In 2-dimensional Euclidean space \( \mathbb{R}^2 \) with volume \( \text{d}x \text{d}y \), I have seen that a "pure" gauge field \( A_x \text{d}x + A_y \text{d}y \) satisfies the field equations when \( F = \partial_x A_y - \partial_y A_x + [A_x, A_y] \) is fixed under parallel translation. This means that the gauge field can be gauge transformed to a field where \( F \) is constant and hence the action won't be finite unless \( F = 0 \), where there is no interesting pure gauge theory.

To get something interesting one has to add a "matter" field which will give sources for the gauge field. Coleman considers the "abelian Higgs model" which consists of a \( U(1) \) gauge field \( A \) and a complex scalar field \( \phi \). One can think of \( A \) as the EM field and \( \phi \) as a charged meson field. The meson mass perhaps arises via the Higgs mechanism.

Work in the Euclidean setup with coordinates \( x, y \) on \( \mathbb{R}^2 \). Then \( A = A_x \text{d}x + A_y \text{d}y \) is a purely imaginary 1-form. Its curvature is

\[
\text{d}A = \frac{(\partial_x A_y - \partial_y A_x) \text{d}x \text{d}y}{F}
\]

and the part of the action due to \( A \) is proportional to

\[
\int |F|^2 \quad \int \quad \int \text{d}x \text{d}y
\]

If \( A \) is varied by \( \delta A \), then

\[
\delta \int |F|^2 = \int \left( \partial_x \delta A_y - \partial_y \delta A_x \right) F + c.c.
\]

\[
= \int \delta \bar{A}_y \left( -\partial_x F \right) + \delta \bar{A}_x \left( \partial_y F \right) + c.c.
\]

\[
= 2 \int \delta A_y \left( \partial_x F \right) - \delta A_x \left( \partial_y F \right)
\]
The part of the action connecting $A, \psi$ is
\[ \int |D\psi|^2 = \sum_I \left| (i\gamma_{\mu} A_\mu + \mu) \psi \right|^2 \]

Its variation w.r.t $A$ is
\[ \delta \int |D\psi|^2 = \int \delta A_\mu \overline{\psi} \cdot D_\mu \psi + \text{c.c.} \]
\[ = \int \delta A_\mu \left( -\overline{\psi} D_\mu \psi + \psi D_\mu \overline{\psi} \right) \]

Its variation w.r.t $\psi$ is
\[ \int \overline{\psi} \delta \psi \overline{D_\mu \psi} + \text{c.c.} \]
\[ = \int \overline{\delta \psi} \left( -D_\mu D_\mu \psi \right) + \text{c.c.} \]

If we combine: \( \frac{1}{2} \int |E|^2 + \int |D\psi|^2 \) then the vanishing of the variation w.r.t $A$ leads to the equation
\[
\left( \partial_\mu F, -\partial_\nu F \right) = \overline{\psi} D_\mu \psi - \psi \overline{D_\mu \overline{\psi}}
\]

This is a linear equation for $A$ given $\psi$. The right side is somehow the current due to the field $\psi$, and it has two parts
\[ \overline{\psi} \partial_\mu \psi - \psi \overline{\partial_\mu \overline{\psi}} + 2A_\mu \overline{\psi} \psi \]

The total action is
\[ \frac{1}{2} \int |E|^2 + \int |D\psi|^2 + \int \lambda \left( |\psi|^2 - a^2 \right)^2 \]

and setting the variation w.r.t $\psi = 0$ leads to
\[ \delta \int \lambda \left( |\psi|^2 - a^2 \right)^2 = \int \overline{\psi} \left( 2(1|\psi|^2 - a^2)^2 \delta \psi \psi + \text{c.c.} \right) \]
The field equation

\[ D_\mu D^\mu \psi = \frac{2\lambda}{|\psi|^2 - a^2} \psi \]

Now I would like to find solutions of these field equations. First of all we should mention gauge transformations:

\[ A, \psi \rightarrow A + g dg^{-1}, g \psi \]

where \( g: \mathbb{R}^2 \rightarrow S^1 \). This obviously doesn’t change the action.

According to Coleman one first looks for finite action solutions. It’s necessary that \( |\psi| \rightarrow 0 \) as \( r \rightarrow \infty \), assuming \( \lambda \psi \rightarrow 0 \). However the phase in the limit is arbitrary. Suppose

\[ \lim_{r \rightarrow \infty} \psi(0, 0) = g(0) a \]

\[ g: S^1 \rightarrow S^1 \quad |g| = 1 \]

Then

\[ D\psi = (d + A) \psi = (dg + Ag) a \]

hence \( A \sim g dg^{-1} \) as \( r \rightarrow \infty \).

(Coleman argues that \( |dg + Ag| \) behaves like an power of \( r \) as \( r \rightarrow \infty \) in any practical case.

If \( dg + gA \sim O(1) \), then \( \int 1 \sim \int \frac{r dr}{\lambda^2} \)

which diverges logarithmically. Hence it must be \( O(\frac{1}{\lambda^2}) \):

\[ A = g dg^{-1} + O(\frac{1}{\lambda^2}) \]

and hence \( dA = O(\frac{1}{\lambda^2}) \) will give rise to \( \int dA^2 < \infty \).

Next he takes a winding number \( \hat{z} \) say 1 and the simplest possible \( g \) with this winding number...
\[
\psi(r, \theta) = f(r) e^{i\theta} a
\]

\[
A = -i \int p(r) \, d\theta
\]

In the following we suppose \( a = 1 \).

\[
iA = p \, d\theta = (\rho \partial_x \theta) \, dx + (\rho \partial_y \theta) \, dy
\]

\[
D_x = \partial_x - i \rho \partial_x \theta \quad \quad D_y = \partial_y - i \rho \partial_y \theta
\]

\[
d\theta = d\tan^{-1}(\frac{y}{x}) = \frac{-\frac{1}{2} \, dy + \frac{1}{2} \, dx}{1 + \frac{y^2}{x^2}} = \frac{-y \, dx + x \, dy}{\lambda^2}
\]

\[
\partial_x \theta = -\frac{y}{\lambda^2} \quad \quad \partial_y \theta = \frac{x}{\lambda^2}
\]

\[
D_x^2 = \partial_x^2 - 2i \rho (\partial_x \theta) \partial_x - \rho^2 (\partial_x \theta)^2 - i \partial_x (\rho \partial_x \theta)
\]

\[
D_y^2 = \partial_y^2 - 2i \rho (\partial_y \theta) \partial_y - \rho^2 (\partial_y \theta)^2 - i \partial_y (\rho \partial_y \theta)
\]

\[
(D_x^2 + D_y^2) \, f(r) = \Delta f - \frac{\rho^2}{\lambda^2} f
\]

