November 20, 1986.

Let's try to replace V by Hilbert space H, and let $\mathcal{U}_p(H) = \{ g \in \mathcal{U}(H) \mid g \equiv -1 \bmod \mathcal{L}^p(H) \}$ where $\mathcal{L}^p$ is the p^{th} Schatten ideal. Here the forms

$$c\,\omega^t_{2k+1} = (4t)^{2k+1} \operatorname{tr}\left(\frac{1}{t(g+1)+(g-1)} \, dg \, \frac{1}{t(g+1)-(g-1)} \right)^{2k+1}$$

are defined only for $(2k+1) \geq p$. Let's call this trace τ^t_{2k+1}. If we differentiate with respect to t we introduce a $g+1$ factor which means the derivatives is defined for $2k+2 \geq p$. A natural question is whether this form is closed.

Let

$$g_t = \frac{t+X}{t-X} = \frac{t(g+1)+(g-1)}{t(g+1)-(g-1)}$$

so that

$$\theta_t = g_t^{-1} dg_t = 4t \, \frac{1}{t(g+1)+(g-1)} \, dg \, \frac{1}{t(g+1)-(g-1)}$$

Suppose we work with $\mathcal{L}^2$ the ideal of Hilbert Schmidt operators. Then is a 1-form on $\mathcal{U}_2(H)$ with values in $\mathcal{L}^2$ since the denominators are just bounded operators and $dg = d(g+1)$ has values in $\mathcal{L}^2(H)$. Thus the form $\operatorname{tr}\theta_t$ is not defined.

However the form $\partial_t\left(\frac{1}{4t}\theta_t\right)$ has a trace and we would like to see if it is closed. For this we want to use that

$$d\theta_t = -\theta_t^2$$

So

$$d\left\{-\partial_t\left(\tfrac{1}{4t}\theta_t\right)\right\} = \partial_t\left\{\tfrac{1}{4t}\theta_t^2\right\}$$

$$= \partial_t\left\{\tfrac{1}{4t}\theta_t\right\}\cdot\theta_t + \tfrac{1}{4t}\theta_t\,\underbrace{\partial_t\theta_t}_{\partial_t\left(4t\frac{1}{4t}\theta_t\right) = 4\frac{1}{4t}\theta_t + 4t\,\partial_t\left(\frac{1}{4t}\theta_t\right)}$$

$$= \partial_t\left\{\tfrac{1}{4t}\theta_t\right\}\cdot\theta_t + \theta_t\cdot\partial_t\left\{\tfrac{1}{4t}\theta_t\right\} + 4\left(\tfrac{1}{4t}\theta_t\right)^2$$

$$= \left[\partial_t\left(\tfrac{1}{4t}\theta_t\right), \theta_t\right] + 2\left[\tfrac{1}{4t}\theta_t, \tfrac{1}{4t}\theta_t\right]$$

Here θ_t has $\mathcal{L}^2$ values while $\partial_t\left(\frac{1}{4t}\theta_t\right)$ has $\mathcal{L}^1$ values. Certainly the trace of the first bracket is zero because $tr(XY) = tr(YX)$ when one is trace class and the other is bounded. However it must be true that $tr(XY) = tr(YX)$ when $X \in \mathcal{L}^p$ and $Y \in \mathcal{L}^q$ with $\frac{1}{p} + \frac{1}{q} \geq 1$. (Note that here, like ℓ^p, one has $\mathcal{L}^1 \subset \mathcal{L}^p \subset \mathcal{L}^\infty$ for $1 \leq p \leq \infty$.)

In fact looking at the appendix on Schatten ideals in Connes paper, one sees that $\mathcal{L}^p . \mathcal{L}^q \subset \mathcal{L}^r$ if $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$, $p, q, r \in [1, \infty]$, and that if $A B$ are bounded with $AB, BA \in \mathcal{L}^1$, then

$$tr(AB) = tr(BA).$$

Narashiman-Ramanan thm. Given a vector bundle E with $(,)$ and a connection D preserving it, there is an isometric embedding $i : E \longrightarrow \tilde{V}$ such that $D = i^* d . i$.

Basic construction: Let $\sum \rho_k^2 = 1$ ~~be~~ a smooth partition

of 1 subordinate to a finite open covering $\{U_k\}$. One constructs this by choosing a partition $\sum_i \psi_k = 1$ as usual, then setting
$$\rho_k = \frac{\psi_k}{\left(\sum_k \psi_k^2\right)^{1/2}}.$$

Next note that $\rho_k \iota_k : E|U_k \longrightarrow \tilde{V}_k|U_k$ can be extended by zero to give a smooth vector bundle map $\rho_k \iota_k : E \longrightarrow \tilde{V}_k$. In effect if $m \in \operatorname{Supp} \rho_k$ then $m \in U_k$ and $\rho_k \iota_k$ is defined + smooth near m; if $m \notin \operatorname{Supp} \rho_k$, then $\rho_k \iota_k$ is zero near m.

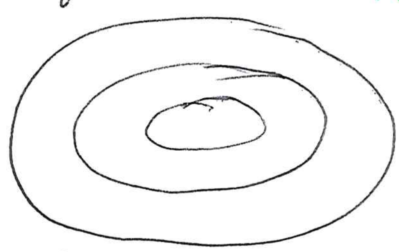
Now form the vector bundle map $\iota : E \longrightarrow \bigoplus_k \tilde{V}_k$ given by the column vector $(\rho_k \iota_k)$. Its adjoint is the row vector $(\iota_k^* \rho_k)$. So
$$\iota^* \iota = \sum_k \iota_k^* \rho_k \rho_k \iota_k = \sum_k \rho_k^2 \overbrace{\iota_k^* \iota_k}^{=1 \text{ on } U_k}$$
$$= \sum \rho_k^2 = 1$$
is an isometric embedding. The induced connection is
$$\iota^* d \iota = \sum_k \iota_k^* \cdot \rho_k \cdot d \cdot \rho_k \cdot \iota_k$$
$$= \sum_k \rho_k^2 \, \iota_k^* d \, \iota_k + \underbrace{\sum_k \rho_k d\rho_k \overbrace{\iota_k^* \iota_k}^{1 \text{ on } U_k}}_{d\left(\frac{1}{2}\sum \rho_k^2\right) = 0}$$

So we see that on averaging embeddings wrt a partition of unity we average the associated connections.

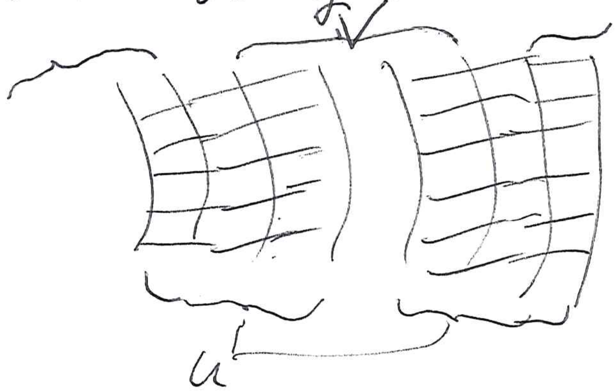
In particular if $D = \iota_k^* d\iota_k$ on U_k for all k, then $i^* di = D$. This shows the Narashiman-Ramanan theorem is local on M.

Here I use that if I know the result is true locally, then it is true in any nbd of a compact as well as for any disjoint of open sets on which it is true. Now use an exhaustion of an open manifold

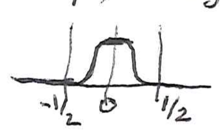
No the dimensions of the V's required might be unbounded.



to reduce to the case where one has two open sets covering the manifold. Union of disjoint annuli.



(Take a proper $f: M \to \mathbb{R}$ and pull back $1 = \varphi + (1-\varphi)$ where $\varphi(x) = \sum \rho(x+n)$ and ρ is



.)

So now we have to construct the embedding locally. First consider the case of a line bundle, which we locally suppose is trivial. The connection is then given by a purely imaginary 1-form θ, which we can write

$$\theta = i \sum_{k=1}^{n} g_k \, df_k$$

where the g_k, f_k are smooth real functions on M. This means that we have a map from M to $\mathbb{R}^{2n}$ with the coordinates x_k, y_k such that θ is

induced from the 1-form
$$\theta = i \sum y_k \, dx_k$$

November 21, 1986

We saw that the NR theorem is a local matter. But recall that given embeddings $\iota_k : E|U_k \longrightarrow \tilde{V}_k$ with $U_k \supset \operatorname{Supp} \rho_k$ and $\sum \rho_k^2 = 1$ we get an embedding $\sum \rho_k \iota_k : E \hookrightarrow \oplus V_k$ with induced connection $\sum \rho_k^2 \, \iota_k^* . d . \iota_k$.

In particular taking $U_1 = U_2 = M$ and $\rho_1 = \rho_2 =$ the constant functions $\frac{1}{\sqrt{2}}$, we see that if connections D_1, D_2 can be realized by embeddings, then so can $\frac{1}{2}(D_1 + D_2)$.

Locally we can trivialize E and a connection is of the form $d + \theta$ where $\theta \in \mathfrak{g} \otimes_{\mathbb{R}} \Omega^1(M)_{\mathbb{R}}$ where $\mathfrak{g}$ = skew hermitian matrices. If T^a is a basis for $\mathfrak{g}$, then $\theta = \sum_a T^a \theta_a$ where $\theta_a \in \Omega^1(M)_{\mathbb{R}}$. By averaging as above if we can realize each of the N connections $d + N T^a \theta_a$, $N = \dim \mathfrak{g}$ by embeddings, then we can realize their average. Thus we are reduced to the case $\theta = T^a \theta_a$ and by diagonalizing T^a, we are reduced to the case of line bundles. This means $\theta = i\omega$ where ω is a real 1-form. Then $\omega = \sum f_k \, dg_k$ and again by averaging we can suppose $\omega = f \, dg$.

Then by naturality we reach the case of $\theta = i \, y \, dx$ on $\mathbb{R}^2$.

Now we can find a map from a nbd.

of 0 in $\mathbb{R}^2$ to the Riemann sphere $\mathbb{CP}^1$ such that the curvature form of $\mathcal{O}(-1)$ pulls back to $d\theta = i\,dy\,dx$. The curvature form of $\mathcal{O}(-1)$ is

$$\frac{1}{1+|z|^2}\,d\bar{z}\;\frac{1}{1+|z|^2}\,dz = \frac{2i\,dx\,dy}{(1+|z|^2)^2}$$

$$= 2i\,\frac{r\,dr\,d\theta}{(1+r^2)^2}$$

so if we put $$2\rho\,d\rho = -\frac{2r\,dr}{(1+r^2)^2}$$ or

$$\rho = \frac{1}{\sqrt{1+r^2}}$$

then the map

$$z = re^{i\theta} \longmapsto \begin{array}{l} x = \rho(r)\cos\theta \\ y = \rho(r)\sin\theta \end{array}$$

has

$$\begin{aligned} dy\,dx &= (d\rho\sin\theta + \rho\cos\theta\,d\theta)(d\rho\cos\theta - \rho\sin\theta\,d\theta) \\ &= -2\rho\,d\rho\,d\theta \\ &= \frac{2r\,dr\,d\theta}{(1+r^2)^2}. \end{aligned}$$

This map is a diffeomorphism of the disk $\rho = \sqrt{x^2+y^2} \leq 1$ in the x,y plane with the disk $0 < r \leq \infty$ in the Riemann spheres. It preserves volume, hence the curvature forms correspond.

Now use the fact that on a simply-connected manifold a line bundle with connection is determined by its curvature.

November 21, 1968 (cont.)

I have to patch a hole in the proof for non-compact manifolds. We of course suppose M countably compact and finite dimensional, and that E has finite rank. Can suppose E has constant rank r. One knows that E is induced from the subbundle over $G_r(\mathbb{C}^{2r+1})$ via a map from M to this Grassmannian. Then as this Grass is compact we get a finite covering over which E is trivial, and so we can suppose E trivial.

Then the connection is given by $\theta \in \mathfrak{g} \otimes \Omega^1(M)$, so using a basis T^a for $\mathfrak{g}$ ~~one can suppose~~ and diagonalizing the T^a one reduces to the case where [redacted] E is the trivial line bundle and $\theta = i\omega$ with ω a real 1-form. If M has dimension $\leq d$, then ω is the sum of at most $2d+1$ 1-forms of the form $f\,dg$ (because M embeds in $\mathbb{R}^{2d+1}$). Then one is done.

Let's now return to the character forms on the group $-U_p(H)$ of unitaries g such that $g+1 \in \mathcal{L}_p(H)$. Set for $\mathrm{Re}(t) > 0$

$$\theta_t = g_t^{-1} dg_t = t \frac{2}{t(g+1)+(g-1)}\, dg\, \frac{2}{t(g+1)-(g-1)}$$

This is a 1-form defined on $-U_p(H)$ with values in $\mathcal{L}_p(H)$. [redacted] Set [redacted] $\eta_t = \frac{1}{t}\theta_t$ and from now on drop the superscript t. Then we have $d\theta = -\theta^2 \implies d\eta = -t\eta^2$

Claim $\partial_t^n(\eta^g)$ has $n+g$ factors in $\mathcal{L}_p$. Better, let us consider
$$\partial_t^n(\eta^g).$$

Now η is a product of the factors $(t(g+1) \pm g-1)^{-1}$ which are invertible bounded operators depending smoothly on g, and $dg = d(g+1)$ which is an $\mathcal{L}^p$ valued 1-form. We have
$$\partial_t \left(t(g+1) \pm (g-1)\right)^{-1} = -\left(t(g+1) \pm (g-1)\right)^{-1} (g+1)\left(t(g+1) \pm (g-1)\right)^{-1}$$
and $g+1 \in \mathcal{L}^p$, so it is clear that $\partial_t^n(\eta^g)$ is a sum of terms each term having g dg factors, $2g+n$ $(t(g+1) \pm (g-1))^{-1}$ factors and n $g+1$ factors. Thus $\partial_t^n(\eta^g)$ has $g+n$ factors in $\mathcal{L}^p$ for each terms, so
$$\partial_t^n(\eta^g) \in \mathcal{L}^{\left(\frac{g+n}{p}\right)^{-1} \vee 1}$$

Better: For $n+g \geq p$, $\partial_t^n(\eta^g) \in \mathcal{L}^1$.

