October 4, 1986

Let's go over some KK-theory. Recall that $KK(A,B)$ is represented by bimodules ${}_A\mathcal{E}_B$ with F-operators, and that ${}_A\mathcal{E}_B$ is a left A-module and right B-module because one wants the cup product to go

$$KK(A,B) \otimes KK(B,C) \longrightarrow KK(A,C)$$

like

$$\mathrm{Hom}(A,B) \times \mathrm{Hom}(B,C) \longrightarrow \mathrm{Hom}(A,C)$$

and to be realized by the tensor product

$$ {}_A\mathcal{E}_B\,,\ {}_B\mathcal{E}'_C \longmapsto {}_A\mathcal{E} \otimes_B \mathcal{E}'_C$$

Next $KK(A,B)$ is contravariant in A, covariant in B, so

$$KK^*(\mathbb{C}, C(X)) = K^*(X) \quad \left\{\begin{array}{l}\text{rep by families of}\\ \text{Fredholms param. by } X\end{array}\right.$$

$$KK^*(C(X), \mathbb{C}) = K_*(X) \quad \left\{\begin{array}{l}\text{rep. by Freds commuting}\\ \text{mod } \mathcal{K} \text{ with } a \in C(X)\end{array}\right.$$

hence ${}_A\mathcal{E}_B$ is to be a Hilbert C^*-module over B (generalizes continuous sections of a Hilbert bundle). Also the F on ${}_A\mathcal{E}_B$ is to commute with B and commute mod $\mathcal{K}$ with A; precisely

$$(F^2-1)a\,,\ [F,a] \in \mathcal{K}_B\,.$$

Now the program will be as follows. Let $\mathcal{G} = U_{res}(H_0, \eta_0)$. We've seen how to define over $\Sigma\mathcal{G}$ a Hilbert bundle with mod $\mathcal{K}$ involution η. This determines a KK class in

$$(*) \qquad KK^1(\mathbb{C}, C(\Sigma\mathcal{G})) \qquad \text{where}$$

$$C(\Sigma\mathcal{G}) = C((0,1)) \otimes C(\mathcal{G})$$

On the other hand there is a canonical

$$\gamma \in KK^1(C((0,1)), \mathbb{C})$$

which determines

$$(**) \qquad \gamma \otimes 1 \in KK^1(C((0,1)) \otimes C(\mathcal{G}), C(\mathcal{G}))$$

The cup product of $(*)$ and $(**)$ lies in

$$KK^0(\mathbb{C}, C(\mathcal{G})).$$

The problem is to get things sufficiently explicit so as to identify this cup product with the K-class represented by $\mathcal{G} = U_{res} \longrightarrow \mathcal{C}_{res}$.

To simplify I want to look at the case of a single $g \in \mathcal{G}$ since the cup product is a single Fredholm operator which can have a nonzero index. So I have to describe a class in

$$KK^1(\mathbb{C}, C((0,1)))$$

attached to $g \in U_{res}(H_0, \eta_0)$

I should have mentioned earlier that B $C((0,1))$ is the space of continuous functions vanishing at ∞, i.e. continuous functions on $[0,1]$ vanishing at the endpoint. We have been thinking of the K^1 class as being represented by a certain Hilbert bundle over $[0,1]$ together with a family of involutions mod $\mathcal{K}$ on the fibres. The Hilbert bundle has the fibre

$$H_t = \operatorname{Im}\begin{pmatrix} \sqrt{1-t} \\ \sqrt{t}\, g \end{pmatrix} \subset H_0 \oplus H_0$$

at t and the involution η_t mod $\mathcal{K}$ is induced by $\begin{pmatrix} \eta_0 & 0 \\ 0 & \eta_0 \end{pmatrix}$.

Now this leads to a <u>paradox</u> because over $[0,1]$ we have the trivialization

$$\iota_t = \begin{pmatrix} \sqrt{1-t} \\ \sqrt{t}\, g \end{pmatrix} : H_0 \xrightarrow{\sim} H_t$$

such that
$$\iota_t^{-1} \eta_t \iota_t = \iota_t^* \begin{pmatrix} \eta_0 & 0 \\ 0 & \eta_0 \end{pmatrix} \iota_t$$
$$= (1-t)\eta_0 + t g^{-1} \eta_0 g = \eta_0$$

since g preserves η_0. Apparently the family is constant.

The paradox is explained by being careful about what $\mathcal{K}_B$ is. In this example we are dealing with the trivial Hilbert bundle, so the Hilbert C^*-module is

$$H_0 \otimes B = \{ f : [0,1] \xrightarrow[\text{cont.}]{} H_0 \mid f(0) = f(1) = 0 \}, \ (\dagger)$$

i.e. the space of sections of the Hilbert bundle vanishing at the ends. Then $\mathcal{K}_B = \mathcal{K} \otimes B$ is the alg. of families of compact operators in the fibres which vanish at the ends. Thus it isn't true that a family of compact operators in the Hilbert bundle is compact in the KK sense, and similarly in working over $(0,1)$ we can't think of the KK F as being the same as a family of involutions mod $\mathcal{K}$ on the fibres. The KK F is more precise.

$(\dagger)$ The reason f has to vanish at the ends is because the inner product $\langle f | f' \rangle$ which is the ptwise inner product must lie in $C((0,1))$.

Anyway the F we want is obtained by lifting η_0 a H_0 to F_0 and then taking the family of self adjoint Fredholms

$$F_t = (1-t)F_0 + t\,g^{-1}F_0 g.$$

Since $F_t^2 - 1 \in \mathcal{K}$ for each t and it vanishes at the ends we have $F^2 - 1 \in \mathcal{K}_B$.

When working with Diracs we could use

$$F_t = \frac{(1-t)\not{D}_0 + t\,g^{-1}\not{D}_0 g}{\sqrt{m(t)^2 + \left((1-t)\not{D}_0 + t g^{-1}\not{D}_0 g\right)^2}}$$

where $m(t) \geq 0$ and is zero near the ends.

Next the element $\gamma \in KK^1(C((0,1)), \mathbb{C}))$ is represented by the Hilbert involution on $L^2(0,1)$. (This maybe becomes clearer if one uses $\mathbb{R}$ instead of $(0,1)$.)

Now to take the cup product we take the tensor product

$$\left(H_0 \otimes C((0,1))\right) \otimes_{C((0,1))} L^2((0,1)$$

and complete (?) to get the Hilbert space

$$H_0 \otimes L^2((0,1)).$$

We next have to put a Fredholm operator on this Hilbert space constructed out of the above F and the Hilbert involution.

What I would like to do is to say that Kasparov's construction should be the same as forming a degree 0 ψDO out of the Dirac

$$i\partial_t + (1-t)\not{D}_0 + t\,g^{-1}\not{D}_0 g$$

but the symbol of this operator is independent of g. So I'm still missing something.

October 5, 1986

Yesterday I learned the KK description, or representation, of an element of $K^1((0,1))$. One has a path A_t $0 \leq t \leq 1$ in the space $\mathcal{F}_1$ of self-adjoint contractions whose image in the Calkin algebra is a non-trivial involution. This path starts and ends at involutions. This path $\{A_t\}$ gives a family of involutions mod $\mathcal{K}$ in the fibres of the trivial Hilbert bundle over $[0,1]$, and this family is what you get if you forget that the K-class is supposed to have "compact support" on $(0,1)$.

The point of asking that the endpts A_0, A_1 are involutions is that then $A_t^2 - 1$ is a family of compacts vanishing at the ends, hence it is $\mathcal{K}_B$ = the compact operators on $H_0 \otimes B$, where $B = C((0,1))$.

If we have such an $\{A_t\}$ we can define its index by using path lifting for $\mathcal{I}_{H_0} \longrightarrow \mathcal{I}(\mathcal{Q})$. This means we get a family F_t of involutions $\ni$ $F_0 = A_0$, $F_t \equiv A_t \mod \mathcal{K}$. Then F_1, A_1 are involutions congruent mod $\mathcal{K}$ so they have an index relative to each other.

All this is pretty clear; the real problem now is how this links up to an operator roughly like $\partial_t + A_t$ on $L^2((0,1), H_0)$, which is supposedly given by the Kasparov cup product construction.

Idea: Instead of worrying about boundary condition at $0,1$ one ought to be able to work with L^2-

boundary conditions over $\mathbb{R}$. Thus suppose we consider the operator

$$* \qquad i\partial_t + (1-\rho(t))\not{D}_0 + \rho(t)g^{-1}\not{D}_0 g$$

acting on sections of $pr_2^*(S\otimes E_0)$ over $\mathbb{R}\times M$ having compact support. Here $\rho(t)$ goes smoothly from 0 to 1 in $(0,1)$. If we closed up this operator in $L^2(\mathbb{R}, H_0) = L^2(\mathbb{R}\times M, pr_2^*(S\otimes E_0))$ we should obtain a Fredholm operator. Actually I should take the direct sum of the above operator $*$ and its formal adjoint, and the assertion should be that it is essentially self-adjoint.

To discuss the index we note that something killed by $*$ for $t\leq 0$ has values in the negative eigenspace H_0^- for $\frac{1}{i}\not{D}_0$, whereas for $t\geq 1$ it has values in the positive eigenspace for $g^{-1}\not{D}_0 g$ which is $g^{-1}H_0^+$. To finish the index picture we need to have the space V of sections over $[0,1]$ killed by $*$ and the boundary map $V\to H_0\oplus H_0$. The index should be that of the map $V\to H_0/H_0^- \oplus H_0/g^{-1}H_0^+$

Next the idea would be to replace $\rho(t)$ by $\rho(t/\varepsilon)$ so that the jump from $\not{D}_0$ to $g^{-1}\not{D}_0 g$ occurs very rapidly. Note that if I substitute into

$$\text{over } [0,\tfrac{1}{\varepsilon}] \qquad \left(i\partial_t + (1-\rho(t/\varepsilon))\not{D}_0 + \rho(t/\varepsilon)g^{-1}\not{D}_0 g\right)\psi = 0$$

the change $t\mapsto \varepsilon t$, then I get

$$(**) \qquad \left(i\partial_t + \varepsilon\left[(1-\rho)\not{D}_0 + \rho g^{-1}\not{D}_0 g\right]\right)\psi = 0$$

over $[0,1]$. This is what I thought was the adiabatic approximation ($\varepsilon\downarrow 0$). I conjecture that as $\varepsilon\downarrow 0$, the space of solutions of $(**)$ approaches the diagonal subspace of $H_0\oplus H_0$.

Finally one can do this over the circle instead of $\mathbb{R}$ by letting the circle size go to infinity as well.

October 7, 1986

Ihara's talk on higher cyclotomic units.

Let $\mathbb{P}^1 = \mathbb{P}^1(\mathbb{C})$ and $S_0 = \{0, 1, \infty\}$. The problem is to construct the maximal pro-l-extension of $\mathbb{Q}(\mu_{l^\infty})$ which is unramified outside of l. Here's a construction which one hopes works:

The first point is that a map $f : \mathbb{P}^1 \to \mathbb{P}^1$ which is a cyclic covering of degree n is ramified at exactly 2 points - this is just the fact that the autos of $\mathbb{P}^1$ are Mobius transformations. Transforming these points to $(0, \infty)$ on both sides, we can assume $f(t) = ct^n$ and then further we can suppose $c = 1$.

So ~~starting~~ with $S_0 = \{0, 1, \infty\}$ we take the covering $f(t) = t^l$. Then $f^{-1}(S_0) = \{0, \zeta^\nu, \infty\}$. Now pick two of these points move them to $0, \infty$ and extract l-th root. In this way we get a tower of ~~cyclotom~~ cyclic coverings

$$\mathbb{P}^1 \longrightarrow \cdots \longrightarrow \mathbb{P}^1 \longrightarrow \mathbb{P}^1 \longrightarrow \mathbb{P}^1$$

such that if f is the final map from $\mathbb{P}^1$ to $\mathbb{P}^1$, this covering is unramified outside of S_0, $f^{-1}(S_0) \supset S_0$, the Galois closure of f is an l-extension. Now consider the subset

$$f^{-1}(S_0) - \{0, \infty\} \subset \mathbb{C}^\times$$

and let E' be the subgroup it generates. It's clear that $E' \subset \overline{\mathbb{Q}}^\times$.

It turns out that E' generates an l-extension of $\mathbb{Q}(\mu_l)$ which is unramified outside of l. Moreover E' consists of l-units meaning an ε such that $\varepsilon, \varepsilon^{-1}$ are integral over $\mathbb{Z}[\frac{1}{l}]$.

Now let E be the subgroup of $\overline{\mathbb{Q}}^\times$ obtained by taking the union of all the E'. One conjectures that $\mathbb{Q}(E)$ is the max pro-l-extension of $\mathbb{Q}(\mu_l)$ which is unramified outside of l. Possibly also E is the full group of l-units.

October 11, 1986

I'm still trying to decide if the model for the loop space of the Grassmannian is useful for the transgression question.

Let $\mathcal{G} = U_{res}(H_0, \eta_0)$. I've been thinking of $B\mathcal{G}$ in terms of projectors on $H_0 \oplus H_0$ which are compatible with $\eta = \eta_0 \oplus \eta_0$. To be precise we consider involutions F on $H = H_0 \oplus H_0$ commuting with η mod $\mathcal{K}$ such that the involutions mod $\mathcal{K}$ induced on the two eigenspaces of F are non-trivial. Thus we have a cartesian square

$$\begin{tikzcd}
B\mathcal{G} \arrow[r] \arrow[d] & \mathcal{I}(H) \arrow[d] \\
\mathcal{I}(\mathcal{Q}_H^+) \times \mathcal{I}(\mathcal{Q}_H^-) \arrow[r] & \mathcal{I}(\mathcal{Q}_H)
\end{tikzcd}$$

where $\mathcal{Q}_H^+ = \frac{\eta+1}{2}\mathcal{Q}_H\frac{\eta+1}{2}$, $\mathcal{Q}_H^- = \frac{\eta-1}{2}\mathcal{Q}_H\frac{\eta-1}{2}$ or better $\mathcal{Q}_H^+ \times \mathcal{Q}_H^- =$ centralizer of η in $\mathcal{Q}_H$.

From this cartesian square, ~~we see~~ and the fact that $\mathcal{I}(H) \sim pt$, $\mathcal{I}(\mathcal{Q}_H^{+}) \to \mathcal{I}(\mathcal{Q})$ is a heq, we conclude that $B\mathcal{G} \sim \mathcal{I}(\mathcal{Q}_H^+)$; this gives the right homotopy type.