Since \( \nabla f \cdot \nabla \theta = 0 \)

\[
\Delta \theta = 0
\]

But I want to apply this to \( f e^{i\theta} \)

\[
e^{-i\theta} D_\mu e^{i\theta} = \Delta \theta_\mu + A_\mu + i \partial_\mu \theta
\]

\[
= \Delta \theta_\mu - i \rho \partial_\mu \theta + i \partial_\mu \theta = \Delta_\mu + i(1 - \rho) \partial_\mu \theta
\]

which has the effect of changing \( \rho \) to \( \rho - 1 \), hence

\[
e^{-i\theta}(D_x^2 + D_y^2) e^{i\theta} f = \Delta f - \frac{(1 - \rho^2)}{\lambda^2} f
\]
Next, \( \bar{f} \left( \partial_\mu + i(1-\rho)\partial_\mu \right) \theta f - f \left( \partial_\mu + i(1-\rho)\partial_\mu \right) \overline{\theta f} \)

\[= \frac{1}{2} \left( \bar{f} \partial_\mu f - f \partial_\mu \bar{f} \right) + 2i(1-\rho)\partial_\mu \theta \overline{f} \left| f \right|^2 \]

Thus the equation \( \partial_\mu \partial_\mu \psi = \psi \left( \partial_\mu \bar{\psi} \right) - \bar{\psi} \left( \partial_\mu \psi \right) \)
becomes

\[ \Delta f - \frac{(1-\rho)^2}{r^2} f = 2i(1-\rho)\partial_\mu \theta \]

Thus the equation \( \partial_\mu \partial_\mu \psi = 2\lambda \left( \left| f \right|^2 - 1 \right) \psi \) becomes

\[ \Delta f - \frac{(1-\rho)^2}{r^2} f = 2\lambda \left( \left| f \right|^2 - 1 \right) f \]

where \( \Delta f(\nu) = \frac{1}{r^2} \frac{\partial}{\partial \nu} \left( r^2 \frac{\partial f}{\partial r} \right) \)

\[iA = \sigma d\theta \]

\[iA = dp \, d\theta = \frac{1}{r} \frac{\partial}{\partial r} \left( r^2 \frac{\partial \phi}{\partial \nu} \right) \]

\[-iF = \frac{1}{r} \frac{\partial \phi}{\partial \nu} \]

Assuming \( f \) is real, we thus get that the equation

\[ (-\partial_y, \partial_x) \bar{F} = \bar{\psi} \partial_\mu \psi - \psi \partial_\mu \bar{\psi} \]

\[ (\frac{-y}{\nu}, x) \]

becomes

\[ (-\partial_y, \partial_x) \frac{i}{r} \frac{\partial \phi}{\partial \nu} h(\nu) = 2i(1-\rho)f^2 \left( \partial_x \theta, \partial_y \theta \right) \]

\[ \left( \frac{-y}{\nu}, \frac{x}{\nu} \right) \]

\[ (-\partial_y, \partial_x) h(\nu) = \left( \frac{-y}{\nu}, \frac{x}{\nu} \right) \frac{\partial h(\nu)}{\partial \nu} \]

\[ = \frac{h'(\nu)}{\nu} \left( \frac{-y}{\nu}, \frac{x}{\nu} \right) \]

So things are consistent and we get

\[ \frac{r^2}{\partial \nu} \left( \frac{\partial \phi}{\partial \nu} \right) = -2(1-\rho)f^2 \]
January 18, 1981

Look at inverse scattering, and review work of '78.

Discrete case: We are given a unitary operator \( U \) on a Hilbert space \( \mathcal{H} \) and "incoming" and "outgoing" representations

\[
L^2(S^1, d\theta) \xleftarrow{\text{in}} \mathcal{H} \xrightarrow{\text{out}} L^2(S^1, d\theta)
\]

These satisfy \( U \cdot \text{in} = \text{in} \cdot \text{in}^* = \text{id} \), so that \( \text{in} \) is a orthogonal projection onto a closed invariant subspace generated by \( e_{\text{in}}^n = \text{in}^*(1) \). We have

\[
\langle e_{\text{in}}^n | U^n e_{\text{in}}^m \rangle = \delta_{nm} \]

\[
in(h) = \sum z^n \langle z^n | in(h) \rangle
\]

\[
= \sum z^n \langle \text{in}^*(z^n) | h \rangle = \sum z^n \langle U^n e_{\text{in}} | h \rangle.
\]

Similarly for \( \text{out} \). One puts

\[
R = \text{out}(e_{\text{in}}) = \sum z^n \langle U^n \text{out} | e_{\text{in}} \rangle
\]

The typical picture is that of a port connected to a line:

\[
\text{in} \quad U^n \text{out} \quad e_{\text{in}} \quad U^n e_{\text{in}} \quad U^n e_{\text{out}}
\]

In this case \( R \) is a series in \( z^{-1} \), so it is analytic outside \( S' \).

Define

\[
F_0^R = (\text{out}, \text{in})^{-1}(H^- \times H^+)
\]

so that in the above picture it is the part of \( \mathcal{H} \) supported in the port. This space is part of an increasing filtration