A more rigorous style of proof would be to argue that by Leibniz
$$\partial_t^n \eta = 4 \sum_{a+b=n} \binom{n}{a} \partial_t^a\left(\frac{1}{t(g+1)+(g-1)}\right) dg \; \partial_t^b\left(\frac{1}{t(g+1)-(g-1)}\right)$$
$$= 4 \sum \binom{n}{a} \frac{(-1)^a a!}{(t(g+1)+(g-1))^{a+1}} (g+1)^a \, dg \, (g+1)^b \frac{(-1)^b b!}{(t(g+1)-(g-1))^b}$$
has $n+1$ $\mathcal{L}^p$ factors, so it lies in $\mathcal{L}^{\left(\frac{n+1}{p}\right)^{-1}}$ and then again to use Leibniz to get
$$\partial_t^n \eta^g = \sum_{a_1+\cdots+a_g=n} \frac{n!}{a_1! \cdots a_g!} \left(\partial^{a_1}\eta\right) \cdots \left(\partial^{a_g}\eta\right).$$

Each term has $(a_1+1)+(a_2+1) \cdots (a_g+1) = n+g$ factors in $\mathcal{L}^p$.

Now having this we know that $\partial_t^n(\eta\delta)$ is of trace class for $n+q \geq p$ and we now want to show that

$$\operatorname{tr} \partial_t^n(\eta\delta) = 0 \qquad q \text{ even}$$

$$d \operatorname{tr} \partial_t^n(\eta\delta) = 0 \qquad q \text{ odd.}$$

Let's try to prove the former using the idea that $\eta\delta = \frac{1}{2}[\eta, \eta\delta^{-1}]$ and that the trace of such a commutator is zero. We have

$$\partial_t^n \tfrac{1}{2}[\eta, \eta\delta^{-1}] = \sum_{a+b=n} \binom{n}{a} \tfrac{1}{2}\left[\partial^a \eta, \partial^b \eta\delta^{-1}\right]$$

has $a+1$, $b+q-1$ factors in $\mathcal{L}^p$

Recall that $\operatorname{tr}(XY) = \operatorname{tr}(YX)$ is X, Y are bdd ops. $\ni XY, YX \in \mathcal{L}^1$. Here $(a+1)+(b+q-1) = n+q \geq p$ so we conclude that the traces of the brackets on the right are 0.

Second formula. We have for q odd

$$d \operatorname{tr} \partial_t^n(\eta\delta) = \operatorname{tr}(d\partial_t^n \eta\delta)$$

$$= \operatorname{tr}(\partial_t^n d(\eta\delta))$$

$$= \operatorname{tr}(\partial_t^n(-t\eta\delta^{+1}))$$

$$= -\cancel{t}\operatorname{tr}(\partial_t^n(\eta\delta^{+1})) - n \operatorname{tr}(\partial_t^{n-1}(\eta\delta^{+1}))$$

since $q+1$ is even both of these traces are 0 by the first part.

Now it is probably desirable to justify the first equality. We are talking about smooth differential forms on $-\mathcal{U}_p(H)$ with values in $\mathcal{L}^p$ and with scalar values. Such forms are closed under multiplication and d. The trace is a continuous linear map from $\mathcal{L}^P$ to $\mathbb{C}$, so it will commute with differentiation.

I will perhaps need eventually to check this calculus of forms.

November 25, 1986

Problem: In order to show that

$$c\,\Phi_A = \frac{u}{2\pi i}\int_C e^{\lambda u}\,\frac{\Gamma(n)}{\lambda^n}\,\omega_n^{\sqrt{\lambda}}\,d\lambda$$

is cohomologous to $u^{\frac{n}{2}}\omega_n^1$ we want to write

$$\omega^t - \omega^1 = d\int_1^t \eta^t\,dt$$

where η^t has polynomial growth in t and thus we need to have

$$* \qquad \partial_t\,\omega^t = d\,\eta^t$$

This can be arranged by using the map $\varphi : \{Re(t) > 0\} \times U(V) \longrightarrow GL(V)$ given by

$$\varphi(t,g) = \frac{1+t^{-1}X}{1-t^{-1}X} = \frac{t(g+1)+(g-1)}{t(g+1)-(g-1)}$$

One has

$$\varphi^*\omega^1 = \omega^t + dt\,\eta^t$$

and the fact that this is closed implies $*$.

Since

$$g^{-1}dg = \frac{2}{1+X}\,dX\,\frac{1}{1-X}$$

we have

$$\varphi^*(g^{-1}dg) = \frac{2}{1+t^{-1}X}\left(-t^{-2}dt\,X + t^{-1}dX\right)\frac{1}{1-t^{-1}X}$$

$$= \frac{2}{t+X}\left(-dt\,X + t\,dX\right)\frac{1}{t-X}$$

$$= \frac{2}{t(g+1)+(g-1)}\left(-dt\,(g^2-1) + 2t\,dg\right)\frac{1}{t(g+1)-(g-1)}$$

Let's consider the odd case where

$$\omega^1_{2k+1} = \underbrace{\frac{(-1)^k k!}{(2k+1)!}}_{c} \operatorname{tr}(g^{-1}dg)^{2k+1}$$

Then $$c^{-1}\varphi^* \omega^1_{2k+1} = \operatorname{tr}\left(\frac{2}{t^2-X^2}(-dt\,X + t\,dX)\right)^{2k+1}$$

$$= \operatorname{tr}(\alpha+\beta)^{2k+1}$$

$\alpha = dt\,\frac{-2X}{t^2-X^2}$

$\alpha^2 = 0.$

$$= \operatorname{tr}(\beta^{2k+1}) + \sum_{i=0}^{2k} \operatorname{tr}(\beta^i \alpha \beta^{2k-i})$$

$$= \underbrace{\operatorname{tr}(\beta^{2k+1})}_{c^{-1}\omega^t} + \underbrace{(2k+1)\operatorname{tr}(\alpha\beta^{2k})}_{dt\, c^{-1}\eta^t}$$

odd no. of 1-forms
$\therefore$ one moves 1 form past 2k forms.

This gives the formula

$$\boxed{\eta^t = -(2k+1)c\operatorname{tr}\left\{\frac{2}{t^2-X^2}X\left(\frac{2t}{t^2-X^2}dX\right)^{2k}\right\}}$$

$$\boxed{\eta^t_{2k} = -(2k+1)c\operatorname{tr}\left\{\frac{2}{t^2(g+1)^2-(g-1)^2}(g^2-1)\left(\frac{4t}{t^2(g+1)^2-(g-1)^2}dg\right)^{2k}\right\}}$$

Next consider the even case where

$$\omega^1_{2k} = \underbrace{\frac{(-1)^k}{k!\,2^{2k+1}}}_{c} \operatorname{tr}(\varepsilon g\,(g^{-1}dg)^{2k})$$

$$\varphi^*\left(c^{-1}\omega^1_{2k}\right) = \operatorname{tr}\left\{\varepsilon\,\frac{t(g+1)+(g-1)}{t(g+1)-(g-1)}\left[\frac{2}{t(g+1)+(g-1)}(-dt(g^2-1)+2t\,dg)\frac{1}{t(g+1)-(g-1)}\right]^{2k}\right\}$$

$$\varepsilon\big(t(g+1)+(g-1)\big) = \varepsilon\big(t(g^{-1}+g^{-1})+(1-g^{-1})\big)\,g$$

$$= \big(t(g+1)-(g-1)\big)\,\varepsilon g$$

So

$$c^{-1}\varphi^*(\omega^1_{2k}) = \operatorname{tr}\left\{\varepsilon g\left[\frac{2}{t^2(g+1)^2-(g-1)^2}\left(-dt\,(g^2-1)+2t\,dg\right)\right]^{2k}\right\}$$

$$= \operatorname{tr}\left\{\varepsilon g\,(\alpha+\beta)^{2k}\right\}$$

$$= \operatorname{tr}\left\{\varepsilon g\,\beta^{2k}\right\} + \sum_{i=0}^{2k-1}\operatorname{tr}\left\{\varepsilon g\,\beta^i\alpha\beta^{2k-1-i}\right\}$$

where
$$\beta = \frac{4t}{t^2(g+1)^2-(g-1)^2}\,dg \qquad \alpha = dt\,\frac{-2(g^2-1)}{t^2(g^2+1)-(g-1)^2}$$
$$= \underbrace{\left(\frac{4t}{t^2(g^{1/2}+g^{-1/2})^2-(g^{1/2}-g^{-1/2})^2}\right)}_{\text{commutes with } \varepsilon\text{, denote this } h} g^{-1}dg$$

Now
$$\varepsilon g\beta = \varepsilon g\,(h g^{-1}dg) = h\,\varepsilon\,dg = -h\,g^{-1}dg\,g^{-1}\varepsilon$$
$$= -(h g^{-1}dg)\,\varepsilon g = -\beta\,\varepsilon g.$$

Thus
$$\operatorname{tr}\left\{\varepsilon g\,\beta^i\alpha\,\beta^{2k-1-i}\right\} = \operatorname{tr}\left\{\varepsilon g\,\alpha\,\beta^{2k-1}\right\}$$
because moving β thru εg gives a sign and then moving β cyclically in the trace also gives a sign. Thus we have

$$c^{-1}\varphi^*(\omega^1_{2k}) = \underbrace{\operatorname{tr}(\varepsilon g\,\beta^{2k})}_{c^{-1}\omega^t_{2k}} + \underbrace{2k\,\operatorname{tr}(\varepsilon g\,\alpha\,\beta^{2k-1})}_{dt\cdot c^{-1}\eta^t_{2k}}$$

$$\boxed{\begin{aligned}\eta^t_{2k} &= -2k\,c\,\operatorname{tr}\left\{\varepsilon g\left(\frac{2(g^2-1)}{t^2(g+1)^2-(g-1)^2}\right)\left(\frac{4t}{t^2(g+1)^2-(g-1)^2}\,dg\right)^{2k-1}\right\}\\ &= -2k\,c\,\operatorname{tr}\left\{\varepsilon\,\frac{2X}{t^2-X^2}\left(\frac{2t}{t^2-X^2}\,dX\right)^{2k-1}\right\}\end{aligned}}$$

Let's go back to the resolvent

$$R_\lambda = \frac{1}{\lambda - X^2 - dX\sigma} = (g+1)\frac{1}{\lambda(g+1)^2-(g-1)^2-2dg\sigma}(g+1)$$

and prove directly that $tr_s(R_\lambda)_n$ is closed when n is large enough that the trace is defined. Note that with $a=\lambda(g+1)^2-(g-1)^2$, $b = 2dg\sigma$

$$R_{\lambda,n} = (g+1)(a^{-1}b)^n a^{-1}(g+1)$$

has $n+2$ factors in $\mathcal{L}^p$ as $(g+1), dg \in \mathcal{L}^p$, so $tr_s(R_{\lambda,n})$ is defined for $n+2 \geq p$. This is an improvement over our previous range I think and it might be important.

Thus previously we started with

$$\eta_t^n = \left(\frac{1}{t}\theta_t\right)^n = \left(\frac{4}{t(g+1)+(g-1)}dg\frac{1}{t(g+1)-(g-1)}\right)^n$$

Each time we differentiate with respect to t one of the inverse factors we get a $g+1$ in the numerator, so $\partial_t^j(\eta_t^n)$ has $n+j$ factors in $\mathcal{L}^p$. However if we work with the conjugate form

$$\tilde{\eta}_\lambda^n = \left(\frac{4}{t^2(g+1)^2-(g-1)^2}dg\right)^n$$

and differentiate with respect to $\lambda = t^2$ so as not to introduce t factors in the numerator, then each ∂_λ brings a $(g+1)^2$ factor into the numerator. Thus

$$\partial_\lambda^j(\tilde{\eta}_\lambda^n) \quad \text{has} \quad 2j+n \quad \text{factors in}$$

the Schatten ideal.

Let's now show $tr_s R_\lambda$ is closed by following the proof in the superconnection game:

$$d\, tr_s\{e^{u(d+X\sigma)^2}\} = tr_s\,[d+X\sigma, e^{u(d+X\sigma)^2}] = 0$$

$$dR_\lambda = d\left(\frac{1}{\lambda-X^2-dX\sigma}\right) = \frac{1}{\lambda-X^2-dX\sigma}\, d(X^2+dX\sigma)\,\frac{1}{\lambda-X^2-dX\sigma}$$

$$= R_\lambda\,(dX\cdot X + X\cdot dX)\,R_\lambda$$

$$[X\sigma, R_\lambda] = \frac{1}{\lambda^2-X^2-dX\sigma}\underbrace{[X\sigma, X^2+dX\sigma]}_{X\sigma\, dX\sigma - dX\sigma\, X\sigma \;=\; -(X\,dX + dX.X)}\frac{1}{\lambda^2-X^2-dX\sigma}$$

Thus $\boxed{{}^*\; dR_\lambda + [X\sigma, R_\lambda] = 0}$. Note that in this calculation R_λ is even in the algebra $\Omega(\mathrm{End}\,E)\hat{\otimes} C_1$, so the above is the usual commutator. Now

$$d\, tr_s(R_\lambda) = tr_s(dR_\lambda) = -\,tr_s([X\sigma, R_\lambda]) = 0.$$

Now we want to establish $*$ using g instead of X. The point is that the formula can be differentiated to make $R_{\lambda,n}$ become trace class.

So we have with $A = \lambda(g+1)^2 - 2dg\,\sigma$

$$[X\sigma, R_\lambda] = \frac{g-1}{g+1}\sigma\,(g+1)A^{-1}(g+1) - (g+1)A^{-1}(g+1)\frac{g-1}{g+1}\sigma$$

$$= (g-1)\,\sigma A^{-1}(g+1) - (g+1)A^{-1}\sigma\,(g-1)$$

As $(A^{-1})_n$ has n factors in the Schatten ideal, we see that provided $n+1 \geq p$, so that each term in is trace class, then the supertrace of this last term is zero. So all

I have to do is to check that

$$dR_\lambda = d\{(g+1)A^{-1}(g+1)\}$$
$$= dg\, A^{-1}(g+1) + (g+1)A^{-1}dg - (g+1)A^{-1}dA\,A^{-1}(g+1)$$

coincides with

$$-(g-1)\,\sigma A^{-1}(g+1) + (g+1)A^{-1}\sigma(g-1).$$

Actually this isn't quite clear. What we want is an identity for dR_λ which can be differentiated and which will show that dR_λ has $\mathrm{tr}_s = 0$.

November 26, 1986

Set
$$R_\lambda = \frac{1}{\lambda - X^2 - dX\sigma}.$$

This is a differential form on the open set of $B(H) =$ bounded operators on H, whose spectrum does not contain $\pm\sqrt{\lambda}$. The values of R_λ lie in $B(H) \otimes C_1$.

