moreover we can say that

$$zg(H^+\otimes V) = (ze+1-e)(zH^+\otimes V) \overset{\text{orth}}{\overset{\downarrow}{\oplus}} (ze+1-e)(1\otimes e_0 V)$$

In other words when we intersect with $(1\otimes V)\oplus(z\otimes V)$ we get the sum of

$$z\otimes(1-e)V \oplus \underbrace{(ze+1-e)(1\otimes e_0V)}$$

e_0V embedded first by splitting V into $eV \oplus (1-e)V$ and embeddings the first in $z\otimes V$ and the second in $1\otimes V$.

It's therefore clear that we are getting exactly the same kind of map to the restricted Grassmannian as before. Except now we have a nice map from the restricted Grassmannian to Fredholm operators we didn't think of before.

We have another explanation for the asymmetry between e, e_0 having to do with non-commutativity in the loop group.

April 30, 1986

I have been using the Bott map

$$\begin{aligned} \mathrm{Grass}(V) &\longrightarrow \Omega(U(V); 1, -1) \\ F &\longmapsto \cos\left(\tfrac{\theta}{2}\right) + i\sin\left(\tfrac{\theta}{2}\right)F \qquad 0 \le \theta \le 2\pi \end{aligned}$$

But a nicer version is

$$\begin{aligned} \mathrm{Grass}(V) &\longrightarrow \Omega(U(V)) \\ e &\longmapsto ze + 1 - e \qquad |z| = 1 \end{aligned}$$

These are essentially the same because of the map

$$\begin{aligned} \Omega(U(V)) &\xrightarrow{\sim} \Omega(U(V); 1, -1) \\ g(z) &\longmapsto e^{-i\frac{\theta}{2}} g(e^{i\theta}) = z^{-1/2} g(z). \end{aligned}$$

In effect, when $F = 2e - 1$.

$$\cos\left(\tfrac{\theta}{2}\right) + i\sin\left(\tfrac{\theta}{2}\right)F = z^{\frac{1}{2}}e + z^{-\frac{1}{2}}(1-e)$$

The next point is that if we pull back [illegible] forms by the map

$$S^1 \times \mathrm{Grass}(V) \longrightarrow U(V) \qquad (z, e) \mapsto ze + 1 - e$$

and integrate over S^1, then the biinvariant forms on $U(V)$ will give invariant forms on $\mathrm{Grass}(V)$, because this map is $U(V)$-equivariant (action is conjugation on [illegible] itself.)

~~[illegible]~~

Let's carry out this process:

$$g = ze + 1 - e$$

$$dg = dz\cdot e + (z-1)de$$

$$g^{-1} = z^{-1}e + 1 - e$$

$$g^{-1}dg = \underbrace{z^{-1}dz\ e}_{\beta} + \underbrace{(1-z^{-1})\,e\,de + (z-1)(1-e)de}_{\alpha}$$

$$\operatorname{tr}(g^{-1}dg)^{2k+1} = \operatorname{tr}(\alpha+\beta)^{2k+1}$$

$$= \operatorname{tr}(\alpha^{2k+1}) + \sum_{i=0}^{2k} \operatorname{tr}(\underset{\sim}{\alpha^i}\,\underline{\beta\,\alpha^{2k-i}}) \qquad ((dz)^2=0)$$

one is of even degree

$$= \operatorname{tr}(\alpha^{2k+1}) + (2k+1)\operatorname{tr}(\beta\,\alpha^{2k})$$

$$\alpha^2 = (1-z^{-1})(z-1)\left(e\,de(1-e)de + (1-e)de.\,e\,de\right)$$

$$= (1-z^{-1})(z-1)\,(de)^2$$

Notice that $\operatorname{tr}\left(\underset{1-e}{(e)}de\,(de)^{2k}\right) = 0$ because de is odd relative to the grading defined by e. Thus

$$\operatorname{tr}(\alpha^{2k+1}) = 0.$$

so

$$\operatorname{tr}(g^{-1}dg)^{2k+1} = (2k+1)\operatorname{tr}\left(\beta(\alpha^2)^k\right)$$

$$= (2k+1)\,dz\ z^{-1}(1-z^{-1})^k(z-1)^k\ \operatorname{tr}\left(e(de)^{2k}\right)$$

$$\int_{S^1} \operatorname{tr}(g^{-1}dg)^{2k+1} = (2k+1)\left(\int_{S^1}\frac{dz}{z}\Big[\underbrace{(z^{\frac12}-z^{-\frac12})\cancel{z^{-\frac12}z^{\frac12}}(z^{\frac12}-z^{-\frac12})}_{(z^{\frac12}-z^{-\frac12})^{2k}}\Big]^k\right)k!\ \operatorname{tr}\left(\frac{e(de^2)^k}{k!}\right)$$

$$= (2k+1)(-1)^k\,2\pi i\,\frac{(2k)!}{k!\,\cancel{k!}}\,\cancel{k!}\quad \operatorname{tr}\left(\frac{e(de^2)^k}{k!}\right)$$

$$= (-1)^k\frac{(2k+1)!}{k!}\quad 2\pi i\ \operatorname{tr}\left(\frac{e(de^2)^k}{k!}\right)$$

$$\boxed{\int_{S^1}(-1)^k\frac{k!}{(2k+1)!}\operatorname{tr}(g^{-1}dg)^{2k+1} = 2\pi i\ \operatorname{tr}\left(\frac{(e\,de^2)^k}{k!}\right)}$$

Natural question. You have the Bott map $e \longmapsto ze+1-e$ from the Grassmannian to $\Omega U(V)$, and on $\Omega U(V)$ you have character forms obtained by mapping this loops group to the restricted Grassmannian in $L^2(S^1)\otimes V$ by letting it act on the subspace $H_+\otimes V$. The question is whether the character forms on $\Omega U(V)$ pull back to the character forms on the Grassmannian of V.

Now this is clear because acting by $ze+1-e$ on $H_+\otimes V$ gives the subspace

$$1\otimes(1-e)V + zH_+\otimes V.$$

So what one has done is to take the vector sub-bundle $e \mapsto (1-e)V$ of $1\otimes V$ and take the direct sum with constant bundle $zH_+\otimes V$. Thus the curvature has values in $\mathrm{End}(1\otimes V)$, and the rest is clear.

Similarly when we consider the map from our model $B\mathcal{G}'$ to the restricted unitary group of $L^2(S^1)\otimes V$ wrt $H_+\otimes V$ given by the map

$$e, e_0 \longmapsto (ze+1-e)(ze_0+1-e_0)^{-1},$$

the induced character forms will be the same as the character forms obtained by the map to the Grassmannian of $z^{-1}\otimes V + 1\otimes V$.

Thus we have so far two sets of character forms on $B\mathcal{G}'$. There's a chance they could agree because of the fact that the character forms of complementary bundles differ by sign. NO

May 2, 1986

Having obtained a nice Laurent polynomial version of the Bott map $\mathbb{Z} \times BU \longrightarrow \Omega U$, namely

$$\begin{aligned} \mathrm{Grass}(V) &\longrightarrow \Omega\, U(V) \\ e &\longmapsto ze + (1-e) \end{aligned}$$

I would like to do the same thing for the map $U \longrightarrow \Omega BU$.