\[
F_n^R = (\text{out}, \text{in})^{-1}(z^n H^- \times H^+)
\]
whose union is \( \text{in}^{-1}(H^1) \) and contains \( \text{in} \). Put
\[
\tilde{x}_n = \text{projection of } \text{in} \text{ on } F_n \mathcal{H}.
\]
\[
\alpha_n = \frac{\tilde{x}_n}{\|\tilde{x}_n\|}
\]
Then we have the picture
\[
\begin{align*}
\beta_0 \cup U^{-n} F_{n+1} \mathcal{H} & \quad U^{-n} \tilde{\beta}_n = \text{projection of } \text{cont and } U^{-n} F_n \mathcal{H} \\
F_n \mathcal{H} & \quad U \beta_n \cup U^{-1} F_{n+1} \mathcal{H} \\
\beta_0 \cup U F_{n-1} \mathcal{H} & \quad U \tilde{\beta}_n \cup U^{-1} F_{n-1} \mathcal{H}
\end{align*}
\]
\[\star\]
\[
\alpha_{n-1} = \frac{\alpha_n - \beta_n \langle \beta_n | \alpha_n \rangle}{\sqrt{1 - \langle \beta_n | \alpha_n \rangle^2}}
\]
\[
U \beta_n = \frac{\beta_n - \alpha_n \langle \alpha_n | \beta_n \rangle}{\sqrt{1 - \langle \alpha_n | \beta_n \rangle^2}}
\]
\[
\begin{pmatrix} 1 & 0 \\ 0 & U \end{pmatrix}
\begin{pmatrix} \alpha_n \cr \beta_n \end{pmatrix}
= \frac{1}{\sqrt{1 - h_n^2}}
\begin{pmatrix} 1 & -h_n \\ -h_n & 1 \end{pmatrix}
\begin{pmatrix} \alpha_n \\ \beta_n \end{pmatrix}
\]
\[
\begin{pmatrix} \alpha_n \\ \beta_n \end{pmatrix}
= \frac{1}{\sqrt{1 - h_n^2}}
\begin{pmatrix} 1 & h_n \\ -h_n & 1 \end{pmatrix}
\begin{pmatrix} 1 & 0 \\ 0 & U \end{pmatrix}
\begin{pmatrix} \alpha_{n-1} \\ \beta_{n-1} \end{pmatrix}
\]

 Worth mentioning is the fact that if we \( \text{assume} \) (out, in) \( \text{injective, then } \mathcal{H} \text{ is spanned by } \text{f}(u) \text{in} + \text{g}(u) \text{cont} \)

 with norm
\[
\|f \text{in} + g \text{cont}\|^2 = \|f\|^2 + \|g\|^2 + \langle g | R f \rangle + \langle R f | g \rangle
\]
\[
= \|R f + g\|^2 + \|f \| (1 - \|R\|^2) f \rangle
\]
Thus \( \|f \text{in} + g \text{cont}\|^2 \geq \|R f \| \cdot \|g\|^2 \geq 2 \|f\|^2 \uparrow \)

 and similarly for \text{g}, which shows that \( \text{provided } \|R\| \leq 1 - \varepsilon \)

(\text{out, in}): \mathcal{H} \rightarrow (L^2)^2

 is a topological isomorphism, Hence each of the squares

above as in (2) are transversal, also
\[ UF_n H \text{ dense in } \text{inv}^{-1}(H^+) \]
which shows that \( x_n \rightarrow e_{in} \) as \( n \rightarrow \infty \).

Next look at what happens as \( n \rightarrow -\infty \).
\[ F_{-\infty} H = (\text{out}, \text{in})^{-1}(0 \times H_+) \subset \text{Ker}(\text{out}) \]
which is spanned by elements \( f(e_{in} - \text{Re}_{out}) \) with norm
\[ \| f(e_{in} - \text{Re}_{out}) \|^2 = \langle f, (1 - |R|^2) f \rangle \]

We have
\[
\begin{array}{ccc}
F_{-\infty} H & \xrightarrow{e_{in}} & UF_{-\infty} H \\
\downarrow & & \downarrow \\
\downarrow & & \downarrow \\
F_{-\infty} H & \xrightarrow{e_{in}} & UF_{-\infty} H \\
\end{array}
\]

We obtain \( \tilde{x}_{-\infty} \) by projecting onto \( F_{\infty} H \) (strictly, one removes a linear combination of \( \beta_n \)). We have
\[ \| \tilde{x}_{-\infty} \|^2 = \text{Tr} (1 - |h_n|^2) \]

since \( \| \tilde{x}_{-\infty} \| = \sqrt{1 - |h_n|^2} \| \tilde{x}_{-\infty} \| \).

If \( e_{in} = \tilde{x}_{-\infty} + \beta \)

where \( \beta \) is the appropriate linear combination of \( \beta_n \), then also
\[ \tilde{x}_{-\infty} = e_{in} - \frac{\beta}{\| \beta \|^2} < \beta, e_{in} > \times \text{proj} \]

Since \( \| \tilde{x}_{n-1} \| = \sqrt{1 - |h_n|^2} \| \tilde{x}_{n} \| \).

If \( e_{in} = \tilde{x}_{-\infty} + \beta \)

where \( \beta \) is the appropriate linear combination of \( \beta_n \), then also
\[ \tilde{x}_{-\infty} = e_{in} - \frac{\beta}{\| \beta \|^2} < \beta, e_{in} > \times \text{proj} \]

\[ \text{in}(\tilde{x}_{-\infty}) = 1 - \text{in}(\beta) \frac{1}{\| \beta \|^2} < \beta, e_{in} > \]
Since \( \|\tilde{x}_{n-1}\| = \sqrt{1 - |h_n|^2} \|\tilde{x}_n\| \).

If \( e_{\infty} = \tilde{x}_{-\infty} + \beta \) where \( \beta \) is the appropriate linear combination of the \( \beta_n \), then
\[
\langle e_{\infty} | \tilde{x}_{-\infty} \rangle = \|\tilde{x}_{-\infty}\|^2 \Rightarrow \text{in}(\tilde{x}_{-\infty})(0) = \|\tilde{x}_{-\infty}\|
\]

Next put \( \tilde{x}_{-\infty} = h(e_{\infty} - \text{Re}_{\text{out}}) \) if this is possible since \( \text{out}(\tilde{x}_{-\infty}) = 0 \). Then because \( \alpha_{-\infty} \) is a unit vector in \( F_{\infty} H \otimes UF_{\infty} H \) we have
\[
\delta_{\infty} = \langle \tilde{x}_{-\infty} | U^n \tilde{x}_{-\infty} \rangle = \langle h \mid U^n h (1 - 1R^2) \rangle
\]

so
\[
|h|^2 (1 - 1R^2) = 1. \text{ Thus if we put}
\]
\[
T = \text{in}(\tilde{x}_{-\infty}) = h (1 - 1R^2) e H^+
\]

we have \( \text{Tr} T = 1 \) so \( h = \frac{1}{T} \). Thus
\[
1R^2 + 1T^2 = 1
\]

\[
\tilde{x}_{-\infty} = \frac{e_{\infty} - \text{Re}_{\text{out}}}{T}
\]

\[
T(0) = \|\tilde{x}_{-\infty}\| = T \sqrt{1 - |h_n|^2}
\]