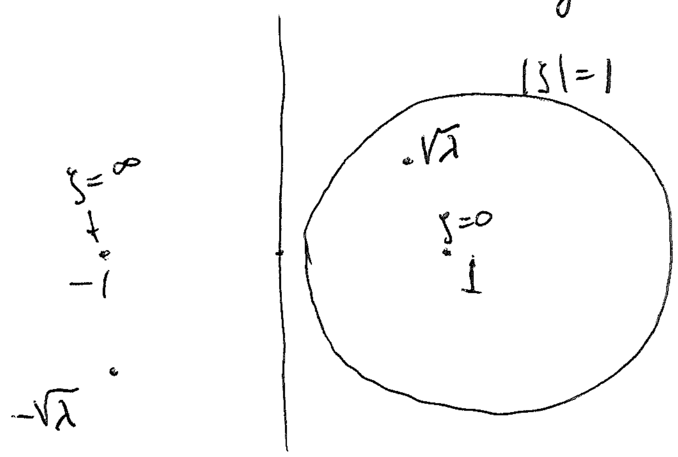
We can do the following manipulations as long as X remains in the open set mentioned.

$$dR_\lambda = R_\lambda\, d(X^2 + dX\sigma)\, R_\lambda = R_\lambda (dX.X + X.dX) R_\lambda$$

$$[X\sigma, R_\lambda] = R_\lambda [X\sigma, X^2 + dX\sigma] R_\lambda = -R_\lambda (dX.X + X.dX) R_\lambda$$

Here we use that $X\sigma\, dX\sigma = -X\,dX$ as σ anti commutes with the 1-form dX. Thus we have
$$dR_\lambda + [X\sigma, R_\lambda] = 0$$

Let us now put $X = \frac{zg-1}{zg+1}$ where g is unitary and z is a complex number such that $|z|$ is close to but not equal to 1. The ~~image of~~ map $\zeta \longmapsto \frac{z\zeta - 1}{z\zeta + 1}$ carries the unit circle into a circle looking like:



If $\mathrm{Re}(\sqrt{\lambda}) > 0$, then for $|z|$ close enough to 1, the image circle will not hit $\pm\sqrt{\lambda}$ and so $\psi_z : g \longmapsto \frac{zg-1}{zg+1}$ will map unitary operators into bounded operators whose spectrum doesn't contain $\pm\sqrt{\lambda}$. The inverse image of R_λ under this map is

$$\psi_z^* R_\lambda = (zg+1)\underbrace{\frac{1}{\lambda(zg+1)^2 - (zg-1)^2 - 2z\,dg\sigma}}_{\text{call this denominator } A_{\lambda z}}(zg+1)$$

$$-\psi_z^*[X\sigma, R_\lambda] = (zg+1)A_{\lambda z}^{-1}(zg+1)\left(\frac{zg-1}{zg+1}\right)\sigma - \left(\frac{zg-1}{zg+1}\right)\sigma(zg+1)A_{\lambda z}^{-1}(zg+1)$$

$$= (zg+1)A_{\lambda z}^{-1}(zg-1)\sigma - (zg-1)\sigma A_{\lambda z}^{-1}(zg+1)$$

Thus we have the identity

$$d\left\{(zg+1)A_{\lambda z}^{-1}(zg+1)\right\} = (zg+1)A_{\lambda z}^{-1}(zg-1)\sigma - (zg-1)\sigma A_{\lambda z}^{-1}(zg+1)$$

between $\mathcal{B}(H)\otimes C_1$ valued forms on $\mathcal{U}(H)$ provided $|z|\neq 1$ and $\pm\sqrt{\lambda}$ does not lie on the image circle $\frac{z\zeta-1}{z\zeta+1}$, $|\zeta|=1$. But both sides are holomorphic in z at $z=1$, so we can let $z\to 1$ and we find

$$\boxed{d\,(g+1)A_\lambda^{-1}(g+1) = (g+1)A_\lambda^{-1}(g-1)\sigma - (g-1)\sigma A_\lambda^{-1}(g+1)}$$

where $A_\lambda = \underbrace{\lambda(g+1)^2 - (g-1)^2}_{a} - \underbrace{2dg\,\sigma}_{b}$.

Pass to forms of degree k

$$A_{\lambda,k}^{-1} = (a^{-1}b)^k a^{-1}$$

Note
$$-\partial_\lambda A_\lambda^{-1} = A_\lambda^{-1}(g+1)^2 A_\lambda^{-1}$$
$$(-\partial_\lambda)^2 A_\lambda^{-1} = 2A_\lambda^{-1}(g+1)^2 A_\lambda^{-1}(g+1)^2 A_\lambda^{-1}$$

and in general

$$(-\partial_\lambda)^q A_\lambda^{-1} = q\left(A_\lambda^{-1}(g+1)^2\right)^q A_\lambda^{-1}$$

$$(-\partial_\lambda)^q A_{\lambda,n}^{-1} = q \sum_{n_0+\cdots+n_q=n} A_{\lambda,n_0}^{-1}(g+1)^2 A_{\lambda,n_1}^{-1}(g+1)^2 \cdots A_{\lambda,n_q}^{-1}$$

Now count $\mathcal{L}^p$ factors using that $(g+1)$, $dg \in \mathcal{L}^p$. We have that

$$A_{\lambda,k}^{-1} = (a^{-1}b)^k a^{-1} \quad \text{has} \quad k \quad \mathcal{L}^p\text{-factors}$$

$$\Longrightarrow (-\partial_\lambda)^q A_{\lambda,n}^{-1} \quad \text{has} \quad n_0+2+n_1+2+\cdots+2+n_q = n+2q$$

$\mathcal{L}^p$ factors.

Let's apply this to showing that $\operatorname{tr}_s R_\lambda(g)$ is closed, where now $g \in -\mathcal{U}^p$. We have

$$\operatorname{tr}_s\left[(g+1)A_\lambda^{-1}(g-1)\sigma\right] = \operatorname{tr}_s\left\{A_\lambda^{-1}(g-1)\sigma(g+1)\right\} = \operatorname{tr}_s\left\{A_\lambda^{-1}(g+1)(g-1)\sigma\right\}$$

$$= \operatorname{tr}_s\left\{(g-1)\sigma A_\lambda^{-1}(g+1)\right\}$$

provided always that when we use $\operatorname{tr}(XY) = \operatorname{tr}(YX)$ we have both XY and YX of trace class.

Consider this manipulation with A_λ^{-1} replaced by $(-\partial_\lambda)^q A_{\lambda,n}^{-1} = Q_n^q$ which has $n+2q$ $\mathcal{L}^p$ factors. The first equality is OK if $n+2q+1 \geq p$ and the last one also. $\therefore$

<u>Thm.</u> If $n+2q+1 \geq p$, then

$$\operatorname{tr}_s (-\partial_\lambda)^q R_{\lambda,n} = \operatorname{tr}_s\left\{(g+1)(-\partial_\lambda)^q A_{\lambda,n}^{-1}(g+1)\right\}$$

is closed.

November 28, 1986

Normalizing character forms:

Let's start from the fact that in the rational cohomology of $U(V)$ and $Gr(V)$ are certain character classes determined up to sign at least. These classes are compatible with Bott maps. We have

$$\frac{1}{k!\,2^{2k+1}} \operatorname{tr} F(dF)^{2k} \quad \text{represents} \quad (-2\pi i)^k ch_k$$

$$\left(\frac{1}{-2\pi i}\right)^k \frac{(-1)^k (k-1)!}{(2k-1)!} \operatorname{tr}(g^{-1}dg)^{2k-1} \quad \text{represents} \quad ch_{k-\frac{1}{2}}$$

In order to have uniform formulas I would like to define ω_n to be the unique invariant form on the symmetric space such that

$$\omega_n \quad \text{represents} \quad (-2\pi i)^{n/2} ch_{n/2}$$

This tells me that in the odd case

$$\boxed{(-2\pi i)^{-1/2} \frac{(-1)^{k-1}(k-1)!}{(2k-1)!} \operatorname{tr}(g^{-1}dg)^{2k-1} = \omega_{2k-1}}$$

Note $(-2\pi i)^{-1/2} = \left(\frac{i}{2\pi}\right)^{1/2}$ so that $\left(\frac{i}{2\pi}\right)^{k-\frac{1}{2}} \omega_{2k-1}$ reps. $ch_{k-\frac{1}{2}}$

This means we now have the formula

$$\left(\frac{i}{2\pi}\right)^{1/2} \frac{2\sqrt{\pi}}{u} \Phi_{2k-1} = \frac{1}{2\pi i} \int e^{-\lambda u} \frac{\Gamma(k-\frac{1}{2})}{\lambda^{k-1/2}} \omega_{2k-1}^{\sqrt{\lambda}} \, d\lambda$$

and therefore it seems appropriate to redefine tr_s on C_1 so that

$$\boxed{tr_s(\sigma) = (2i)^{1/2}}$$

Recall there is an isomorphism

$$C_1 \hat{\otimes} C_1 = C_2 \qquad \begin{aligned} \sigma \hat{\otimes} 1 &= \gamma^1 \\ 1 \hat{\otimes} \sigma &= \gamma^2 \end{aligned}$$

and that for the natural supertrace on C_2 given by the representation on the spinors one has $tr(\varepsilon \gamma^1 \gamma^2) = 2i$. Thus this definition of $tr_s(\sigma)$ is consistent with

$$tr_s(\underbrace{\sigma \hat{\otimes} \sigma}_{\gamma^1\gamma^2}) = tr_s(\sigma)\, tr_s(\sigma)$$

But this isn't exactly what one wants since $tr_s : C_1 \longrightarrow \mathbb{C}$ is of odd degree so we would prefer to have

$$* \qquad (tr_s \hat{\otimes} tr_s)(\sigma \hat{\otimes} \sigma) = -\, tr_s(\sigma)\, tr_s(\sigma)$$

Maybe we can reconcile this by being careful about tr_s on $\Omega(M, \operatorname{End} E) \hat{\otimes} C_1$. Set $A = \Omega(M, \operatorname{End} E)$, so that $tr : A \to \Omega$ is a supertrace. Then

$$(tr \hat{\otimes} tr_s)(\omega \hat{\otimes} \sigma) = (-1)^{\deg \omega}\, tr(\omega)\, tr_s(\sigma)$$

But this doesn't help as it changes $tr_s(\sigma)$ from $(2i)^{1/2}$ to $-(2i)^{1/2}$ which is not going to produce the $-$ in $*$.

You should take a more functorial approach

The Clifford algebra $C(V)$ is attached to a vector space V with quadratic form and we know there is a canonical additive isomorphism of $C(V)$ and $\Lambda(V)$. The map

$$C(V) \simeq \Lambda(V) \longrightarrow \Lambda^{max}(V)$$

is the universal supertrace for the Clifford algebra. On the other hand $\Lambda^{max}(V)$ has two generators of opposite sign and a choice of one of them is an orientation of V. Note that the quadratic form on V extends to one on $\Lambda(V)$ and the two elements of $\Lambda^{max}(V)$ in question are those α such that $\alpha \cdot \alpha = 1$.

Suppose now that V is even dim'l. In this case $C(V)$ is simple and there is a unique irreducible module S, the spinors. The grading on $C(V)$, which is defined by $v \to -v$ extended to an algebra autom, is induced by an autom ε of S unique to scalars. Then we can require $\varepsilon^2 = 1$ and there are two choices which are the two possible gradings of S. Each gives rise to a supertrace on $C(V)$ which we have seen is the same as a map $\Lambda^{max}(V) \to \mathbb{C}$. Thus in even dims. we get another pair of generators for $\Lambda^{max}(V) = C(V)/[C(V),C(V)]$. This is the $\pm(2i)^m$, $2m = \dim V$ factors.

Thus on C_2 we have two natural maps $\overbrace{C_2/[C_2,C_2]}^{\overline{C}_2} \to \mathbb{C}$ namely $\gamma^1\gamma^2 \to \pm 2i$, and if we then want to factor this in terms of $C_2/[C_2,C_2] = \overline{C}_2 \simeq \overline{C}_1 \otimes \overline{C}_1 \xrightarrow{\mu\otimes\mu} \mathbb{C}$ we want $\mu(\sigma) = (\pm 2i)^{1/2}$. Total of four possibilities $\mu(\sigma) = \sqrt[4]{-4}$.

December 1, 1986

It seems that given a smooth function $\varphi(z)$ defined on S^1 then one gets a smooth map $g \longmapsto \varphi(g)$ from U^p to $\varphi(1)+\mathcal{L}^p$. Here $\varphi(g)$ is defined within the group $U(H)$ by the spectral theorem. Even better by Fourier series: If $\varphi(z) = \sum a_n z^n$, then

$$\varphi(g) = \sum a_n g^n$$

Then one has

$$\varphi(g) - \varphi(1) = \sum a_n \left(\frac{g^n-1}{g-1}\right)(g-1)$$

$$\frac{g^n-1}{g-1} = \begin{cases} 1+g+\cdots+g^{n-1} & n>0 \\ 0 & n=0 \\ -g^{-1}-g^{-2}-\cdots-g^{+n} & n<0 \end{cases}$$

so $\left\|\frac{g^n-1}{g-1}\right\|_\infty \le |n|$ $\qquad$ ∞ norm = usual norm on $\mathcal{L}(H)$.

Thus $$\|\varphi(g)-\varphi(1)\|_p \le \left(\sum |a_n||n|\right)\|g-1\|_p$$

showing at least that $\varphi(g) \in \varphi(1)+\mathcal{L}^p$ when the derivative of φ has an absolutely convergent F.S.

Now I want to refine this idea to show smoothness. Let X belong to the Lie algebra of U^p, i.e. X is a skew adjoint operator belonging to $\mathcal{L}^p$. Each X gives rise to a left-invariant vector field on U^p, and it makes sense to take the Lie derivative L_X of a form or function on U^p. For example the function $U^p \longrightarrow \mathcal{L}^p(H)$ given by inclusion has

$$\mathcal{L}_X g = \frac{d}{dt}\Big|_{t=0} g e^{tX}$$

$$\therefore \quad \mathcal{L}_X g = gX.$$

Better: $\quad \mathcal{L}_X \varphi(g) = \frac{d}{dt}\Big|_{t=0} \varphi(ge^{tX})$

so that when $\varphi(g) = g$ we get

$$\mathcal{L}_X g = gX$$

Here X is a constant function on U^p

(Check: $\mathcal{L}_X g = i_X dg = \iota_X g\theta = g\,\iota_X\theta = gX$.

