Recall that for a simple harmonic oscillator with a finite time perturbation
\[ H = \frac{p^2}{2} + \frac{1}{2}(\omega^2 + \varepsilon(t)) \frac{q^2}{2} \]
one gets an S-operator
\[ \langle e^{-i\mu} | S | e_{\pm} \rangle = \langle 0 | S | 0 \rangle \ e^{\frac{i}{\hbar}(R\lambda^2 + 2TA\mu + \tilde{R}\mu^2)} \]
where one has the asymptotic behavior
\[ Re^{i\omega t} + e^{-i\omega t} \longleftrightarrow Te^{-i\omega t} \]
\[ Te^{i\omega t} \longleftrightarrow e^{i\omega t} + Re^{-i\omega t} \]
for the solutions of the classical equation of motion
\[ \left( \frac{d^2}{dt^2} + \omega^2 + \varepsilon \right) q = 0 \]
(see p.247).

Next I want to apply this to the infinite dimensional oscillator represented by the wave eqn.
\[ \left[ \partial_t^2 + \partial_x^2 \right] \phi = 0 \]
and the perturbation given by the wave equation with potential:
\[ \phi \left[ \partial_t + \left( -\partial_x^2 + V \right) \right] \phi = 0 \]
The idea is that because of the continuous spectrum of \((-\partial_x)^2\) we can deal with \(V\) independent of \(t\).

Let us first examine the asymptotic behavior of the solutions of the classical equation of motion (1). Let's work on \(x > 0\) with boundary condition \(\partial_x \phi = 0\) at \(x = 0\). Then a solution of (1) is of the form
\[
\phi(x, t) = \int \frac{d\omega}{2\pi} u(\omega) e^{-i\omega t} f(\omega)
\]

where
\[
(-\omega^2 - \partial_x + V) u_\omega(x) = 0 \quad \Rightarrow \quad \partial_x u_\omega = 0 \text{ at } x = \infty.
\]
and
\[
u_\omega(x) \sim e^{-i\omega x + R(\omega)} e^{-i\omega x} \quad x \to \infty.
\]

Notice that \( \phi \) can be written
\[
\phi(x, t) = \int_0^\infty \frac{d\omega}{2\pi} e^{-i\omega t} u(x, \omega) f(\omega) + \int_0^{\infty} \frac{d\omega}{2\pi} e^{-i\omega t} u(x, -\omega) f(-\omega)
\]

\[= e^{-iPt} \int_0^\infty \frac{d\omega}{2\pi} u_\omega f(\omega) + e^{iPt} \int_0^\infty \frac{d\omega}{2\pi} u_\omega f(-\omega)
\]

where \( P = +\sqrt{-\partial_x^2 + V} \).

Because of the Riemann - Lebesgue lemma, we know that \( \phi(x, t) \) has the following asymptotic behavior
\[
\phi(x, t) \sim \int \frac{d\omega}{2\pi} \left( e^{-i\omega t} + e^{i\omega t} \right) e^{-i\omega t} f(\omega) \quad t \to \infty
\]

\[\sim \int \frac{d\omega}{2\pi} \left( e^{-i\omega t} R(\omega) f(\omega) \quad t \to \infty
\]

where on the right are solutions of the free wave equation. Therefore if we use these asymptotics we have
\[e^{-iPt} \int \frac{d\omega}{2\pi} u_\omega f(\omega) \quad \Rightarrow \quad e^{-iPt} \int \frac{d\omega}{2\pi} u_\omega R(\omega) f(\omega)
\[+ e^{iPt} \int \frac{d\omega}{2\pi} u_\omega f(-\omega) \quad + e^{iPt} \int \frac{d\omega}{2\pi} u_\omega R(-\omega) f(\omega)
\]

Hence we have
\[e^{-iPt} f \quad \Rightarrow \quad e^{-iPt} Rf
\]
\[e^{iPt} g \quad \Rightarrow \quad e^{iPt} Rf
\]
where \( \mathring{f} = \int \frac{d\omega}{2\pi} u_\omega f(\omega) \)
Thus there is no reflection, only transmission, and the $T$ operator is multiplication by $R(\omega)$.