Let's review how I handled Laurent polynomial loops in symmetric spaces in 1975. Let G be compact connected with involution σ. One has the smooth model of the fibration

$$\Omega(G) \longrightarrow \Omega(G; 1, G) \longrightarrow G$$

given by

$$(*) \qquad \mathcal{G}' \longrightarrow \mathcal{A} \longrightarrow G$$

where $\mathcal{G}$ = free smooth loop group of G, and $\mathcal{G}'$ = based loops. Thus a [redacted] path $h(t) \in \Omega(G; 1, G)$ is [redacted] identified with a map

$$h : \mathbb{R} \longrightarrow G \qquad \text{satisfying}$$

$$h(0) = 1 \qquad h(1+t) = h(1)\,h(t)$$

and the smooth h's are given by connections:

$$\begin{cases} h'(t) = h(t)A(t) \\ h(0) = 1 \end{cases} \qquad A : S^1 \to \mathfrak{g}$$

Inside the smooth model sits the Laurent polynomial model where $\mathcal{G}_{poly}$ consists of algebraic maps* $S^1 \to G$ and $\mathcal{A}$ consists of $A(t)$ [redacted] integrating to give

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* One identifies $S^1 = \mathbb{R}/\mathbb{Z}$ with $\{z \mid |z|=1\}$ via $z = e^{2\pi i t}$

$h(t)$ of the form

$$h(t) = e^{tX} g(z) \qquad \begin{array}{l} X \in \mathfrak{g} \\ g \in \mathcal{G}'_{poly} \end{array}$$

Next consider the involution σ on G and extend it to $\mathcal{G}$, $\mathcal{A}$ by

$$\tilde{\sigma}(g(z)) = \sigma g(\bar{z})$$
$$\tilde{\sigma}(h(t)) = \sigma h(-t)$$

whence

$$\tilde{\sigma}(A(t)) = [\tilde{\sigma}(h(t))]^{-1} \cdot [\tilde{\sigma}(h(t))]'$$
$$= \sigma\left(h(-t)^{-1} h(-t)' \right)$$
$$= \sigma(- h(-t)^{-1} h'(-t)) = -\sigma A(-t)$$

~~We now want to identify the fixpoints of $\tilde{\sigma}$. $\mathcal{A}^{\tilde{\sigma}}$ consists of $h(t) : R \to G$ such that $h(0) = 1$, $h(t+1) = h(1)h(t)$, $\sigma h(t) = h(-t)$.~~

It's clear that this involution preserves $\mathcal{G}'$ and commutes with right multiplication of $\mathcal{G}'$ on $\mathcal{A}$. In fact recall $\mathcal{G}$ acts on $\mathcal{A}$ to the right by

$$(h * g)(t) = g(1)^{-1} h(t) g(z).$$

and

$$\tilde{\sigma}(h*g)(t) = \sigma((h*g)(-t))$$
$$= \sigma(g(1)^{-1} h(-t) g(\bar{z}))$$
$$= \sigma g(1)^{-1} \sigma h(-t) \sigma g(\bar{z}) = (\tilde{\sigma} h * \tilde{\sigma} g)(t)$$

Thus the is an involution $\tilde{\sigma}$ on G compatible with the endpoint map $h \longrightarrow h(1)$. This is

$$\begin{tikzcd}
h(t) \arrow[r, mapsto] \arrow[d, "\tilde\sigma"] & h(1) \\
\sigma h(-t) \arrow[r, mapsto] & \sigma h(-1) = \sigma h(1)^{-1}
\end{tikzcd}$$

Thus $\tilde\sigma(g) = (\sigma g)^{-1}$ and the fixpoints of $\tilde\sigma$ on G are the elements of G which are reversed by σ. Now recall that we have a twisted action of [redacted] G on itself given by

$$g * x = g x \sigma(g)^{-1}$$

and so [redacted]

$$G/G^\sigma \xrightarrow{\sim} \{g\sigma(g)^{-1}\} \subset \{g \mid \sigma g = g^{-1}\} = G^{\tilde\sigma}.$$

In fact it is easy to see [redacted] using the fact that $G^{\tilde\sigma}$ is a submanifold, that this identifies:

$$G/G^\sigma = \text{identity component of } G^{\tilde\sigma}.$$

Consequently we get a principal fibration

$$\mathcal{G}^{\tilde\sigma} \longrightarrow \mathcal{A}^{\tilde\sigma} \longrightarrow G/G^\sigma$$

The second map sends $h(t)$ to $h(\frac{1}{2})G^\sigma$ since

$$h(\tfrac{1}{2}) = h(1)\,h(-\tfrac{1}{2}) = h(1)\,\sigma h(\tfrac{1}{2})$$

$$h(1) = h(\tfrac{1}{2}) \cdot \sigma h(\tfrac{1}{2})^{-1}$$

Note that the $\mathcal{G}^{\tilde\sigma}$-action on $\mathcal{A}^{\tilde\sigma}$ is compatible with the G^σ action on G/G^σ since

$$(h*g)(\tfrac{1}{2})\,G^\sigma = g(1)^{-1}\,h(\tfrac{1}{2})\,g(-1)\,G^\sigma = g(1)^{-1} \cdot h(\tfrac{1}{2})\,G^\sigma$$

$$(g \in \mathcal{G}^{\tilde\sigma} \Rightarrow \sigma g(\bar z) = g(\bar z) \Rightarrow g(1), g(-1) \in G^\sigma.)$$

Next we should discuss the homotopy meaning of the above construction. What we are doing is to replace $\Omega(G/G^\sigma)$, which is not a group, by the group $\Omega(G)^{\tilde{\sigma}}$ consisting of based loops $g(z)$ in G such that $\sigma(g(z)) = g(\bar{z})$. Thus:

$$\Omega(G^\sigma) = \Omega(G)^\sigma \qquad \text{but}$$

$$\Omega(G/G^\sigma) \sim \Omega(G)^{\tilde{\sigma}}$$

The reason this works is as follows. An element of $\Omega(G)^{\tilde{\sigma}}$ is a based loop $g : S^1 \longrightarrow G$ such that $\sigma g(z) = g(\bar{z})$. Such a g is determined by its values for $z = e^{2\pi i t}$ $0 \leq t \leq \frac{1}{2}$ and any path $[0, \frac{1}{2}] \xrightarrow{h} G$ such that $1 = h(0)$, $h(\frac{1}{2}) \in G^\sigma$ can occur. Thus

$$\Omega(G)^{\tilde{\sigma}} = \Omega(G; 1, G^\sigma)$$

and by homotopy theory if $p : G \longrightarrow G/G^\sigma$ is the canonical map, one has a homotopy equivalence

$$\Omega(G; 1, G^\sigma) \longrightarrow \Omega(G/G^\sigma)$$

induced by p.

Now let's apply this to the Grassmannian

$$G_n(\mathbb{C}^{2n}) = G/G^\sigma \qquad G = U(2n)$$

$$\sigma g = \varepsilon g \varepsilon \qquad \varepsilon = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

We are ultimately interested in the space of paths $\Omega(G/G^\sigma; \varepsilon, -\varepsilon)$ where we are thinking of the Grassmannian as being involutions.

Now $-\varepsilon$ is also a fixed point for G^{σ}, so the subset of $\mathcal{A}^{\tilde{\sigma}}$ consisting of h mapping to $-\varepsilon$ should be an orbit under $\mathcal{G}^{\tilde{\sigma}}$ with stabilizer $\simeq$ G^{σ}.

Let $h(t) \in \mathcal{A}^{\tilde{\sigma}}$ map to $-\varepsilon$. This means that

$$\begin{array}{c} h(\tfrac{1}{2})\,\varepsilon\, h(\tfrac{1}{2})^{-1} = -\varepsilon \\ \| \\ h(\tfrac{1}{2})\, h(-\tfrac{1}{2})^{-1}\,\varepsilon = h(1)\,\varepsilon \end{array}$$

and so $h(1) = -1$. Thus we have a model for $\Omega(G/G^{\sigma}; \varepsilon, -\varepsilon)$ consisting of $h \in \tilde{\mathcal{A}}^{\sigma}$ such that $h(1) = -1$.

For example let $h(t) = e^{tX}$, where $\varepsilon X \varepsilon = -X$ and $X \in \text{Lie}\, U(2n)$. Then $h(1) = e^{X}$ is -1 when the eigenvalues of X are $\equiv \pm i\pi$ mod $(2\pi i \mathbb{Z})$. So we get a space of minimal geodesics by taking $X = i\pi F$ where $F = \begin{pmatrix} 0 & g^* \\ g & 0 \end{pmatrix}$ is an involution anti-comm. with ε.

Notice that a path e^{tX}, $X \in \mathfrak{g}^{-}$ satisfies

$$\sigma(e^{tX}) = (e^{tX})^{-1}$$

so it is a path in $G^{\tilde{\sigma}}$. Normally an element $h(t) \in \mathcal{A}^{\tilde{\sigma}}$ becomes the path $h(t)\cdot(\sigma h(t))^{-1}$ in $G^{\tilde{\sigma}}$. ?

Review: I am studying the Bott map

$$U(n) \longrightarrow \Omega\left(\mathrm{Grass}_n(\mathbb{C}^{2n})\,;\, \varepsilon, -\varepsilon\right)$$

$$g \longmapsto (\cos\theta)\varepsilon + (\sin\theta)\begin{pmatrix} 0 & g^* \\ g & 0 \end{pmatrix}$$

I thought it might be possible to use the theory of polynomial loops to obtain some sort of nice algebraic model for the above path space.