Also $\quad \mathcal{L}_X \mathcal{L}_Y g = \mathcal{L}_X gY = gXY \implies [\mathcal{L}_X, \mathcal{L}_Y]g = \mathcal{L}_{[X,Y]}g$)

Now $\quad \mathcal{L}_X g^n = \sum_{i=1}^{n} g^i X g^{n-i}$

and similarly we see that

$$\mathcal{L}_{X_k} \cdots \mathcal{L}_{X_1} g^n$$

is a sum of n^k terms each of which is a product of n g factors and $X_1, \dots, X_k$ in some order. Thus

$$\| \mathcal{L}_{X_k} \cdots \mathcal{L}_{X_1} g^n \|_p \le n^k \prod_1^n \| X_j \|_p$$

and since the Fourier coefficients of φ decrease rapidly this implies

$$* \qquad \mathcal{L}_{X_k} \cdots \mathcal{L}_{X_1} \varphi(g) \in \mathcal{L}_p .$$

Actually you have to say that we have a sequence of continuous functions $a_n \mathcal{L}_{X_k} \cdots \mathcal{L}_{X_1} g^n$ of $g, X_1, \dots, X_k$ and the series is uniformly convergent so the function $*$ is continuous with values in $\mathcal{L}_p$.

December 3, 1986

Let's discuss the Connes-Moscovici form from my viewpoint. We consider the form $\varphi^* \omega_{2k}$ over $\mathbb{R}_{>0} \times Gr^p(H, -\varepsilon)$ where $\varphi(t, g) = g_t$ as usual and ω_{2k} is the character form on $Gr^p(H, -\varepsilon)^c$. Here $2k \geq p$. Up to constants and restricting to $g = \frac{1+X}{1-X}$ we have

$$\varphi^* \omega_{2k} = c \operatorname{tr} \varepsilon \left(\frac{1}{1 - t^{-2} X^2} d(t^{-1} X) \right)^{2k}$$

$$= \omega_{2k}^t + dt \underbrace{\left\{ -2kc \operatorname{tr} \varepsilon \left(\frac{1}{t^2 - X^2} X \left(\frac{t}{t^2 - X^2} dX \right)^{2k-1} \right\}\right.}_{\eta_{2k-1}^t}$$

Note that η_{2k-1}^t is a smooth form on $Gr^p(H, -\varepsilon)$ depending smoothly on t. Because $\varphi^* \omega_{2k}$ is closed we have

$$\boxed{d\eta_{2k-1}^t = \partial_t \, \omega_{2k}^t}$$

What I now want to do is to integrate η_{2k-1}^t over $0 < t < \infty$, so we have to impose conditions at $0, \infty$. As $t \to 0$ we will have trouble unless X is invertible. In this case $g = \frac{t+X}{t-X}$ approaches -1 smoothly and so there is no problem.

So we next look at $t \to +\infty$. The main example is where the operators X are the Dirac operators associated to different connections. In this case the variations

dX are bounded operators (in $\mathcal{L}$)
and the operators $\frac{1}{t+X}$ $\frac{1}{t-X}$ are in $\mathcal{L}^p$
where $p > n =$ dimension of the manifold. For
example over a torus with the constant
coefficient operator one has eigenvalues $\frac{1}{t+\lambda}$
λ running over $i\Gamma$, Γ a lattice. So for $t>0$

$$\left\| \frac{1}{t+X} \right\|_p = \sum_{x \in \Gamma} \frac{1}{|t+ix|^p} = \frac{1}{t^p} \sum_{x \in \frac{1}{t}\Gamma} \frac{1}{|1+ix|^p}$$

$$\sim \frac{1}{t^p} \frac{t^n}{\mathrm{cov}(\Gamma)} \int_{\mathbb{R}^n} \frac{1}{|1+ix|^p} d^n x$$

In this example

$$\lim_{t \to \infty} \left\| \frac{1}{t+X} \right\|_p = 0$$

but the convergence can be slow if p is close to n.

In general if λ_k are the eigenvalues of X
then we have

$$\left\| \frac{1}{t+X} \right\|_p^p = \sum \frac{1}{|t+\lambda_k|^p} = \sum \frac{1}{(t^2+|\lambda_k|^2)^{p/2}}$$

This series is dominated by $\sum |\lambda_k|^{-p} < \infty$

$$\frac{1}{t^2+|\lambda_k|^2} \leq \frac{1}{|\lambda_k|^2}$$

so by dominated convergence we see that

$$\left\| \frac{1}{t \pm X} \right\|_p \to 0 \quad \text{as} \quad t \to +\infty$$

Notice that

$$\left|\frac{\lambda_k}{t \pm \lambda_k}\right| = \frac{|\lambda_k|}{(t^2 + |\lambda_k|^2)^{1/2}} \leq 1 \implies \left|\frac{X}{t \pm X}\right| \leq 1$$

Similarly $\left\|\frac{t}{t \pm X}\right\| \leq 1$.

Now look at η_{2k-1} again

$$\operatorname{tr}\left\{\varepsilon \, \frac{1}{t^2 - X^2} X \left(\frac{t}{t^2 - X^2} dX\right)^{2k-1}\right\}$$

Assume dX is a bounded operator. Then we have

$$\underbrace{\frac{1}{t-X}}_{\mathcal{L}^p} \underbrace{\left(\frac{1}{t+X} X\right)}_{\mathcal{L}} \left(\underbrace{\frac{1}{t-X}}_{\mathcal{L}^p} \underbrace{\frac{t}{t+X} dX}_{\mathcal{L}}\right)^{2k-1}$$

and so there are $2k$ $\mathcal{L}^p$ factors each of which by dominated convergence goes to zero. Thus we see that for $2k \geq p$ the form η^t_{2k-1} tends to zero as $t \longrightarrow +\infty$.

Unfortunately I seem to have no control over the rate of convergence, certainly not enough to integrate.

However if $2k-1 \geq p$, then I don't need the first factor $\frac{1}{t-X}$ in $\mathcal{L}^p$ in order to insure the trace is defined, so I can write it as

$$\frac{1}{t} \, \frac{t}{t-X}.$$

This gains, but still there isn't enough to integrate.

Let's assume $2k-1 > p$ and write

$$\frac{t^{2a}}{t^2-X^2}\,dX = \underbrace{\frac{t^{2a}}{(t^2-x^2)^a}}_{\in \mathcal{L}}\ \underbrace{\frac{1}{(t^2-x^2)^b}\,dx}_{\in \mathcal{L}^c}$$

We will choose b so that raising this to the $2k-1$ will give us something in $\mathcal{L}^1$. Note that

$$\frac{1}{t^2-X^2} \in \mathcal{L}^{p/2} \qquad \frac{1}{(t^2-X^2)^b} \in \mathcal{L}^{p/2b}$$

provided $\frac{p}{2b} \geq 1$. Now choose b so that

$$\frac{p}{2b(2k-1)} = 1. \qquad b = \frac{1}{2}\frac{p}{2k-1} < \frac{1}{2}$$

So $a > \frac{1}{2}$. Thus η is

$$\operatorname{tr}\left\{ \frac{t}{t-X}\ \frac{X}{t+X}\ \underbrace{\left(\frac{t^{2a}}{t^2-X^2}\,dX\right)^{2k-1}}_{\mathcal{L}^1 \text{ and } \to 0 \text{ as } t\to\infty} \right\} \underbrace{\frac{1}{t}\left(\frac{t}{t^{2a}}\right)^{2k-1}}_{\frac{1}{t^{g}}\quad g>1.}$$

which assures that it can be integrated.

December 5, 1986

The idea to show $U^p(H)$ and $Gr^p(H,\varepsilon)$ are Banach manifolds by explicitly showing they are smooth nbd retracts of a Banach space. Now

$$U^p(H) = \{ u \in U(H) \mid u \equiv 1 \mod \mathcal{L}^p \}$$

sits inside

$$GL^p(H) = \{ g \in GL(H) = \mathcal{L}(H)^\times \mid g \equiv 1 \mod \mathcal{L}^p \}$$

which is an open subset of $1 \oplus \mathcal{L}^p$.

Polar decomposition gives a retraction of $GL(H)$ onto $U(H)$. The formula is

$$(*) \qquad g \longmapsto g\,(g^*g)^{-1/2}$$

where $(g^*g)^t = \frac{1}{2\pi i}\int \lambda^t \frac{1}{\lambda - g^*g}\, d\lambda$. Since we know that $g \equiv 1 \mod \mathcal{L}^p \implies g^*g \equiv 1 \mod \mathcal{L}^p$

$$\Rightarrow \quad \frac{1}{\lambda - g^*g} - \frac{1}{\lambda - 1} = \frac{1}{\lambda - g^*g}\left(\lambda - 1 - (\lambda - g^*g)\right)\frac{1}{\lambda - 1}$$

$$= \underbrace{\frac{1}{\lambda - g^*g}}_{\mathcal{L}} \underbrace{(g^*g - 1)\frac{1}{\lambda - 1}}_{\mathcal{L}^p} \in \mathcal{L}^p$$

we see that this polar decomposition map, perhaps I should say phase map, is a smooth retraction of $GL^p(H)$ onto $U^p(H)$.

Next consider the Grassmannians. There are two ways to proceed it seems. We have inclusions

$$\begin{array}{ccc} Gr(H) & \subset & Grc(H) \\ \cap & & \cap \\ U(H) & \subset & GL(H) \end{array}$$

Here $Grc(H)$ = space of involutions in $GL(H)$. Now we have retractions in the horizontal direction. Consider the open subset W of $GL(H)$ consisting of invertibles g whose spectrum does not meet $i\mathbb{R}$. Then spectrum of any g divides into two disjoint pieces and so by contour integration one gets a retraction of W onto $Grc(H)$ which retracts $U(H) \cap W$ onto $Gr(H)$. These are actually deformation retractions.

However it seems unlikely that these vertical retractions commute with the horizontal ones. In effect if $g \in W$, then I see no reason why $g(g^*g)^{-1/2}$ also belongs to W.

So it appears that the good thing to do is to use the retraction on the unitary side. I want a formula:

$$r(g) = \frac{1}{2\pi i} \oint\limits_{\substack{\text{around} \\ \operatorname{spec}(g) \cap RHP}} \frac{1}{\lambda - g}\, d\lambda \;-\; \frac{1}{2\pi i} \oint\limits_{\substack{\text{around} \\ \operatorname{spec}(g) \cap LHP}} \frac{1}{\lambda - g}\, d\lambda$$

It's clear that if $g \equiv \varepsilon \mod \mathcal{L}^p$, then $r(g) \equiv r(\varepsilon) = \varepsilon \mod \mathcal{L}^p$. Thus $Gr^p(H, \varepsilon)$ is a smooth neighborhood retract of $U^p(H)\varepsilon \simeq U^p(H)$.

Next I'd like some formulas for charts which comes from this approach. The idea is that when one has a submanifold given as a smooth nbd retract, then at a point the tangent space to the submanifold is mapped by the retraction onto the manifold and this is a diffeomorphism near the point by the implicit function theorem.

So look at the point $1 \in U(H)$; the tangent space is found by looking to the retraction to first order

$$g(g^*g)^{-1/2} = (1+T)(1+T^*+T+T^*T)^{-1/2}$$

$$= 1 + T - \tfrac{1}{2}(T^*+T) + O(T^2)$$

$$= 1 + \tfrac{1}{2}(T-T^*) + O(T^2)$$

Thus the tangent space is $\{1+X \mid X=-X^*\}$. This set is contained in $GL(H)$ and so the retraction maps it to unitaries. But

$$g = 1+X \implies g(g^*g)^{-1/2} = \frac{1+X}{\sqrt{1-X^2}}$$

which we've seen is the square root of the Cayley transform.

Leave Grass case until later.

What does the phase retraction do to the tangent space at a point u of $U(H)$. This space of the form $g = u(1+X)$ with $X=-X^*$ and

$$g \longmapsto u(1+X)\{(1-X)u^*u(1+X)\}^{-1/2} = u\,\frac{1+X}{\sqrt{1-X^2}}$$

Alternatively if we write $g = (1+Y)u$ we get

$$g(g^*g)^{-1/2} = (1+Y)u\left\{u^{-1}(1-Y^2)u\right\}^{-1/2}$$
$$= \frac{1+Y}{\sqrt{1-Y^2}}\,u$$

Thus what happens at the point u is either left or right translation by u of what happens at the identity, and what happens at the identity is the square root of the C.T.

Let's now look at $Gr(H)$, which we are thinking of as embedded in the obvious way in $U(H)$. Pick a point ε of $Gr(H)$. At this point we have the tangent space which I will identify with the fixpoints of the retraction on the tangent space to $U(H)$ at ε.

No. I think the way to proceed is as follows. One has an embedding and nbd retraction of $Gr(H)$ into $U(H)$, and an embedding and nbd. retraction of $U(H)$ into $\mathcal{L}(H)$. Combining I get an embedding + nbd. retraction of $Gr(H)$ into $\mathcal{L}(H)$. Now we look at the composite projection on the tangent space to $\mathcal{L}(H)$ at a point ε. This gives the tangent space to $Gr(H)$ and we use retraction to map it to $Gr(H)$.

Let then $\varepsilon(1+X)$ be tangent to $Gr(H)$. Then

$$(\varepsilon(1+X))^* \equiv \varepsilon(1+X) \qquad + O(X^2)$$
$$(\varepsilon(1+X))^2 \equiv 1 \qquad + O(X^2)$$

The second equation implies $\varepsilon X \varepsilon = -X$, the first that $X^*\varepsilon = \varepsilon X$, so $X = -X^*$ anti-commutes with ε. Now given such a tangent vector $\varepsilon(1+X)$ we apply the retraction to $U(H)$ which gives $g = \varepsilon \frac{1+X}{\sqrt{1-X^2}}$ as we have seen. Now because ε anti-commutes with X this element g is already an involution and lies in $Gr(H)$.

So we conclude that the tangent space to $Gr(H)$ viewed as a submanifold of $\mathcal{L}(H)$ at ε is $\{\varepsilon(1+X) \mid X = -X^*, \ \varepsilon X \varepsilon = -X\}$ and that the nbd. retraction maps the tangent vector $\varepsilon(1+X)$ to the involution

$$\varepsilon \frac{1+X}{\sqrt{1-X^2}}.$$

A natural <u>question</u> is whether there is a nice embedding with nbd retraction of the unitary group which leads to the Cayley transform.

It seems more natural to embed $Gr(H)$ into the space of invertible self adjoint operators and to use the phase retraction. A tangent vector at ε of the form $\varepsilon + A$ where A is self-adjoint and anti commutes with ε. The phase retraction is:

$$(\varepsilon + A)^2 = \varepsilon^2 + \varepsilon A + A\varepsilon + A^2 = 1 + A^2$$

$$\frac{\varepsilon + A}{|\varepsilon + A|} = \frac{\varepsilon + A}{\sqrt{1+A^2}} = \frac{1+A\varepsilon}{\sqrt{1-(A\varepsilon)^2}}\,\varepsilon$$

Let's check this works well with $Gr^P(H)$. This is the set of F with $F \equiv \varepsilon \mod \mathcal{L}^P$ and we embed it into space of self-adjoint operators $A \equiv \varepsilon \mod(\mathcal{L}^p)$. Then the phase

$$A \longrightarrow \frac{A}{|A|} = \frac{1}{2\pi i}\int \operatorname{sgn}(\lambda)\frac{1}{\lambda - A}\,d\lambda$$

is a smooth retraction. At a point of $Gr^p(H)$ which we may take to be ε, the tangent space ~~should be~~ to this submanifold of $\varepsilon + \mathcal{L}^p_{s.a.}$, should be $\{\varepsilon + A \mid A \in \mathcal{L}^p,\ A = A^*,\ \varepsilon A \varepsilon = -A\}$ and the retraction maps this onto those F such that $F\varepsilon$ has its spectrum in $\{e^{i\theta} \mid \theta \in (-\frac{\pi}{2}, \frac{\pi}{2})\}$.