It follows that the $S$-operator on the quantum level is essentially trivial in the sense that there is no creation and annihilation of particles, only a phase shift.

But it would be interesting to ask about $\langle 0|s|0 \rangle$.

We have

$$\langle e^{-i} | S | e^{+i} \rangle = \langle 0|s|0 \rangle e^{i \mu T}$$

and also

$$|\langle 0|s|0 \rangle|^2 = |\text{det } T| = 1.$$
Consider the fermion analogue of the perturbed oscillator. One is given $H_0$ on $H$, then $H_0$ is extended to Fock space $\Lambda(H)$ as a derivation. Assuming $H_0$ does not have 0 as eigenvalue, the ground state of $H_0$ on $\Lambda(H)$ is the element belonging to the subspace $H^- \subset H$ on which $H_0 < 0$. Next suppose $H_0$ is perturbed:

$$H = H_0 + V(t)$$

where $V(t)$ has compact support. We want to compute the $S$-matrix of this perturbation on $\Lambda H$.

Because $H$ is extended as a derivation, the operators $e^{-i\theta H_0}$, $U(t)$, $U_0(t)$ are the automorphisms of $\Lambda H$ induced by the corresponding automorphisms of $H$. Then the $S$-matrix on $\Lambda H$ is the extension of the $S$-matrix on $H$.

By analogy with boson case, we would like to write $S$ in terms of creation & annihilation operators. The idea will be to work with particles and hole operators as follows. Let $H_0$ have the eigenvalues $\omega_\alpha$ and $H_0 \phi_\alpha = \omega_\alpha \phi_\alpha$, and let $a_\alpha, a^*_\alpha$ be the corresponding annihilation and creation operators. Let $b_\alpha = a_\alpha$ if $\omega_\alpha > 0$ and $c_\alpha = a^*_\alpha$ if $\omega_\alpha < 0$. Then $V(t)$ lies in the Lie algebra of operators spanned by $a_\alpha^* a_\beta$ and these fall into four types:

$$b^* c^* b c c^* c b$$
Consequently we expect the S-matrix to appear in the form

\[ S = \left< 0|s|0 \right> \quad e^{ab^*c^* \xi \eta \zeta \delta} \]

Furthermore we know that \( S: \mathcal{H}^+ \otimes \mathcal{H}^- \rightarrow \mathcal{H}^+ \otimes \mathcal{H}^- \) breaks up into 4 pieces, hence we expect these pieces will be essentially \( \alpha, \beta, \delta, \gamma \).

Here's how to determine \( \alpha \). Note that

\[ \left< 0|s|0 \right> = \left< 0|s|0 \right> e^{ab^*c^*} \]

Now if the eigenvectors of \( H_0 \) in increasing order of eigenvalue are \( \nu_1, \nu_2, \ldots, \nu_n \) with \( \omega_1 < \ldots < \omega_p < 0 < \omega_{p+1} < \ldots < \omega_n \), then

\[ |0\rangle = \nu_1 \cdots \nu_p \]

\[ S|0\rangle = (S\nu_1) \cdots (S\nu_p) \]

Assume that we are in the nice case where \( S\mathcal{H}^- \cap \mathcal{H}^+ = 0 \). Then \( S\mathcal{H}^- \) is the graph of a linear map \( T \) from \( \mathcal{H}^- \) to \( \mathcal{H}^+ \), and so we can find orthonormal basis \( e_1, \ldots, e_p \) for \( \mathcal{H}^- \) and \( e_{p+1}, \ldots, e_n \) for \( \mathcal{H}^+ \) so that

\[ Te_i = \lambda_i e_{p+i} \]

Then

\[ S(e_i) = \frac{e_i + \lambda_i e_{p+i}}{\sqrt{1 + \lambda_i^2}} \]

and

\[ |0\rangle = \frac{1}{\lambda_1} \frac{1}{\sqrt{1 + \lambda_2^2}} \left( e_1 + \lambda_1 e_{p+1} \right) \cdots \left( e_p + \lambda_p e_{2p} \right) \]

\[ \lambda_1 e_{p+1} \cdots + \lambda_p e_{2p} (c_1 \cdots c_p) \]

for the \( e_i \) basis.
But it is even simpler than this, maybe:
\[ e^{b^{*}c^{*}} \mid 0 \rangle = e^{\sum_{i=p}^{d} \alpha_{i} a_{i}^{*} a_{i}^{(v_{i} = \ldots v_{p})}} \]
\[ = (e^{\sum_{i=p}^{d} \alpha_{i} a_{i}^{*} e_{i}}) \cdots (e^{\sum_{i=p_{q}}^{d} \alpha_{i}^{q} e_{i}}) \]
\[ = (e_{i} + \sum_{i=p}^{d} \alpha_{i} e_{i}) \cdots (e_{p} + \sum_{i=p}^{d} \alpha_{i} e_{i}) \]