The theory gives the following: Set $G = U(2n)$ $\varepsilon = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $\sigma =$ conjugation by ε, $G^\sigma = U(n) \times U(n)$, $G/G^\sigma = \mathrm{Grass}_n(\mathbb{C}^{2n})$. Consider

$$\mathcal{A}^{\tilde\sigma} = \left\{ h : \mathbb{R} \to U(2n) \;\middle|\; \begin{matrix} h \text{ smooth}, & \overset{h(0)=1}{h(t+1) = h(1)h(t)} \\ & \sigma h(t) = h(-t) \end{matrix} \right\}$$

The map $h \longmapsto h(t)^{-1} h'(t)$ sets up an isom. of $\mathcal{A}^{\tilde\sigma}$ with the space of smooth maps $A : \mathbb{R}/\mathbb{Z} \longrightarrow \mathfrak{g}$ such that $\sigma A(t) = -A(-t)$. Thus $\mathcal{A}^{\tilde\sigma}$ is contractible. Consider

$$\mathcal{G}^{\tilde\sigma} = \left\{ g : S^1 \to U(2n) \;\middle|\; \begin{matrix} g \text{ smooth} \\ \sigma g(t) = g(-t) \end{matrix} \right\}$$

and the subgroup of based loops $\mathcal{G}'^{\tilde\sigma}$. $\mathcal{G}^{\tilde\sigma}$ acts to the right on $\mathcal{A}^{\tilde\sigma}$ by

$$(h * g)(t) = g(0)^{-1} h(t) g(t)$$

& the action of $\mathcal{G}'^{\tilde\sigma}$ is free. We have

$$\begin{tikzcd}
\mathcal{A}^{\tilde\sigma}/\mathcal{G}'^{\tilde\sigma} \arrow[rr, "\sim"] \arrow[dr] & & G/G^\sigma \arrow[dl] \\
& G^{\tilde\sigma} = \{ g \mid \sigma g = g^{-1} \} &
\end{tikzcd}$$

$$\begin{tikzcd}
h \arrow[r, mapsto] \arrow[d, mapsto] & h(\tfrac{1}{2})G^\sigma \arrow[dl, mapsto] \\
h(1) = h(\tfrac{1}{2})h(-\tfrac{1}{2})^{-1} &
\end{tikzcd}$$

This can be summarized by saying that we have a principal bundle with contractible total space

$$\mathcal{G}'^{\tilde\sigma} \longrightarrow \mathcal{A}^{\tilde\sigma} \longrightarrow G/G^{\sigma}$$

and hence we have a model for $\Omega(G/G^{\sigma})$, given by the group $\mathcal{G}'^{\tilde\sigma}$.

(It seems to be worthwhile noting that $\mathcal{A}^{\tilde\sigma}$ & $\mathcal{G}^{\tilde\sigma}$ consists of paths in $G = U(2n)$, and that the obvious way to go from a path in $U(2n)$ to one in G/G^{σ} is the wrong map.)

Now I am interested above all in the path space $\Omega(G/G^{\sigma}; \varepsilon, -\varepsilon)$, because this is associated to the Bott map. I propose to consider instead of this space the fibre of $\tilde{\mathcal{A}}^{\sigma}$ over the point $-\varepsilon$ of the Grassmannian. This fibre is

$$(\mathcal{A}^{\tilde\sigma})_{-\varepsilon} = \left\{ h : \mathbb{R} \to U(2n) \;\middle|\; \begin{array}{l} h(0)=1, \ h \text{ smooth} \\ \varepsilon h(t)\varepsilon = h(-t) \\ h(t+1) = -h(t) \end{array} \right\}$$

and it should be compared with

$$\mathcal{G}'^{\tilde\sigma} = \left\{ g : \mathbb{R} \to U(2n) \;\middle|\; \begin{array}{l} g(0)=1, \ g \text{ smooth} \\ \varepsilon g(t)\varepsilon = g(-t) \\ g(t+1) = g(t) \end{array} \right\}$$

The point probably is that these are different homogeneous spaces over $\mathcal{G}^{\tilde\sigma}$. Can we find two inequivalent ways of embedding $G^{\sigma} = U(n) \times U(n)$ into $\mathcal{G}^{\tilde\sigma}$?

I'd like to see that $(\mathcal{A}^{\tilde\sigma})_{\varepsilon}$ and $(\mathcal{A}^{\tilde\sigma})_{-\varepsilon}$

are different homogeneous spaces of $\mathcal{G}^{\tilde\sigma}$.

To do this note the former is $\mathcal{G}^{\sigma}/G^{\sigma}$ where G^{σ} is the constant loops, and hence it is enough to show that G^{σ} has no fixpoints on $(a^{\tilde\sigma})_{-\varepsilon}$.

[redacted] Suppose $h : \mathbb{R} \longrightarrow G$ is an elt. of $(a^{\tilde\sigma})_{-\varepsilon}$, whence $h(0)=1$, $\varepsilon h(t)\varepsilon = h(-t)$, $h(1+t) = -h(t)$. For h to be fixed under G^{σ} means

$$g^{-1}h(t)\,g = h(t) \qquad \text{all } g \in G^{\sigma}$$

In particular $\varepsilon h(t)\varepsilon = h(t)$, so $h(t) = h(-t)$. Thus $h(\frac{1}{2}) = h(-\frac{1}{2})$ contradicting $h(1+t) = -h(t)$. QED.

[redacted]

Summary: The Bott map

$$U(n) \longrightarrow \Omega\left(\mathrm{Grass}_n(\mathbb{C}^{2n})\,;\ \varepsilon, -\varepsilon\right)$$

can't be defined into the appropriate loop space for the Grassmannian as symmetric space without destroying the $U(n)\times U(n)$ symmetry. The setup is analogous to the case of $\Omega SU(n)$ where there are various types of lattices in the building. In our case [redacted] we have some kind of loop group $\mathcal{G}^{\tilde\sigma}$ and two homogeneous spaces which are [redacted] free over $\mathcal{G}'^{\tilde\sigma}$ but not equivalent.

Possible ideas: Kac's theory of autos of finite order of Lie algebras; [redacted] with loop groups the periodicity game might be much richer as one has an interesting Galois group, also skew-fields.

Motivation for $\mathcal{G}^{\tilde\sigma}$ as the good loop associated to

G/G^{σ}. The natural geodesics in the symmetric space are the paths e^{tX} where $X \in \mathfrak{g}^{-} = \{X \in \mathfrak{g} \mid \sigma X = -X\}$. These really do lie in $G^{\tilde{\sigma}} = \{g \mid \sigma g = g^{-1}\}$. If these are to be in a principal bundle over G/G^{σ}, then the loops $g(t) = e^{tX} e^{-tY}$ for $e^{X} = e^{Y}$ should lie in the loop group. But then $\sigma g(t) = g(-t)$.

Further points to discuss.

1) Is transgressing left-invariant forms on $\mathcal{G}$ to $B\mathcal{G}$ a reasonable program? Is it consistent with the van Est picture of the different cohomologies?

May 5, 1986

Real periodicity

First Bott's starting point: $\Omega(SO(2n))$. maximal torus $SO(2)^n$. If J is a complex structure, i.e. J orthogonal & $J^2=-1$, then

$$e^{\theta J} = \cos\theta + (\sin\theta)J \qquad 0 \leq \theta \leq \pi$$

is a geodesic going from 1 to -1 in $SO(2n)$. Bott map:

$$O(2n)/U(n) \longrightarrow \Omega(SO(2n); +1, -1)$$

In general fix a large C_k-module, $C_k =$ Clifford algebra with generators $e_1, \ldots, e_k$ & $e_ie_j + e_je_i = 0$ $i \neq j$, $e_i^2 = -1$. Let $J_j =$ mult by e_j. Put

$$X_k = \{ J \mid J \text{ complex structure anti-comm. with } J_1, \ldots, J_k \}$$

We have a Bott map

$$X_k \longrightarrow \Omega(X_{k-1}; J_k, -J_k)$$

$$J \longmapsto (\cos\theta)J_k + (\sin\theta)J$$

$$\overset{\shortparallel}{e^{-\theta(JJ_k)} \cdot J_k}$$

and the periodicity thm. says this is a homot. equiv. in the stable range.