December 9, 1986

Here's some midpoint geometry connected with the unitary group and Grassmannian. I want to work in infinite dimensions where I thought of it first.

I look at the following two sets

$$Gr^{p} = \{F \in \varepsilon + \mathcal{L}^{p}_{sa} \mid F^2 = 1\}$$

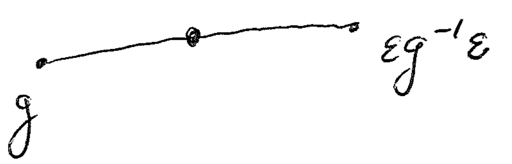
$$Gr^{p\prime} = \{g \in U^{p} \mid \varepsilon g \varepsilon = g^{-1}\}$$

There's a bijection between them: $g \longmapsto g\varepsilon$, $F \mapsto F\varepsilon$. The former is a smooth retract of

$$GL^{p}_{sa} = GL \cap (\varepsilon + \mathcal{L}^{p}_{sa})$$

the retraction is given by the "phase" map.

I would like to show the latter is a smooth nbd retract in U^p. This can be done as follows. Given g in U^p compare it to $\varepsilon g^{-1} \varepsilon$ which also belongs to U^p. Geodesics in



the unitary group are supposed to be one parameter subgroups essentially. So assuming g and $\varepsilon g^{-1}\varepsilon$ are suff. close there should be a midpoint given by

$$\mu = u \cdot \varepsilon g^{-1} \varepsilon$$

where $u^2 = g\varepsilon g\varepsilon = (g\varepsilon)^2$. So let's define an open set of U^p by

$$W = \{g \in U^p \mid (g\varepsilon)^2 + 1 \text{ is invertible}\}$$

The condition that $(g\varepsilon)^2$ doesn't have -1 in its spectrum means that

$$(g\varepsilon)^2 = \frac{1+X}{1-X} \qquad X \in \mathcal{L}_{sk}$$

whence it has a square root

$$u = \frac{1+X}{\sqrt{1-X^2}}$$

which is the unique square root with its spectrum in $Re(z) > 0$. It's better to say that by the spectral theorem any $\hat{g} \in \mathcal{U}$ with -1 outside its spectrum has a unique square root u with $\frac{1}{2}(u+u^{-1}) > 0$.

But then $g\varepsilon$ commutes with u for $g \in W$ so

$$\varepsilon\mu\varepsilon = \varepsilon u (g\varepsilon)^{-1} = \varepsilon (g\varepsilon)^{-1} u$$

$$\mu^{-1} = \varepsilon g \varepsilon u^{-1} = \varepsilon (g\varepsilon)^{-1} u^2 u^{-1} = \varepsilon (g\varepsilon)^{-1} u$$

so this midpoint belong to $Gr^{P'}$.

But another way of proceeding is to first map $\mathcal{U}^P$ into $\varepsilon + \mathcal{L}^P_{sa}$ by

$$g \longmapsto \frac{1}{2}(g\varepsilon + \varepsilon g^{-1}) = \frac{1}{2}(g\varepsilon + (g\varepsilon)^{-1}) = A$$

and to define W as the open set of $\mathcal{U}^P$ such that A is invertible. This is the same condition that $g\varepsilon$ doesn't have the eigenvalue -1. Then take the phase of A to get an F.

But $$A^2 = \frac{1}{4}\left((g\varepsilon)^2 + 2 + (g\varepsilon)^{-2}\right) = \left(\frac{1}{2}(u+u^{-1})\right)^2$$

$$|A| = \frac{1}{2}(u+u^{-1})$$

$$F = A/|A| = (u+u^{-1})^{-1}\left((g\varepsilon) + (g\varepsilon)^{-1}\right) =$$

$$= \left((u^2+1)u^{-1}\right)^{-1}\left((g\varepsilon)^2+1\right)(g\varepsilon)^{-1}$$
$$= u(g\varepsilon)^{-1}$$

Thus we see $F\varepsilon = \mu$, the midpoint.

I want to check now that Gr^P and $Gr^{P'}$ are diffeomorphic manifolds. We have a smooth map

$$i : Gr^P \longrightarrow U^P \qquad F \longmapsto F\varepsilon$$

and the reason it's smooth is that it is the restriction to Gr^P of the map

$$\varepsilon + \mathcal{L}^P_{sa} \xrightarrow{\cdot\varepsilon} 1 + \mathcal{L}^P$$

We also smooth maps

$$g \longmapsto \tfrac{1}{2}(g\varepsilon + \varepsilon g^*)$$

$$\begin{array}{ccc}
GL^P & \longrightarrow & \varepsilon + \mathcal{L}^P_{sa} \\
\cup & & \| \\
U^P & \longrightarrow & \varepsilon + \mathcal{L}^P_{sa} \\
\cup & & \cup \\
W & \longrightarrow & GL^P_{sa}
\end{array}$$

hence $g \to \frac{1}{2}(g\varepsilon + (g\varepsilon)^{-1})$ is a smooth map from W to GL^P_{sa}. Following this by the phase gives a smooth map $r : W \longrightarrow Gr^P$. It's clear that $ri = id$ so the image of i is a submanifold. This shows Gr^P and $Gr^{P'}$ are diffeomorphic.

December 13, 1986

Let $\Omega \subset \mathbb{C}$. Say that a fn. $f(\lambda, g)$ defined on $\Omega \times (-U^P)$ has poly growth in $\mathcal{L}^{p'}$ when f is a smooth map from $\Omega \times (-U^P)$ to $\mathcal{L}^{p'}$ such that $\forall$ integer $k \geq 0$ $\exists$ integer $N_k \geq 0$ and a cont. fn. $C_k(g) \geq 0$ on $-U^P$ such that

$$\| \mathcal{L}_{X_1} \cdots \mathcal{L}_{X_k} f \|^{(\lambda, g)}_{p'} \leq C_k(g)(1+|\lambda|)^{N_k} \|X_1\|_p \cdots \|X_k\|_p$$

Let us now check that if $f_1(\lambda, g)$ has poly growth in $\mathcal{L}^{p_1}$ and if $f_2(\lambda, g)$ has poly growth in $\mathcal{L}^{p_2}$, then $f_1 f_2$ has poly growth in $\mathcal{L}^{p'}$ for any p' such that $(p')^{-1} \leq p_1^{-1} + p_2^{-1}$.

$$\| \mathcal{L}_{X_1} \cdots \mathcal{L}_{X_k} f_1 f_2 \|_{p'} \leq \sum_{I \subset \{1, \dots, k\}} \Big\| \Big(\prod_{i \in I} \mathcal{L}_{X_i} \Big) f_1 \Big\|_{p_1} \Big\| \Big(\prod_{i \notin I} \mathcal{L}_{X_i} \Big) f_2 \Big\|_{p_2}$$

$$\leq \underbrace{\Big\{ \sum_I C^1_{|I|}(g)(1+|\lambda|)^{N^1_{|I|}} C^2_{|I'|}(g)(1+|\lambda|)^{N^2_{|I'|}} \Big\}}_{\leq \left(\sum_I C^1_{|I|}(g)\, C^2_{|I'|}(g) \right) (1+|\lambda|)^{\sup(N^1_{|I|} + N^2_{|I'|})}} \prod_i^k \|X_i\|_p$$

a

January 4, 1987

Program: To see if we can get a simple proof of the index theorem by embedding methods.

The first case to understand would be where M is a torus and we have a vector bundle E with connection D. The idea would be to pick an isometric embedding $i: E \hookrightarrow \tilde{V}$ inducing D. Then $\not{D} = \gamma^\mu D_\mu$ on $\Gamma(M, S\otimes E)$ is just $i^* \gamma^\mu \partial_\mu i$ on $\Gamma(M, S\otimes\tilde{V})$.

From the superconnection theory, in particular from the family obtained by letting E vary over subbundles of $\tilde{V}$ I feel an important thing to consider is the extension by -1 of the C.T. of $\gamma^\mu D_\mu$. On the other hand I can also take (E, D, g) where $g = 1$ and extend to $(\tilde{V}, d, \tilde{g})$ where $\tilde{g} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ relative to $\tilde{V} = E \oplus E^\perp$. I then know that the character form of E, D is the super conn. char. form of $(\tilde{V}, d, \tilde{g})$.

This last sentence isn't clear because of grading problems. I should be thinking of E as a graded bundle $E = E^+ \oplus 0$ and V as graded $V = V^+ \oplus 0$. Then $\tilde{g}$ corresponds to the subbundle $E^+ \oplus 0 \subset \tilde{V} \oplus 0$. It's confusing but OK.

I should digress to understand this better. Let's look at the superconnection resolvent when we are given a (E, D, g) with $g^2 = 1$. To keep it simple suppose $(E, D) = (\tilde{V}, d)$. We then have

$$\frac{1}{\lambda - D'^2} = i^* R(\tilde{V}, d, g)\, i = i^* (g+1) \frac{1}{\lambda(g+1)^2 - (g-1)^2 - 2dg\sigma} (g+1)\, i$$

In the end this amounts to $D'^2 = (i^*di)^2$

$= -(i^*dj)(j^*di)$

Let's return to the main line of the investigation. We see that the superconnection theory treats E, D via $(\tilde{V}, d, \tilde{g})$. Consequently it is natural to look for an index problem attached to any (E, D, g). We know how to set up an index problem, I should say, we know how to associate a Dirac operator with coefficients in a bundle with superconnection (E, D, X). The natural question is whether the C.T. of $\not{D} + X$ can be written in terms of the C.T. of X.

A key case to understand is where $M = S^1$ and $g : S^1 \longrightarrow U(V)$. We know this can lead to a nontrivial index. X in this case is if where f is real and the Dirac operator is

$$L = \begin{pmatrix} 0 & \partial_x - f \\ \partial_x + f & 0 \end{pmatrix}$$

Then the question is whether $\frac{1+L}{1-L}$ can be written in terms of $g = \frac{1+if}{1-if}$.

We should figure out where σ is to enter. It seems reasonable that $h\not{D} + X$ should become $h\not{D} + X\sigma = h\gamma^\mu D_\mu + X\sigma$, so we want σ to anticommute with the γ^μ. In other words when M has odd dimension $2m-1$ we look at the Clifford algebra with the $2m$ generators $\gamma^1, \ldots, \gamma^{2m-1}, \sigma$

and we use the corresponding module of spinors with its grading ε.

Let's start with $L = h\gamma^\mu \partial_\mu + \sigma X$ and consider the resolvent

$$\frac{1}{\lambda - L^2} = \frac{1}{\lambda - X^2 - h\gamma^\mu\sigma\,\partial_\mu(X) - h^2\partial_\mu^2}$$

$$= (g+1)\,\frac{1}{\lambda(g+1)^2 - (g-1)^2 - 2h\gamma^\mu\sigma\,\partial_\mu(g) - (g+1)h^2\partial_\mu^2(g+1)}\,(g+1)$$

There is a problem with the denominator being invertible - it is a singular operator where $g=-1$. Maybe it would help to write

$$\circledast \qquad \frac{1}{\lambda - L^2} = (g+1)\,\frac{1}{(g+1)(\lambda - h^2\partial_\mu^2)(g+1) - (g-1)^2 - 2h\gamma^\mu\sigma\,\partial_\mu g}\,(g+1)$$

Here one might hope to be able to invert

$$(g+1)(\lambda - h^2\partial_\mu^2)(g+1) - (g-1)^2$$

because its leading term is

$$(\lambda + p_\mu^2)(g+1)^2 - (g-1)^2$$

$$\lambda \longrightarrow \lambda + p_\mu^2$$

and $\lambda \notin (-\infty, 0] \Longrightarrow \lambda + p_\mu^2 \notin (-\infty, 0]$.

Conclude: The inverse on the RHS of $\circledast$ is probably well-defined for any unitary g because if we set $h=0$ it is invertible, and this is the **leading** term in Getzler's sense.

January 6, 1987

Goal is to extend the Dirac operator $L = \not{\partial} + X\sigma$ to the Cayley transform of X. The idea is to show that the Cayley transform $\frac{1+L}{1-L}$ can be written in terms of $g = \frac{1+X}{1-X}$.

Now von Neumann solved the problem of handling unbounded operators via their graphs and the Cayley transform. One can't write down $\frac{1+L}{1-L}$ for a partially defined operator. Some argument is needed to see it is well-defined.