Another description of this $B\mathcal{G}$ is as a homogeneous space

$$U_{res}(H,\eta) \big/ U_{res}(H,\eta)^{\varepsilon} \xrightarrow{\sim} B\mathcal{G} \qquad g \mapsto g\varepsilon g^{-1}$$

where $\varepsilon = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ relative to $H = H_0 \oplus H_0$. To see this it suffices to check transitivity of the action of $U_{res}(H,\eta)$, and here one uses that there is only one Hilbert space with nontrivial mod $\mathcal{K}$-splitting up to isomorphism.

~~Now I would like to study~~

Notice that the above can be written

$$U_{res}(H,\eta) \big/ U_{res}(H_0,\eta_0)\times U_{res}(H_0,\eta_0) \xrightarrow{\sim} B\mathcal{G}$$

and so as $U_{res}(H_0,\eta_0) \longrightarrow U_{res}(H,\eta)$ is a heq, one sees again this Grassmannian is a classifying space for $\mathcal{G}$.

Next I would like to study the loop space in this Grassmannian, specifically paths going from ε to $-\varepsilon$. By homotopy theory one has a heq

$$\Omega\left(U_{res}(H,\eta);\ 1,\ K_{\varepsilon}\right) \xrightarrow{\sim} \Omega\left(B\mathcal{G};\ \varepsilon, -\varepsilon\right)$$

$$K_{-\varepsilon} = \{ g \mid g\varepsilon g^{-1} = -\varepsilon \} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} U_{res}(H,\eta)^{\varepsilon}$$

In finite dimensions we have seen how to go from a cont. path $h : [0,\frac{1}{2}] \longrightarrow U(V)$ such that $h(0)=1$ and $h(\frac{1}{2})\,\varepsilon\, h(\frac{1}{2})^{-1} = -\varepsilon$, to a continuous $h : R \longrightarrow U(V)$ satisfying

$$h(0)=1, \quad \varepsilon h(t)\varepsilon = h(-t), \quad h(t+1) = -h(t).$$

Also at least for smooth $h(t)$ (over all of R) one can assign an involution on $L^2(S^1)\otimes V$ which anti commutes with $\sigma : f(t) \longmapsto \varepsilon f(-t)$

Thus in finite dimensions at least we have a kind of monodromy map from the loop space in the Grassmannian to a big unitary group. This construction ought to work in infinite dims, so that to a path h in $U_{res}(H,\eta)$ going from $\mathbf{1}$ to $K_{-\varepsilon}$ belongs a **unitary** transformation between the eigenspaces of σ on $L^2(S^1)\otimes H$, i.e. an odd involution on $L^2(S^1)\otimes H$ relative to σ. The problem is to see what the significance of the fact that $h(\frac{1}{2})$ preserves η means.

What I would like is ~~to~~ find an $\tilde{\eta}$ on $\tilde{H} = L^2(S^1)\otimes H$ commuting with σ such that $\tilde{\eta}$ commutes mod $\mathcal{K}$ with the involution associated to any h. Then I would have constructed a map

$$\Omega(U_{res}(H,\eta);\ 1,\ K_{-\varepsilon}) \longrightarrow U_{res}(\tilde{H}^+,\tilde{\eta}^+)$$

Next I would like to discuss the graded case. Here one starts with a graded Hilbert space $H_0 = H_0^+ \oplus H_0^-$ equipped with an odd mod $\mathcal{K}$ involution

$$\eta_0 = \begin{pmatrix} 0 & \alpha^* \\ \alpha & 0 \end{pmatrix}$$

and there is an index $\in \mathbb{Z}$, namely the index of α, which distinguishes different types.

Here we have a K-theory with non-trivial K_0 and perhaps we should not think in terms of BG, where $G = U(A)$, A some algebra, but rather BG is the classifying space of the category of $P(A)$ and isomorphisms.

Consider $H_0 = H_0^+ \oplus H_0^-$ equipped with an odd degree involution $\eta_0 = \begin{pmatrix} 0 & \alpha^* \\ \alpha & 0 \end{pmatrix}$ mod $\mathcal{K}$. $\mathcal{G}$ is its group of automorphisms:

$$\mathcal{G} = U_{res}(H_0, \eta_0)^{\varepsilon}$$

We consider $H = H_0 \oplus H_0$ with $\eta = \eta_0 \oplus \eta_0$, and then we consider the Grassmannian

$$U_{res}(H,\eta)^{\varepsilon} \Big/ \left(U_{res}(H_0,\eta_0)^{\varepsilon}\right)^2 .$$

This can be identified with the set of involutions F on H which commute with ε, which commute with η mod $\mathcal{K}$ and such that both eigenspaces for F have the same index as (H_0, η_0).

Now all we have is a pair of Grassmannians tied together via η. When we do the loops in the Grassmannian, it will be two copies of the ungraded case.

One of the reasons I am having difficulties is that I don't have a hold on compact operators on $L^2(S^1) \otimes H$. The Kasparov cup product that I need to be able to work with involves the Hilbert involution on S^1 and a continuous family $\{A_t\}$ of self-adjoint Fredholms on S^1. This cup product is either a Fredholm operator on $L^2(S^1) \otimes H$ or a point in a restricted Grass on a similar Hilbert space; in either case one needs to know the compacts on $\tilde{H} = L^2(S^1) \otimes H$.

When H is finite dimensional we

can associate to any loops in $U(H)$ a Fredholm operator by the Toeplitz construction. What does this become when H is infinite dimensional? It is necessary to understand this business carefully.

Let us begin with the basic fact that on $L^2(S^1)$ the commutator of the Hilbert involution F with a continuous function multiplication operator is compact. To prove this one uses uniform approximation of continuous functions by trig polys (Fejer) to reduce to checking for polynomials.

Nicer version: Consider the space of bdd measurable f such that $[F,f]$ is compact. It is a closed $*$ subalgebra of $L^\infty(S^1)$ — ($[F,fg] = [F,f]g + f[F,g]$ + compacts form an ideal.) It contains z, z^{-1} and so must contain $C(S^1)$.

Next look at $L^2(S^1)\hat{\otimes} H$. It's clear I think that the ideal of compacts here is the closure of compacts in $L^2(S^1)$ tensored with compacts in H. Then the above style argument shows that if f is a cont. function on S^1 with values which are compact operators on H, then $[F,f]$ is compact on $L^2(S^1)\hat{\otimes} H = \tilde{H}$.

Next suppose we have a continuous family g_t of unitary operators on H, whence we get a unitary g on $\tilde{H}$. If $g_t \equiv 1 \mod \mathcal{K}$, then $[F,g] = [F,g-1]$ is compact on $\tilde{H}$ by the above. Hence we get a map from the loops in $U(\mathcal{K})$ to the restricted unitary group of $\tilde{H}$ wrt F.

<u>October 12, 1986</u> (Becky is 20)

Yesterday we saw that if $t \mapsto K(t)$ is a continuous map $S^1 \longrightarrow \mathcal{K}(H)$, then the associated mult. op. K on $\tilde{H} = L^2(S^1) \otimes H$ has the property that $[F, K] \in \mathcal{K}(\tilde{H})$. As a corollary, given $t \longrightarrow g(t)$, $S^1 \longrightarrow U(\mathcal{K}(H)) = \{\text{unitaries on } H \equiv 1 \bmod \mathcal{K}(H)\}$ one has that the associated mult. op. g on $\tilde{H}$ belongs to $U_{res}(\tilde{H}, F)$:

$$\mathcal{L}\{U(\mathcal{K}(H))\} \hookrightarrow U_{res}(\tilde{H}, F)$$

But now I want understand what happens for the loop group of $U_{res}(H, \eta)$. Take η and lift it to A on H. Let $g(t)$ be a continuous loop in $U_{res}(H, \eta)$, i.e. ~~g(t)~~

$$g(t)^{-1} A g(t) - A = g(t)^{-1}[A, g(t)] \in \mathcal{K}(H)$$

so $g^{-1}[A, g]$ is a continuous loop in $\mathcal{K}(H)$. Then

$$\cancel{[gFg^{-1}, A] = [g, A] F g^{-1} + g F(-g^{-1}[}$$

$$\begin{aligned}
[A, gFg^{-1}] &= [A,g] F g^{-1} + gF(-g^{-1}[A,g]g^{-1}) \\
&= g\left(g^{-1}[A,g] F - F(g^{-1}[A,g])\right) g^{-1} \\
&= g \underbrace{[g^{-1}[A,g], F]}_{\in \mathcal{K}(\tilde{H})} g^{-1}
\end{aligned}$$

and we conclude we have a map from $\mathcal{L}\{U_{res}(H,\eta)\}$ to the Grassmannian of involutions which commute mod $\mathcal{K}(\tilde{H})$ with A.

If I pick A to be an involution, then the Grassmannian of involutions commuting mod $\mathcal{K}(\tilde{H})$ with A is a model for a classifying space of the restricted unitary group. Notice that the operator A on $\tilde{H}$ is really $1 \otimes A$ and that modifying A by a compact on H leads to a non compact modification of $1 \otimes A$. It's not clear that this non uniqueness has any significance.

Let us next consider the space of paths $h: \mathbb{R} \longrightarrow U_{res}(H, \eta)$ such that $h(0) = 1$, $h(t+1) = -h(t)$. To such an h belongs an involution F_h on $\tilde{H}$ which gives the positive, + negative eigenspaces for the operator $h \cdot \frac{1}{i}\partial_t \cdot h^{-1}$. In fact if we take $h_0(t) = e^{-i\pi t}$, then

$$h_0 \partial_t h_0^{-1} = \partial_t + i\pi$$

corresponds to the Hilbert involution F. Then we can write any other h in the form $g h_0$ with $g(0) = 1$, $g(t+1) = g(t)$, and we have $F_h = g F g^{-1}$. So we conclude for any h having its values in $U_{res}(H, \eta)$ that $[F_h, A] \in \mathcal{K}(\tilde{H})$.

So far I have been looking at the periodicity map going from loops of unitaries to the Grassmannian in various guises.

Next I want to continue and consider the case where H has a grading ε commuting with η, i.e. $H = H_0 \oplus H_1$, $\varepsilon = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $\eta = \begin{pmatrix} \eta_0 & 0 \\ 0 & \eta_1 \end{pmatrix}$. In this case we can define σ on $\tilde{H}$ and assuming $\varepsilon h(t) \varepsilon = h(-t)$ we know σ anticommutes with F_h. Let's lift η to an involution A commuting with ε: $A = \begin{pmatrix} A_0 & 0 \\ 0 & A_1 \end{pmatrix}$

Recall that our space of h is a model for the loops in the Grassmannian of involutions in H commuting with A mod K. Then we have assigned to such an h an involution F_h which is odd relative to σ, therefore is equiv. to a unitary between the two eigenspaces of σ; moreover this unitary commutes with A modulo $\mathcal{K}(\tilde{H})$.

But this is equivalent to having a restricted unitary transf. Let

$$F_h = \begin{pmatrix} 0 & \varphi_h^{-1} \\ \varphi_h & 0 \end{pmatrix} \qquad \varphi_h : \tilde{H}^+ \xrightarrow{\sim} \tilde{H}^-$$

and choose $\rho : \tilde{H}^+ \longrightarrow \tilde{H}^-$ so that ρ identifies the A's on these pieces. Then $\rho^{-1}\varphi_h$ is a unitary on $\tilde{H}^+$ preserving the A on $\tilde{H}^+$ mod compacts.

October 14, 1986

In KK theory an important example of the cup product comes from ΨDO's on a product $X \times Y$ of two manifolds. Recall that a ΨDO of order zero is a bounded operator on L^2. To be more precise let E be a v.b. over X compact having a volume elt.; then one has $L^2(X,E)$ a Hilbert space, as well as a whole family of Sobolev spaces L^2_s. ΨDO's of order zero are operators on each of these, and ΨDO's of order -1 are compact operators. So we have

$$\begin{tikzcd} 0 \arrow[r] & \Psi^{-1} \arrow[r] \arrow[d, phantom, "\cap"] & \Psi^{0} \arrow[r] \arrow[d, phantom, "\cap"] & \Psi^0/\Psi^1 \arrow[r] \arrow[d, dotted] & 0 \\ 0 \arrow[r] & \mathcal{K} \arrow[r] & \mathcal{B} \arrow[r] & \mathcal{Q} \arrow[r] & 0 \end{tikzcd}$$

and Ψ^0/Ψ^1 = sections of $p^*(\mathrm{End}\,E)$ over $S^*X = S(T^*X)$.

The point is that for ΨDO's of order zero and more generally singular integral operators (Calderon-Zygmund operators), the operator modulo compacts is determined by its symbol which is a continuous function on the cosphere bundle.

Thus if we are dealing with involutions mod $\mathcal{K}$ we are going to be looking at families of involutions parametrized by S^*X. For example the Dirac operators will give rise to the involution

$$\frac{c(\xi)}{|\xi|}.$$

This will be ungraded for X odd, and graded for X even.

Here is the *problem* I want to understand well, as I posed it to Roe:

In $KK^1(C(S^1), \mathbb{C})$ there is a canonical class β which is represented by the Hilbert space $L^2(S^1)$ with its natural module structure over $C(S^1)$ together with the Hilbert involution. (this involution mod $\mathcal{K}$ is completely canonical).

In $KK^1(\mathbb{C}, C(S^1))$ let α be the class represented by a continuous map $t \longmapsto A(t)$ from S^1 to self-adjoint Fredholm operators on the Hilbert space H in the following way: The space $C(S^1, H)$ is a Hilbert C^*-module over $C(S^1)$ and the family $A(t)$ defines an operator A on this Hilbert module. I will suppose each $A(t)$ is such that $-1 \leq A(t) \leq 1$ and that $A(t)$ has essential spectrum $\{+1, -1\}$. Then A commute with $B = C(S^1)$, and $A^2 - 1$ is a cont. family of compact operators, hence $A^2 - 1 \in \mathcal{K}_B$.

The first *problem* is to describe the cup product $\alpha \cup \beta \in KK^0(\mathbb{C}, \mathbb{C})$. It is represented by a Fredholm operator on

$$L^2(S^1) \otimes_{C(S^1)} C(S^1, H)$$

which should be (possibly after completion)

$$L^2(S^1, H).$$

So we have to construct this Fredholm operator.

On the other hand we have another construction as follows: By applying a map which wraps $[-1,1]$ around S^1, e.g.

$$x \mapsto e^{i\pi x} \qquad \text{or} \qquad x \mapsto \left(\sqrt{1-x^2}+ix\right)^2$$

to each $A(t)$ we obtain a continuous family $g(t)$ of unitaries on H we are congruent to -1 mod $\mathcal{K}$. It follows that the multiplication operator by g on $L^2(S^1,H)$ will give rise to a Toeplitz operator on $H^2(S^1,H)$ which is Fredholm.