Also one knows \( T \) has no zeros for \( |z| < 1 \), hence \( \log |T| \) is a harmonic function in the disk with only values \( \frac{1}{2} \log (1 - 1R^2) \). So
\[
\log T(0) = \int \frac{1}{2} \log (1 - 1R^2) \frac{d\theta}{2\pi}
\]

hence
\[
T \sqrt{1 - |h_n|^2} = \exp \int \log (1 - 1R^2) \frac{d\theta}{2\pi}
\]
Next consider the continuous case. Suppose given $R(k)$ of modulus \( \leq 1-\varepsilon \) and form a Hilbert space $\mathcal{H}$ of $f \in \mathcal{H}$ where $f, g \in L^2(\mathbb{R}, \frac{dk}{2\pi})$ with
\[
\| f \oplus g \| = \| f \| + \| g \|^2 + \langle f, Rf \rangle + \langle Rf, g \rangle
\]
On $\mathcal{H}$ we have a 1-parameter unitary group $U(t)$ unitary by $e^{-ikt}$. We use physics conventions, hence must change roles of $H^+$, in this situation $H^+$ is spanned by $e^{ikx}$ with $x > 0$, hence is analogous to span of $e^{-it}$ since $\varepsilon = e^{-it}$.

Filtration is
\[
F_x \mathcal{H} = (\text{out}, \text{in})^{-1} (e^{-ikx} H^+ \times H^-)
\]
This increases with closed union $\text{in}^{-1} (H^+)$. We define (heuristically) $\lambda_x$ as the projection of $e_{\text{in}}$ onto $F_x \mathcal{H}$. Thus
\[
\lambda_x = e_{\text{in}} + \bar{f} x e^{-ikx} e_{\text{out}}
\]
where $\bar{f}, g \in H^+$. Note that
\[
(F_x \mathcal{H})^\perp = e^{-ikx} H^- e_{\text{out}} + H^+ e_{\text{in}}
\]
so $\lambda_x$ differs from $e_{\text{in}}$ by an element of $F_x \mathcal{H}^\perp$. For $\lambda_x$ to belong to $F_x \mathcal{H}$ means formally
\[
\text{out}(\lambda_x) = R(1+f_x) - g_x e^{-ikx} \in e^{-ikx} H^+
\]
\[
\text{in}(\lambda_x) = 1 + \bar{f} x - g_x e^{-ikx} R \in H^-
\]
and we can make these precise as follows
\[
\begin{align*}
\begin{cases}
P_- R_x (1+f_x) = \bar{g}_x \\
f_x = P_+ R_x \bar{g}_x
\end{cases}
\end{align*}
\]
$R_x = Re^{ikx}$
where \( \hat{P}_\pm \) are the projectors onto \( H^\pm \) respectively. We assume that \( R \to \infty \) as \( |k| \to \infty \) sufficiently fast. Solve

\[
f_x = \hat{P}_+ \overline{R_x} \hat{P}_- R_x (1 + f_x)
\]

\[
1 + f_x = \frac{1}{1 - \Gamma_x} 11 >
\]

\[
\Gamma_x = \hat{P}_+ \overline{R_x} \hat{P}_- R_x
\]

\[
f_x = \hat{P}_- \overline{R_x} \left( \frac{1}{1 - \Gamma_x} 11 > \right)
\]

Assuming that \( |R| \) is bounded below 1, the operator \( \Gamma_x \) on \( L^2 \) has norm < 1 and so the Neumann series for \((1 - \Gamma_x)^{-1}\) converges.

Next we want to compute derivatives w.r.t. \( x \).

\[
\Gamma_x = \hat{P}_+ \overline{R} e^{-ikx} \hat{P}_- e^{ikx} \overline{R}
\]

Take \( f \in L^2(\mathbb{R}, \frac{dk}{2\pi}) \), \( f = \int dy e^{iky} \langle e^{iky} | f \rangle \)

\[
P_- e^{ikx} f(k) = \int_{-\infty}^{\infty} dy e^{iky} \tilde{f}(y-x)
\]

\[
(e^{-ikx} P_- e^{ikx} f)(k) = \int_{-\infty}^{\infty} dy e^{iky} f(y)
\]

\[
\frac{d\langle e^{-ikx} P_- e^{ikx} f \rangle}{dx} = \langle e^{-ikx} f(-x) \rangle = -e^{-ikx} \langle e^{-ikx} f \rangle
\]

Thus

\[
\frac{d}{dx} \Gamma_x = -\hat{P}_+ \overline{R_x} 11 > \langle 11 | R_x \rangle
\]

\[
P_- R_x f = e^{ikx} \int_{-\infty}^{\infty} dy e^{iky} \langle e^{iky} | Rf \rangle
\]
Thus
\[
\frac{d}{dx} f_x = \frac{d}{dx} \frac{1}{1 - \Gamma_x} 11\rangle
\]
\[
\begin{align*}
&= \frac{1}{1 - \Gamma_x} \frac{d\Gamma_x}{dx} \frac{1}{1 - \Gamma_x} 11\rangle \\
&= \frac{1}{1 - \Gamma_x} (- p_+ R_x 11\rangle <1 \mid R_x \rangle \frac{1}{1 - \Gamma_x} 11\rangle
\end{align*}
\]
Notice that
\[
\tilde{g}_x = (p_- R_x + p_+ R_x p_+ R_x p_- R_x + \ldots) 11\rangle
\]
\[
= p_- R_x \left\langle \frac{1}{1 - p_+ R_x p_- R_x} \right\rangle 11\rangle
\]
\[
= \frac{1}{1 - p_+ R_x p_- R_x} \left\langle \frac{1}{\Gamma_x} \right\rangle 11\rangle
\]
Thus
\[
g_x = \frac{1}{1 - \Gamma_x} p_+ R_x 11\rangle
\]
and we get
\[
\frac{df_x}{dx} = - g_x \langle 1 \mid R_x \frac{1}{1 - \Gamma_x} 11\rangle
\]
Next
\[
\frac{dg_x}{dx} = \frac{d}{dx} \left( p_- R_x \frac{1}{1 - \Gamma_x} 11\rangle \right)
\]
\[
\begin{align*}
&= i\hbar (p_- R_x \frac{1}{1 - \Gamma_x} 11\rangle - 11\rangle <1 \mid R_x \frac{1}{1 - \Gamma_x} 11\rangle \\
&\quad + p_- R_x (- g_x \langle 1 \mid R_x \frac{1}{1 - \Gamma_x} 11\rangle)
\end{align*}
\]
\[
- f_x
\]
\[
\frac{d\bar{g}_x}{dx} = i\bar{k} \bar{g}_x - (1 + f_x) \langle 1 | R_x \frac{1}{1 - f_x} | 1 \rangle
\]