The point therefore seems to be that any element of \( N^{\circ} \) belonging a subspace complementary to \( H^{+} \) is uniquely represented in the form \( e^{b^{*}c^{*}} \mid 0 \rangle \).

In fact this is clear \( \Delta b^{*}c^{*} = \sum_{i=p}^{d} \alpha_{i} a_{i}^{*} a_{i} \) is a nilpotent (square zero) operator on \( H \) carrying \( H^{-} \) to \( H^{+} \).

It seems clear then that writing \( S \) in the form
\[ S = \langle 0 \mid \delta \mid 0 \rangle \]
\[ = e^{a^{*} a} \langle 0 \mid \delta \mid 0 \rangle e^{b^{*} \beta} \]
\[ = e^{a^{*} a} e^{b^{*} \beta} e^{c^{*} \gamma} e^{d^{*} \delta} \]
corresponds to factoring \( S : H^{-} \otimes H^{+} \rightarrow H^{-} \otimes H^{+} \) in the form
\[ S = \left( \begin{array}{cc} 1 & 0 \\ \alpha & 1 \end{array} \right) \left( \begin{array}{cc} \beta & 0 \\ 0 & \gamma \end{array} \right) \left( \begin{array}{cc} 1 & \delta \\ 0 & 1 \end{array} \right) \]
Yesterday we looked at $H = H_0 + V$ on $\mathcal{H}$. Then $S$ on $\Lambda^q \mathcal{H}$ is the automorphism extending

$$S = T \{ e^{-i \int dt V(t)} \} \quad \text{on } \mathcal{H}$$

The perturbation will create pairs when

$$S: \mathcal{H}^- \otimes \mathcal{H}^+ \longrightarrow \mathcal{H}^- \otimes \mathcal{H}^+$$

does not preserve this decomposition, because $S|_{\mathcal{H}^-}$ is the element of $\Lambda^q \mathcal{H}$ corresponding to the subspace $S(\mathcal{H}^-)$.

So now let us consider an infinite dimensional situation, say where $H_0$ is the Dirac operator and $V$ is the perturbation represented by a weak external time-independent EM field. Then we know the scattering operator $S$ exists on $\mathcal{H}$ and that it commutes with $H_0$. In particular $S$ has to preserve the decomposition into positive and negative eigenspaces. Thus $S$ cannot create pairs. So what happens is that $S$ separately moves the particles and holes around.
Consider two oscillators described by

\[ H_0 = \omega_1 q_1^* q_1 + \omega_2 q_2^* q_2 \]

and a coupling between them, say

\[ H_{int} = \varepsilon q_1 q_2 \]

\[ q_i = \frac{a_i + a_i^*}{\sqrt{2\omega_i}} \]

In practice, \( \omega_1, \omega_2 \) are large relative to \( \omega_1 - \omega_2 \), hence the terms involving \( q_1^* q_2, q_2 q_1 \) in \( H_{int} \) have rapidly changing phase in the Dirac picture. So if we were to compute

\[ U_0(t, t') = \exp\left(-i \int_{t'}^t dt' \varepsilon q_1(t') q_2(t')\right) \]

to first order these terms would be small. Thus we make the "rotating wave" approximation and drop them.