The <u>second problem</u> is to link these two constructions.

Note that the second construction makes sense when H is finite dimensional, because there are topologically non trivial loops in the unitary group in this case, even though there are none in self-adjoint operators.

I envisage a similar thing happening for the Kasparov product. I should be able to allow $A(t)$ to acquire infinite eigenvalues.

Let's return to the product $X \times Y$. One has $T^*_{X\times Y} = pr_1^*(T^*_X) \oplus pr_2^*(T^*_Y)$, so the sphere in $T^*_{X\times Y}$ over (x,y) is the sphere in $T^*_{X,x} \oplus T^*_{Y,y}$ which is the join of $S^*_{X,x}$ with $S^*_{Y,y}$.

The way to think is that the unit sphere in $V \oplus W$ is the join:

$$S(V\oplus W) = S(V) * S(W)$$

Specifically, a unit vector in $V \oplus W$ can be expressed in the form

$$\sqrt{1-t}\, v + \sqrt{t}\, w \qquad \|v\| = \|w\| = 1,\ 0 \le t \le 1.$$

Supposing we have involutions F'_v, F''_w depending on $v \in S(V)$, $w \in S(W)$, we then form the family of unitaries

$$\sqrt{1-t}\; F'_v \otimes 1 + i\sqrt{t}\; 1 \otimes F''_w$$

parametrized by the join.

For example, let's consider the Dirac operator (made self adjoint) $\frac{1}{i}\partial_x$ on $X = \mathbb{R}$ and the associated Hilbert involution whose symbol is $\frac{\xi}{|\xi|}$. Consider the same operator $\frac{1}{i}\partial_y$ on $Y = \mathbb{R}$. Then on $\mathbb{R}^2$ we have the operator

$$\frac{1}{i}\partial_x + i\frac{1}{i}\partial_y \qquad \text{with symbol } \xi + i\eta$$

and the associated unitary has symbol

$$\frac{\xi + i\eta}{|\xi + i\eta|} = \underbrace{\frac{|\xi|}{|\xi+i\eta|}}_{\sqrt{1-t}} \frac{\xi}{|\xi|} + i \underbrace{\frac{|\eta|}{|\xi+i\eta|}}_{\sqrt{t}} \frac{\eta}{|\eta|}.$$

What's important here can be summarized by saying that the Kasparov cup product for involutions attached to Dirac operators is the involution belonging to the <u>Dirac operator</u> on the product manifold.

Let's review the problem. I want to combine the Hilbert involution on $L^2(S^1)$ with a loop A_t of self adjoint Fredholm operators. If the loop comes from a family of Dirac operators: $A_t = \dfrac{\frac{1}{i}D_t}{\sqrt{1+D_t^2}}$, then what

I get is the phase of the Dirac op. $\partial_t + \frac{i}{i}\not{D}_t$ on $S^1 \times M$. However, I feel there ought to be a way to include the case where A_t is replaced by a loop in a finite diml unitary group. This suggests I can perhaps make some analytical sense out of $\partial_t + X(t)$, where $X(t)$ is self-adjoint but is allowed to have infinite eigenvalues.

Analytically we have to make sense of the sum of two unbounded operators. The first case to look at is $\partial_x + p(x)$ where $x \mapsto p(x)$ is a map $S^1 \longrightarrow P_1(\mathbb{R})$. The kernel of this operator is

$$u = C\, e^{-\int^x p}$$

on the open set where $p \neq \infty$. It's not clear how to handle things where p is infinite, since changing the sign of p causes u to goes from rapid decay to rapid growth as we approach a point where $p(x) = \infty$.

An idea comes from KdV where we factor

$$-\partial_x^2 + q = (-\partial_x - p)(\partial_x - p)$$

whence $p = \frac{u'}{u}$ where $(-\partial_x^2 + q)u = 0$. This gives some sort of control over the operator

$$\begin{pmatrix} 0 & -\partial_x - p \\ \partial_x - p & 0 \end{pmatrix}$$

where p has simple pole singularities.

1

October 16, 1986

Jaced Brodzki showed me a Springer LNS by Patrick Shanahan on the A.S. Index thm. In it is a discussion of the index thm. for S^1 which seems worthwhile to review.

Let's recall that A.S. prove their thm in the context of ψDO's, and mainly ψDO's of order zero. The symbol of a ψDO D of order zero from $\Gamma(E)$ to $\Gamma(F)$ over X is a map $\sigma_D : \pi^*E \longrightarrow \pi^*F$, where $\pi: S^*X \to X$ is the cosphere bundle. D is elliptic $\Leftrightarrow$ σ_D invertible, and in this case one gets

$$[\sigma_D] \in K^1(S^*X).$$

Because of the exact sequence

$$K^1(X) \xrightarrow{\delta} K^1(S^*X) \longrightarrow K^0(T^*X) \longrightarrow K^0(X)$$

(recall this means compact supps)

one gets a class in $K^0(T^*X)$. Because T^*X is symplectic it has an almost complex structure unique up to homotopy, so there is a Gysin map

$$K^0(T^*X) \longrightarrow K^*(pt) = \mathbb{Z}$$

and the index thm. says that the image of $\delta[\sigma_D]$ under this Gysin map is the index of D.

Now let's take $X = S^1$, whence $T^*S^1 = S^1 \times \mathbb{R}$ and $S^*S^1 = S^1 \times \{-1, 1\}$. As bundles over S^1 are trivial the symbol of a ψDO of order zero over S^1 is a map from $S^1 \times \{-1,1\}$ to invertible matrices, so it is two loops of invertible matrices. The first question is how to realize a map $S^1 \times \{-1,1\} \longrightarrow GL_n$

as the symbol of a ψDO of order zero.

Consider the Hardy projector

$$P(z^n) = \begin{cases} z^n & n \geq 0 \\ 0 & n < 0 \end{cases}$$

Apparently it is possible to verify that P is a ψDO using Hörmander's defn: asymptotic expansion of

$$e^{-it\varphi} P\{e^{it\varphi} f\}(x)$$

where $\operatorname{grad}(\varphi) \neq 0$ at x. In any case using Hormander's thm, that it is enough to check for one φ, we ought to be able to determine the symbol with $e^{i\varphi} = z$ or z^{-1}. Clearly from the Fourier series of f we have

$$z^{-n} P(z^n f) = f + O\left(\frac{1}{n^N}\right) \qquad \text{as } n \to +\infty$$

$$= O\left(\frac{1}{n^N}\right) \qquad \text{as } n \to -\infty$$

for any N. Thus

$$\sigma(P)(x,\xi) = \begin{cases} +1 & \xi > 0 \\ 0 & \xi < 0 \end{cases}$$

Now suppose we have the two loops call them g_{-1}, g_1 in GL_n. Then these loops give rise to multiplication operators on $L^2(S^1)^{\oplus n}$. It's clear that a ψDO with the symbol (g_{-1}, g_1) is

$$g_{-1}(1-P) + g_1 P$$

Without changing the index we can change it to

$$(1-P) + (g_{-1}^{-1} g_1)P$$

or ultimately to

$$P(g_{-1}^{-1} g_1)P \oplus (1-P).$$

(Here we use that mod $\mathcal{K}$ the operator P and any multiplication operator commute.)

There are two algebra extensions involving the circle. One concerns operators on $H^2(S^1)$ and is

$$0 \longrightarrow \mathcal{K} \longrightarrow \left\{ \begin{matrix} \text{alg. gen} \\ \text{by } T = z\cdot \\ \text{and } T^* = Pz\cdot \end{matrix} \right\} \longrightarrow C(S^1) \longrightarrow 0$$

The other concerns operators on $L^2(S^1)$ and is

$$0 \longrightarrow \mathcal{K} \longrightarrow \left\{ \begin{matrix} \text{singular integral} \\ \text{operators on } L^2(S^1) \end{matrix} \right\} \longrightarrow C(S^1 \times \{-1,1\}) \longrightarrow 0$$

$$C(S^1 \times \{-1,1\}) = C(S^1) \times C(S^1)$$

The former be identified with the reduction of the latter with respect to P.

Now that I have some control over the 4DO's on the circle, I should look again at the Kasparov product.

October 17, 1986

Here it seems is the critical situation:

We start on the families index side which we have under control. Let E be a v.b. over M (odd) and consider $Grass(E)$, the space of subbundles of E. Suppose given a connection D on E, whence we have $\not{D}$ on $H = L^2(M, S \otimes E)$. For each $e \in Grass(E)$ we have an induced connection eDe on eE and the corresponding Dirac $e\not{D}e$ on eH. As e varies over $Grass(E)$ we get a family of elliptic operators.

We've seen how the superconnection character forms for this family, where the Hilbert bundle $\{eH\}$ is equipped with the Grass. connection, are closely related to the index map

$$Grass(E) \longrightarrow -\mathcal{U}(\mathcal{K}_H)$$

defined by taking the Cayley transform of $e\not{D}e$ on eH and extending it by -1 on $(1-e)H$.

Now take a loop in $Grass(E)$. This will give us a subbundle $\mathcal{E}$ of $pr_2^*(E)$ over $S^1 \times M$, and $\mathcal{E}$ has an induced connection from D. So we can form the total Dirac operator $\mathcal{D}$ on $L^2(S^1 \times M, S(S^1 \times M) \otimes pr_2^* E)$.

The total Dirac operator is the closest I can come so far to the Kasparov product of Dirac on S^1 cupped with a family of self-adjoint Freds param. by S^1. The index of $\mathcal{D}$ turns out to be equal to the integer in $\pi_1(-\mathcal{U}(\mathcal{K}_H)) = \mathbb{Z}$ represented by the map

$$S^1 \longrightarrow \mathrm{Grass}(E) \longrightarrow -\mathcal{U}(\mathcal{K}_H)$$

This is the spectral flow theorem and is a K-theory theorem discussed in A[redacted], Patodi, S III.

[redacted] It appears we can give an analytical proof as follows. First the winding no. of the index map over the loop is the integral of the superconnection 1-form on $-\mathcal{U}$. 2nd the index of the Dirac operator $\mathcal{D}$, when we take its analytical expression and let the S^1 directions become classical, is the integral of the super connection 1-form over the circle.

But [redacted] I really want a good version of the K-theoretic proof. [redacted] What seems important are: 1) [redacted] If $E_0 = e_0 E$ is a basepoint of Grass(E) and we restrict to based loops, then up to homotopy the index map

$$\Omega\, \mathrm{grass}(E) \longrightarrow -\mathcal{U}(\mathcal{K}_H)$$

factors through the monodromy

$$\Omega\, \mathrm{Grass}(E) \longrightarrow \mathcal{G} = \mathrm{Aut}(E_0)$$

2) The Dirac operator $\mathcal{D}$ over $S^1 \times M$ is somehow linked to the Toeplitz operator attached to the index loop $S^1 \to -\mathcal{U}(\mathcal{K})$.

In the above I [redacted] used Dirac operators because I as yet have no control over the Kasparov cup product. However I feel the essential difficulties are K-theoretic. Let's see if we can isolate the important features.

We have the Hilbert space H with the nontrivial mod $\mathcal{K}$ involution η, and we consider the Grassmannian B consisting of projectors e on H such $[e,\eta]=0$ mod $\mathcal{K}$. Ultimately we want to restrict to those e such that the involutions mod $\mathcal{K}$ induced by η on eH and $(1-e)H$ are nontrivial.

Let us fix a basepoint e_0 in the Grassmannian B and consider based loops in B. Given such a loop $t \mapsto e_t$ we then get over S^1 a Hilbert space bundle $\mathcal{H}$ with $\mathcal{H}_t = e_t H$ which is embedded in the trivial Hilbert bundle $\tilde{H}$. Moreover the Hilbert bundle $\mathcal{H}$ has a canonical mod $\mathcal{K}$ splitting $\eta_{\mathcal{H}}$ induced by η.

Let's now fix our attention on this $(\mathcal{H}, \eta_{\mathcal{H}})$ over S^1. Because $\mathcal{H}$ is embedded in the trivial bundle it has a connection, hence we have parallel transport. It should be easy to see that parallel transport preserves $\eta_{\mathcal{H}}$. Thus the monodromy at the basepoint is a unitary of g on $\mathcal{H}_{e_0} = e_0 H$ which preserves $\eta_0 = (\eta_{\mathcal{H}})_{e_0}$. It's clear that up to isomorphism the bundle $(\mathcal{H}, \eta_{\mathcal{H}})$ over S^1 is determined with the connection on $\mathcal{H}$ by this $g \in U_{res}(e_0 H, \eta_0)$.

Thus we have a monodromy map

$$\Omega(B, e_0) \longrightarrow U_{res}(e_0 H, \eta_0) = \mathcal{G}$$

which is consistent with the idea that B is a classifying space for $\mathcal{G}$.

But we have another ingredient, namely an index map

$$B \longrightarrow -U(\mathcal{K}_H)$$

defined as follows. Given $e \in B$, let η_e be the nontrivial mod $\mathcal{K}$ involution on eH induced by η, and extend it to a nontrivial mod $\mathcal{K}$ involution on H by letting it be 1 (or -1) on $(1-e)H$. Call this extension $\tilde{\eta}_e$. We get then a map

$$B \longrightarrow \mathcal{I}(\mathcal{Q}_H)$$

which is canonical up to the choice of ± 1. Then we can use the Atiyah-Singer heq's

$$\mathcal{I}(\mathcal{Q}_H) \longleftarrow \mathcal{F}_1 \longrightarrow -U(\mathcal{K}_H)$$

to obtain a map which is unique up to homotopy and which is independent of the choice of ± 1.

A better way to say this is to use the AS heq fibrewise in the bundle $\mathcal{H}$, so as to get a unitary op on $\mathcal{H}$ congruent to -1 mod $\mathcal{K}_{\mathcal{H}}$ and then extend it by -1 to a unitary on $\tilde{H}$.

Now comes the fundamental problem of relating the canonical maps

$$\Omega(B) \longrightarrow U_{res}(e_0 H, \eta_0)$$

$$\Omega(B) \longrightarrow \Omega(-U(\mathcal{K}_H))$$

using periodicity.

The main idea involves bringing in the Dirac operator or a suitable KK analogue [redacted] on $L^2(S^1, H)$.

Now I notice that the total Dirac operator on $S^1 \times M$ is obtained by reducing $i\partial_t + \not{D}$ on $L^2(S^1, H) = L^2(S^1 \times M, pr_2^*(S \otimes E))$ with respect to the idempotent operator e on $pr_2^*(E)$ defined by the family e_t, $t \in S^1$.