Let's put \( h(x) = \langle 1 | R_x \frac{1}{1 - f_x} | 1 \rangle \). Then

\[
\frac{df_x}{dx} = -g_x h(x)
\]

\[
\frac{dg_x}{dx} = -ik g_x - (1 + f_x) h(x).
\]

\( h(x) = \langle 1 | R_x \frac{1}{1 - f_x} | 1 \rangle \)

Let's now consider a wave equation

\[
\partial_t^2 u = (\partial_x^2 - q(x)) u
\]

on the line with \( q \in C^\infty \) to simplify. Temporarily think of \( H \) as consisting of all solutions which for each \( x \) are rapidly decreasing \( C^\infty \) functions of \( t \). Then we can replace \( u \) by its FT with \( t \)

\[
u(x, t) = \int \frac{dk}{2\pi} \hat{u}(x, k) e^{-ikt}
\]

Then

\[
[-\partial_x^2 + q(x)] \hat{u} = k^2 \hat{u}
\]

so \( \hat{u} \) is a section of a 2-dimensional vector bundle over the \( k \)-line; it is smooth and rapidly decreasing as a function of \( k \) for \( x \) fixed. For \( x \to \infty \) we have

\[
\hat{u}(x, k) = A(k) e^{-ikx} + B(k) e^{ikx}
\]

and hence each element of \( H \) determines two functions of \( k \), except that there are slight problems defining \( A, B \) at \( k = 0 \), which I would like to avoid facing as long as
possible. I will assume $-\Delta_x^2 + g(x)$ has no bound states, so that we don't lose solutions of the wave equation by requiring there are rapidly decreasing $u$ for fixed $x$. Also I will try to use in $\mathcal{H}$ only solutions given by a pair $A(k), B(k)$ of $C^\infty$ rapidly-decreasing functions.

If

$$\hat{u}(x,k) = A(k)e^{-ikx} + B(k)e^{ikx} \quad x \gg 0$$

then

$$u(x,t) = \hat{A}(x+t) + \hat{B}(t-x) \quad x \gg 0$$

and as $t \to +\infty$, the first term decays leaving only $\hat{B}(t-x)$ for $x > 0$. Thus we have a natural in and out maps

$$\text{out}(\hat{u}) = B(k)$$
$$\text{in}(\hat{u}) = A(k).$$

We can define $e_{in}$ to be the element of $\mathcal{H}$ which for $t < 0$ is entirely an incoming $\delta$-function from the right:

$$e_{in} = \hat{T}(k)e^{-ikx} \quad e_{in} \leftrightarrow e^{-ikx} + R(k)e^{ikx}$$

and

$$e_{out} = \hat{T}(k)e^{ikx} \quad e_{out} \leftrightarrow e^{ikx} + \overline{R(k)}e^{-ikx}$$

similarly.

We can draw pictures of elements of $\mathcal{H}$ in the $(x,t)$ plane. Thus $e_{in}$ looks as follows:

[Diagram]

Thus $e_{in}$ looks as follows:
Next consider \( F_0 H = (\text{out, in})^{-1}(H^+ \times H^-) \).

This consists of solutions with

\[
\text{out}(\hat{u}) = B(k) = \int_0^\infty e^{i k y} \hat{B}(y) \, dy
\]

\[
\int \frac{dk}{2\pi} B(k) e^{i k \hat{x}} = \hat{B}(x = x) = 0 \quad t < x
\]

Hence \( F_0 H \) consists of solutions 0 for \( |t| < x \), i.e. whose Cauchy data at \( t = 0 \) has support \((-\infty, 0]\).

\[
e^{-ikx} F_x H = (\text{out, in})^{-1}\left( e^{-ik \frac{x}{2}} H^+ \times e^{ik \frac{x}{2}} H^- \right)
\]

contains solutions which look like
and \( F_x H = (\text{out, in})^{-1}(e^{+ikx} H^+ x H^-) \) looks like

\[ H_2 \]

Notice that \( \cap F_x H = F_\infty H \) consists of solutions supported in \( x + t < 0 \), in particular the solution

\[ e^{-ikx} + \bar{R} e^{ikx} \leftrightarrow T e^{-ikx} \]

**Question:** We've seen how \( e^{ik\frac{x}{2}} F_x H \) is an increasing filtration of \( H \). Are the elements \( e^{ik\frac{x}{2}} \) and \( e^{+ik\frac{x}{2}} \) orthogonal to \( e^{ik\frac{y}{2}} F_y H \) for \( y < x \)?

And do these elements correspond to solutions with \( S \)-function Cauchy data at \( \alpha \)?

Consider \( \alpha_0 = (1 + f_0) e_{\text{in}} - \bar{f}_0 e_{\text{out}} \). Now \( f_0 \) is a linear combination of \( e^{iky} \) \( y > 0 \) so \( (1 + f_0)e_{\text{in}} \) is a linear combination of

\[ (e^{ik} e_{\text{in}})(x, t) = \int \frac{dk}{2\pi} e^{i(k(y-t))} e_{\text{in}}(x, k) e^{-ikx} + \cdots \]

\[ = \delta(x - (y - t)) + \cdots \]
Thus the support of $e^{iky}e^{it}$ is as follows.

Similarly, $\tilde{g}_0e^{out}$ is a linear combination of $e^{-iky}e^{out}$

$$(e^{-iky}e^{out})(x, t) = \int_{2\pi} e^{-ik(y+t)} e^{out(x, k)} e^{ikx} + \ldots$$

$$= \delta(x-(y+t)) + \ldots$$

whose support is as follows.
The conclusion is that, \((1 + f_0)e_{in} - f_0e_{out}\) has support

So therefore, \(\alpha_0\) has support exactly at \(x = 0\).
Recall we are considering solutions of the wave eqn.
\[ \frac{\partial^2 u}{\partial t^2} = (\partial_x^2 - \beta^2)u \]

In general given a solution of \( \frac{\partial^2 u}{\partial t^2} = Lu \) we can form its Laplace transform
\[ L(u) = \int_0^\infty dt \, e^{-st} u(t) \]

and
\[ L(u') = \int_0^\infty dt \, [e^{-st} u'(t) + s e^{-st} u(t)] \]

\[-L(u) = -u'(0) - su(0) + s^2 L(u) \]

so
\[ L(u) = \frac{u'(0) + su(0)}{s^2 + L} \]

Similarly
\[ L_-(u) = \int_0^\infty dt \, e^{-st} u(t) \]

will be given by
\[ L_-(u) = \frac{-u'(0) + su(0)}{s^2 + L} \]

Here \((s^2 + L)^{-1}\) is computed in a right half-plane \( \text{Re}(s) > \text{Re}(L) \) for \( L_+(u) = L(u) \) and a left half-plane for \( L_- \).