Structure of Clifford algebras:

$$(e_1 \cdots e_k)^2 = (-1)^{k+\frac{1}{2}k(k-1)} = (-1)^{\frac{1}{2}k(k+1)} = \begin{cases} -1 & k \equiv 1,2 \ (4) \\ +1 & k \equiv 3,4 \ (4) \end{cases}$$

Isom. $\boxed{C_4 \otimes C_k = C_{k+4}}$

Pf: Set $\varepsilon = e_1e_2e_3e_4 \in C_4$. Then in $C_4 \otimes C_k =$ the ord. tensor product where the factors commute the elements

$$e_i \otimes 1 \qquad 1 \leq i \leq 4$$

$$\varepsilon \otimes e_j \qquad 1 \leq j \leq k$$

anti-commute and have square -1.

The same kind of argument shows:

$$\boxed{C_4 = C_2 \otimes M_2(\mathbb{R})}$$

$$e_1 \longleftrightarrow e_1 \otimes 1$$

$$e_2 \longleftrightarrow e_2 \otimes 1$$

$$e_1e_2e_3 \longleftrightarrow 1 \otimes \varepsilon$$

$$e_1e_2e_4 \longleftrightarrow 1 \otimes \varepsilon'$$

So we get the following table:

k	C_k	G_k	$(G_k/G_{k+1})_*$	$\pi_0(G_k/G_{k+1})$
0	$\mathbb{R}$	O	SO/U	$\mathbb{Z}_2$
1	$\mathbb{C}$	U	U/Sp	0
2	$\mathbb{H}$	Sp	BSp	$\mathbb{Z}$
3	$\mathbb{H} \times \mathbb{H}$	$Sp \times Sp$	Sp	0
4	$\mathbb{H} \otimes M_2(\mathbb{R})$	Sp	Sp/U	0
5	$\mathbb{C} \otimes M_4(\mathbb{R})$	U	U/O	0
6	$M_8(\mathbb{R})$	O	BO	$\mathbb{Z}$
7	$M_8(\mathbb{R}) \times M_8(\mathbb{R})$	$O \times O$	SO	$\mathbb{Z}_2$
8	$M_{16}(\mathbb{R}$	O		

In order to the periodicity without the shift apparent in the above table one wants to use graded C_k-modules:

$KO^{-k}(X) =$ relative theory of graded C_k-modules$/X$ modulo graded C_{k+1}-modules$/X$.

$= [X, \mathcal{F}_k]$

where $\mathcal{F}_k$ is the space of skew-adjoint contraction operators on the graded Hilbert C_k-module which are of odd degree, anti-commute with $J_1, \ldots, J_k$ and which are nontrivial mod $\mathcal{K}$ in some sense. So if $k=0$ we have the space of $\begin{pmatrix} 0 & -\alpha^* \\ \alpha & 0 \end{pmatrix}$ where $\alpha : H^+ \longrightarrow H^-$ is an essentially orthogonal contraction operator.

The following explains the graded setup:

$\{ J \mid J$ complex ~~structure~~ anti-comm. with $\varepsilon \} = \left\{ \begin{pmatrix} 0 & -g^* \\ g & 0 \end{pmatrix} \middle| \ g \text{ orthogonal} \right\} = O$

$\left\{ J \middle| \ J \text{ cx. st. anti-comm. with } \varepsilon,\ J_1 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \right\} = \left\{ \begin{pmatrix} 0 & J' \\ J' & 0 \end{pmatrix} \middle| \ J' \text{ cx. st.} \right\} = O/U$

May 7, 1986

First observation: Consider the model we have for $B\mathcal{G}'$ consisting of pairs of projectors e, e' which are congruent mod $\mathcal{K}$. Then over $B\mathcal{G}'$ we have a $(\mathbb{Z}/2)$-graded vector bundle with superconnection, hence we do get even forms on our model. **At least** this works in finite dimensions, but mod $\mathcal{K}$ we [redacted] know that the odd endom. $L = i\begin{pmatrix} 0 & ee' \\ e'e & 0 \end{pmatrix}$ is an involution which is constant relative to the connection, hence the supertrace should be defined ($\mathcal{K}$ being replaced by some Schatten ideal).

The observation is that this formalism gives us even forms which are symmetric in e, e'.

One of the unsolved problems from 2 years ago is to link superconnections with Grassmannian graph methods. The problem is that the graph method seems unsymmetrical in the two bundles. Actually I found a difficulty before in that the super-conn. forms [redacted] are definitely different from the forms constructed using the graph methods. But this may be because the graph method is inherently asymmetrical. ?

Anyway one should begin with the 2-form on $B\mathcal{G}'$. But before this one can look at the 1-form on $B\mathcal{G}$, which is constructed analogously using the superconnection formalism in the odd case.

[redacted]

May 8, 1986

I propose to study the 1-form on BG from various angles. First of all the superconnection stuff gives the 1-form

$$(*) \qquad \mathrm{tr}\left(e^{L^2}[\nabla, L]\right)$$

for a family of skew-adjoint operators; this is up to some scalar factor. Next the Cayley transform

$$L \longrightarrow \frac{1+L}{1-L} = g$$

from skew-adjoint ops. to unitaries provides the 1-form

$$(**) \qquad \mathrm{tr}(g^{-1}dg) = \mathrm{tr}\left(\frac{2\,dL}{1-L^2}\right) \qquad dL = [\nabla, L]$$

We have viewed $(*)$ and $(**)$ as linked by the Laplace transform in some sense.

Let's review the superconnection stuff in the odd case. One is interested in elements of $K^1(X,U)$; such an element is represented by a vector bundle E over X together with a unitary operator g such that $+1$ is not an eigenvalue of g over U. (This is a fancy way of requiring that over U there is a canonical deformation of g to a constant.) One way of obtaining a unitary operator is from a skew-adjoint operator L by $g = e^L$, or by the Cayley transform. For the latter the 0 eigenvalues of L correspond to the eigenvalue 1 of g. So we have various classes in $K^1(X,U)$ which can be represented by a vector bundle E over X together with a skew-adjoint operator L which is invertible over U. Such classes become zero in $K^1(X)$ because L can be deformed to zero. From the exact sequence

$$K^0(X) \to K^0(U) \to K^1(X,U) \to K^1(X)$$

the classes in question come from classes in $K^0(U)$.

Specifically suppose [illegible] we start with (E,g) representing a class in $K^1(X,U)$. By adding to E a complement with the identity map, we can suppose E trivial. Then we have

$$\begin{tikzcd}
X \arrow[r] & X/U \arrow[r] \arrow[d, "g"] & \Sigma U \\
 & U(N) &
\end{tikzcd}$$

so if g becomes null-homotopic on X we have an induced map $\Sigma U \to U(N)$ which by periodicity determines a class in $K^0(U)$.

Now I have the feeling that working with ~~supports in~~ finite dimensional vector bundles and supports captures important aspects of the infinite diml setup.

May 9, 1986

Berry (of Bristol) talk: Quantum chaos of the Riemann zeta function?

Background - old work of Dyson + extensive calculations by Odlysko. These establish a link between the statistics of the Riemann zeroes and the grand unitary ensemble. This is the ensemble of hermitian matrices of size N with Gaussian weight with norm $tr(A^2)$, as $N \to \infty$.

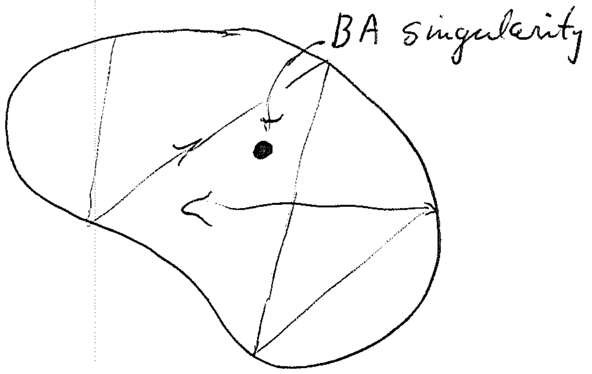
Earlier work by physicists dealt with random orthogonal matrices (grand orthogonal ensemble) because the systems studied had time reversal symmetry. There are three examples kept in mind - Poisson, GOE, GUE.

The conjecture is that there is a [redacted] system whose quantum energy levels are given by the Riemann zeroes, and whose classical limit exhibits chaotic behavior.