I have already seen how superconnection forms [redacted] behave relative to such a reduction, and I know that the superconnection forms are a limit of certain analytical expressions for the [redacted] index of the total Dirac operator. So this leads to the following:

<u>Question</u>: Given $\not{D}$ on $L^2(M, S \otimes E)$ and an idempotent e on E, can you explain the operator $e\not{D}e$ on $L^2(M, S \otimes eE)$ as somehow the result of letting something to do with $\not{D}$ on $(1-e)E$ go to infinity?

October 19, 1986

Review the K-theory: One starts with (H,η) and lets B be the Grass. of projectors e on H such that $[e,\eta]=0$ and $\eta_e = \bar{e}\eta = \eta\bar{e}$ is a non-trivial mod $\mathcal{K}$ involution on eH. Thus one has a Hilbert bundle $\mathcal{H} = \{eH\}$ over B embedded in $\tilde{H}$ and equipped with an $\eta_{\mathcal{H}} = \{\eta_e\}$. By the AS homotopy equivalence

$$\mathcal{I}(\mathcal{Q}(eH)) \longleftarrow \mathcal{F}_1(eH) \longrightarrow -\mathcal{U}(\mathcal{K}(eH))$$

$\eta_{\mathcal{H}}$ determines a unitary transf. g on $\mathcal{H}$ $\equiv -1$ mod $\mathcal{K}$ which is unique up to homotopy. Extending g by -1 on $\mathcal{H}^\perp \subset \tilde{H}$, we get an index map

$$\circledast \qquad B \longrightarrow -\mathcal{U}(\mathcal{K}(H)).$$

Next suppose given a loop $S^1 \longrightarrow B$. Then over S^1 we get a Hilbert bundle $\mathcal{H}$ with $\eta_{\mathcal{H}}$ and an embedding into $\tilde{H}$. The pair $(\mathcal{H},\eta_{\mathcal{H}})$ over S^1 has an index defined as follows. If one ~~uses Kuiper~~ ~~proceeds to~~ trivializes $\mathcal{H}$ (say by Kuiper although this only uses the connectedness of the big unitary group), then one gets a loop in $\mathcal{I}(\mathcal{Q})$. Lifting this path in

$$\mathcal{U}_{res}(H_0,\eta_0) \longrightarrow \mathcal{U}(H_0) \longrightarrow \mathcal{I}(\mathcal{Q}(H_0))$$

and taking the endpoint gives an elt. of $\mathcal{U}_{res}(H_0,\eta_0)$.

But this path lifting amounts to a new trivialization of $\mathcal{H}$ over $[0,1]$ such that $\eta_{\mathcal{H}}$ is constant relative to the trivialization. Moreover the embedding of $\mathcal{H}$ in $\tilde{H}$ defines a connection on $\mathcal{H}$ whose parallel transport should preserve $\eta_{\mathcal{H}}$ and hence produce the required path lifting.

What's important here is that a based loop in B with basepoint e_0 has a monodromy lying in $\mathcal{U}_{res}(H_0,\eta_0)$.

so we have the monodromy map

$$\Omega(B,e_0) \longrightarrow U_{res}(H_0,\eta_0)$$

Now the problem is to relate this to the index map ⊛, somehow using a periodicity map $\Omega U \sim \mathbb{Z}\times BU$. The only one that comes to mind is a Toeplitz type operator.

Let's return to a fixed loop in B. We have over this loop, say over S^1, the Hilbert bundle $\mathcal{H}$ with compatible $\eta_{\mathcal{H}}$ and connection. Suppose we choose a g on $\mathcal{H}$ as above. Then g is a unitary operator in $L^2(S^1,\mathcal{H})$ and a natural question is whether there is a Hilbert involution on $L^2(S^1,\mathcal{H})$ preserved mod compacts by g.

The problem is that there doesn't seem to be a canonical F. For example if $\mathcal{H}$ is trivialized, whence $L^2(S^1,\mathcal{H}) = L^2(S^1)\otimes H_0$, then for **any** g, and any Hilbert F on $L^2(S^1)$, we have

$$[\tilde{F},g] = [\tilde{F},g-1] \quad \text{is compact}$$

where $\tilde{F} = F\otimes \mathrm{id}_{H_0}$.

What maybe we want to do is to concoct an $\tilde{\eta}$ on $L^2(S^1,\mathcal{H})$ out of the connection and $\eta_{\mathcal{H}}$ (which is preserves by the connection).

It seems that the K-theory story is pretty simple. Over B we have the Hilbert bundle with mod K involution $(\mathcal{H},\eta_{\mathcal{H}})$ and this defines the odd K index class on B.

It might be better to use the embedding of $\mathcal{H}$ in $\tilde{H}$ for then we actually get a map $B \longrightarrow \mathcal{I}(\mathcal{Q}(H))$ which is canonical except for the choice of ± 1 on the complement.

Now restricting to a loop in B gives a loop in $\mathcal{I}(\mathcal{Q}(H))$. Part of the Atyiah–Singer approach is to use the fibration

$$\mathcal{I}_{res} \longrightarrow \mathcal{I}(\mathcal{B}(H)) \longrightarrow \mathcal{I}(\mathcal{Q}(H))$$

together with Keiper's thm. in order to identify the loop space of $\mathcal{I}(\mathcal{Q})$ with $\mathcal{I}_{res}$. Therefore it is naturally part of their approach to periodicity to lift the path in $\mathcal{I}(\mathcal{Q}(H))$ up to a path of involutions in H and then take the endpoint. This explicitly gives the periodisity map

$$\Omega\, \mathcal{I}(\mathcal{Q}(H)) \sim \mathbb{Z} \times BU$$

I require.

Unfortunately because I want diff'l forms, it is necessary to realize K-classes by maps to either a unitary group or a Grassmannian.

October 20, 1986

I have to get hold of the adiabatic expansion. Consider over $\mathbb{R}\times M$ the operator

$$\partial_t + \frac{1}{i}\gamma^\mu D_\mu$$

where $D_\mu = \partial_\mu + A_\mu(t,x)$, and to keep things simple suppose that M is a torus, e.g. a circle. I suppose that outside of $-1 \leq t \leq 1$, $A_\mu(t,x)$ is constant in t. This operator acts on $L^2(\mathbb{R})\otimes L^2(M, S\otimes E_0)$

We want to do the following transformations on this operator. First I want to change the operator to

$$\circledast \qquad \frac{1}{\varepsilon}\partial_t + \frac{1}{i}\gamma^\mu D_\mu(t)$$

which is equivalent to scaling the metric on $\mathbb{R}\times M$ in the t direction. Then I want to apply the diffeomorphism $t \longrightarrow \frac{t}{\varepsilon}$ of $\mathbb{R}$, which gives

$$\partial_t + \frac{1}{i}\gamma^\mu D_\mu\left(\frac{t}{\varepsilon}\right).$$

This is the same as the original operator except the transition between $A_\mu(t)$ for $t \ll 0$ and $t \gg 0$ now takes place in the interval $[-\varepsilon,\varepsilon]$.

Let's give more details about the diffeomorphism. On $L^2(\mathbb{R}, H)$ we have a unitary given by

$$(Tf)(t) = \frac{1}{\sqrt{\varepsilon}} f\left(\frac{t}{\varepsilon}\right)$$

Check: $$\|Tf\|^2 = \int_{-\infty}^{\infty} \left\|\frac{1}{\sqrt{\varepsilon}} f\left(\tfrac{t}{\varepsilon}\right)\right\|_H^2 dt = \int_{-\infty}^{\infty} \left\|f\left(\tfrac{t}{\varepsilon}\right)\right\|_H^2 \frac{dt}{\varepsilon}$$
$$= \int_{-\infty}^{\infty} \|f(t)\|_H^2\, dt = \|f\|^2.$$

Then we want to complete

$$T\left(\partial_t + B(t)\right) T^{-1} f$$

$$\left(T^{-1}f\right)(t) = \sqrt{\varepsilon}\, f(\varepsilon t)$$

$$\left[\left(\partial_t + B(t)\right) T^{-1} f\right](t) = \sqrt{\varepsilon}\Big(\underbrace{\partial_t f(\varepsilon t)}_{\varepsilon f'(\varepsilon t)} + B(t) f(\varepsilon t)\Big)$$

$$\left[T\left(\partial_t + B(t)\right) T^{-1} f\right](t) = \varepsilon f'(t) + B(t/\varepsilon) f(t)$$
$$= \left[\left(\varepsilon \partial_t + B(t/\varepsilon)\right) f\right](t).$$

I am interested in the limit as $\varepsilon \downarrow 0$, I think, because this should concentrate the index into the half spaces of positive and negative energy solutions for the operators $\frac{1}{i} D\!\!\!/(t)$ for $t \ll 0$ and $t \gg 0$. However I think the adiabatic limit is the opposite one where $\varepsilon \to \infty$, ~~[illegible]~~ where the transition between the operators takes place very slowly.

Supposedly the adiabatic limit is the one where the metric on the circle is scaled so that the length of the circle becomes very long. The Dirac operator on the circle $\mathbb{R}/\mathbb{Z}$ with constant metric of length L has eigenvalues $\frac{2\pi i}{L}\mathbb{Z}$ so it is the operator

$$\frac{1}{L}\partial_t$$

Thus length $L \to \infty$ is the $\varepsilon \to \infty$ limit in ⊛.

Another check: If on the line we use the metric $ds^2 = g\,dt^2$, e.g. $g = (\dot{x}^2 + \dot{y}^2)$ where $t \to (x(t), y(t))$ is an embedding, then the unit tangent vector in the positive t direction is

$$\frac{1}{\sqrt{g}}\partial_t \quad \left(\text{becomes } \frac{1}{\sqrt{g}}(\dot{x}\,\partial_x + \dot{y}\,\partial_y)\right)$$

Thus the corresponding coframe is $\omega = \sqrt{g}\,dt$ which means that the Dirac operator is $\frac{1}{\sqrt{g}}\partial_t$. So again if we want a long circle $g = L^2$, we get $\frac{1}{L}\partial_t$ with $L \to \infty$.

So the conclusion is that in order to make the link with cyclic cohomology I want the opposite of the adiabatic limit.

2nd thoughts: The adiabatic limit concerns

$$\varepsilon \partial_t + A(t)$$

as $\varepsilon \downarrow 0$, or by changing $t \to \varepsilon t$ the operator

$$\partial_t + A(\varepsilon t).$$

If ε is small this operator is nearly constant in any normal-sized interval. The adiabatic principle or theorem says that an eigenstate for $A(-\infty)$ will evolve in time as follows. One assume the eigenvalues of $A(t)$ are non degenerate. Then over the real axis one has eigenline line bundles and these inherit connections. The adiabatic theorem then says that the eigenstate evolves via this connection.

In my case I would like to see how the negative eigenspace of $A(t)$ for $t \ll 0$ evolves. Presummably the adiabatic thm. might say that the is a nice smooth family of projectors $E(t)$ commuting with $A(t)$ such that $E(t) =$ neg. projector of $A(t)$ for $t \ll 0$. Then the index for $\varepsilon\partial_t + A(t)$ would be given by comparing $E(t)$ with the negative projector for $A(t)$, $t \gg 0$.

However to get the index all we have to do is to find an $E(t)$, which mod compacts is a negative projector for the Dirac $A(t)$. And I can take $E(t)$ constant if I wish.

It's time to return to Diracs on the circle. Thus we take $M = S^1$ and let E_0 be the trivial bundle $\tilde{\mathbb{C}}^n$.

I want to start the discussion from the viewpoint I have been using recently, which means that I want a bundle E over S^1 and I want to look at a Grassmannian B of subbundles of E. E_0 is the basepoint of B. Let us take $E = \tilde{\mathbb{C}}^{2n}$ and restrict to n dimensional subbundles with over the basept of S^1 have the fibre $\mathbb{C}^n \oplus 0 \subset \mathbb{C}^{2n}$ (first n-coords.) Thus

$$B = \Omega(Gr_n(\mathbb{C}^{2n}))$$

We have a map

$$(*) \qquad U_n \longrightarrow \Omega(Gr_n(\mathbb{C}^{2n}))$$

in which one uses the graph of $g \in U_n$ to go from ε to $-\varepsilon$ and then back again by $1 \in U_n$. We have seen this map $(*)$ corresponds to choose a transverse connection in the canonical bundle over $U_n \times S^1$.

Next let $\mathcal{G} = \Omega U_n$. The evaluation map + $(*)$

$$S^1 \times \mathcal{G} \xrightarrow{ev} U_n \xrightarrow{(*)} \Omega(Gr_n(\mathbb{C}^{2n}))$$

combine to a map

$$\mathcal{G} \times T^2 \longrightarrow Gr_n(\mathbb{C}^{2n}).$$

What's important is that g is an automorphism of E_0 over S^1 so we can use g as a clutching function to define a vector bundle E over $S^1 \times_M S^1$ which has a natural connection in the transverse direction. Using ⊛ puts a longitudinal connection on E.

In any case what we have is a nice family of Diracs on T^2 parametrized by G. So we can now play adiabatic limit games.

~~So what happ~~

Somehow I have to bring in the odd forms on $\Omega(G_n(\mathbb{C}^{2n}))$ which result from the superconnection formalism. Or another way to say it is to bring in the map defined by the Cayley transform to unitaries $\equiv -1 \mod \mathcal{K}$.

Notice that we can also construct, by clutching along the hemisphere, a family of vector bundles over S^2. These bundles are holomorphic and have metrics, but maybe there is a discontinuity in the connection.

It seems we could get insight into the non-adiabatic limit by looking carefully at this example.

In any case, the important point seems to be that however we do the joining we end up with a family of 2 diml Dirac operators parametrized by G. It's formally analogous to having a family of 1-diml Diracs parametrized by U_n and leads ~~to~~ problems concerning the comparison of the family forms and the invariant forms.

October 21, 1986

Yesterday I started looking at transgressing the index of the family of Diracs on M, which is a K-class on BG, G = group of gauge transformations, to obtain a K-class on G. The K-class on G is represented by a family of Diracs on $S^1 \times M$ parametrized by G. Even If I suppose that I could connect up the superconnection forms on BG and G, this leaves the problem of relating the superconnection forms for the family parametrized by G to the natural left invariant forms on G.

This problem appeared first in the following context. Let $G = LU_n$. Then we know how using the clutching function idea to construct a family of $\bar\partial$ operator on a Riemann surface param. by G. What have the superconn. forms associated to this family (and the extra connection data required) to do with the standard left-invariant forms on G?

A simpler version of this problem has been encountered before, namely, the natural family of Dirac's on S^1 parametrized by U_n. However we should go one step further in the sequence: Riemann surface, S^1, and look at a zero-diml. gauge group.