In good cases we can analytically continue to a strip around the imaginary \( s \)-axis, whence the sum
\[ u = L_+(u) + L_-(u) \]

is an analytic function along the imaginary axis contained in the kernel of \( s^2 + L \).

When \( L \) is hermitian so that the resolvent \((s^2 + L)^{-1}\)

is defined off \( i \mathbb{R} \), a solution is determined by its Fourier transform \( \hat{u} \). Think of it as like a Laurent series which
then gets split into parts analytic in the inner and outer disks.

When \( L \) has discrete spectrum, the functions \( L_+(u) \) and \( L_-(u) \) are the same except for sign. The Fourier transform involves appropriate residue contributions at the eigenvalues.

Given the reflection coefficient \( R(k) \) for \((-\partial_x^2 + q)\psi = k^2 \psi\) with \( q \in C^o \) say, we can define

\[ x_\pm = (1 + f_x) e_{in} - \bar{g}_x e^{-ikx} e_{out} \]

so that

\[
\begin{align*}
\text{out}(x_\pm) &= (1 + f_x) R - \bar{g}_x e^{-ikx} e_{in}^- & \in e^{-ikx} H^+ \\
\text{in}(x_\pm) &= 1 + f_x - \bar{g}_x e^{-ikx} R^- & \in e^{ikx} H^-
\end{align*}
\]

for suitable \( f_x, g_x \in H^+ \). Arguments given above suggest that as a solution of the wave equation \( x_\pm \) has Cauchy data on \( t = 0 \) supported at \( x \). However this cannot be correct because \( x_\pm \) depends only on \( R(k) \), hence the above \( x_\pm \) belong to \( q \) without bound states.

Let's consider an example \( q = -\hbar \delta(x) \). Then

\[
(-\partial_x^2 + q) \psi = k^2 \psi
\]

can be integrated in a small interval \([-\varepsilon, \varepsilon]\) to give

\[
-(\partial_x \psi)^{\text{out}}_{\text{in}} - \hbar \psi'(0) = 0
\]

or

\[
[\partial_x \psi]^{\text{out}}_{\text{in}} = -\hbar \psi'(0)
\]

Thus if

\[ e^{-ikx} \psi \rightarrow A e^{-ikx} + B e^{ikx} \]

gives

\[
\begin{align*}
1 &= A + B \\
-ik(A - B - 1) &= -\hbar \\
A &= 1 + \frac{\hbar}{2ik} \\
B &= -\frac{\hbar}{2ik}
\end{align*}
\]

Thus \( A - B = 1 + \frac{\hbar}{ik} \)

\( A + B = 1 \)
Notice if $h > 0$, then $A = 0 \Rightarrow k = \frac{h}{2} i \ c\ h\ p$
so we have the bound state $e^{-\frac{h}{2}|x|}$.

Next we want $c_{in}$ which I thought should be

$$
\frac{2i k}{2i k + h} \ e^{-ikx} \leftrightarrow \ e^{-ikx} + \frac{-h}{2i k + h} \ e^{-ikx}
$$

$$
\frac{1}{R(k)}
$$

But let's look at the solution of the wave equation:

$$
\int \frac{dk}{2\pi} \frac{h}{2i k + h} \ e^{-ik(x-t)} = \Theta(x-t) \frac{\frac{h}{2}(x-t)}{2\pi - \frac{2i}{2}}
$$

$$
= \Theta(x-t) \frac{h}{2} \ e^{-\frac{h}{2}(x-t)}
$$

Thus for $x > 0$ we have the support

and similarly in the other side.

Thus the solution is

$$
\delta(x+t) - \Theta(x-t) \frac{h}{2} e^{\frac{h}{2}(x+t)} \leftrightarrow \delta(x+t) - \Theta(x-t) \frac{h}{2} e^{-\frac{h}{2}(x-t)}
$$

Note that if we add to this the bound state solution

$$
\frac{h}{2} e^{-\frac{h}{2}|x|} e^{\frac{h}{2}t}
$$
we get

\[ \delta(x+t) + \Theta(x+t) \frac{\hbar}{2} e^{\frac{\hbar}{2}(x+t)} \]

\[ \Longleftrightarrow \delta(x+t) + \Theta(t-x) \frac{\hbar}{2} e^{-\frac{\hbar}{2}(x-t)} \]

which has the support

I expect for ein.
January 23, 1981

Let us consider a wave equation

$$\partial_t^2 u = (\partial_x^2 - q) u$$

with $q \in C_0^\infty(\mathbb{R})$. I want to identify the solution $e_{in}(x,t)$ which is equal to an incoming $\delta$-function disturbance for $t \ll 0$. Thus for $t \ll 0$ we have

$$e_{in}(x,t) = \delta(x+t).$$

The support of $e_{in}$ should look like

![Graph](image)

and hence the Fourier-Laplace transform of $e_{in}$

$$e_{in}(x,k) = \int dt \, e^{ikt} e_{in}(x,t)$$

should be analytic for $\text{Im}(k) > 0$. Moreover it should be a solution of

$$(k^2 + \partial_x^2) \psi = q(x) \psi$$

My candidate for $e_{in}(x,k)$ is as follows. Start with

$$e^{-ikx} \xrightarrow{\phi(x,k)} A(k) e^{-ikx} + B(k) e^{ikx}$$

and then form

$$\frac{1}{A(k)} e^{-ikx} \xrightarrow{\phi} e^{-ikx} + \frac{B(k)}{A(k)} e^{ikx}$$
We know that $A(k)$ is analytic in the UHP with zeroes belonging to the bound state and that (roughly)

$$A(k) = 1 + \frac{1}{2ik} \int g(x) dx + O\left(\frac{1}{k^2}\right) \quad \text{as } k \to \infty.$$ 