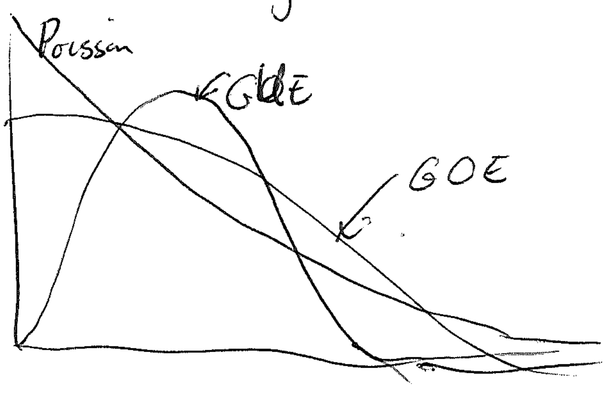
There is a semi-classical theory of quantum systems with chaotic classical behavior. Related to Selberg's work.

Standard examples of [redacted] chaotic classical systems are billiard ball problems. These exhibit time reversal [redacted] symmetry. To break this symmetry put in a magnetic field, especially a Bohm-Aharonov singularity which doesn't change the [redacted] Newtonian mechanics but which is not time-reversal symmetric on the Hamiltonian level.

Pictures:



Probability dists.



May 9, 1986 (cont)

Idea: I have found that there is another model for $\Omega(Gr)$ which is natural from the viewpoint of loop groups and the Bott map

$$U(n) \longrightarrow \Omega(Gr_n(\mathbb{C}^{2n}); \varepsilon, -\varepsilon)$$

The question is whether there is another way of viewing the inverse map in K-theory, that is, the map given by the Dirac operator.

I have seen that mixing with the Dirac operator on S^1 gives a map up to homotopy

$$\Omega(Gr) \longrightarrow U$$

which is essentially the map associating to a loop in the Grassmannian the ~~monodromy~~ holonomy of the Grassmannian connection along the loop. This holonomy map is more precisely a map

$$\Omega(Gr_n(\mathbb{C}^{2n}); \varepsilon, +\varepsilon) \longrightarrow U(n) \times U(n)$$

since one has parallel transport in both the sub and quotient bundles.

(More generally, one can consider the principal bundle

$$H \longrightarrow G \longrightarrow G/H$$

with G acting on the left. A G-invariant connection in this principal bundle is the same as a splitting of

$$0 \longrightarrow \mathfrak{h} \longrightarrow \mathfrak{g} \longrightarrow \mathfrak{g}/\mathfrak{h} \longrightarrow 0$$

which is H-invariant. If H is connected this means a complement $\mathfrak{m}$ for $\mathfrak{h}$ in $\mathfrak{g}$ such that $[\mathfrak{h}, \mathfrak{m}] \subset \mathfrak{m}$. The curvature is probably given by elements in $\mathfrak{g}/\mathfrak{h}$ to $\mathfrak{m}$, taking bracket and projecting into $\mathfrak{h}$. The last step is not needed for a symmetric space as $[\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{h}$.)

Let's review our model for $\Omega(Gr; \varepsilon, -\varepsilon)$. We have the principal bundle

$$\mathcal{G}^{\tilde{\sigma}} \longrightarrow \mathcal{A}^{\tilde{\sigma}} \xrightarrow{\pi} Gr$$

where $\mathcal{A}^{\tilde{\sigma}}$ is the space of smooth $A(t) : \mathbb{R}/\mathbb{Z} \longrightarrow \text{Lie}\, U(2n)$ such that

$$(*) \qquad \varepsilon A(t) \varepsilon = -A(-t).$$

By integrating [illegible] the initial value prob.

$$h'(t) = h(t) A(t)$$
$$h(0) = 1$$

such an $A(t)$ can be identified with a path $h(t) : \mathbb{R} \longrightarrow U(2n)$ satisfying

$$h(0) = 1 \qquad h(t+1) = h(1) h(t) \qquad \varepsilon h(t) \varepsilon = h(-t)$$

The map π [illegible] is

$$\pi(h) = h(\tfrac{1}{2}) \varepsilon h(\tfrac{1}{2})^{-1} = h(1) \varepsilon$$

Our model for $\Omega(Gr; \varepsilon, -\varepsilon)$ is the space $\pi^{-1}(-\varepsilon)$; it consists of connections satisfying $(*)$ with monodromy $h(1) = -1$.

Now we have a map

$$\pi^{-1}(-1) \longrightarrow \Omega(Gr; \varepsilon, -\varepsilon)$$
$$h \longmapsto h(t) \varepsilon h(t)^{-1} \qquad 0 \le t \le \tfrac{1}{2}$$

which we have seen is a homotopy equivalence. We want the connection in $U(2n) \longrightarrow Gr$ to construct ~~as~~ lifting.

Suppose given a smooth path F_t in Gr starting with $F_0 = \varepsilon$. We want a path g_t in $U(2n)$ such that

$$g_t \varepsilon g_t^{-1} = F_t .$$

Then

$$\dot{g}_t \varepsilon g_t^{-1} + g_t \varepsilon \left(-g_t^{-1} \dot{g}_t g_t^{-1} \right) = \dot{F}_t$$

or

$$[g_t^{-1} \dot{g}_t , \varepsilon] = g_t^{-1} \dot{F}_t g_t \quad \leftarrow \begin{array}{l} \text{anti comm.} \\ \text{with } g_t^{-1} F_t g_t = \varepsilon \end{array}$$

There is a unique solution for $A_t = g_t^{-1} \dot{g}_t$ provided we require it to anti-commute with ε, namely

$$g_t^{-1} \dot{g}_t = \frac{1}{2} g_t^{-1} \dot{F}_t g_t \varepsilon$$

Thus we can lift F_t to g_t in $U(2n)$ uniquely such that $g_0 = 1$ and $g_t^{-1} \dot{g}_t$ anti-commutes with ε. This is undoubtedly the [redacted] horizontal lift for the connection.

Now suppose F_t is given for $0 \leq t \leq \frac{1}{2}$ and goes from ε to $-\varepsilon$. Also suppose that $\dot{F}_t = 0$ for t near $0, \frac{1}{2}$. [redacted] Then A_t will be smooth on $0 \leq t \leq \frac{1}{2}$ and will vanish near the ends, so it can be extended to the real line so as to be periodic and satisfy $(*)$. Thus we obtain an [redacted] element of [redacted] $\pi^{-1}(-\varepsilon)$ which lifts the path F_t.

Thus we have constructed (modulo smoothness) a lifting

$$\begin{tikzcd} \pi^{-1}(-\varepsilon) \arrow[r] & \Omega(Gr; \varepsilon, -\varepsilon) \arrow[l, bend right, dashed] \\ h \arrow[r, mapsto] & h(t) \varepsilon h(t)^{-1} \end{tikzcd}$$

The holonomy of the path F_t is the element $h(\frac{1}{2})$ which satisfies

$$h(\tfrac{1}{2})\,\varepsilon\, h(\tfrac{1}{2})^{-1} = -\varepsilon$$

so $h(\frac{1}{2}) \in \left\{ g \in U(2n) \,\middle|\, \varepsilon g \varepsilon = -g \right\} = \left\{ \begin{pmatrix} 0 & g_2 \\ g_1 & 0 \end{pmatrix} \in U(2n) \right\}$

At this point I understand the monodromy: Given a path in the Grassmannian I take its holonomy $h(\frac{1}{2})$ which is of the form $\begin{pmatrix} 0 & g_2 \\ g_1 & 0 \end{pmatrix}$ and I take either g_1 or $g_2 \in U(n)$. The question now is whether there is a map of $\pi^{-1}(-1)$ to Dirac operators which is natural and has the right behavior in K-theory.