Thus I want to consider $G = U(V^0) \times U(V^1)$ on which I can put left invariant forms by choosing a odd contraction matrix $A = \begin{pmatrix} 0 & \alpha^* \\ \alpha & 0 \end{pmatrix}$. Recall that we dilate A to an odd involution F on a graded Hilbert space $H^0 \oplus H^1$ containing $V^0 \oplus V^1$. We then get an action of G on the symmetric space

$U(H^0, H^1)$ under $U(H^0) \times U(H^1)$ and we pull back the odd invariant forms on this symmetric space to G.

It is natural to consider then a fixed $H = H^0 \oplus H^1$, $F = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and to consider the Grassmannian of graded subspaces $eH = e^0H^0 \oplus e^1H^1$ isomorphic to $V^0 \oplus V^1$. This is the base of ~~the~~ the principal G-bundle whose points are the isometric embeddings of V in H. Call this Grassmannian $\mathcal{B}$; it is $Gr_k(H^0) \times Gr_\ell(H^1)$.

Now I think it ought to be the case that we have a canonical index map

$$\mathcal{B} \longrightarrow \text{Grass}(H)$$

associated to F. This is what I discovered in the spring.

Let's now consider the various ways we have encountered for constructing a map

$$Gr_k(H^0) \times Gr_\ell(H^1) \longrightarrow \text{Grass}(\tilde{H})$$

which represents the difference bundle. The simplest approach would be to assign to (e^0, e^1) the projector $e^0 \oplus (1-e^1)$. But recall that we are interested in the case where one is given a unitary mod compacts isomorphism η of H^0 and H^1 and that we want e^0 and e^1 to be compatible wrt η. When η has index zero it lifts to an isom. F of H^0 and H^1, so we are looking at the space of (e^0, e^1) which are congruent mod K. In this case we want a map to the restricted Grassmannian.

Let's set the goals a bit more carefully.

One thing we have that's fairly canonical is a family of left-invariant forms on $\mathcal{G}$ parametrized by the space of odd self-adjoint contractions on $V = V^0 \oplus V^1$. It's this family of forms which we would like to relate via transgression to some character form on $\mathcal{B}$. Note that we also have a canonical principal $\mathcal{G}$-bundle over $\mathcal{B}$ consisting of embeddings of V into H, and that once an A is given on H it contracts to one on V.

The natural conjecture is that starting from an A on H we could construct a transgression form on P which would restrict at each fibre embedding $\xi : \mathcal{G} \longrightarrow P$ to the left-invariant form on $\mathcal{G}$ associated to the embedding ξ. We would like everything to be natural, so there should be no loss of generality in taking A to be an involution.

So the problem is to start with an isom $H^0 \simeq H^1$ and to construct a transgression form over the bundle P of embeddings of V into H.

Now I recall and would like to review ways of going from $\mathcal{B}$ to a Grassmannian.

The first method takes the subspace eH contracts F to it to obtain $A = \begin{pmatrix} 0 & e^0e^1 \\ e^1e^0 & 0 \end{pmatrix}$, then one uses the "Cayley" map $\left(\sqrt{1-A^2}+iA\right)^2$ to get a unitary reversed by ε, i.e. a subspace of eH which is not graded. One extends this unitary by -1 on $(1-e)H$, which means one adds $(1-e')H'$ to the subspace.

Let's go over this more carefully. We have

$$\begin{array}{ccc} H^0 & = & H^1 \\ \cup & & \cup \\ e^0 H^0 & \xrightarrow{\lambda} & e^1 H^1 \end{array}$$

where λ is the contraction operator $e^1 e^0$. We have seen that we get better formulas with the skew-adjoint operator

$$A\varepsilon = \begin{pmatrix} 0 & -\lambda^* \\ \lambda & 0 \end{pmatrix}$$

and to use $(\sqrt{1-A^2} + A\varepsilon)^2$ instead of $(\sqrt{1-A^2} + iA)^2$. Recall how this works.

We want to spread $(-1, 1)$ to $\mathbb{R}$:

$$X = \frac{A\varepsilon}{\sqrt{1-A^2}}$$

The Cayley transform is

$$g = \frac{1+X}{1-X} = \frac{\sqrt{1-A^2} + A\varepsilon}{\sqrt{1-A^2} - A\varepsilon}$$

and the involution corresponding is

$$g\varepsilon = \left(\sqrt{1-A^2} + A\varepsilon\right) \varepsilon \left(\sqrt{1-A^2} + A\varepsilon\right)^{-1}$$

As $\sqrt{1-A^2} + A\varepsilon = \begin{pmatrix} \sqrt{1-\lambda^*\lambda} & -\lambda^* \\ \lambda & \sqrt{1-\lambda\lambda^*} \end{pmatrix}$ this involution is $+1$ on the subspace

$$\operatorname{Im} \begin{pmatrix} \sqrt{1-\lambda^*\lambda} \\ \lambda \end{pmatrix}$$

<u>October 22, 1986</u>

Last spring (April 86 pp. 355-360, 377-379) we constructed a map

$$Gr(V^0) \times Gr(V^1) \longrightarrow Gr(V^0 \oplus V^1)$$

starting from an odd F on V, i.e. an isom. $V^0 = V^1$. One of the defects in this construction was its asymmetry in (e^0, e^1).

It turns out that the map constructed yesterday is different, and apparently symmetrical.

Recall the previous construction (p. 356): ~~embed V^0 into $V^0 \oplus V^1$ as~~ ~~map V^0 to V^1 by~~ The map we are looking for is supposed to classify the bundle

$$(e^0, e^1) \longmapsto e^0V^0 \oplus (1-e^1)V^1$$

representing $[pr_1^*(S)] - [pr_2^*(Q)]$ over $Gr(V^0) \times Gr(V^1)$. So we want to embed this bundle into $\tilde{V}$. Moreover we don't want the obvious embedding into $V^0 \oplus V^1$ because we want a constant subspace where $e^0 = e^1$. Now we have

$$e^0V^0 \oplus (1-e^1)V^1 \xrightarrow{\begin{pmatrix} e^1e^0 & 0 \\ 0 & 1 \end{pmatrix}} \underset{\displaystyle V^1}{\underset{\|}{e^1V^1 \oplus (1-e^1)V^1}}$$

or better

$$e^0V^0 \oplus (1-e^1)V^1 \xrightarrow{(e^1e^0 \;\; 1)} V^1$$

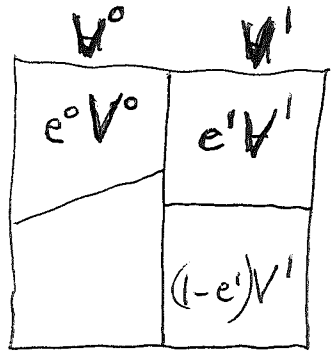
which is a contraction. So we get an embedding upon adding a mapping $e^0V^0 \xrightarrow{\beta} V^0$ such that $\begin{pmatrix} \text{[redacted]}\ \beta \\ e^1e^0 \end{pmatrix} : e^0V^0 \longrightarrow V^0 \oplus V^1$

is isometric.

Now last spring I chose β to be $(1-e^1_{\text{[redacted]}})\, e^0 : e^0V^0 \longrightarrow \text{[redacted]}\, V^1 \simeq V^0$. But a much better choice it seems is to take

$$\beta = \sqrt{1 - e^0e^1e^0}$$

In other words we have the picture



and the subspace we want is [redacted] the sum of $(1-e^1)V^1$ and a graph-like subspace of $e^0V^0 \oplus e^1V^1$. When we take the orthogonal complement we get something containing $(1-e^0)V^0$ and contained in $V^0 \oplus e^1V^1$. I think the symmetry is clear.

~~[illegible]~~

The above construction of taking the graph and extending by $(1-e^1)V^1$ is obvious related to earlier work with Grassmannians. Let's recall the space

$$\tilde{V}_s = \left\{ (\overset{s}{K}, \overset{n-s}{I}, \overset{n}{W}) \in Gr_s(V^0) \times Gr_s(\hat{V}^1) \times Gr_n(V) \;\middle|\; \begin{array}{ccccc} & & \overset{n}{V^1} & & \\ & \nearrow & & \searrow & \\ \overset{n-s}{I} & & & & \overset{s}{K} \oplus \overset{n}{V^1} \\ & \searrow & & \nearrow & \\ & & \overset{n}{W} & & \end{array} \right\}$$

which is a resolution of the descending submanifold from $F_s = Gr_s(V^0)\times Gr_s(\hat{V}^1)$.

We have
$$\tilde{Y}_s^- = \mathrm{Grass}_s(\mathcal{K} \oplus \tilde{V}^1/\mathcal{I})$$
over $Gr_s(V^0)\times Gr_{n-s}(V^1)$. Moreover the virtual bundle over this Grass bundle of interest is
$$[\mathcal{W}/\mathcal{I}] - [\tilde{V}^1/\mathcal{I}] = [\mathcal{W}] - [\tilde{V}^1].$$

The point is that we have seen how given any $A=\begin{pmatrix}0&\alpha^*\\ \alpha&0\end{pmatrix}$ an odd s.a. contraction, we can construct a section of the Grass bundle $\tilde{Y}_s^-$ over F_s, namely, we have an induced contraction $e^1\alpha e^0 : e^0V^0 \longrightarrow e^1V^1$, which one converts by the Cayley process to a subspace W of V containing $I=(1-e^1)V^1$.

Thus for each A we get a section of the Grass bundle. The induced bundle from $\mathcal{W}$ is isom. to $\{e^0V^0\}\oplus\{(1-e^1)V^1\}$.

Next we want to look at the transgression problem. We have a K-class on $B=F_s$ which goes to zero on $\mathcal{P}$

At this point I want to change notation and call the above space H instead of V, and to let $V^0, V^1 = \mathbb{C}^s$. $\mathcal{P}$ is then the space of embeddings of V into H preserving the grading.

Now when we work over P the bundles [redacted] $\{e^0 H^0\}$ and [redacted] $\{e^1 H^1\}$ become trivial. This fact has to be used to produce a null homotopy of the map from P to $Gr(H)$.

It is natural to first look at a single Grassmannian. Let $P = Inj(V, H)$ be the space of embeddings (isometric) of V into H, and $B = Gr_s(H)$, $s = \dim V$. Then is the map $P \to B$ null-homotopic? No: the Hopf map $S^3 \to S^2$. However it is true that the map $P \to B$ becomes null-homotopic if B is replaced by a larger Grassmannian.

[redacted] Thus we have

$$\begin{array}{ccccc} P & \subset & pt * P & \subset & Inj(V, V \oplus H) \\ \downarrow & & & & \downarrow \\ B = Gr_s(H) & & \subset & & Gr_s(V \oplus H) \end{array}$$

where the inclusion of the cone sends

$$(t, \xi) \quad \text{to} \quad [redacted] \begin{pmatrix} \sqrt{t} \\ \sqrt{1-t}\,\xi \end{pmatrix} : V \to V \oplus H$$

Note that this leads to the following deformation of connections on the trivial bundle $\tilde{V}$ over P:

$$\begin{pmatrix} \sqrt{t} \\ \sqrt{1-t}\,\xi \end{pmatrix}^* \cdot d \cdot \begin{pmatrix} \sqrt{t} \\ \sqrt{1-t}\,\xi \end{pmatrix} = t\,d + (1-t)\,\xi^* \cdot d \cdot \xi$$

$$= d + t\,\xi^*[d, \xi]$$

Let's make clear what this means. I am looking at the principal bundle

$$\begin{array}{ccccc} G & \longrightarrow & P & \longrightarrow & B \\ \parallel & & \parallel & & \parallel \\ U(V) & & \mathrm{Inj}(V,H) & & Gr_s(H) \end{array}$$

On the base is a canonical K-class represented by the subbundle; there are also differential forms representing its character obtained from the Grass. connection. I am trying to understand the transgression process for all of these.

A key fact is that the subbundle lifted to P is canonically trivial. This I can express concretely by explicitly giving a null-homotopy of the map $P \longrightarrow B \longrightarrow$ infinite Grassmannian. One gets an explicit map

$$\mathrm{Cone}(P) \longrightarrow Gr_s(V \oplus H)$$

In some sense this is the transgression cochain on the K-theory level, namely, it expresses the triviality of the class on the base pulled up to P. Now you restrict to a fibre getting a map from $\mathrm{Cone}(G)$ to $Gr_s(V \oplus H)$ which in fact factors through the map $\mathrm{Cone}(G) \longrightarrow \Sigma(G)$ to give a map

$$\Sigma(G) \longrightarrow Gr_s(V \oplus H)$$

Now we can follow this whole process on the level of differential forms and because of what we said about the connection forms we get the standard odd forms on G. At some point we integrate over the [redacted] time parameter and use that this converts a $2k$-form to a $(2k-1)$-form. The similar process in K-theory is more subtle and brings in the operator ∂_t.

Next I would like to try to do something similar [redacted] where V, H are graded. In this case $B = Gr_s(H^0) \times Gr_s(H^1)$ and the K-class of interest is the difference $[pr_1^*(S)] - [pr_2^*(S)]$.

Digression: Go back to the idea that the K class on $\Sigma(G)$ is to be integrated to get a K class on G, and that this is supposed to involve the Dirac or d_t on the line. Then we have [redacted] the persistent problem of the family of Diracs on the circle parametrized by $G = U_n$. The problem here is how to link superconnection forms for the family which depend on a (transverse) connection to the biinvariant forms.

First note that over $G \times S^1$ is a canonical vector bundle obtained by clutching. This bundle is canonically trivial over $G \times \{1\}$ and $\{1\} \times S^1$, hence it is a vector bundle over [redacted] $G \wedge S^1$. In fact where g has eigenvalues $= 1$, there is a canonical trivialization over the corresponding part of the bundle. In fact if we remove the -1 eigenvalue part we should get a trivialization of what's left.

Explain more carefully. The can vector bundle E over $G \times S^1$ comes with canonical partial connection in the S^1-direction. So we can look at those elements of E [redacted] which are periodic (or anti-periodic) with respect to the parallel transport. This gives a subset of E which if removed, the rest ~~[illegible]~~ sits inside the trivial bundle.

Does this mean there might be a natural character form on E concentrated over the set where the eigenvalue is 1.

October 23, 1986

In spite of Bismut's paper on transgression and cyclic cocycles for Dirac operators, I would still like to carry out my program of linking the process with transgression in K-theory.