Let's review von Neumann's construction. Let T be a closed densely defined operator from H^+ to H^-. The graph $\Gamma_T = \binom{1}{T}\mathcal{D}_T \subset H^+ \oplus H^-$ is a closed subspace such that $pr_1 : \Gamma_T \longrightarrow H^+$ is injective and has dense image. The adjoint T^* is essentially the same as the orthogonal complement, i.e.

$$(\Gamma_T)^\perp = \binom{-T^*}{1}\mathcal{D}_{T^*}.$$

Precisely $\mathcal{D}_{T^*}$ is the set of $w \in H^-$ such that $v \mapsto \langle Tv|w\rangle$ is a bdd linear functional on $\mathcal{D}_T$ ~~It is then represented by a unique elt T^*w~~ with respect to the norm on H^+. This linear functional extends to H^+ hence is represented by a unique elt. $T^*w \in H^+$. Then

$$\left\langle \binom{v}{Tv} \middle| \binom{-T^*w}{w} \right\rangle = -\langle v|T^*w\rangle + \langle Tv|w\rangle = 0$$

Conversely if $\binom{v'}{w} \in (\Gamma_T)^\perp$, then $\langle Tv|w\rangle = \langle v|v'\rangle$ such by the uniqueness of v' (as $\mathcal{D}_T$ is dense) we have

$\sigma' = T^* \omega$. Thus we see from the denseness of $\mathcal{D}_T$ that is the graph of a map from a subspace of H^- to H^+. It would be nice to know $\mathcal{D}_{T^*}$ is dense, but it seems one must assume this. <u>NO</u> see below

Summarizing we see that T densely defined $\Longrightarrow$ T^* well-defined & closed. If T^* also densely defined, then we have $T^{**} \supset T$. It seems then that T closely densely defined + T^* densely defined $\Longrightarrow$ $T = T^{**}$. In any case we have a nice closed subspace Γ_T of $H \oplus H$ which we can convert to a unitary operator g inverted by ε. Formulas are the usual ones

$$g = \frac{1+X}{1-X} \qquad X = \begin{pmatrix} 0 & -T^* \\ T & 0 \end{pmatrix}$$

$$= \frac{(1+X)^2}{1-X^2} = \begin{pmatrix} 1-T^*T & -2T^* \\ 2T & 1-TT^* \end{pmatrix} \begin{pmatrix} \frac{1}{1+T^*T} & 0 \\ 0 & \frac{1}{1+TT^*} \end{pmatrix}$$

Suppose now that $T = -T^*$. Then X commutes with $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ which means it preserves the eigenspaces $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$. We have

$$g\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{1+T^2}{1-T^2} & \frac{2T}{1-T^2} \\ \frac{2T}{1-T^2} & \frac{1+T^2}{1-T^2} \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{(1+T)^2}{1-T^2} \\ \frac{(1+T)^2}{1-T^2} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1+T}{1-T} \\ \frac{1+T}{1-T} \end{pmatrix}$$

Similarly

$$g\begin{pmatrix}1\\-1\end{pmatrix} = \begin{pmatrix}\frac{1+T^2}{1-T^2} & \frac{2T}{1-T^2}\\ \frac{2T}{1-T^2} & \frac{1+T^2}{1-T^2}\end{pmatrix}\begin{pmatrix}1\\-1\end{pmatrix}$$

$$= \begin{pmatrix}\frac{(1-T)^2}{1-T^2}\\ -\frac{(1-T)^2}{1-T^2}\end{pmatrix} = \begin{pmatrix}\frac{1-T}{1+T}\\ -\frac{1-T}{1+T}\end{pmatrix}$$

$$\therefore \quad g\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix} = \begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}\begin{pmatrix}\frac{1+T}{1-T} & 0\\ 0 & \frac{1-T}{1+T}\end{pmatrix}$$

Summarizing, these manipulations show that starting from a skew-adjoint operator T on $H^+ = H^-$ where skew-adjoint is defined in terms of graphs we get its Cayley transform instantly, by taking the involution F defining the graph which satisfies $\gamma F = -F\gamma$, converting F to a unitary g, which then commutes with γ, and then looking at $g = F\varepsilon$ restricted to the $+1$ eigenspace of γ.

This will be the method I probably want to use to ~~[illegible]~~ make sense of $\frac{1+L}{1-L}$ when I replace X by g. The goal will be to describe explicit subspaces of $H \oplus H$.

If ~~[illegible]~~ T is closed + densely defined the same is true for T^*. If not you would find ~~[illegible]~~ a nonzero $\begin{pmatrix}0\\w\end{pmatrix} \perp \begin{pmatrix}-T^*\\1\end{pmatrix}\mathcal{D}_{T^*}$. As $\Gamma_T{}^{\perp} = \begin{pmatrix}-T^*\\1\end{pmatrix}\mathcal{D}_{T^*}$ it follows $\begin{pmatrix}0\\w\end{pmatrix} \in \Gamma_T$ contradiction.

I now understand the v.N. approach, which is based on closed subspaces. But I want to do perturbation theory, i.e. deduce what I need about the unitary $\frac{1+L}{1-L}$ from what I know about $\frac{1+\gamma}{1-\gamma}$.

January 7, 1987

The problem is to see if the Cayley transform $U = \frac{1+L}{1-L}$ $\quad L = \not{\partial} + X\sigma$ when expressed in terms of $g = \frac{1+X}{1-X}$, does it make sense for an arbitrary unitary transf. g? If this is the case then

$$U + 1 = \frac{2}{1-L} \qquad U^{-1} + 1 = \frac{2}{1+L}$$

have an extension as bounded operators. I think that the existence of U is equivalent to the existence of the resolvents $\frac{1}{t \pm L}$ for $Re(t) > 0$. as bounded operators.

So

$$\frac{1}{t - L} = \frac{1}{t - \not{\partial} - X\sigma} = \frac{1}{t - \not{\partial} - \left(\frac{g-1}{g+1}\right)\sigma}$$

$$= (g+1)\frac{1}{(t-\not{\partial})(g+1) - (g-1)\sigma}$$

$$= \frac{1}{(g+1)(t-\not{\partial}) - (g-1)\sigma}\,(g+1)$$

One might hope to define $\left[(t-\not{\partial})(g+1) - (g-1)\sigma\right]^{-1}$, but this is apt to be difficult since points where $g+1$ vanishes are singular points for this differential operator.

It is clearly important to look at the case of the circle, say where $g : S^1 \longrightarrow U(1)$.

$$L = \not{\partial} + X\sigma = \begin{pmatrix} 0 & \partial - iX \\ \partial + iX & 0 \end{pmatrix} = \begin{pmatrix} 0 & \partial + f \\ \partial - f & 0 \end{pmatrix}$$

where f is a real-valued function.

I am still not focusing on the real problem. It is unwise to try to prove the existence of operators before understanding well the von Neumann ideas.

The key case remains the circle, since in this case you are dealing with ordinary differential operators, which are singular unfortunately. So from now on $M = S^1$ or maybe $\mathbb{R}$. Over S^1 we consider the trivial [redacted] vector bundle $\tilde{V}$ and we consider the space of **unitary** automorphisms (or gauge transformations) g of $\tilde{V}$. The process I am trying to understand [redacted] is a concrete version of the Bott periodicity isom.

$$(*) \qquad K^{-1}(S^1) \longrightarrow K^0(pt).$$

There are other ways to obtain this map, the most well-known being the Toeplitz operator construction.

The map $*$ is a kind of Kasparov product. Better, the map $*$ is the cap product with the canon. K-homology class on S^1. It is a question of mixing the Dirac operator on S^1, or possibly the Hilbert transform with the gauge transformation g.

The method I am trying is to first assume $g = \frac{1+X}{1-X}$ where X is skew-adjoint, and then look at the operator $L = \partial\!\!\!/ + X\sigma$ and take its Cayley transform $U = \frac{1+L}{1-L}$. The hope is that U can be rewritten in terms g. ~~Now L is an odd operator relative to $+$~~

I have to find a way to handle the operators $\not{D} + X\sigma$. A key idea seems to be to invent an algebra first and then worry about its modules later.

Taking this viewpoint we see that to form the operator $\not{D} + X\sigma = \gamma^\mu D_\mu + \sigma X$ over a torus M to fix the ideas, we start out with D_μ, X operating on $\Gamma(M,E)$, and then we adjoin to these operators the Clifford algebra with generators γ^μ, σ. So we are working in $\text{Cliff} \otimes \mathcal{L}$ $\qquad \mathcal{L} = \text{End}(\Gamma(M,E))$. Irreducible modules over $C \otimes \mathcal{L}$ are of the form $S \otimes H$, $\quad H = \Gamma(M,E)$ where S is an irred. repn of the Cliff alg C.

Now let us discuss the different cases.

odd-odd: This means we have an odd no. of γ_μ's and (E, D, X) is ungraded. Then C has an even no. of generators so there is only one irred. S. This S is graded by $\pm i^{-m}\gamma^1 \cdots \gamma^{2m-1}\sigma$ and the operator $\gamma^\mu D_\mu + \sigma X$ is odd relative to this grading. So we obtain two graded case situations.

even-odd. Here C has an odd no. of generators, so there are two possible S. This gives two ungraded situations.

even-even. Here C has two irreducible repns. depending on the sign of the central involution $i^{-m}\gamma^1 \cdots \gamma^{2m}\sigma$. But E comes with the grading ε, so $i^{-m}\gamma^1 \cdots \gamma^{2m}\overbrace{\varepsilon}^{=\pm\sigma\varepsilon}$ is an involution and $\gamma^1, \ldots, \gamma^{2m}, \sigma X$ are odd relative to this. Thus we are in the graded case, and there are two possibilities.

odd – even: Here C has one irred. repn. Look at $i^{-m+1}\gamma^1 \cdots \gamma^{2m-1} \varepsilon$; this is an involution commuting with $\gamma^1, \ldots, \gamma^{2m-1}, \sigma X$ so you get two ungraded situations.

Summarizing, I think I have straightened the odd & even nonsense with the Clifford algebras. I learn that the basic σ notation is very suitable for handling the basic operator $\not{D} + \sigma X$. Moreover I have in each case a specific way of representing this operator. A very simple way over the torus.

The next step involves looking at the Cayley transform of $\not{D} + \sigma X$

January 10, 1987

The main problem of interest to me at the moment is to determine whether the Cayley transform of $\not{\partial} + \sigma X$ can be extended to arbitrary gauge transformations in analogy with the way the superconnection forms extend from E, D, X to E, D, g. What would be really nice is for the actual C.T. $\frac{1+L}{1-L}$, $L = \not{\partial} + \sigma X$ as a unitary operator to admit a construction starting with the C.T. $g = \frac{1+X}{1-X}$ and then for the construction to make sense for an arbitrary g.

So let us review the method by which one constructs $\frac{1+L}{1-L}$ when $L = \not{\partial} + \sigma X$. L is an unbounded operator which is skew adjoint in the von Neumann sense. Now I have to be precise about what L is. So far all I have is a differential operator. To fix the ideas think of the circle. So the differential operator is defined on smooth vector functions over the circle. One closes it up. This means one considers the closure Γ of the graph of the operator on smooth functions in $H \oplus H$. This is a well-defined operator with dense domain. To see this we have to check that that the closure Γ is a graph, i.e. $\Gamma \cap (0 \oplus H) = 0$. This follows from the fact that L has a "formal" adjoint. So that if $\binom{0}{v} \in \Gamma$, i.e. $\exists\ \varphi_n \to 0$, $L\varphi_n \to v$ and then

$$\langle \psi | v \rangle = \lim \langle \psi | L\varphi_n \rangle = \lim \langle L^* \psi | \varphi_n \rangle = 0$$

for all smooth ψ, whence $v = 0$.

Now we have a closed densely defined operator L defined. Skew-adjointness means that the closed graph $\Gamma = \binom{1}{L}\mathcal{D}_L$ and $\sigma\Gamma = \binom{L}{1}\mathcal{D}_L$ are orthogonal complements. Put another way, recall that the adjoint in the Hilbert space sense of the minimal closed operator defined by a differential operator D is the maximal closed operator defined by the formal adjoint D^*. Thus skew-adjointness means that the maximal and minimal closed extensions coincide. In practice it means that there are no missing boundary conditions.

I want to make some comments concerning the von Neumann method. It seems to me that the techniques in my paper for handling the Grassmannian are quite efficient.

For example a closed subspace W of $H = H^+ \oplus H^-$ immediately gives rise to ~~as~~ unitary g inverted by ε to which we can apply spectral theory to. What does it mean for g not to have the eigenvalue -1? If $g+1$ has a non-zero kernel K, then we decompose it under $\varepsilon = -F$. We have

$$W \cap H^- = K \cap \{\varepsilon = -1\}$$

$$W^\perp \cap H^+ = K \cap \{\varepsilon = +1\}$$

$W \cap H^- \neq 0$ means that W as a correspondence: $H^+ \dashrightarrow H^-$ is indeterminate and $W^\perp \cap H^+ \neq 0$ means that the correspondence is not densely defined. So $K = 0$ $\Longleftrightarrow$ W is the graph of a closed densely defined operator!

Then changing signs $F \to -F$, $\varepsilon \to -\varepsilon$ we see that $W^\perp$ is the graph of a closed densely defined operator from H^- to H^+, namely $-T^*$. I think one also gets out of this the fact that the operators $(1+T^*T)^{-1}$, $T(1+T^*T)^{-1}$ etc. are defined, just by writing down the involution F.

Next I want to discuss the case where $T = -T^*$. More generally we consider $W \subset H \oplus H$ such that $W^\perp = \sigma W$, $\sigma = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Then F anti-comm. with σ, so $g = F\varepsilon$ commutes with σ. Then g is a unitary commuting with σ and reversed ~~anti-commuting with~~ by ε. g has to preserve the eigenspaces $\begin{pmatrix} 1 \\ 1 \end{pmatrix} H$, $\begin{pmatrix} 1 \\ -1 \end{pmatrix} H$ of σ. Thus we have a unitary u on $\begin{pmatrix} 1 \\ 1 \end{pmatrix} H$ and ~~the~~ inverse of its transform via $\varepsilon : \begin{pmatrix} 1 \\ 1 \end{pmatrix} H \to \begin{pmatrix} 1 \\ -1 \end{pmatrix} H$.

In the case of $T = -T^*$ we calculate u on $\begin{pmatrix} 1 \\ 1 \end{pmatrix} H$ as follows $\qquad X = \begin{pmatrix} 0 & T \\ T & 0 \end{pmatrix}$

$$
\begin{aligned}
g \begin{pmatrix} 1 \\ 1 \end{pmatrix} &= (1+X)(1-X)^{-1} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \\
&= (1+X) \begin{pmatrix} 1 \\ 1 \end{pmatrix} (1-T)^{-1} \\
&= \begin{pmatrix} 1 \\ 1 \end{pmatrix} (1+T)(1-T)^{-1}
\end{aligned}
$$

Thus $\qquad u = \dfrac{1+T}{1-T}$.

One can see directly that $1 \pm T : \mathcal{D}_T \to H$ is invertible just by

$$\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 1 & T \\ T & 1 \end{pmatrix} = \begin{pmatrix} 1+T & 1+T \\ 1-T & -1+T \end{pmatrix}$$

$$= \begin{pmatrix} 1+T & 0 \\ 0 & 1-T \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

so that

$$\begin{tikzcd}
\mathcal{D}_T \oplus \mathcal{D}_T \arrow[r, "\begin{pmatrix} 1 & T \\ T & 1 \end{pmatrix}", "\sim"'] \arrow[d, "\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}"', "\simeq"] & H \oplus H \arrow[d, "\simeq"', "\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}"] \\
\mathcal{D}_T \oplus \mathcal{D}_T \arrow[r, "\begin{pmatrix} 1+T & 0 \\ 0 & 1-T \end{pmatrix}"] & H \oplus H
\end{tikzcd}$$

because one is assuming $\Gamma_T \oplus \sigma\Gamma_T = H \oplus H$

commutes.