Let's go back to $\mathcal{G}$ = free loop group of $U(2n)$ acting on $H = L^2(S^1) \otimes \mathbb{C}^{2n}$ and on $\mathcal{A} = C^\infty(S^1, \mathrm{Lie}\, U(2n))$, better $\mathcal{A}$ = space of connections. Then we have defined $\tilde{\sigma}$ on $\mathcal{G}, \mathcal{A}$ by

$$(\tilde{\sigma} g)(t) = \varepsilon g(t) \varepsilon$$

$$(\tilde{\sigma} A)(t) = -\varepsilon A(-t) \varepsilon$$

But these are compatible with, and induced by, the involution $\tilde{\sigma}$ on $L^2(S^1) \otimes \mathbb{C}^{2n}$ given by

$$(\tilde{\sigma} f)(t) = \varepsilon f(t) \qquad (\tilde{\sigma}^2 f)(t) = \varepsilon (\tilde{\sigma} f)(t)$$
$$= \varepsilon\, \varepsilon f(t) = f(t)$$

$$\tilde{\sigma}(gf)(t) = \varepsilon\, g(t) f(t) = \varepsilon g(t) \varepsilon\, \varepsilon f(t) = (\tilde{\sigma} g \cdot \tilde{\sigma} f)(t).$$

$$\tilde{\sigma}\Big(\big(\partial_t + A(t)\big) f \Big)(t) = \varepsilon\, (f' + Af)(-t)$$
$$= -\partial_t\big(\varepsilon f(-t)\big) + \varepsilon A(-t) \varepsilon\ \varepsilon f(-t)$$

$$\tilde{\sigma}\Big(\big((\partial_t + A)f\big)(t)\Big) = -\Big[\big(\partial_t + \tilde{\sigma}(A)\big)\tilde{\sigma}(f)\Big](t)$$

$$\therefore \quad \tilde{\sigma}\big((\partial_t + A)f\big) = -\big(\partial_t + \tilde{\sigma}A\big)\,\tilde{\sigma}(f)$$

Thus we see that for $A \in \mathcal{A}^{\tilde{\sigma}}$, the involution $\tilde{\sigma}$ on $L^2(S^1)^{\oplus 2n}$ and the operator $\partial_t + A$ anti-commute.

Therefore we have a family of self-adjoint operators anti-commuting with a fixed involution $\tilde{\sigma}$ parametrized by $\mathcal{A}^{\tilde{\sigma}}$, the whole setup equivariant for the action of $\mathcal{G}^{\tilde{\sigma}}$. So we obtain over the quotient $\mathcal{A}^{\tilde{\sigma}}/\mathcal{G}'^{\tilde{\sigma}} \cong G_n(\mathbb{C}^{2n})$ ~~a family of graded Hilbert spaces equipped with a~~ a graded Hilbert bundle with a family of odd degree self-adjoint Fredholm operators.

May 11, 1986

Recall that the problem is to transgress the cyclic cocycles on [redacted] the restricted unitary group to [redacted] differential forms on [redacted] a ~~classify~~ suitable model of its classifying space. The obvious model to use is the Milnor model, as this fits naturally into the Hilbert space context.

Let's fix notation. Let $\mathcal{G} = U_{res}(V, \eta)$, $\eta \in \mathcal{I}(\mathcal{Q}(V))$. Then the Milnor model for $E\mathcal{G}$ is a space of embeddings of V into $H = V \oplus V \oplus \cdots$ such that η on V is induced by $\eta \oplus \eta \oplus \cdots$ on H. It is the space of embeddings of the form

$$\sum_{j \geq 0} \sqrt{t_j}\, i_j\, g_j$$

Now let $V = L^2(S^1, \mathbb{C}^n)$ with Hilbert η, and let us change notation [redacted] and let $\mathcal{G}$ be the subgroup of $U_{res}(V, \eta)$ [redacted] given by the loop group:

$$\mathcal{G} = C^\infty(S^1; U_n)$$

We then have an obvious map from the Milnor model for $\mathcal{G}$ [redacted] to the ~~space of free loops in~~ free loop space of the Milnor model for BU_n. We have a map [redacted] from

$$\mathcal{G} \longrightarrow E\mathcal{G} \longrightarrow B\mathcal{G}$$

to

$$C(S^1, U_n) \longrightarrow C(S^1, EU_n) \longrightarrow C(S^1, BU_n)$$

The point of the preceding is not too clear. I wanted to stress the idea that [redacted] the models used for $B(U_{res})$ so far, such as projectors in the restricted operator algebra, are analogues of the free loop space of the Grassmannian. Now we have a different model for $\mathcal{L}(\text{Grass})$, namely the group $\mathcal{G}^{\tilde{\sigma}}$.

There are **two** interesting features:

1) It should be possible to define cyclic cocycles on $\mathcal{G}^{\tilde{\sigma}}$ in the standard way. These will be odd degree classes because the relevant Hilbert space is graded.

2) We have over the circle a graded Fredholm module situation. I'm used to thinking this occurs only for even diml Dirac operators.

May 14, 1986

Let $\mathcal{G} = \mathcal{L}U(n)$ act in the usual way on $H = L^2(S^1, \mathbb{C}^n)$ and on the space of connections $\mathcal{A}$. To each connection A we associate the Dirac operator

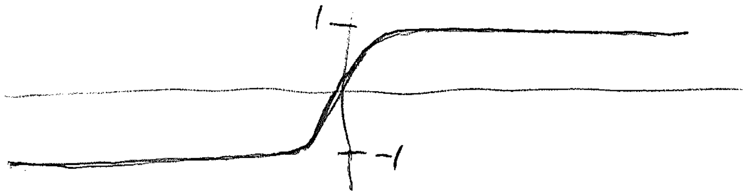
$$D_A = \frac{1}{2\pi i}(\partial_t + A)$$

on H. Its spectrum is the set of λ in $\mathbb{R}$ such that $e^{2\pi i\lambda}$ is an eigenvalue of the monodromy of A, and there are n eigenvalues in any fund. domain for $\mathbb{R}/\mathbb{Z}$.

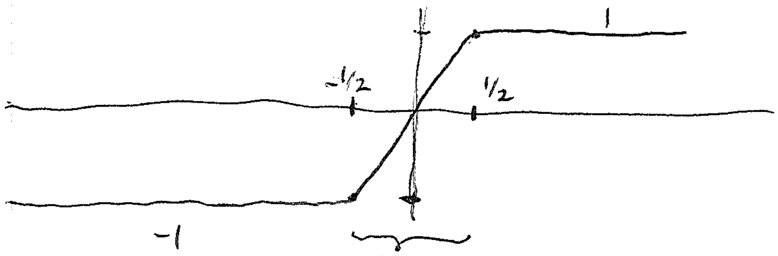
[In my mind there is a close link between $\mathcal{A}$ and the space $\mathcal{F}_{1,\eta}$ of self-adjoint contractions which are congruent to the Hilbert involution mod $\mathcal{K}$. Both are convex and there are various maps $\mathcal{A} \to \mathcal{F}_{1,\eta}$ which are $\mathcal{G}$-equivariant, e.g. $A \longmapsto D_A/\sqrt{m^2+D_A^2}$. Also $\mathcal{A}$ has a kind of building structure, a kind of simplicial structure, ~~It seems that~~ which is described by flags of outgoing subspaces.

I mention this because I am trying to motivate a construction to be presented below. Another line to take is that I ~~[illegible]~~ am concerned with constructing left-invariant forms on $\mathcal{G}$ by picking points in the restricted Grassmannian. This means I would like a natural way to go from a connection to an outgoing subspace ~~by breaking the~~ which roughly takes the positive eigenspace of the Dirac operator. This latter ~~doesn't~~ is discontinuous but can be smoothed at the expense of having a map from $\mathcal{A}$ to $\mathcal{F}_{1,\eta}$.

If one thinks of the ideal as a map from $\mathcal{A}$ to Ires, then the best one could do is to have a self-adjoint contraction with as few eigenvalues $\neq \pm 1$ as possible. One way to do this is to use the map $A \longrightarrow \varphi(D_A)$ where $\varphi(x)$



is a smoothing of $\text{sgn}(x)$. The building picture suggests taking φ to be



spacing between eigenvalues (if monodromy is scalar)

as I have seen.

The nice thing about this choice of φ is that $\varphi(D_A)$ is an involution precisely when the monodromy of A is -1, i.e. when the spectrum lies in $\frac{1}{2}+\mathbb{Z}$. We have seen that monodromy -1 is of special interest for the Bott map.]

The above discussion is intended to suggest there might something ~~canonical~~ special about associating to a connection with monodromy -1 the involution

$$F_A = D_A / |D_A|$$

although this works for any A whose monodromy doesn't have the eigenvalue 1.