Let us consider $G = U(V)$ ~~where~~ $V = \mathbb{C}^n$. Then we have a canonical map

$$\Sigma(G) \longrightarrow G_n(V \oplus V)$$

which gives a canonical K-class on this suspension. Now one has to somehow integrate over the circle or unit interval. This means introducing an operator family parametrized by G.

Yesterday I tried to do things over $[0,1]$. What we have over $\Sigma(G)$ is really a K-class with compact support on $(0,1) \times G$. I don't know how to formulate this very well, but maybe Kasparov theory gives some ideas.

We saw in the case of self-adjoint operators that a K^1-class with compact support of the open interval $(0,1)$ is a family of self-adjoint Freds A_t starting and ending with involutions. Similarly a K^0-class should be represented by a Fred operator family α_t starting and ending with isos.

The point somehow is that this bundle over $[0,1] \times G$ has given trivializations at the ends, and this must be taken into account in forming the cup product.

October 24, 1986

Question: What does the non adiabatic limit look like in a finite dimensional setting?

Let's consider $V = V^0 \oplus V^1 = \mathbb{C}^n \oplus \mathbb{C}^n$ (this is a graded Hilbert space) with the group $\mathcal{G} = U(V^0) \times U(V^1)$ of "gauge transformations" and the "Dirac" operator $\not{D} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ (which is invertible).

Now set $H = V \oplus V$. There is a natural embedding

$$\Sigma(\mathcal{G}) \subset Gr^{\varepsilon}_{(n,n)}(H) = Gr_n(H^0) \times Gr_n(H^1)$$

Also we equip H with $\not{D} \oplus \not{D}$ which we shall call F. Then for each $(g,t) \in \Sigma(\mathcal{G})$ we get an induced operator on

$$\Gamma_{(gt)} = \operatorname{Im}\left\{ \begin{pmatrix} \sqrt{1-t} \\ \sqrt{t}\, g \end{pmatrix} : V \longrightarrow H \right\}$$

namely

$$\begin{pmatrix} \sqrt{1-t} \\ \sqrt{t}\, g \end{pmatrix}^* F \begin{pmatrix} \sqrt{1-t} \\ \sqrt{t}\, g \end{pmatrix} = (1-t)\not{D} + t\, g^* \not{D} g$$

Thus we have described pretty clearly the graded Hilbert bundle over $\Sigma(\mathcal{G})$ with its odd self-adjoint endomorphism. Now the next stage is to bring in, or mix in, the operator ∂_t. This uses the connection in the bundle $\{\Gamma_{gt}\}$ in the t-direction. Also one has to put a metric on the t-axis.

The way I propose to work is to use the line $\mathbb{R}$ and to use L^2 boundary conditions. Thus we form the operator

$$\frac{1}{i}\partial_t + (1-\rho(t))\not{D} + \rho(t)\, g^{-1}\not{D} g$$

on [redacted] $L^2(\mathbb{R}, V)$, where $\rho(t)$ is a smooth approximation to the Heaviside function. Far out the self-adjoint operator $(1-\rho(t))\not{D} + \rho(t)\, g^{-1}\not{D} g$ is constant and invertible so the above operator with L^2 boundary conditions should be a self-adjoint Fredholm op.

Some remarks on this construction: In some sense we have succeeded in incorporating the fact that our K-class on $\Sigma(\mathcal{G})$ is really a family of compactly supported K-elements on $\mathbb{R}$ parametrized by $\mathcal{G}$. The other point I like is that the actual Hilbert bundle in which the operators work is a trivial Hilbert bundle.

I better check carefully that the operator is Fredholm. I am dealing with an ordinary diff'l equation on the line, so things should be straightforward.

Obviously we want to look at the operator

$$\frac{1}{i}\partial_t + \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

or

$$\partial_t + \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

on $L^2(\mathbb{R})^{\oplus 2}$. Spectrum: Look for solution $e^{i\omega t}v$ and get

$$\begin{pmatrix} \omega & +1 \\ 1 & \omega \end{pmatrix} v = \lambda v \qquad v \neq 0,$$

This implies

$$(\omega-\lambda)^2-1=0 \qquad \omega-\lambda=\pm 1$$

and so the spectrum is the whole line.

Probably I have the wrong formula for the Dirac operator and I should be using

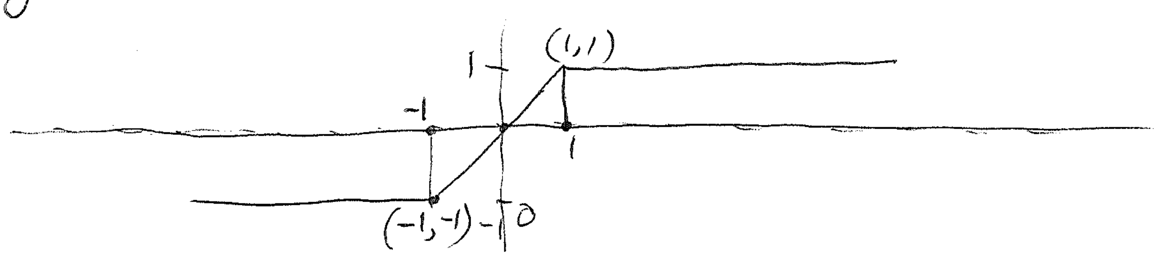
$$\begin{pmatrix} \frac{1}{i}\partial_t & 1 \\ 1 & -\frac{1}{i}\partial_t \end{pmatrix}$$

which ~~apple~~ has eigenfunctions $e^{i\omega t}v$ where

$$\begin{pmatrix} \omega & 1 \\ 1 & -\omega \end{pmatrix} v = \lambda v .$$

Thus $(\omega-\lambda)(-\omega-\lambda)=1$, $\lambda^2 = 1+\omega^2$, $\lambda = \pm\sqrt{1+\omega^2}$ and so the spectrum does indeed have a gap around zero. Thus the operator is Fredholm.

Unfortunately this Dirac operator has the continuous spectrum $(-\infty,-1] \cup [1,\infty)$, so it will be difficult to apply the superconnection stuff to it. But for K-theory purposes we have good control because we can chop off the spectrum, i.e. apply the function



to the operator or a smooth analogue. Then by Cayley one obtained a unitary congruent to -1 mod K.

I am thinking here of the variable Dirac operator where one shifts between $\not{D}$ and $g^{-1}\not{D} g$.

The problem is now to get control of the index of this family as g varies. So I have to look for bound states. We know something about the exponentially decaying solutions for $\begin{pmatrix} \frac{1}{i}\partial_t & 1 \\ 1 & -\frac{1}{i}\partial_t \end{pmatrix}$. These are of the form $e^{i\omega t}v$ where $\operatorname{Im}\omega > 0$ and where $\lambda = \sqrt{1+\omega^2}$ is real. ?

Recall that for $\lambda \notin \mathbb{R}$ there is an n-diml spase of eigenfunctions of $\begin{pmatrix} \frac{1}{i}\partial_t & 1 \\ 1 & -\frac{1}{i}\partial_t \end{pmatrix}$ decaying as $t \to \infty$, and another n-diml subspace decaying as $t \to -\infty$. Suppose $n=1$, then these are of the form $e^{i\omega t}v$ where

$$\begin{pmatrix} \omega & 1 \\ 1 & -\omega \end{pmatrix} v = \lambda v \qquad \Longrightarrow \qquad \lambda^2 = 1+\omega^2$$

If λ is real then precisely for $-1 < \lambda < 1$ one gets decaying solutions.

Notice that the operators

$$\circledast \qquad \begin{pmatrix} \frac{1}{i}\partial_t & g^{-1} \\ g & -\frac{1}{i}\partial_t \end{pmatrix}$$

where g is unitary are self adjoint and they are all conjugate, so to understand them it's enough to take $g = 1$. Also if one combines an ungraded involution A with ∂_t to get the graded Dirac

$$\begin{pmatrix} 0 & -\partial_t + A \\ \partial_t + A & 0 \end{pmatrix} = \gamma^2 \frac{1}{i}\partial_t + \gamma^1 A$$

then this is conjugate to

$$\varepsilon \frac{1}{i}\partial_t + \gamma^1 A = \begin{pmatrix} \frac{1}{i}\partial_t & A \\ A & -\frac{1}{i}\partial_t \end{pmatrix}$$

which is of the form $\circledast$. Thus we really only have to worry about the operator

$$\begin{pmatrix} \frac{1}{i}\partial_t & 1 \\ 1 & -\frac{1}{i}\partial_t \end{pmatrix}$$

Notice that for F any involution anti-commuting with ε

$$\left(\varepsilon \frac{1}{i}\partial_t + F\right)^2 = -\partial_t^2 + 1 \geq 1$$

and so the Dirac operator is Fredholm.

Now I want to analyze the operator

$$\varepsilon \frac{1}{i}\partial_t + F_t \qquad F_t = \begin{cases} F_0 & t < 0 \\ F_1 & t > 0 \end{cases}$$

October 25, 1986

On $L^2(\mathbb{R}, \mathbb{C}^n \oplus \mathbb{C}^n)$ we have the self-adjoint operator

$$\circledast \qquad \varepsilon \frac{1}{i}\partial_t + F_t \qquad\qquad F_t = \begin{pmatrix} 0 & g_-^{-1} \\ g_- & 0 \end{pmatrix} \quad t<0$$

$$= \begin{pmatrix} 0 & g_+^{-1} \\ g_+ & 0 \end{pmatrix} \quad t>0$$

having continuous spectrum $(-\infty,-1] \cup [1,\infty)$ and a finite number of "bound states" with eigenvalues $\lambda \in (-1,1)$. I want to take this operator on the space of bound states, call it A, + convert it to a unitary $-(A+i\sqrt{1-A^2})^2 = (\sqrt{1-A^2}+i\lambda)^2$

Given $\lambda \in (-1,1)$ we need to find the eigenfunctions for $\varepsilon\frac{1}{i}\partial_t + \begin{pmatrix} 0 & 1 \\ 1 & 0\end{pmatrix}$ which decay as $t \to \infty$. Such an eigenfunction is of the form $e^{i\omega t}v$ where

$$\begin{pmatrix} \omega & 1 \\ 1 & -\omega \end{pmatrix} v = \lambda v \qquad\qquad \operatorname{Im}\omega > 0.$$

$$\Rightarrow \quad -\omega^2+\lambda^2-1=0 \Rightarrow \omega = i\sqrt{1-\lambda^2}$$

$$\begin{pmatrix} \omega-\lambda & 1 \\ 1 & -\omega-\lambda \end{pmatrix}\begin{pmatrix} \lambda+\omega \\ 1 \end{pmatrix} = 0$$

So if we put

$$W_\lambda^\pm = \operatorname{Im}\begin{pmatrix} \lambda \pm i\sqrt{1-\lambda^2} \\ 1 \end{pmatrix}$$

Then the eigenfunctions with eigenvalue λ which decay at $\pm\infty$ are $e^{\mp\sqrt{1-\lambda^2}\,t}\,w$ with $w \in W_\lambda^\pm$.

Next we look at the original operator ⊛. The λ-eigenfunctions decaying as $t \to +\infty$ for the operator

$$\varepsilon \frac{1}{i}\partial_t + \begin{pmatrix} 0 & g_+^{-1} \\ g_+ & 0 \end{pmatrix} = \begin{pmatrix} g_+^{-1} & 0 \\ 0 & 1 \end{pmatrix}\left(\varepsilon\frac{1}{i}\partial_t + \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right)\begin{pmatrix} g_+ & 0 \\ 0 & 1 \end{pmatrix}$$

have their values at $t=0$ in the subspace

$$\begin{pmatrix} g_+ & 0 \\ 0 & 1 \end{pmatrix}^{-1} W_\lambda^+ = \operatorname{Im} \begin{pmatrix} g_+^{-1} & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \lambda + i\sqrt{1-\lambda^2} \\ 1 \end{pmatrix}.$$

The space of λ-eigenfunctions for ⊛ can thus be identified (via evaluation at $t=0$) with

$$\begin{pmatrix} g_- & 0 \\ 0 & 1 \end{pmatrix}^{-1} W_\lambda^- \cap \begin{pmatrix} g_+ & 0 \\ 0 & 1 \end{pmatrix}^{-1} W_\lambda^+ .$$

A typical element of this subspace is given by $a, a' \in \mathbb{C}^n \ni$

$$\begin{pmatrix} g_-^{-1} & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \lambda - i\sqrt{1-\lambda^2} \\ 1 \end{pmatrix} a = \begin{pmatrix} g_+^{-1} & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \lambda + i\sqrt{1-\lambda^2} \\ 1 \end{pmatrix} a' .$$

Thus $a = a'$ and

$$g_+ g_-^{-1} a = \frac{\lambda + i\sqrt{1-\lambda^2}}{\lambda - i\sqrt{1-\lambda^2}} a$$

$$= \left(\lambda + i\sqrt{1-\lambda^2}\right)^2 a$$

Thus we see that by taking a λ-eigenfunction for ⊛ into its value at $t=0$ and then taking the projection in V', we get an isomorphism of this eigenspace with the eigenspace for $g_+ g_-^{-1}$ for the eigenvalue $\left(\lambda + i\sqrt{1-\lambda^2}\right)^2$.

What this means is that if we take the

operator * apply the function

to it to obtain an A and then take the unitary $(A+i\sqrt{1-A^2})^2$, then we obtain the unitary $g_+ g_-^{-1}$ extended by -1.

October 26, 1986

Return to the model for the loop space of the Grassmannian. It consists of smooth $h : \mathbb{R} \to U(2n)$ such that

$$h(0) = 1, \quad h(t+1) = -h(t), \quad \varepsilon h(t) \varepsilon = h(-t).$$

To h we can assign a Dirac on $H = L^2(S^1, \mathbb{C}^n \oplus \mathbb{C}^n)$ namely

$$h \cdot \partial_t \cdot h^{-1}$$

On H we have the involution σ:

$$(\sigma f)(t) = \varepsilon f(-t)$$

and the Dirac anti-commutes with σ. As the Dirac operator is invertible (monodromy is $h(1) = -1$ which has no eigenvalues $= 1$), by taking its phase we obtain an odd involution on H wrt σ, i.e. a unitary operator from H^+ to H^-.

Let us use h as a sort of gauge transformation

$$\underset{\text{anti-periodic}}{f(t)} \longmapsto \underset{\text{periodic}}{h(t) f(t)}$$

to identify the operator $h \cdot \partial_t \cdot h^{-1}$ (together with periodic boundary conditions) with the operator ∂_t and anti periodic boundary conditions. This isomorphism commutes with σ

$$\sigma(hf)(t) = \varepsilon h(-t) f(-t) = h(t) \varepsilon f(-t)$$
$$= (h \sigma f)(t).$$

This means that our operator splits into the direct sum of the operators ∂_t, anti periodic b.c., on $L^2(S, \mathbb{C}^n) \oplus L^2(S, \mathbb{C}^n)$.