So we consider

\[ H_{int} = \varepsilon (q_1^* q_2 + q_2^* q_1). \]

The classical equations of motion are

\[ \dot{q}_1 = i [H, q_1] = i (-\omega_1 q_1 - \varepsilon q_2) \]

\[ \dot{q}_2 = i (-\omega_2 q_2 - \varepsilon q_1) \]

or simply

\[ i \left( \frac{a_1}{a_2} \right) = \left( \begin{array}{cc} \omega_1 & \varepsilon \\ \varepsilon & -\omega_2 \end{array} \right) \left( \begin{array}{c} a_1 \\ a_2 \end{array} \right) \]

Notice that

\[ H = \omega_1 q_1^* q_1 + \omega_2 q_2^* q_2 + \varepsilon (q_1^* q_2 + q_2^* q_1) \]

preserves "particle number". Hence the quantum situation is the symmetric algebra on the 1-particle space which is the same for both \( H_0 \) and \( H \). The 1-particle space is described by the "classical" equations (\#) which happens to be a Schrödinger equation for a 2-state
Think of the states of a 2-particle system as corresponding to points on the Riemann sphere, with \((0) \leftrightarrow \text{north pole}\) \((\tilde{0}) \leftrightarrow \text{south pole}\). If \(\varepsilon = 0\), these points stay fixed and
\[
\frac{a_1}{a_2}(t) = \frac{e^{-i\omega_1 t}a_1}{e^{-i\omega_2 t}a_2} = e^{i(\omega_2 - \omega_1)t} \frac{a_1}{a_2}
\]
rotates with angular frequency \(\omega_2 - \omega_1\).

Suppose \(\omega_1 = \omega_2\). The eigenvalues of \((\omega, \varepsilon)\) are \(\omega \pm \varepsilon\) and the eigenvectors are
\[
\begin{pmatrix} -1 \\ 1 \end{pmatrix} \quad \text{for} \quad \omega - \varepsilon \\
\begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \text{for} \quad \omega + \varepsilon
\]
Thus the perturbed system is described by a rotation leaving 1, -1 fixed. The state \((0)\) evolves as
\[
\psi(t) = e^{-i\omega t} \begin{pmatrix} \cos \varepsilon t \\ -i \sin \varepsilon t \end{pmatrix}
\]
and alternates between N and S. It's a mixture of the N,S states. It seems the the probability splitting
\[
\cos^2 \varepsilon t + \sin^2 \varepsilon t = 1
\]
is the relative height, or level, between N and S.

On the other hand if \(\varepsilon \ll |\omega_1 - \omega_2|\), then the eigenstates for the perturbed system are very close to those of the unperturbed system. Thus the perturbed system is described by rotation about an axis very close to NS. So we see that the states N, S are nearly invariant in this case.

These two cases exhibit the transitions \(N \rightarrow S\) when \(\omega_1 = \omega_2\), and the impossibility of such transitions when \(\omega_1 \neq \omega_2\) or more precisely \(\varepsilon \ll |\omega_1 - \omega_2|\).
Review Nyquist's relation: Suppose we have a transmission line described by
\[ \frac{\partial I}{\partial x} = -C_0 \frac{\partial V}{\partial t} \quad \frac{\partial V}{\partial x} = -L_0 \frac{\partial I}{\partial t} . \]

The travelling waves are
\[ \left( \begin{array}{c} V \\ I \end{array} \right) = \left( \begin{array}{c} \hat{V} \\ \hat{I} \end{array} \right) e^{i(kx-\omega t)} \]
with \( k \hat{I} = C_0 \omega \hat{V} \quad k \hat{V} = L_0 \hat{I} \omega \) or
\[ \frac{\hat{V}}{\hat{I}} = \frac{k}{C_0 \omega} = \frac{L_0 \omega}{k} \quad \text{speed} = \frac{\omega}{k} = \frac{1}{\sqrt{L_0 C_0}} \quad R_0 = \text{imped.} = \sqrt{\frac{L_0}{C_0}} \]

Suppose the transmission line is terminated by a circuit at \( x=0 \) with impedance \( Z(\omega) \):

The normal mode are
\[ V(x,t) = \Re \left( A \left( e^{-ikx} + S(\omega) e^{-ikx} \right) e^{-i\omega t} \right) \quad \hat{k} = \frac{\sqrt{L_0}}{\omega} \]
\[ I(x,t) = \frac{1}{R_0} \Re \left( A \left( -e^{-ikx} + S(\omega) e^{-ikx} \right) e^{-i\omega t} \right) \]
where the reflection coefficient \( S(\omega) \) satisfies
\[ + Z(\omega) = R_0 \frac{S+1}{S-1} \]