In any case on the $\mathcal{G}$ orbit of connections with monodromy -1 there is a natural map to the Grassmannian of outgoing subspaces. This $\mathcal{G}$ orbit is a smoothed version of $\Omega(U(n); 1, -1)$, and it receives the Bott map

$$\begin{tikzcd} \mathrm{Grass}(\mathbb{C}^n) \arrow[r] & \Omega(U(n);\, 1, -1) \\ F \arrow[r, mapsto] & e^{i\pi t F} \end{tikzcd} \qquad 0 \le t \le 1.$$

The corresponding Dirac operator is

$$D_F = e^{-i\pi t F}\left(\frac{1}{2\pi i}\partial_t\right)e^{i\pi t F} = \frac{1}{2\pi i}\partial_t + \frac{1}{2}F$$

Put $\mathbb{C}^n = V = V^+ \oplus V^-$ for the eigenspace decomposition relative to F. If $z = e^{2\pi i t}$, then on $z^n V^+$ the Dirac operator has the value $n + \frac{1}{2}$, and on $z^n V^-$ it has the value $n - \frac{1}{2}$. Thus

$$\begin{aligned} \text{positive space for } D_F &= H^2 \otimes V^+ \oplus zH^2 \otimes V^- \subset L^2(S^1) \otimes V \\ &= \big(e + z(1-e)\big) \cdot H^2 \otimes V \end{aligned}$$

where $e = \frac{F+1}{2}$.

So we have the following picture

$$\boxed{\begin{tikzcd} \mathrm{Grass}(V) \arrow[r, hook] & \mathfrak{a}_{(\mathrm{mon}=-1)} \arrow[r, hook] & \mathcal{I}_{\mathrm{res}} \\ W \arrow[r, mapsto] & i\pi F_W \arrow[r, mapsto] & W + z(H^2 \otimes V) \end{tikzcd}}$$

Next we pass to the case of the Grassmannian which is a graded version of the above. Suppose given a fixed grading $V = V^+ \oplus V^-$, $\varepsilon = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. We define an involution σ on $L^2(S^1, V)$ by

$$(\sigma f)(t) = \varepsilon f(-t).$$

There is then an induced involution on $\mathcal{I}$ given by

$$(\sigma g)(t) = \varepsilon g(-t)\varepsilon$$

and an involution on $\mathcal{A}$ given by

$$(\sigma A)(-t) = -\varepsilon A(-t)\varepsilon .$$

These satisfy

$$\sigma(g \cdot f) = \sigma(g)\,\sigma(f)$$

$$\sigma \circ D_A = -D_{\sigma A} \circ \sigma \qquad \text{(see p. 403)}$$

Hence $\mathcal{G}^\sigma$ acts on H preserving the σ-grading and to each $A \in \mathcal{A}^\sigma$ we have an odd operator D_A relative to the σ-grading.

Recall that a connection $A \in \mathcal{A}^\sigma$ integrates to a path $h : \mathbb{R} \longrightarrow U(n)$ satisfying

$$h(0) = 1, \quad h^{-1}h' = A, \quad h(t+1) = h(1)\,h(t), \quad h(-t) = \varepsilon h(t) \varepsilon .$$

and that it gives rise to a path in the Grassman. by

$$h \longmapsto F_t = h(\tfrac{t}{2})\,\varepsilon\, h(\tfrac{t}{2})^{-1} = h(\tfrac{t}{2})\,h(-\tfrac{t}{2})^{-1}\varepsilon .$$

such that $F_1 = h(\frac{1}{2})\,h(-\frac{1}{2})^{-1}\varepsilon = h(1)\varepsilon$. In this way [redacted] $\mathcal{A}^\sigma_{mon=-1}$ is a smoothed version of $\Omega\left(U(n)/U(n)^\varepsilon ; \varepsilon, -\varepsilon\right)$.

For the Bott map we need odd involutions F on V relative to ε, so change notation and let n become $2n$. Then

$$\mathcal{I}(V)^- = \{ F \in \mathcal{I}(V) \mid \varepsilon F + F \varepsilon = 0 \} = U(V^+, V^-)$$

and we have the Bott map

$\mathcal{I}(V)^- \longrightarrow \Omega($ [redacted] $Gr_n(\mathbb{C}^{2n}) ; \varepsilon, -\varepsilon)$

$$F \longmapsto (\cos\theta)\varepsilon + (\sin\theta)F = e^{\theta F \varepsilon} \cdot \varepsilon \qquad 0 \le \theta \le \pi$$

This lifts to the path

$$h(t) = e^{\pi t F\varepsilon}$$

and the connection

$$A(t) = (h^{-1}h')(t) = \pi F\varepsilon$$

and Dirac operator

$$\frac{1}{2\pi i}\left(\partial_t + \pi F\varepsilon\right) = \frac{1}{2\pi i}\partial_t + \frac{1}{2}\left(\frac{1}{i}F\varepsilon\right).$$

Essentially what we have done is to apply the automorphism $F \longrightarrow \frac{1}{i}F\varepsilon$ on involutions anti-commuting with ε. So perhaps I should change notation to

$$h(t) \longmapsto F_t = h\left(\tfrac{t}{2}\right)\varepsilon\, h\left(+\tfrac{t}{2}\right)^{-1} = h\left(\tfrac{t}{2}\right)h\left(-\tfrac{t}{2}\right)^{-1}\varepsilon$$

and [redacted] let the Bott map be

$$\mathcal{I}(V)^- \longrightarrow \Omega\left(Gr_n(\mathbb{C}^{2n});\varepsilon,-\varepsilon\right)$$

$$F \longmapsto e^{i\pi t F}\varepsilon \qquad 0 \le t \le 1$$

Then the lifting is

$$h(t) = e^{i\pi t F} \qquad A(t) = \frac{1}{2\pi i}\partial_t + \frac{1}{2}F$$

consistent with the notation on page 408.

A better way then is to write down:

(*)

$$\begin{tikzcd}
\mathcal{I}(V) \arrow[r, hook] & \mathcal{A}_{(mon=-1)} \arrow[r, hook] & \mathcal{I}_{res} \\
F \arrow[r, mapsto] & \frac{1}{2\pi i}(\partial_t)+\frac{1}{2}F \arrow[r, mapsto] & \otimes\{F=1\}\oplus zH^2\otimes V
\end{tikzcd}$$

and then to examine what happens under the involution σ. In particular we want to describe

$\mathcal{I}^{\sigma,-}_{res}$, by which we mean involutions anti-commuting with σ. (Recall σ changes the sign of the Dirac operator.)

Let's write

$$H = H_{\sigma=1} \oplus H_{\sigma=-1}$$

and note that an involution anti-comm. with σ is the same as a unitary isomorphism of $H_{\sigma=1}$ with $H_{\sigma=-1}$. So its clear that $\mathcal{I}^{\sigma}_{res}$ consists of all unitary isos. of $H_{\sigma=1}$ with $H_{\sigma=-1}$ which are congruent mod $\mathcal{K}$ to a given one. Thus $\mathcal{I}^{\sigma,-}_{res} \simeq U(\mathcal{K})$.

We can also think of $\mathcal{I}^{\sigma,-}$ as all subspaces of H which are transformed into their orthogonal complements by σ. Now look at the fixpoints for σ on $\circledast$:

$$(**) \qquad \mathcal{I}(V)^{\varepsilon,-} \hookrightarrow \mathcal{A}^{\sigma}_{(mon=-1)} \hookrightarrow \mathcal{I}^{\sigma,-}_{res}$$

Note that if ~~ε anti-commutes with F on V~~ F on V anti-commutes with ε, then σ carries

$$1 \otimes \{F=1\} \oplus zH^2 \otimes V$$

into $\quad 1 \otimes \{F=-1\} \oplus \overline{zH^2} \otimes V \quad$ which is its complement.

As F varies only the unitary transformation between V^+ and V^- moves, the part outside $1 \otimes V$ remains fixed. This should mean that the composition (**) is essentially the inclusion of a finite unitary group into $U(\mathcal{K})$.

May 16, 1986

I have been concerned with Dirac operators over the circle S^1 where the auxiliary bundle has rank n. Any connection over S^1 is flat, so it is described mod gauge equivalence by its monodromy (or holonomy).

There is a canonical vector bundle E over $U_n \times S^1$ which is equipped with a partial connection in the S^1 direction, and which is a universal family of flat rank n bundles over S^1 trivialized over the baseproint $0 \in S^1$. Sections of E are smooth functions $f: U_n \times \mathbb{R} \longrightarrow \mathbb{C}^n$ satisfying

$$f(g,t+1) = g f(g,t)$$

Thus E_g is the flat line bundle over $S^1 = \mathbb{R}/\mathbb{Z}$ with monodromy g; its sections are $f: \mathbb{R} \to \mathbb{C}^n$ $\ni$ $f(t+1) = g f(t)$, and the flat connection is $\nabla f = f' dt$.