Next point: Once you have the unitary operator g you automatically get the analytic continuation. Notice that

$$\|[t(g+1)+(g-1)]v\| = \|[(t+1)g+(t-1)]v\|$$

$$\geq \|(t+1)gv\| - \|(t-1)v\|$$

$$= (|t+1| - |t-1|)\|v\|$$

so that for $\mathrm{Re}(t) > 0$ we have

$$\left\| \frac{1}{t(g+1)+(g-1)} \right\| \leq \frac{1+|t|}{2\,\mathrm{Re}(t)}$$

$$\left\| \frac{t(g+1)+(g-1)}{t(g+1)-(g-1)} \right\| \leq \frac{(1+|t|)^2}{\mathrm{Re}(t)}$$

The reason this is interesting is that

$$\frac{1}{t+X} = \frac{(g+1)}{t(g+1)+(g-1)}$$

so that control over the denominator is stronger than control over $\dfrac{1}{t+X}$.

January 11, 1987

More on $\gamma^\mu\partial_\mu + \sigma X$. Let's consider the odd-even case where the manifold is odd-diml and where $(\tilde{V}, X)$ are graded. Then the γ^μ and σ generate an even Clifford algebra which has only one irreducible module S; S has two gradings $\pm i^{-m}\gamma^1 \cdots \gamma^{2m-1}\sigma$. Consider $\gamma^1 \cdots \gamma^{2m-1}\varepsilon$; up to a constant it's an involution. It commutes with $\gamma^\mu\partial_\mu$ and with σX.

Take $m=1$, whence we are looking at the ± 1 eigenspaces of $\gamma\otimes\varepsilon$ on $S\otimes V$. The $+1$ eigenspace is $S^+\otimes V^+ \oplus S^-\otimes V^-$ ($S^\pm$ eigenspaces of σ) and the operator on this eigenspace given by $\gamma\partial_x + \sigma X$ is when we take $\gamma = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ $\sigma = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ $X = \begin{pmatrix} 0 & -T^* \\ T & 0 \end{pmatrix}$ is

$$\begin{pmatrix} \partial_x & -T^* \\ T & -\partial_x \end{pmatrix}$$

This is a general type of 1-diml Dirac operator unlike

$$\begin{pmatrix} \partial_x & T \\ T & -\partial_x \end{pmatrix} = \varepsilon\partial_x + \gamma^1 T$$

which we can conjugate into

$$\begin{pmatrix} 0 & -i\partial_x + T \\ i\partial_x + T & 0 \end{pmatrix}$$

and which is related to a Schroedinger operator.

An obvious question is whether de Branges analysis tells us anything. I recall that his methods and string methods in the sense of Krein tell us a great deal about singular Dirac type operators. This suggests ~~an~~ interesting possibility, namely, one might be able to settle the existence question by these methods.

Before following this route one first ought to try to understand the K-theoretic meaning of the construction you are looking at. The case of ungraded $(\tilde{V}, X)$ we have discussed already. When X is generalized to g, then g represents an element of $K^1(S^1)$ and it is to be capped with the generator of $K_1(S^1)$ given by the Dirac operator or Hilbert involution.

In the graded case, when X is generalized to a g inverted by ε, we have simply an involution F on $\tilde{V}$. Now there is an obvious way to proceed, take the $+1$ eigenprojection e of F and reduce $\not{\partial}$ by e.

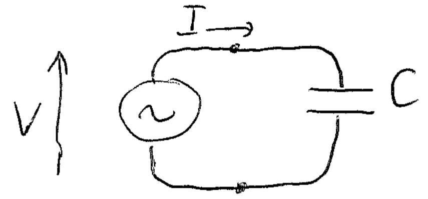
This reduction by an idempotent is confusing. In the past what I tried to do is to let e vary and then look at the family $e\not{\partial}e$ on eH. Then you take the Cayley transform of $e\not{\partial}e$ and extend by -1. This gives a family of unitaries on H depending on e. The problem has been that I have no way to relate the C.T. of $e\not{\partial}e$ with the C.T. of $\not{\partial}$. The original idea was to work not with unitaries but with essential involutions. To be more specific one converts the Dirac operator to a ψDO of order zero, which is an involution modulo some Schatten ideal. Unfortunately I still can't relate $e\not{D}e/\sqrt{m^2+(e\not{D}e)^2}$ to the corresponding operator without e.

Now there is another approach as follows. Given e on $\tilde{V}$ we form $D' = ede$ on $E' = e\tilde{V}$, and we have then (E', D', g') with $g' = 1$. We then extend g' by -1 to get $(\tilde{V}, d, g)$. And now it seems that the sort of "operator" I wish to attach to $(\tilde{V}, d, g)$ is exactly the Cayley transform of $\not{D}'$ extended by -1.

a

January 20, 1987

Circuit review:



$$Q = CV \qquad V(t) = \hat{V} e^{\overset{s}{i\omega} t}$$

$$I = C \frac{dV}{dt} \qquad \hat{I} = Cs\,\hat{V}$$

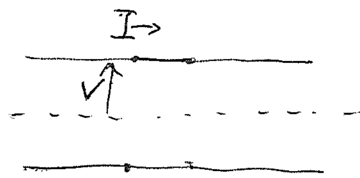
$$Z = \frac{\hat{V}}{\hat{I}} = \frac{1}{Cs}$$


L

$$V = L \frac{dI}{dt} = Ls \cdot I \qquad Z = Ls$$

Transmission line

$$-\partial_x I = C \partial_t V$$
$$-\partial_x V = L \partial_t I$$



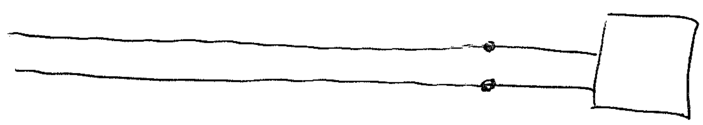
Solutions with dependence $e^{-i(kx-\omega t)}$

$$k I = C \omega V$$
$$k V = L \omega I$$

$$\therefore \quad \frac{V}{I} = \frac{k}{C\omega} = \frac{L\omega}{k} \qquad \left(\frac{\omega}{k}\right)^2 = \frac{1}{LC}$$

$\therefore$ speed is $\frac{1}{\sqrt{LC}}$ and the impedance is $\frac{L}{\sqrt{LC}} = \sqrt{\frac{L}{C}}$

Take $L = C = 1$. Find scattering by a circuit attached to a transmission line



Solution of
$$\left.\begin{aligned} -\partial_x I &= \partial_t V \\ -\partial_x V &= \partial_t I \end{aligned}\right\} \Longrightarrow \begin{aligned} (\partial_x + \partial_t)(V+I) &= 0 \\ (\partial_x - \partial_t)(V-I) &= 0 \end{aligned}$$

are $(V+I)(x,t) = \underset{\text{forward moving}}{f(x-t)} \qquad (V-I)(x,t) = \underset{\text{backward moving}}{g(x+t)}$

$$(V+I)(x,t) = Be^{+i\omega(t-x)}$$
$$(V-I)(x,t) = Ae^{i\omega(t+x)}$$

At $x=0$ we have

$$V+I = Be^{i\omega t}$$
$$V-I = Ae^{i\omega t}$$

so the scattering is

$$S = \frac{A}{B} = \frac{V-I}{V+I} = \frac{Z-1}{Z+1}$$

E.g. if $Z=\infty$ (open circuit) then $I=0$ at $x=0$ and so $S=1$.

Recall that a typical impedance fn. $Z(s)$ is real $\overline{Z(s)} = Z(\bar{s})$ and satisfies a positivity condition which ~~[illegible]~~ is due to the fact that it absorbs energy. Probably this condition is

$$\text{Re } Z(s) \geq 0 \qquad \text{for} \qquad \text{Re}(s) \geq 0.$$

Also $Z(s)$ is analytic for $\text{Re}(s) > 0$.

Variable transmission line. Here L, C depend on x. Let us assume the parameter x chosen so that signals travel at unit speed. This probably means $LC=1$. We want to change variables so as to write the transmission line equations in "Dirac" form.

Recall that the energy in the trans. line is

$$E = \int \tfrac{1}{2}(CV^2 + LI^2)\,dx$$

so that

$$\partial_t E = \int (V C \partial_t V + I L \partial_t I)\, dx$$

$$= \int \{V(-\partial_x I) + I(-\partial_x V)\}\, dx$$

$$= \int (-\partial_x)(VI)\, dx = -[VI]_a^b$$

The energy gives an inner product which we must convert to the usual L^2 product in order ~~that~~ to obtain equations in Dirac form. Assuming $LC = 1$ work with L instead of C because the impedance is $L/\sqrt{LC} = L$. Set

$$\tilde{V} = C^{1/2} V = L^{-1/2} V \qquad V = L^{1/2}\tilde{V}$$

$$\tilde{I} = L^{1/2} I.$$

Then

$$-\partial_x I = C\,\partial_t V \implies L^{-1}\partial_t(L^{1/2}\tilde{V}) = -\partial_x(L^{-1/2}\tilde{I})$$

$$-\partial_x V = L\,\partial_t I \implies L\,\partial_t(L^{-1/2}\tilde{I}) = -\partial_x(L^{1/2}\tilde{V})$$

$$\boxed{\partial_t \begin{pmatrix} \tilde{V} \\ \tilde{I} \end{pmatrix} = -\begin{pmatrix} 0 & L^{1/2}\partial_x L^{-1/2} \\ L^{-1/2}\partial_x L^{1/2} & 0 \end{pmatrix}\begin{pmatrix} \tilde{V} \\ \tilde{I} \end{pmatrix}}$$

This is in the Dirac form. i.e. $\begin{pmatrix} 0 & \partial_x - f \\ \partial_x + f & 0 \end{pmatrix}$

where $f = \partial_x\left(\frac{1}{2}\log L\right)$

The interest in this calculation is due to the fact the Krein string theory provides a theory of singular transmission lines.

January 22, 1987

Let's describe the manifold of Lagrangian subspaces L in a real symplectic vector space V of dimension $2n$. We fix a metric in V such that the operator J representing the symplectic form satisfies $J^2 = -1$. Thus V becomes a complex n diml v.s. with hermitian inner product and the symplectic form is the imaginary part of the inner product. Thus $V \simeq \mathbb{C}^n$.

Let $L \subset V$ be Lagrangian. Pick an orthonormal basis $e_1, \dots, e_n$ for L. Then $e_1, \dots, e_n$ is an orthonormal basis for V over $\mathbb{C}$. In effect

$$\langle v | v' \rangle = \underbrace{\operatorname{Re}\langle v | v' \rangle}_{\substack{\text{inner product} \\ \text{on } V \text{ over } \mathbb{R}}} + i \underbrace{\operatorname{Im}\langle v | v' \rangle}_{\text{symp. form}}$$

so $\quad \langle e_i | e_j \rangle = \delta_{ij} + 0 \quad$ as L Lagragian means that the symplectic form vanishes in L. Conversely the real subspace generated by an orthonormal basis of V over $\mathbb{C}$ is Lagrangian.

Thus we can conclude

$$\boxed{\{L \mid L \text{ Lagrangian in } V\} \xleftarrow{\sim} U(n)/O(n)}$$

Thus the Lagrangian Grassmannian is a symmetric space. Natural ~~questions~~ problems are to describe the Cayley transform and "eigenvalues" in this context. Notice that

$$\dim U(n)/O(n) = n^2 - \frac{n(n-1)}{2} = \frac{n(n+1)}{2}$$

is the dimension of the space of real symmetric $n \times n$ matrices.

Let's now think in terms of involutions.

Let F be an involution on V over $\mathbb{R}$. Then F corresponds to a Lagrangian subspace $\Longleftrightarrow$ $FJ = -JF$. (We can think of L as being a real reduction of V over $\mathbb{C}$ and F as the associated conjugation.)

Let's fix a basept. of the Lagrangian Grassmanian ε. Then $g = F\varepsilon$ commutes with J, so g is a unitary transformation inverted by ε, where ε is the conjugation on $\mathbb{C}^n$. Thus

$$\varepsilon g \varepsilon = \bar{g} \qquad\qquad \varepsilon g \varepsilon = g^{-1} = \bar{g}^{t}$$

so g is symmetric: $g = g^t$. Thus

$$\boxed{\text{Lagrangian Grassmannian} \quad = \quad \left\{\begin{matrix}\text{symmetric}\\ \text{unitary matrices}\end{matrix}\right\}}$$

A more direct way to see this is to note that $O(n)$ is the fixpt subgroup of the involution $g \longrightarrow \bar{g}$ on $U(n)$, hence $U(n)/O(n)$ can be identified with (the identity component of) the set of g inverted by this involution, which is the space of symmetric unitary matrices. Actually we would still have to see the latter space is connected for this to work.

Next we want to look at eigenvalues. ε, J are fixed, and then given F we want to decompose V as a real representation of the group generated by F, ε, J.

Actually it is probably better to look first at the C.T. We are given ε which corresponds to the decomposition $\mathbb{C}^n = \mathbb{R}^n \oplus i\mathbb{R}^n$ into complementary Lagrangian subspaces. Relative to this ε we have a Cayley transform from skew-symmetric real X on V anti-commuting with ε to orthogonal transf. $g = \frac{1+X}{1-X}$ inverted by ε. In the Lagrangian case we want g to commute with J, hence X should also, i.e. X is skew-hermitian. Conjugating by ε in this case is conjugation: $\varepsilon X \varepsilon = \bar{X}$, so we want $\bar{X} = -X$ i.e. X is purely imaginary. Thus X is purely imaginary and symmetric; hence

~~Sign? Relative to the decomposition~~

$$\cancel{\mathbb{C}^n = \mathbb{R}^n \oplus i\mathbb{R}^n \simeq \mathbb{R}^n \oplus \mathbb{R}^n}$$

$X = iA$ where A is real & symmetric. Thus relative to the decomposition

$$\mathbb{C}^n = \mathbb{R}^n + i\mathbb{R}^n \simeq \mathbb{R}^n \oplus \mathbb{R}^n$$

we have $X = \begin{pmatrix} 0 & -A \\ A & 0 \end{pmatrix}$, and the C.T. of X corresponds to the graph of A.

<u>Conclude:</u> The C.T. in the case of the Lagrangian Grassmannian can be identified with the map associating to a symmetric map $A : W \to W^*$ its graph $L \subset W \oplus W^*$.

Next look at eigenvalues. The eigenvalues of A are real numbers. The interesting question is what happens at ∞. In the case of the complex Grassmannian we saw that two kinds of behavior were possible at $g=1$, $g=-1$.

Let's suppose given F, ε, J on $V = \mathbb{C}^n$. Here $J =$ mult. by i, $\varepsilon =$ complex conjugation, F a $\mathbb{R}$-linear orthogonal involution such that $FJF = -J$. Then replace F by $g = F\varepsilon$ which commutes with J. The group generated by ε, g, J is a semi-direct product of the abelian group generated by g, J with the cyclic group of order 2 generated by ε. It's irreducible repns over $\mathbb{R}$ are found by looking at irreducible repns. of $\langle g, J \rangle$, ~~which are 2-dim'l as $J^2 = -1$. A list of irred repns.~~ and taking orbits under ε.