Moreover, $\phi(x,k)$ is analytic in the UHP because $e^{-ikx}$ is the small solution there. Thus for $\text{Im}(k) > 0$ the bound state region, the solution $\frac{1}{A(k)} \Phi(x,k)$ is analytic.
January 24, 1981

I want to work out the details of scattering on the line from the wave equation viewpoint. Consider

$$L = -\partial_x^2 + q(x)$$

where \( q \in C^\infty_0(\mathbb{R}) \) and suppose there are no bound states. Then Fourier transform allows us to pass between solutions of the wave equation

$$\partial_t^2 u = (\partial_x - q) u = -Lu$$

and functions \( \psi \) for \( k \) real satisfying

$$(-\partial_x^2 + q) \psi = k^2 \psi$$

We have for \( x \gg 0 \)

$$\psi = A(k)e^{-ikx} + B(k)e^{ikx}$$

and we can define

$$\text{in}(\psi) = A(k)$$

$$\text{out}(\psi) = B(k)$$

Actually we have

$$A'e^{-ikx} + B'e^{-ikx} \Rightarrow Ae^{-ikx} + Be^{ikx}$$

and one knows that

$$|A|^2 + |A'|^2 = |B|^2 + |B'|^2$$

for example

$$\begin{cases} Te^{-ikx} \leftrightarrow e^{-ikx} + Re^{-ikx} \\ |T|^2 + |R|^2 = 1. \end{cases}$$

We shall define a norm on the solutions to the wave equation by putting

$$\|\psi\|^2 = \frac{1}{2\pi} \int dk \left( |A|^2 + |A'|^2 \right)$$

This is not the energy norm.
\[ E(u) = \frac{1}{2} \int \left( |u|^2 + \bar{u} \text{Lu} \right) dx \quad \text{any fixed } t \]

which can be evaluated by letting \( t \to -\infty \) to get

\[ E(u) = \int \frac{dk}{2\pi} k^2 \left( |A|^2 + |A'|^2 \right) \]

Thus we have

\[ E(u) = \| Ky \|^2 \]

Next define the solutions \( e_{in}, e_{out} \) by

\[ T(k) e^{-ikx} \leftrightarrow e_{in}(x,k) + R(k) e^{ikx} \]

\[ e_{out}(x,k) = \overline{e_{in}(x,k)} = e_{in}(x,-k) \]

Hence \( e_{out}(x,t) = e_{in}(x,-t) \).
January 27, 1981

I want to understand how to compute the terms of the asymptotic series for $G_k(x,x)$ as $k \to \infty$ in terms of the potential. Here I am considering the operator

$$L = -\partial_x^2 + u(x) = D^2 + u$$

$$D = \frac{i}{\partial x}$$

Proceeding formally put

$$A = e^{-(D^2+u)} e^{tD^2}$$

Then

$$\partial_t A = -(D^2+u)A + \Box AD^2$$

hence if

$$A = \sum \frac{t^l}{l!} a_l$$

we get the recursion formula

$$a_{l+1} = -[D^2, a_l] - u a_l$$

Starting with $A|_{t=0} = a_0 = 1$ we find

$$a_1 = -u$$

$$a_2 = -[D^2, -u] - u(-u) = 2u'D + u'' + u^2$$

It's clear from the recursion formula that $a_l$ is a differential operator of order $\leq l-1$.

$$a_l = (\frac{-ad(D^2) - u}{1})^l$$

Heat kernel: $K(t,x,x') = \langle x | e^{-tL} | x' \rangle$

$$= \sum \phi_i(x) \overline{\phi_i(x')}$$

supposedly has some kind of asymptotic expansion as $t \to 0$. 
My first idea is that for each $k$, $K_k$, which is a smooth function on $\mathbb{R}^2$, has an asymptotic expansion in the space of distributions on $\mathbb{R}^2$. However, this seems to be the formal expansion

$$e^{-tL} = I - tL + \frac{t^2L^2}{2!} + \cdots$$

The terms are distributions supported in the diagonal.

So instead what seems to be the good result is that one takes the restriction to the diagonal first: $K_k(x,x)$ and then asks for an asymptotic expansion.

In the same way, we can form the resolvent kernel

$$\langle x | \frac{1}{s-L} | x' \rangle$$

and ask about an asymptotic expansion in $s$ as $s \to -\infty$. If we work in distributions on $\mathbb{R}^2$, we seem to get just

$$\frac{1}{s-L} = \frac{1}{s} + \frac{L}{s^2} + \frac{L^2}{s^3} + \cdots$$

However, the interesting case arises when we set $x = x'$ and then ask for an asymptotic expansion.

Here is a possible method for computing the expansions of $\langle x | e^{-tL} | x \rangle$ or $\langle x | \frac{1}{s-L} | x \rangle$. Begin with

$$e^{-tL} = e^{-t(0^2+u)} = \sum_{\ell} \frac{t^\ell}{\ell!} a_\ell(x, D)$$

where $a_\ell = (-ad 0^2-u)^\ell(1)$. Then

$$\langle x | e^{-tL} | x \rangle = \int \frac{dk}{2\pi} \langle x | e^{-tL} | k \rangle \langle k | x \rangle$$

$$= \int \frac{dk}{2\pi} e^{ikx} e^{-tk^2} e^{-ikx'}$$
Thus

\[
\langle x | e^{-U} | x \rangle = \int \frac{dk}{2\pi} A(t, x, k) e^{-tk^2}
\]

\[
\sim \sum \frac{t^l}{l!} \int \frac{dk}{2\pi} a_e(x, k) e^{-tk^2}
\]

Since \( a_e(x, k) \) is a poly. in \( k \), one can do these Gaussian integrals and get the required asymptotic expansion. Unfortunately, this introduces \( \frac{1}{k} \) factors.