At this point I want to bring in the idea that h determines a path in the Grassmannian $Gr_n(\mathbb{C}^{2n})$:

$$F_t = h(t)\,\varepsilon\, h(t)^{-1}$$

which goes from ε to $-\varepsilon$ as $0 \le t \le \frac{1}{2}$, which is periodic $F_{t+1} = F_t$ and $\varepsilon F_t \varepsilon = F_{-t}$.

Example:

$$h(t) = e^{t\pi\begin{pmatrix} 0 & -g^{-1} \\ g & 0 \end{pmatrix}} = \begin{pmatrix} \cos \pi t & -(\sin\pi t)g^{-1} \\ (\sin\pi t)g & \cos\pi t \end{pmatrix}$$

Here F_t is the involution associated to the subspace

$$\operatorname{Im}\begin{pmatrix} \cos\pi t \\ (\sin\pi t)g \end{pmatrix} = \text{graph of } \tan(\pi t)\, g$$

Moreover for this example $h(t)$ is the parallel transport in the direct sum of $S \oplus Q = \widetilde{\mathbb{C}}^{2n}$ associated to the two Grassmannian connections.

So in this example H, $h\cdot\partial_t h^{-1}$ + period. b.c. is simply the direct sum of the Dirac ops. on the canonical sub and quotient bundles but over the whole loop which starts at ε goes to $-\varepsilon$ and returns to ε.

In the above we've seen that the Dirac operator on the subbundle is isom. to ∂_t on $L^2(S,\mathbb{C}^n)$ with anti-p.b.c. and that this space is stable under σ and ∂_t is odd relative to σ. Thus we have attached to an h (which is roughly equivalent to a path from ε to $-\varepsilon$ in $Gr_n(\mathbb{C}^{2n})$) an invertible Dirac operator on Hilbert space which is roughly double that of the Hilbert space of sections of the subbundle over the path from ε to $-\varepsilon$. I have to recall that I already have a Dirac operator attached to such a path, and that its spectrum depends on the holonomy along the path together with the natural identification of the ends given by $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

What is striking is the idea of doubling the non-invertible Dirac so as to get an invertible odd operator. Another example is to send A to

$$A\gamma^1 + \gamma^2 = \begin{pmatrix} 0 & A-i \\ A+i & 0 \end{pmatrix};$$

this is the Cayley transform essentially

October 27, 1986

It seems desirable to understand well the operator on $L^2(\mathbb{R}, H)^{\oplus 2}$ given by

$$\begin{pmatrix} 0 & -\partial_t + F_t \\ \partial_t + F_t & 0 \end{pmatrix} = \gamma^2 \frac{1}{i}\partial_t + \gamma^1 F_t$$

where $F_t = \begin{cases} F_0 & t<0 \\ F_1 & t>0 \end{cases}$ are involutions on H.

This operator anti-commutes with ε. It has cont. spectrum $(-\infty,-1] \cup [1,\infty)$, and we obtain a unitary by collapsing it to an A, $-1 \leq A \leq 1$, and applying the transform $(iA + \sqrt{1-A^2})^2$. This unitary is reversed by ε hence correspond to a subspace of $H^{\oplus 2}$ depending on the two involutions F_0, F_1.

We first need the eigenfunctions for $\gamma^2 \frac{1}{i}\partial_t + \gamma^1 F$ which decay as $t \to +\infty$.

Let's return to p. 239-242 where we considered

$$\varepsilon \frac{1}{i}\partial_t + F_t \qquad F_t = \begin{cases} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} & t<0 \\ \begin{pmatrix} 0 & g^{-1} \\ g & 0 \end{pmatrix} & t>0 \end{cases}$$

We want the space of bound states for this operator i.e. eigenfunctions with eigenvalue $\lambda \in (-1,1)$ of the form

$$e^{-\sqrt{1-\lambda^2}\,t}\begin{pmatrix} g & 0 \\ 0 & 1 \end{pmatrix}^{-1}\begin{pmatrix} \lambda + i\sqrt{1-\lambda^2} \\ 1 \end{pmatrix} v \qquad t>0$$

$$e^{\sqrt{1-\lambda^2}\,t}\begin{pmatrix} \lambda - i\sqrt{1-\lambda^2} \\ 1 \end{pmatrix} v' \qquad t<0$$

Continuity requires $v = v'$

$$gv = (\lambda + i\sqrt{1-\lambda^2})^2 v = -(\sqrt{1-\lambda^2} + i\lambda)^2 v$$

Let us consider the mapping from the space of bound states to $\mathbb{C}^n$ given by

$$\psi(t) = \begin{pmatrix} \psi_0(t) \\ \psi_1(t) \end{pmatrix} \longmapsto \psi_1(0)$$

We have seen this map carries the eigenspace H_λ of the Dirac operator ~~isomorphism iso~~ isomorphically onto the eigenspace of $-g$ with the eigenvalue $(\sqrt{1-\lambda^2}+i\lambda)^2$. Since $\lambda \longmapsto (\sqrt{1-\lambda^2}+i\lambda)^2$ is a diffeo of $(-1,1)$ with $S^1 - \{-1\}$, we see that the space of bound states with the operator $(A+i\sqrt{1-A^2})^2$, A being the restriction of the Dirac operator, is isomorphic to the subspace of $\mathbb{C}^n$ where $g \neq 1$ with the operator $-g$. This isomorphism is isometric up to a factor of $\sqrt{2}$, because $|\lambda + i\sqrt{1-\lambda^2}| = 1$ and the eigenspaces are orthogonal in both spaces.

The question then arises of how to start with g on $\mathbb{C}^n$ and find the space of bound states for the operator, say their values at $t=0$. Not really important perhaps, since in the end we only expect to see $-g$ extended by -1 under some embedding.

Now to get some insight let us suppose that $V = V^0 \oplus V^1$ and we don't identify V^0 and V^1. Again on $L^2(\mathbb{R}, V)$ we consider

$$\varepsilon \frac{1}{i}\partial_t + F_t \qquad F_t = \begin{cases} \begin{pmatrix} 0 & g_0^{-1} \\ g_0 & 0 \end{pmatrix} & t<0 \\ \begin{pmatrix} 0 & g_1^{-1} \\ g_1 & 0 \end{pmatrix} & t>0 \end{cases}$$

Again the bound state subspace

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Consider again

$$\varepsilon \frac{1}{i}\partial_t + F_t \qquad F_t = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad t<0$$
$$= \begin{pmatrix} 0 & g^{-1} \\ g & 0 \end{pmatrix} \quad t>0$$

on $L^2(\mathbb{R}, V\oplus V)$. We define a unitary operator on $V\oplus V$ as follows. Let $K = \oplus K_\lambda$ $\lambda\in(-1,1)$ be the space of bound states for the Dirac operator. We know that by taking initial values one has an embedding of K into $V\oplus V$ which is not necessarily isometric. In more detail we have

$$K_\lambda \xrightarrow{\sim} W_\lambda^- \cap \begin{pmatrix} 1 & 0 \\ 0 & g \end{pmatrix} W_\lambda^+$$

$$W_\lambda^- = \operatorname{Im}\begin{pmatrix} \lambda - i\sqrt{1-\lambda^2} \\ 1 \end{pmatrix} = \operatorname{Im}\begin{pmatrix} 1 \\ \lambda + i\sqrt{1-\lambda^2} \end{pmatrix}$$

$$W_\lambda^+ = \operatorname{Im}\begin{pmatrix} \lambda + i\sqrt{1-\lambda^2} \\ 1 \end{pmatrix} = \operatorname{Im}\begin{pmatrix} 1 \\ \lambda - i\sqrt{1-\lambda^2} \end{pmatrix}$$

so that

$$K_\lambda \xrightarrow{\sim} W_\lambda^- \cap \begin{pmatrix} 1 & 0 \\ 0 & g \end{pmatrix} W_\lambda^+$$
$$= \left\{ \begin{pmatrix} 1 \\ \lambda + i\sqrt{1-\lambda^2} \end{pmatrix} v \;\middle|\; gv = \left(\lambda + i\sqrt{1-\lambda^2}\right)^2 v \right\}$$

For different λ the latter spaces are orthogonal, hence one gets an embedding of K into $V\oplus V$.

Next let A be the Dirac operator restricted to the bound state space K; it has eigenvalue λ on K_λ. We apply the transform $(\sqrt{1-A^2} + iA)^2$ to obtain a unitary on K which we can then transport via the above embedding to the image of K in $V\oplus V$ and then extend by -1. This defines the unitary operator on $V \oplus V$. (It's unitary because the eigenspaces are orthogonal.)

I now want to describe this operator directly starting from $-g$. The operator has the eigenvalue $(\sqrt{1-\lambda^2}+i\lambda)^2$ on the space

$$W_\lambda^- \cap \begin{pmatrix} 1 & 0 \\ 0 & g \end{pmatrix} W_\lambda^+ = \left\{ \begin{pmatrix} \sqrt{1-\lambda^2}+i\lambda \\ i \end{pmatrix} v \;\middle|\; (-g)v = (\sqrt{1-\lambda^2}+i\lambda)^2 v \right\}$$

So we see that we take the operator $-g$ on V remove the -1 eigenspace, and then extract a square root so that the spectrum lies on the RH of S^1, then we use the graph embedding essentially of $(-g)^{1/2}$

Thus you take the embedding

$$V \xrightarrow{\frac{1}{\sqrt{2}}\begin{pmatrix} \sqrt{-g} \\ i \end{pmatrix}} V \oplus V$$

which is well-defined on the complement of the -1 eigenspace of $-g$, and you push g forward and extend by -1.

At this point I have several examples and I would like to analyze in general terms what happens.

Suppose to fix the ideas that $\mathcal{G}$ is a group of gauge transformations acting on $H = L^2(M, S \otimes E)$, M Riemannian odd diml. Over $\Sigma\mathcal{G}$ I get a canonical K-class which is represented by a Hilbert bundle $\mathcal{H}$ with $\eta_{\mathcal{H}}$. Then by taking a Kasparov cup product with the canonical K-homology class on $\mathbb{R}$ we obtain a K-class on $\mathcal{G}$ which is represented by a graded Hilbert bundle with a graded η.

To make this concrete I guess I want to

take the opposite parity example, where M is a point and E is a graded vector space $V = V^0 \oplus V^1$ equipped with $F_0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and $G = U(V^0) \times U(V^1)$. Then after doing the non-adiabatic approximation I find the following representative for the (odd) K-class on G. We have the Hilbert space $H = L^2(\mathbb{R}, V^0 \oplus V^1)$ with the Dirac operator

$$L^g = \varepsilon \frac{1}{i} \partial_t + F_t^g$$

where
$$F_t^g = F_0 \qquad \text{for } t < 0$$
$$= g F_0 g^{-1} \qquad \text{for } t > 0$$

where $g \in G$. In this case H is a representation of G and we are just taking the orbit of a fixed Dirac operator on H to get the family. NO see below.

This seems to be important - to find a representation H of G and a fixed operator L_0 such that the family representing the K-class is the family $g \mapsto g L_0 g^{-1}$ in H.

In any case we still somehow have to get from this family to the obvious family represented by the representation of G on V and the operator F_0. But because of the G-action all we have to do is to find an equivariant embedding $V \hookrightarrow H$ and a link between F_0 and L_0.

However it is clear we have made a mistake as L^g is not equal to $g L_0 g^{-1}$; the spectrum is wrong as L^g has bound states. Thus we still haven't reached the desired setup.

Here's a nice idea:

Let's recall that we have a K-class on G which is represented by a Hilbert bundle together with an involution modulo compacts. The problem is to show how to link this to a suitable pair consisting of a representation of G on a Hilbert space H together with an involution F_0 on H which is preserved by G mod compacts.

I want to be more specific. The K-class on G can be realized by a family of self-adjoint Fred. operators depending on $g \in G$. By means of a Cayley type transform one converts this to a family of unitaries $\equiv -1$ mod compacts, allowing eigenvalue analysis. Actually the eigenvalues make sense for the original self-adjoint Fredholm operators.

Notice that if we associate to a representation of G on H and an F_0 on H, the family gF_0g^{-1}, $g \in G$, then there are no zero modes to consider, and no variation of eigenvalues. However if we look at the family $F_0 g F_0 g^{-1}$, then this is a unitary congruent to 1 mod $\mathcal{K}$, and so we do have eigenvalues.

The idea is that I have to think of $F_0 g F_0 g^{-1}$ as being linked to the Cayley transform of the operator associated to g.

Let us now try to carry out this idea in some examples. Let's first look at the operator $\partial_t + F_t$ on $L^2(\mathbb{R}, H)$ direct sum with its adjoint, where

$$F_t = \begin{cases} F_0 & t<0 \\ gF_0g^{-1} & t>0 \end{cases}$$

This is a graded self-adjoint operator on $L^2(\mathbb{R}, H)^{\oplus 2}$. The kernel of $\partial_t + F_t$ can be identified with

$$H^- \cap gH^+$$

where $H^{\pm}$ are the eigenspaces of F_0. The kernel of $-\partial_t + F_t$ can be identified with

$$H^+ \cap gH^-.$$

Actually we are comparing two involutions F_0 and $F_1 = gF_0g^{-1}$. $H^- \cap H_1^+$ is where $F_0 = -1$, $F_1 = +1$ while $H^+ \cap H_1^-$ is where $F_0 = +1$ and $F_1 = -1$. Thus we are looking at the -1 eigenspace of F_0F_1 with the $-F_0$ grading.

Notice that $F_0F_1 = F_0gF_0g^{-1}$ is a unitary which is reversed by F_0, hence its spectrum will be symmetric under $\zeta \mapsto \zeta^{-1}$. Similarly the Dirac operator $\begin{pmatrix} 0 & -\partial_t + F_t \\ \partial_t + F_t & 0 \end{pmatrix}$ is odd relative to ε, so its spectrum is symmetric under $\lambda \mapsto -\lambda$. With a little more work one ought to be able to link the type of Cayley transform considered above for this Dirac operator to the unitary F_0F_1.