The energy density for this mode is
\[ \langle \frac{1}{2} C_0 V^2 + \frac{1}{2} L_0 I^2 \rangle = \frac{1}{2} C_0 \left| A \right|^2 \left( e^{-ikx} + S e^{-ikx} \right)^2 \]
\[ + \frac{1}{2} \left| A \right|^2 \left( e^{-ikx} + S e^{-ikx} \right)^2 \]
\[ = C_0 \left| A \right|^2 \]
assuming the circuit is lossless, so that \( |S| = 1 \).

Fix a length \( l \) of the line and terminate it by a short circuit. Then the normal modes are given by frequencies \( \omega \) satisfying
\[
e^{-i\omega l} + S(\omega)e^{-i\omega l} = 0 \quad \text{or} \quad S(\omega)e^{-i\omega l} - e^{-2i\omega l} = 0
\]

If \( l \) is large and we want the density of modes around a fixed frequency, the variation of \( S(\omega) \) can be neglected, and so the modes are distributed for small change
\[
k\omega = n\pi + S(\omega) \quad n \in \mathbb{Z}
\]

hence the number in the range \( \omega \) to \( \omega + dw \) is
\[
dn = \frac{\ell}{\pi} \frac{1}{V_{\omega}} \, dw.
\]

Let's now suppose the system is at the temperature \( T \). Then the situation is described by the Maxwell-Boltzmann distribution in phase space. Since we have an oscillator, this means each amplitude \( A_\omega \) is a Gaussian variable such that the average energy of the mode is \( kT \). The energy is \( C_0 \langle |A_\omega|^2 \rangle \) plus the energy inside the circuit. If \( l \) is large then approximately we have
\[
C_0 l \langle |A_\omega|^2 \rangle = kT
\]

Now
\[
V_{x=0} = \sum_\omega \text{Re} \left( A_\omega (1 + S(\omega)) \right)
\]

so
\[
\langle V^2 \rangle = \sum_\omega \frac{1}{2} \langle |A_\omega|^2 \rangle \left| 1 + S(\omega) \right|^2 \frac{dn}{\pi \sqrt{V_{\omega}}} \frac{\ell}{C_0} \frac{1}{\sqrt{2\pi} \omega} \, dw
\]
\[
\langle V_o^2 \rangle = \int_0^{\infty} \frac{dw}{\pi} \cdot \frac{1}{2} kT R_o \cdot \left| 1 + S(w) \right|^2
\]

On the other hand, we can think of the line as a resistor \( R_o \) plus noise generator

\[
\begin{array}{c}
\begin{array}{c}
|Z| \quad V_o \quad \frac{R_o}{Z} \\
\hline
V_o \quad V_i \\
\end{array}
\end{array}
\]

Whence

\[
\frac{Z}{R_o + Z} = \frac{S+1}{S-1} = \frac{S+1}{2S}
\]

\[
\left| \frac{Z}{R_o + Z} \right|^2 = \left| \frac{S+1}{2} \right|^2 \quad \text{since} \quad |S| = 1.
\]

\[
\therefore \quad \langle V_o^2 \rangle = \int_0^{\infty} \frac{dw}{\pi} \frac{2kTR_o}{\pi} \left| \frac{S+1}{2} \right|^2 \left| \hat{V}_i(w) \right|^2
\]

and so one concludes that

\[
\langle V_o^2 \rangle = \int_0^{\infty} \frac{dw}{\pi} 2kTR_o
\]

which is the Nyquist relation.

The above is sloppy in many respects at the
First of all one has used the Wiener-Khintchine theorem to get
\[
\langle V_o^2 \rangle = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} |V_\omega(\omega)|^2
\]
and secondly one argues that one can take various impedances $Z$. For example, Wannier takes $Z$ to be a tuned circuit
\[
\frac{1}{L} \frac{1}{C}
\]
with a sharp resonance frequency. By equipartition one knows that
\[
\frac{1}{2} C \langle V_o^2 \rangle = \frac{1}{2} kT,
\]
doing so one can calculate $\frac{Z}{R_0 + Z}$ by computing $s^2$ one can check the Nyquist formula at the resonant frequency.