Different procedure: Put

\[
(D^2 + u)^n = \left( \sum_{k=0}^{\infty} a_k^n D^{-k} \right) D^{2n}
\]

Then we can derive recursion relations

\[
(D^2 + u)^{n+1} = \sum_k (D^2 + u) a_k^n D^{-k+2n}
\]

\[
= \sum_k \left( a_k^n + 2(Da_k^n) + (D^2 + u)a_k^n \right) D^{-k+2(n+1)}
\]

So

\[
a_k^{n+1} = a_k^n + 2Da_k^n + (D^2 + u)a_k^{n-2}
\]

This can be used to grind out \( a_k^n \) recursively starting from \( a_0^n = 1 \), \( a_0^0 = 0 \), \( k > 0 \). Thus

\[
a_0^n = 1
\]

\[
a_1^n = 0
\]

\[
a_2^n = nU
\]

\[
a_3^n = n(n-1)(Du)
\]

In general \( \frac{a_k^n}{k} \) is a polynomial of degree \( \leq k-1 \) in \( n \).
and hence can be expanded in terms of the basic binomial polynomials
\[ \phi_\ell(n) = \frac{u(n-1)...(n-\ell+1)}{\ell!} \]
\[ \Delta \phi_\ell = \phi_{\ell-1} \]

Hence we get
\[ (D^2 + u)^n = \sum_{0 \leq \ell < k} \alpha_{k\ell}(x) \phi_\ell(n) D^{-\ell + 2n} \]

Now we can use this to define the symbol of a pseudo-differential operator: \((D^2 + u)^{-s}\). One has
\[ \Gamma(s) L^{-s} = \int_0^\infty dt e^{-tL} t^{s-1} D^{-k+2s-2(s+1)} \]
\[ \Gamma(s) (D^2 + u)^{-s} = \sum_{0 \leq \ell < k} \alpha_{k\ell}(x) \phi_\ell(s) D^{-\ell + 2s} \]
\[ \Gamma(s) \frac{(-1)^{s+1} \ell!}{s(s+1)...(s+\ell-1)} \Gamma(s+1) = \frac{\Gamma(s+1)}{\ell!} \]

So
\[ e^{-t(D^2 + u)} = \sum_{0 \leq \ell < k} \alpha_{k\ell}(x) \frac{(-tD^2)^\ell}{\ell!} D^{2\ell-k} \frac{t^\ell}{\ell!} e^{-tD^2} \]
\[ e^{-t(D^2 + u)} = \sum_{0 \leq \ell < k} \alpha_{k\ell}(x) \frac{(-1)^{\ell+1}}{\ell!} t^\ell D^{2\ell-k} e^{-tD^2} \]

This is a formal expression which gives a new way of computing the operators \(a_\ell(x,D)\), but it’s not really new. I need the Gaussian moments
\[ \sum \sum \frac{u^{2n}}{m!} \int dx \ x^n \ e^{-(x^2)} = \sum \int dx \ e^{-\frac{2tx^2}{t} + ux} = \sqrt{\frac{2\pi}{2t}} e^{\frac{u^2}{4t}} \]
\[ = \sqrt{\frac{\pi}{t}} \sum \frac{u^{2n}}{(4t)^n n!} \]
\[ \therefore \int \frac{du}{2\pi i} e^{\frac{-u}{4t}} \frac{2^k-k^2}{t^{k/2}} = \frac{1}{\sqrt{4\pi t^k}} \frac{(2t-k)!}{(4t)^{k/2} (t-k/2)!} \quad \text{for } k \text{ even} \]
Hence it seems we get the asymptotic expansion

\[ \langle x | e^{-t(D^2+u)} | x \rangle \sim \sum_{0 \leq l < k \atop k \text{ even}} \frac{a_l(x) (l!)^{\frac{1}{2}}}{\sqrt{4\pi t}} \frac{(2l-k)!}{4^{-l-k^2} l! (l-k)^{l-k^2}} \]

here \( \frac{k}{2} \leq l < k \).

It's clear this derivation can be streamlined quite a bit.
\[ A = e^{-t(D^2 + u)} e^{tD^2} \]

\[ \partial_t A = -(D^2 + u)A + AD^2 = -[D^2, A] - uA \]

If

\[ A \sim \sum_{\alpha \leq \ell} \frac{t^{\ell}}{\ell!} a_{\ell m}(x) D^m \]

then

\[ a_{\ell+1, m} = -2(D_{\ell, m-1}) - (D^2 + u) a_{\ell, m} \]

e.g.

\[ a_0 = 1 \]
\[ a_1 = -u \]
\[ a_2 = \frac{1}{2} (2D_D) D + (D^2 + u) u \]
\[ a_3 = -4(D^2_D) D^2 + \ldots. \]

\[ \langle 0 | D^m e^{-tD^2} | 0 \rangle = \int \frac{dk}{2\pi} k^m e^{-tk^2} = \partial_x^m \int \frac{dk}{2\pi} e^{kx - tk^2} \bigg|_{x=0} \]

\[ = \partial_x^m \left[ \frac{1}{\sqrt{4\pi t}} \frac{x^2}{4t} \right]_{x=0} = \frac{m!}{(m/2)!} \left( \frac{1}{4t} \right)^{m/2} \]

\[ = \left\{ \begin{array}{ll}
\frac{m!}{(m/2)!} \frac{1}{2^m} t^{-m/2} & m \text{ even} \\
0 & m \text{ odd}
\end{array} \right. \]
Thus
\[ \langle x \mid e^{-t(D^2+u)} \mid x \rangle \sim \sum_{0 \leq m < l \text{ even}} \frac{l^m}{l!} a_m(x) \frac{1}{V_{4\pi t}} \frac{m!}{(m/2)!^2} t^{-m/2} \]

We want to collect according to powers of \( t \), i.e., we want \( l - m/2 = k \) or \( m = 2l - 2k \), which gives us

\[ \langle x \mid e^{-t(D^2+u)} \mid x \rangle \sim \frac{1}{V_{4\pi t}} \left( 1 + t (-u) + \frac{t^2}{2!} (D^2u + u^2) \right. \]

\[ + \frac{t^2}{3!} (-4D^2u) \frac{2^k}{1! 3^2} \left. \right) \]

\[ = \frac{1}{V_{4\pi t}} \left( 1 - tu + \frac{t^2}{2!} (u^2 + \frac{1}{3} D^2u) + \ldots \right) \]

Now we have
\[ \int_0^\infty dt \ e^{-t(L - s)} = \frac{1}{s + L} \]

\[ \int_0^\infty dt \ e^{-st} \frac{t^{k-1/2}}{\sqrt{\pi}} = \frac{\Gamma(k+1/2)}{s^{k+1/2} \Gamma(1/2)} = \frac{1}{2} \frac{3}{2} \ldots \frac{2k-1}{2} \]

so that
\[ \langle x \mid \frac{1}{s + D^2 + u} \mid x \rangle \sim \left( \frac{1}{2\sqrt{s}} - \frac{u}{4s^{3/2}} + \frac{1}{2} \frac{13}{22} \frac{(u^2 + \frac{1}{3} D^2u)}{s^{5/2}} + \ldots \right) \]