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Let's find the bound states for $\begin{pmatrix} 0 & -\partial_t + F \\ \partial_t + F & 0 \end{pmatrix} = \gamma^2 \frac{1}{i}\partial + \gamma^1 F$. If $\psi(t) = e^{i\omega t}\alpha$ is an eigenfn. with eigenvalue λ, then

$$(\gamma^2\omega + \gamma^1 F)\alpha = \lambda\alpha$$

and as $(\gamma^2\omega + \gamma^1 F)^2 = \omega^2 + 1$ we have $\lambda^2 = \omega^2 + 1$, $\omega = \pm\sqrt{1-\lambda^2}$. For $\lambda \neq 0$, $\frac{1}{\lambda}(\gamma^2\omega + \gamma^1 F)$ is an involution odd rel ε so is of the form $\begin{pmatrix} 0 & g^{-1} \\ g & 0 \end{pmatrix}$ and its $+1$ eigenspace is $\operatorname{Im}\begin{pmatrix} 1 \\ g \end{pmatrix}$. Thus

$$\alpha = \begin{pmatrix} \lambda \\ i\omega + F \end{pmatrix} v$$

and from this we see that an eigenfn. for $\lambda \neq 0$ is of the form

$$e^{\sqrt{1-\lambda^2}\,t} \begin{pmatrix} \lambda \\ \sqrt{1-\lambda^2} + F_0 \end{pmatrix} v \qquad t < 0$$

$$e^{-\sqrt{1-\lambda^2}\,t} \begin{pmatrix} \lambda \\ -\sqrt{1-\lambda^2} + F_1 \end{pmatrix} v' \qquad t > 0$$

where continuity requires $v = v'$ and

$$\left(\sqrt{1-\lambda^2} + F_0\right) v = \left(-\sqrt{1-\lambda^2} + F_1\right) v$$

Now we analyze this equation using the fact that F_0, F_1 generate a dihedral group. So we can take an irreducible repn. and suppose

$$F_0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad F_1 = \begin{pmatrix} 0 & e^{-i\theta} \\ e^{i\theta} & 0 \end{pmatrix}$$

Then

$$(F_1 - F_0)v = \begin{pmatrix} 0 & e^{-i\theta} - 1 \\ e^{i\theta} - 1 & 0 \end{pmatrix} v = 2\sqrt{1-\lambda^2}\, v$$

$$\Rightarrow \quad (e^{i\theta} - 1)(e^{-i\theta} - 1) = 4\sin^2\frac{\theta}{2} = \left(2\sqrt{1-\lambda^2}\right)^2 \quad \Rightarrow \lambda = \cos\frac{\theta}{2}$$

$$\Rightarrow \quad \left(\lambda + i\sqrt{1-\lambda^2}\right)^2 = e^{i\theta}.$$

~~I think the rest is more or less the same.~~

Let F_0, F_1 be two involutions on V and consider the operator

$$\gamma^2 \frac{1}{i}\partial_t + \gamma^1 F_t \qquad \text{on } L^2(\mathbb{R}, V\oplus V)$$

where $F_t = F_0$ for $t<0$, and $F_t = F_1$ for $t>0$. This Dirac operator is odd relative to ε. Let Z be the space of bound states for this operator; taking the transform $(\sqrt{1-A^2}+iA)^2$ of the operator on Z given a unitary on Z without the eigenvalue -1. We have an injection of Z into $V\oplus V$ by taking initial values. This unitary on Z can be extended to $V\oplus V$ by defining it to be -1 on the complement. In this way one obtains a unitary on $V\oplus V$ reversed by ε, hence a subspace of $V\oplus V$.

Thus we have a map from pairs of involutions on V to subspaces of $V\oplus V$. The problem is to compute what it is.

Given $\lambda \in (-1,1)$ we have identified the ~~space of~~ initial values of eigenfunctions for the Dirac operator with eigenvalue λ with vectors

$$\begin{pmatrix} \lambda \\ \sqrt{1-\lambda^2}+F_0 \end{pmatrix} v$$

where v satisfies

$$\left(\sqrt{1-\lambda^2}+F_0\right)v = \left(-\sqrt{1-\lambda^2}+F_1\right)v$$

i.e.

$$(F_1 - F_0)v = 2\sqrt{1-\lambda^2}\, v$$

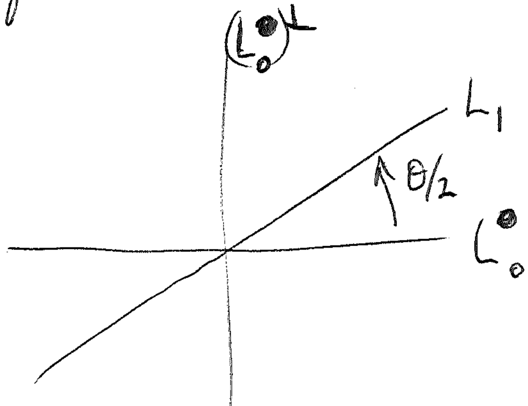
$$2\sqrt{}\, F_0 v = (F_0F_1 - 1)v$$

$$2\sqrt{}\, F_1 v = (1 - F_1F_0)v$$

$$\underbrace{2\sqrt{\quad}\;\underbrace{(F_0-F_1)v}_{-2\sqrt{\quad}\;v}}_{-4(1-\lambda^2)v} = (F_0F_1+F_1F_0-2)v$$

Thus $\quad \frac{1}{2}(F_0F_1+F_1F_0)v = (2\lambda^2-1)v$

Now let us suppose V is an irreducible repn. of the dihedral group generated by F_0 and F_1. In other words V is 2 dimensional, F_0 and F_1 correspond to lines ~~xxxx~~ L_0, L_1 and the only invariant of interest is the angle L_1 makes with L_0



$0 < \theta < \pi$

In this picture F_1F_0 is rotation through θ, and thus $\frac{1}{2}(F_0F_1+F_1F_0) = \cos\theta$. Note that $\frac{1}{2}(F_0F_1+F_1F_0)$ commutes with F_0, F_1 and so is a scalar in an irreducible representation. So we have

$$2\lambda^2-1 = \cos\theta = 2\cos^2\left(\frac{\theta}{2}\right)-1$$

and hence $\quad \lambda = \pm\cos\left(\frac{\theta}{2}\right) \qquad$ note $\cos\left(\frac{\theta}{2}\right) \in (0,1)$

Suppose that θ is given we get two values of $\lambda = \pm\cos\left(\frac{\theta}{2}\right)$ and $\sqrt{1-\lambda^2} = \sin\left(\frac{\theta}{2}\right)$. There is one possible v up to scalar factors satisfying

$$(F_1-F_0)v = \left(2\sin\frac{\theta}{2}\right)v$$

so the space of bound states is 2 diml spanned by

$$\circledast \quad \begin{pmatrix} \pm \cos\frac{\theta}{2} \\ \sin\frac{\theta}{2} + F_0 \end{pmatrix} v$$

On the two lines spanned by these vectors the Cayley transform has the eigenvalues respectively

$$\left(\sin\frac{\theta}{2} \pm i\cos\frac{\theta}{2}\right)^2 = -e^{\mp i\theta}$$

(Check: Let $\theta \downarrow 0$ then the operator becomes -1, whereas as $\theta \uparrow \pi$ the operator becomes $+1$.

Lets now take $F_0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $F_1 = \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}$; note the $F_1 = 1$ eigenspace is spanned by $\begin{pmatrix} \cos\frac{\theta}{2} \\ \sin\frac{\theta}{2} \end{pmatrix}$. The projection from this line L_1 to the line L_0 where $F_0 = 1$ is essentially multiplication by $\cos\frac{\theta}{2}$.

Calculation shows that

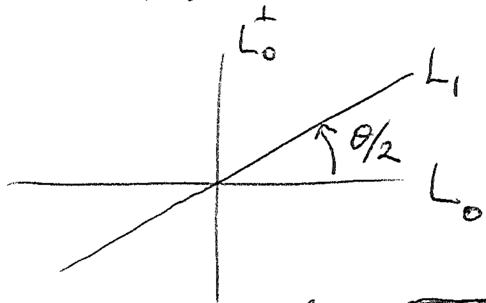
$$v = \begin{pmatrix} \cos\frac{\theta}{2} \\ \sin\frac{\theta}{2} + 1 \end{pmatrix}$$

We know the vectors $\circledast$ span the space $W \subset V \oplus V$ where the Cayley transform is different from -1. Notice that this contains the line spanned by $\begin{pmatrix} v \\ 0 \end{pmatrix}$ and also the line spanned by $\begin{pmatrix} 0 \\ (\sin\frac{\theta}{2} + F_0) v \end{pmatrix}$.

$$\left(\sin\frac{\theta}{2} + F_0\right) v = \begin{pmatrix} \sin\frac{\theta}{2} + 1 & 0 \\ 0 & \sin\frac{\theta}{2} - 1 \end{pmatrix} \begin{pmatrix} \cos\frac{\theta}{2} \\ \sin\frac{\theta}{2} + 1 \end{pmatrix} = \begin{pmatrix} (\sin\frac{\theta}{2} + 1)\cos\frac{\theta}{2} \\ -\cos^2\frac{\theta}{2} \end{pmatrix}$$

$$\sim \begin{pmatrix} \sin\left(\frac{\theta}{2}\right) + 1 \\ -\cos\frac{\theta}{2} \end{pmatrix} \quad \perp \quad v$$

These calculations are a bit confusing however certain general features are worth pointing out. [redacted] It is enough to worry about the case where V is two-dimensional and where the two involutions F_0, F_1 correspond to lines L_0, L_1 in V. Let $\theta/2$ be the angle



[redacted] it is the unique angle [redacted] in $[0, \pi/2]$ such that the projection of [redacted] L_1 on L_0 has norm $\cos\frac{\theta}{2}$. Suppose $0 < \frac{\theta}{2} < \pi/2$. [redacted] We get the initial values of bound states; [redacted] this is a subspace $W \subset V \oplus V$ which is stable under ε and hence is of the form $L \oplus L'$. Now within W we use operator to construct a unitary & reversed by ε, which is equivalent to a subspace of $L \oplus L'$. Then we extend the unitary to $V \oplus V$ by letting it be -1 on $L^\perp \oplus (L')^\perp$, which means we add to the subspace $0 \oplus (L')^\perp$.

Now in the calculations we found that W is $\mathbb{C}v \oplus (\mathbb{C}v)^\perp$ where

$$v = \begin{pmatrix} \cos\frac{\theta}{2} \\ \sin\frac{\theta}{2} + 1 \end{pmatrix} \quad \text{if} \quad F_0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad F_1 = \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}$$

v is the result of adding the unit vector $\begin{pmatrix} \cos\frac{\theta}{2} \\ \sin\frac{\theta}{2} \end{pmatrix}$ in L_1 to the unit vector $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ in $L_0^\perp$ so it is half way between L_1 and $L_0^\perp$. This is well defined - think of the half circle through the antipodal points corresp. to $L_0, L_0^\perp$ which passes through L_1. It's defined as long as $L_1 \neq L_0$. One can also see this from the equation

$$(F_1 - F_0)v = 2\sin\left(\frac{\theta}{2}\right) v$$

which specifies the line spanned by v as long as $F_1 \neq F_0$.

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There are two questions to consider now that you have made a bit of progress on transgression using the non-adiabatic idea: First is whether the loop group model for $\Omega(Gr_n \mathbb{C}^{2n}; \varepsilon, -\varepsilon)$ is going to useful. Secondly, can you now say something about the S operator and periodicity?

Let's review the loop group setup. Let $\mathcal{a}^\sigma_{-1}$ be the paths $h: \mathbb{R} \longrightarrow U_{2n}$ satisfying $h(0) = 1$, $h(t+1) = -h(t)$, $\varepsilon h(t) \varepsilon = h(-t)$. To such a path we associate the Dirac operator $h\, \partial_t\, h^{-1}$ on $L^2(S, \mathbb{C}^{2n})$ which is odd rel. to $\sigma: f(t) \longmapsto \varepsilon f(-t)$. This operator is invertible as the monodromy is -1 so taking its phase gives an involution F_h odd relative to σ, i.e. a unitary operator $H^+ \xrightarrow{\sim} H^-$. All these F's are congruent modulo compacts, so if we pick a basepoint F_0, e.g. corresponding to

$$h_0(t) = e^{\pi t \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}} = \begin{pmatrix} \cos \pi t & -\sin \pi t \\ \sin \pi t & \cos \pi t \end{pmatrix}$$

then $h \longrightarrow F_0^{-1} F_h$ is a map from $\mathcal{a}^\sigma_{-1}$ to unitaries congruent to 1 modulo compacts.

Notice that $\mathcal{G}^\sigma = \{ g: \mathbb{R}/\mathbb{Z} \to U_{2n} \mid \varepsilon g(t) \varepsilon = g(-t) \}$ acts on $\mathcal{a}^\sigma_{-1}$ by $(g * h)(t) = g(t) h(t) g(0)^{-1}$. The action is transitive.

Now we have a map

$$\begin{aligned} \mathcal{a}^\sigma_{-1} &\longrightarrow \Omega(Gr_n(\mathbb{C}^{2n}), \varepsilon, -\varepsilon) \\ h &\longmapsto h(t)\, \varepsilon\, h(t)^{-1} \qquad 0 \le t \le \tfrac{1}{2} \end{aligned}$$

Over this ~~loop~~ path space we have a family of Dirac operators on the circle, namely you take the Dirac operator over the path on sections of the subbundle together with periodic boundary conditions using the identification $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ of the subbundle at the ends of the path. This is the same as pulling back the canonical family parametrized by U_n for the monodromy map.

A basic problem is whether there is a natural link between these two families. It is not so far-fetched because we can use h as a gauge transformation

$$f(t) \longrightarrow h(t) f(t)$$

to set up an isomorphism of the bundle over S^1 whose sections are the anti-periodic fns. with values in $\mathbb{C}^n$ with the direct sum of the subbundle and quotient bundle over the loop $t \longrightarrow h(t)\varepsilon h(t)^{-1}$, $0 \le t \le 1$. $h(t)$ isn't quite the parallel transport for the direct sum of the Grass connections, but it should be for the paths $g h_0(t) g^{-1}$, $g \in U_n \times U_n$.

In any case the problem is really to relate the Dirac operator on the path from ε to $-\varepsilon$ which is not invertible to the operator one gets by looking at the loop going from ε to ε.

?

So it seems that I don't get anywhere using this family of invertible Dirac operators on the circle parametrized by $\tilde{\mathcal{O}}_{-1}$. This means maybe that I ought to go back to the family of Diracs on S^1 param. by U_n.

Ideas: Kasparov's stability thm. might be used to set up the equivalence

One can produce a family depending on elements of $U(V^0)\times U(V^1)$ and an $F_0 : V^0 \overset{\sim}{\rightleftarrows} V^1$, namely take a ~~Dirac operator~~ connection on $\tilde{V}_0$ over I whose holonomy is g^0 and piece it together with the connection on $\tilde{V}_1$ over I with holonomy g^1 using F_0 to go from V^0 to V^1 and back again.