Another way to obtain E is to use the principal bundle

$$\mathcal{G}_* \longrightarrow \mathcal{A} \longrightarrow U_n$$

and the natural action of $\mathcal{G}_*$ on $S^1 \times \mathbb{C}^n$ over S^1.

This canonical family E of flat rank n vector bundles over S parametrized by U_n provides a family of Dirac operators on S^1 parametrized by U_n. The Hilbert bundle is induced by the representation of $\mathcal{G}_*$ on $H = L^2(S^1, \mathbb{C}^n)$.

The problem is to link the odd character forms on U_n with the even forms on $\mathcal{G}$. Perhaps it is easier to first treat the K-theory and identify the index of the family with the canonical class in $K^1(U_n)$. I discussed this before (see April 2, 1986, p. 311).

I should also bring in the index theorem for families. In order to do this I need to have a connection in the bundle E extending the partial connection in the S^1-direction. I need this in order to make sense of the differential forms giving the character of the index. ~~Recall that the [illegible]~~

Once I have a connection in E extending the partial S^1-connection, I can take its character forms and integrate them over the circle, thereby obtaining odd forms on U_n. Any chance these are biinvariant?

Construction of the desired connection in E over $U_n \times S^1$

We've seen before that we have to construct a partial connection in the U_n-direction, and that this is the same as a connection in the principal bundle

$$G_* \longrightarrow \mathcal{A} \longrightarrow U_n$$

Given an $h(t)$ with $h(1)=g$:

$$h(t+1) = g\,h(t) \qquad\qquad h(0)=1$$

and a variation δg of g a connection gives a corresponding δh $\ni$

$$\delta h(t+1) = g\,\delta h(t) + \delta g\, h(t) \qquad\qquad \delta h(0)=0$$

or

$$\star \qquad (h^{-1}\delta h)(t+1) - (h^{-1}\delta h)(t) = h(t)^{-1} g^{-1}\delta g\, h(t)$$

Thus we are led to solve the diffce eqn.

$$F(t+1) - F(t) = f(t)$$

A formal solution is

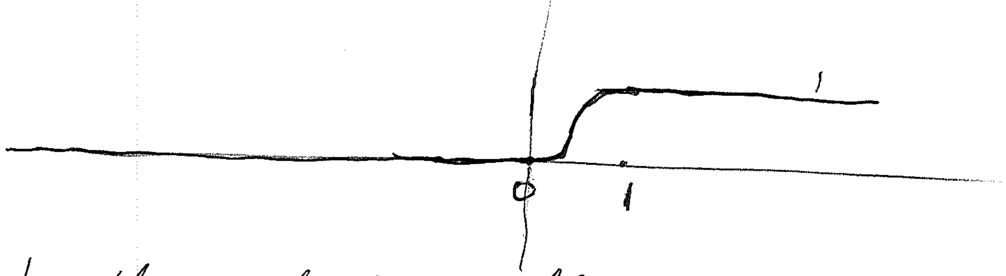
$$F(t) = f(t-1) + f(t-2) + \cdots$$

but this has problems unless $f(t)$ decays at $t \to -\infty$. The general solution differs by a periodic fn. Thus

$$F(t) = \sum_n \left(\mathbb{1}_{n>0} - \alpha(n-t)\right) f(n-t)$$

where $\alpha(t) = \begin{cases} 1 & t \gg 0 \\ 0 & t \ll 0 \end{cases}$

will be a solution which is always well-defined. A typical choice for α is a smooth approx. to the Heaviside fn:



and this choice will give $F(0) = 0$.

I won't write down the solution of $*$, because I ~~only~~ wrote the above to motivate the introduction of a function like α. It seems that such a choice is natural — rather as natural as choosing a partition of unity.

Let's now give a more direct construction of a connection on E. We recall that E is the quotient of an action of $\mathbb{Z}$ on $U_n \times \mathbb{R} \times \mathbb{C}^n$ over $U_n \times \mathbb{R}$. A section of E over $U_n \times S^1$ lifts to a map $f(g,t) : U_n \times \mathbb{R} \longrightarrow \mathbb{C}^n$ such that

$$g f(g,t+1) = f(g,t)$$

(For fixed $g \in U_n$, E_g is the vector bundle associated to the principal $\mathbb{Z}$-bundle $\mathbb{R} \to S^1$ and the representation $n \to g^n$. So a section of E_g is a map $f : \mathbb{R} \to \mathbb{C}^n$ such that $f(t+1) = g^{-1} f(t)$.)

Thus the $\mathbb{Z}$ action on sections of $U_n \times \mathbb{R} \times \mathbb{C}^n / U_n \times \mathbb{R}$ is generated by

$$(Tf)(g,t) = g f(g,t+1)$$

We now want an invariant connection on this vector bundle which has the form

$$d_{U_n \times \mathbb{R}} + A = \delta + dt\, \partial_t + A$$

where A is a 1-parameter family of 1-forms on U_n. This last condition means that the connection restricts to the canonical partial conn. $dt\, \partial_t$ in the S^1-direction.

Invariance means

$$(d + A)\, Tf = T(d+A) f$$

$$[\delta + dt\, \partial_t + A(g,t)](g f(g,t+1))$$

$$= \delta g\, f(g,t+1) + g[\delta \overset{\checkmark}{f}(g,t+1) + dt\, \partial_t \overset{\checkmark}{f}(g,t+1)] + A(g,t) \underset{\wedge}{\overset{g}{}} f(g,t+1)]$$

$$\overset{?}{=} g[(\overset{\checkmark}{\delta} + dt\, \partial_t) f(\overset{\checkmark}{g}, t+1) + A(g,t+1) f(g,t+1))]$$

So A must satisfy

$$* \quad \boxed{\delta g + A(g,t) g = g A(g,t+1)}$$

We also want $A(g,0)=0$. This means

$$A(g,0) = 0$$
$$A(g,1) = g^{-1}\delta g$$
$$A(g,2) = g^{-1}\delta g + g^{-1}(g^{-1}\delta g)g$$

Now to solve all we have to do is to interpolate smoothly. [redacted] We define $A(g,t)$ for $0 \le t \le 1$ so that when extended [redacted] so as to satisfy $*$ it is smooth at the ends.

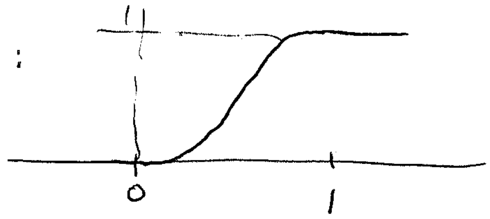
In the abelian ~~case~~ we can use

$$A(g,t) = t\, g^{-1}\delta g$$

but this doesn't work in general. The simplest kind of general choice is to take

$$A(g,t) = \rho(t)\, g^{-1}\delta g \qquad -\varepsilon \le t \le 1+\varepsilon$$

where $\rho(t)$:



Then this A is constant at the ends.

Ideas: 1) We have three models for $B\mathcal{G}_*$. First there is $\mathcal{A}/\mathcal{G}_* = U_n$, [redacted] secondly ΩBU_n (projectors over $C^\infty(S^1)$ of rank n), + finally the loop group $\mathcal{G}^\sigma$ attached to the Grassmannian as a symmetric space. On U_n and $\mathcal{G}^\sigma$ one has explicit left-invariant odd degree forms.

2) Go from [redacted] $\Omega\,\mathrm{Grass}_n \longrightarrow \mathcal{G}^\sigma$ and then $\mathcal{G}^\sigma \longrightarrow U(\mathcal{H})$. This defines odd forms on $\Omega\,\mathrm{Grass}_n$. (The first map is lifting using the Grassmannian connection, the 2nd involves a choice of involution.) The problem would be

to understand these odd forms on $\Omega\,\mathrm{Grass}_n$

3) Recall how to define an index map for a Hilbert bundle E with K-splitting. One embeds E in $\tilde{H}$, then extends η on E by $\eta = +1$ or -1 on the complement. There's an asymmetry here. Can it be avoided by an infinite repetition trick?