We decompose V into irreducibles under the group $\langle \varepsilon, g, J \rangle$. Then each irreducible $W \subset V$ is actually a complex subspace of V with J acting as i and it is preserved by the conjugation ε.

Better. Let $V_\zeta = \{ v \in V \mid gv = \zeta v \}$. Then V_ζ is stable under $J = i$ and ε:

$$g\,\varepsilon(v) = \varepsilon(g^{-1} v) = \varepsilon(\zeta^{-1} v) = \zeta\,\varepsilon(v).$$

So we conclude the irred subspaces are 1-dim'l over $\mathbb{C}$.

Conclusion: The eigenspaces of a symmetric unitary matrix are stable under conjugation. The eigenvalues of a point of the Lagrangian Grass are just the usual eigenvalues of the associated symmetric unitary matrix.

January 23, 1987

Let's describe the program. Let's consider two real symplectic vector spaces V, W of the same dimension. Given a symplectic transf. $T: V \longrightarrow W$ there is associated a unitary $\mathcal{H}_V \longrightarrow \mathcal{H}_W$ where $\mathcal{H}_V$ denotes the irreducible repn. of the Weyl algebra of V. Now a symplectic transformation can be identified with a kind of Lagrangian subspace of $V \oplus W'$. In fact just like $GL(V)$ for V complex is dense in $Gr_n(V \oplus V)$, I think it's true that the space of symplectic transformations $T: V \rightarrow W$ is dense in the space Lagrangian subspaces of $V \oplus W'$. Now the first problem is to explain what corresponds on the Hilbert space level to a general Lagrangian subspace of $V \oplus W'$. Maybe it is some sort of correspondence between $\mathcal{H}_V$ and $\mathcal{H}_W$.

Now we know, more or less, that the space of Lagrangian subspaces is the boundary of the Siegel UHP, and that points of the Siegel UHP correspond to Gaussian states in the irreducible representation. ~~This means that the~~ Now the irred. repn. of $\mathrm{Weyl}(V \oplus W')$ is roughly $\mathcal{H}_V \otimes \overline{\mathcal{H}_W}$. So a point of the Siegel UHP should determine a Hilbert-Schmidt operator from $\mathcal{H}_V$ to $\mathcal{H}_W$. We have to see what happens in the limits as the point in the UHP goes to the boundary.

There is something else which is perhaps going to be important for the calculations. Instead of working in the symplectic category it might be easy to work with finite diml complex vector spaces equipped with inner product and involution ε. The symplectic form is then $\mathrm{Im}\langle x|\varepsilon y\rangle$. Put another way if one equips V with the complex structure $J = i\varepsilon$, then for the new complex structure we have the standard situation.

The first thing to do is to work out the kind of Lagrangian subspaces which are appropriate to this situation.

Let us recall that if V is real equipped with scalar product $(\,,\,)$ and J orthogonal $\ni$ $J^2 = -1$, then Lagrangian $L \subset V$ can be identified with orthogonal involutions $F \ni FJ = -JF$. So if I take $V = \mathbb{C}^n$ + the usual $(\,,\,) = \mathrm{Re}\langle\,|\,\rangle$, and $J = i\varepsilon$, then I want orthogonal involutions F $\ni$ $Fi\varepsilon = -i\varepsilon F$. It's natural to look for $F \ni Fi = iF$; these corresp. to cx subspaces of V. Thus we want $F\varepsilon = -\varepsilon F$ which means $F = \begin{pmatrix} 0 & g^{-1} \\ g & 0 \end{pmatrix}$ where $g : V^+ \longrightarrow V^-$ is unitary. The the Lagrangian subspaces of (V, J) which are complex subspaces are just the graphs of unitary transfs.

Let's now return to the case where we have two symplectic vector spaces V, W. We form $V \oplus W'$ where the prime $'$ means we reverse the sign of the symplectic form on W. Suppose to simplify that V, W have complex structures i, gradings $\varepsilon_V, \varepsilon_W$ and herm. inner products. Then $V \oplus W'$ has the direct sum complex Hilbert space structure, but it is graded via $\varepsilon_V \oplus (-\varepsilon_W)$. I now look at Lagrangian subspaces of $V \oplus W'$ which are complex subspaces. According to what I have seen above, these are the graphs of unitary maps

$$V^+ \oplus W^- \longrightarrow V^- \oplus W^+$$

January 24, 1987: (David is 23)

The problem is to make sense of the operator

$$L = \begin{pmatrix} 0 & \partial_x - f(x) \\ \partial_x + f(x) & 0 \end{pmatrix}$$

where $f: S^1 \longrightarrow \mathbb{P}_1(\mathbb{R})$ is smooth. The natural ~~thing~~ to do is to look at this operator over an interval (a,b) where $f(x)$ is finite and such that $f(a) = f(b) = \infty$, and to ask whether L is essentially skew-adjoint, and if not, what sort of boundary conditions should be imposed. There is a limit point - limit circle analysis to be made at each endpoint.

Today I noticed for the first time the fact that as $x \downarrow a$ one has $|f(x)| \geq \frac{c}{x-a}$ and hence it might be possible to conclude that one has always the limit point situation.

Let's consider some examples. Take $f(x) = \frac{n}{x}$ and work on $(0,\infty)$. We want the behavior as $x \downarrow 0$ of eigenfunctions of L. Now the equation to be solved

$$\begin{pmatrix} 0 & \partial_x - \frac{n}{x} \\ \partial_x + \frac{n}{x} & 0 \end{pmatrix} \psi = \lambda \psi$$

is essentially Bessel's equation.

It has a regular singular point at $x = 0$, and the Frobenius method gives the form of solutions around zero. Thus one looks for solutions of the form $x^s \varphi(x)$ where $\varphi(x)$ is analytic near $x = 0$.

The equations are

$$(x\partial_x - r)\varphi_2 = \lambda x \varphi_1$$
$$(x\partial_x + r)\varphi_1 = \lambda x \varphi_2$$

from which we can see $s = \pm r$. If $2r \notin \mathbb{Z}$, there will be two independent solutions

$$x^r \begin{pmatrix} O(x) \\ 1 + O(x) \end{pmatrix} \qquad x^{-r} \begin{pmatrix} 1 + O(x) \\ O(x) \end{pmatrix}.$$

Suppose $r > 0$. Then the former solution is L^2 as $x \downarrow 0$ and the latter is also when $r < 1/2$. So we have the limit circle case where $0 < r < \frac{1}{2}$ and the limit point case for $r > \frac{1}{2}$.

January 25, 1987

Let us return to our transmission line and change the sign of I so that the equations become

$$\partial_x I = C\, \partial_t V$$
$$\partial_x V = L\, \partial_t I$$

or

$$\begin{pmatrix} C & 0 \\ 0 & L \end{pmatrix} \partial_t \begin{pmatrix} V \\ I \end{pmatrix} = \begin{pmatrix} 0 & \partial_x \\ \partial_x & 0 \end{pmatrix} \begin{pmatrix} V \\ I \end{pmatrix}.$$

Recall that the energy norm is

$$\frac{1}{2} \int \left(C V^2 + L I^2 \right) dx$$

I would like to understand the following Dirac operator problem. Let $g : \mathbb{R} \longrightarrow U(1)$ be such that $g \neq -1$ on (a,b) and $g = -1$ on $(-\infty, a] \cup [b, \infty)$. Then on (a,b) we have the Dirac operator

(*) $$\gamma^1 \partial_x + \gamma^2 \left(\frac{g-1}{g+1} \right)$$

It seems likely that on $L^2(a,b)$ this operator is essentially skew-adjoint. This seems reasonable on the basis of what I did yesterday. If so, then we ought to be able to extend the resolvent of the Dirac operator by zero to get an operator R_λ on $L^2(\mathbb{R})$.

Link between Krein strings and transmission lines. The Krein string has an equation of motion

$$m \partial_t^2 u = \partial_x^2 u \qquad m = m(x)$$

and the energy

$$E = \int \tfrac{1}{2}\left(m\dot{u}^2 + (\partial_x u)^2\right) dx$$

Comparing with the transmission line equations on the preceding page we could set

$$m = L \quad , \qquad \dot{u} = I \quad , \qquad \partial_x u = V$$
$$1 = C$$

Thus it appears that $u(x)$ is the total charge in the line over $(-\infty, x]$.

The nice thing about the transmission line equations is their symmetry. But x is subject to reparametrization. Changing parametrization multiplies C, L by a common factor. One can thus in the case where C, L are smooth >0 normalize either C or L to be one, obtaining either the string or the dual string.

Changing the parametrization doesn't affect the quantities

$$\int C\,dx \quad , \qquad \int L\,dx$$

which are the length and total mass of the string. I think that if either of these is infinite then there is no need to tie the string at the endpoint. That is, one has the limit point case.

It is pretty clear now that the problem I am interested in, namely the Dirac operator $\gamma^1 \partial_x + \gamma^2 \frac{g-1}{g+1}$ is not the same as a Krein string problem. When $g=-1$ near a point x, there is no connection between the operators on either side of x. It is not clear what corresponds to there being "point" or discrete capacitances or inductances on the Dirac side. The natural energy norm Hilbert space makes sense in general for the trans. line, but it is not clear what this corresponds to in the singular case.

<u>Summary</u>: We have a nice correspondence between transmission lines and Diracs in the smooth case, but the singular versions of each seem to be different.

It seems that by making use of the theorems in the string case we can see that the Dirac operator $(*)$ is really well-defined in the case where $g : \mathbb{R} \longrightarrow U(1)$. Let's now try to write this out.

We are going to need formulas to pass from a Dirac operator to a transmission line. In fact mainly I need to be able to compute the length and mass $\int C dx$, $\int L dx$.

Recall that the natural parameter to use on a transmission line in the smooth case is the "proper" time, that is, the time in which the signal speed is 1. This means $LC = 1$, so that the eigenvalue equation is

$$\begin{pmatrix} C & 0 \\ 0 & L \end{pmatrix} \lambda \begin{pmatrix} V \\ I \end{pmatrix} = \begin{pmatrix} 0 & \partial_x \\ \partial_x & 0 \end{pmatrix} \begin{pmatrix} V \\ I \end{pmatrix}$$

$$\lambda \begin{pmatrix} L^{-1/2} V \\ L^{1/2} I \end{pmatrix} = \begin{pmatrix} 0 & L^{1/2} \partial_x L^{-1/2} \\ L^{-1/2} \partial_x L^{1/2} & 0 \end{pmatrix} \begin{pmatrix} L^{-1/2} V \\ L^{1/2} I \end{pmatrix}$$

Thus $\quad L^{-1/2} \partial_x L^{1/2} = \partial_x + f \quad$ means

$$\partial_x \left(\tfrac{1}{2} \log L \right) = f$$

or $\quad L = e^{\int^x 2f} \qquad C = e^{-\int^x 2f}$

Example: $\quad f = \dfrac{r}{x}$. Then

$$L = x^{2r} \qquad C = x^{-2r}$$

and

$$\int_{0+}^{1} x^{2r}\, dx < \infty \implies 2r > -1$$

$$\int_{0+}^{1} x^{-2r}\, dx < \infty \implies 2r < 1$$

so if $-\frac{1}{2} < r < \frac{1}{2}$ is excluded, then we should have the limit point case as $x \downarrow 0$. This is the conclusion reached before.

Now we look at the closed set where $g+1 \doteq 0$. I assume that $g+1=0$ outside $|x| \leq R$ so as to keep things simple. In the set Z where $g+1=0$ there are point where $g' \neq 0$, i.e. such that g crosses -1 tranversally. These are isolated points of Z, and they fall into 2 types depending on the slope g'. If this is sufficiently large then we have limit circle behavior otherwise limit point behavior.

The idea is that we remove from Z those points where we have limit circle behavior and look at the complement. This is an open set U and is a disjoint union of open intervals such that there is limit point behavior at each endpt. This means that on each interval (a,b) the Dirac $\begin{pmatrix} 0 & \partial_x - f \\ \partial_x + f & 0 \end{pmatrix}$ is essentially skew-adjoint. Thus on $L^2(U)$ we have an essentially skew-adjoint operator and we can extend it by "$i\infty$" on the orthogonal complement.

Lessons learned today: Transmission line or Krein string theory are not equivalent to Diracs when we pass to singular cases.

Does there exist a common generalization of both setups? A continuous linear chain of 2 ports?

January 28, 1987

I have reached the conclusion that to try to make sense of the operator $\gamma^\mu \partial_\mu + \sigma X$ where X is allowed to become infinite is not a sensible program. Already in dimension 1 we encounter problems related to the singularities of X. In particular even when X crosses $i\infty$ transversally there is a subtle difference in the [redacted] analysis depending on [redacted] the slope.

So it is necessary to adopt a better more general approach. The problem still remains [redacted] of investigating the integration map in K-theory. To be more precise we have a manifold M and a K-homology class on it which is represented by a Dirac operator. We have K-cohomology classes represented by maps from M to unitary groups and Grassmannians. The problem is to understand on a deep level the pairing of the K-homology and cohomology.

There are various ideas. The first is to [redacted] use constructions like the one in your recent paper where you lift to the bundle $U(E)$ in order to study unitary autos of E. Recall the advantages. A general g has a rather singular set where $g=-1$. The universal $\tilde{g}$ on π^*E over $U(E)$ [redacted] is such that the set $\tilde{g}=-1$ is a divisor whose singularities can be resolved rather easily. Moreover locally on $U(E)$ the universal $\tilde{g}$ can be approximated by Cayley transforms.

In the same way we probably want to do something about the Dirac operator. Probably I want to do something ~~[illegible]~~ like work, as Atiyah + Singer do, with T^*M and its fundamental K class.

The first thing to understand is the circle case, i.e. you want to do all the analysis with Fourier series but with arbitrary coefficients.

Take $M = S^1$ and consider the trivial bundle $\tilde{V}$ over M. Then $U(\tilde{V}) = M \times U(V)$. Over $U(\tilde{V})$ we have $\bar{g}$ which defines a canonical K-cohomology class. ~~[illegible]~~ Any autom. g of $\tilde{V}$ ~~[illegible]~~ determines a section $M \xrightarrow{g} U(\tilde{V})$. ~~[illegible]~~ Now pulling back $[\bar{g}]$ by g and integrating over M is the same as integrating $[\bar{g}]$ over the image $g_![M]$.

Try again: On ~~[illegible]~~ M we have a K-hom. class, denote it γ. On $U(V)$ we have a canonical K-coh. class $[g_{univ}]$. Then to any $g: M \longrightarrow U(V)$ we have

$$\langle \gamma, g^*[g_{univ}] \rangle = \langle g_! \gamma, [g_{univ}] \rangle$$

Probably $g_!\gamma$ has to be constructed via the graph of g.