Let's go over this in detail. For the tuned circuit
\[
Z(\omega) = \frac{1}{Cs + \frac{1}{Ls}}
\]
we have
\[
\frac{s+1}{2s} = \frac{Z}{R_0 + Z} = \frac{1}{R_0(Cs + \frac{1}{Ls}) + 1} = \frac{Ls}{R_0} \frac{1}{LCS^2 + \frac{Ls}{R_0} + 1}
\]
Thus
\[
\langle V_o^2 \rangle = 2kT R_0 \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \left| \frac{s}{R_0 C (s^2 + \frac{1}{R_0 C} s + \frac{1}{LC})} \right|^2
\]
\[
= \frac{2kT R_0}{(R_0 C)^2} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \frac{\omega^2}{(\omega^2 - \omega_0^2)^2 + g^2 \omega^2}
\]
where $\omega_0 = \frac{1}{1/RC}$ and $g = \frac{1}{R_0 C}$. 

The integral can be evaluated by residues and yields
\[ \mathcal{V} = \frac{2kTR_0}{(R_0C)^2} \frac{R_0C}{2} = \frac{kT}{C} \]
so that \( \frac{1}{2}kT = \frac{1}{2}C\langle V_o^2 \rangle \) as required.

Let's go over why this has to hold. We are calculating \( \langle V_o^2 \rangle \) by the Maxwell-Boltzmann distribution. The phase space involved has points given by the pair of functions \( \left( \frac{V(x)}{I(x)} \right) \), together with the current through the inductance, call it \( I_L \). Then on this vector space we have the energy
\[ H = \frac{1}{2}L I_L^2 + \frac{1}{2}C V_o^2 + \int \left\{ \frac{1}{2}C_0 V(x)^2 + \frac{1}{2}L_0 I(x)^2 \right\} dx \]
Therefore on this vector space goes the Gaussian measure \( e^{-\beta H} \), and hence \( \langle \frac{1}{2}I_L^2 \rangle = \langle \frac{1}{2}CV_o^2 \rangle = \frac{1}{2}kT \).
Consider a line of density \( \lambda \), tension \( \lambda \) connected to an oscillator.

\[ \lambda \partial_x^2 u = \lambda \partial_x^2 u \quad x \geq 0 \]

\[ M(u_0 + \omega_0^2 u_0) = \lambda (\partial_x u)_0 \]

Suppose the string is tied down at \( x = b \), where \( b \) is very large. The motion is of the form

\[ u(x, t) = \sum_{\omega} \text{Re} \left( A_\omega (e^{-i\omega x + i\omega x} + s(\omega) e^{i\omega x}) e^{-i\omega t} \right) \]

where \( \omega \) runs over solutions \( > 0 \) of

\[ e^{-i\omega b} + s(\omega) e^{i\omega b} = 0 \]

and \( s(\omega) \) is determined by

\[ M(\omega^2 + \omega_0^2)(1 + s) = 2\omega(1 - s) \quad \omega = -i\omega \]

\[ \frac{1 + s}{1 - s} = \frac{\lambda a}{M(\omega^2 + \omega_0^2)} \]

The energy density along the string is

\[ \frac{1}{2} \lambda u^2 + \frac{1}{2} \lambda (\partial_x u)^2 \]

For the \( \omega \)-th mode, the time-averaged energy density is

\[ \lambda |A_\omega|^2 \omega^2 \]

so proceeding as in the Nyquist problem

\[ \langle u_0^2 \rangle = \int_0^\infty \frac{d\omega}{\pi} \frac{2kT}{\lambda \omega^2} \left| \frac{1 + s_\omega}{2} \right|^2 \]
Here \( \frac{1+S}{2} = \frac{Z}{Z+1} = \frac{\Delta \nu}{M(o^2 + o_e^2) + \Delta \nu} \)

and a residue calculation again gives

\[
\frac{1}{2} M o_e^2 \left< u_e^2 \right> = \frac{1}{2} kT
\]

as required by equi-partition.

It seems to be possible to understand emission classically as follows. Consider the string as an absorber so that only outgoing waves are allowed. Then we have a damped harmonic oscillator. In effect

\[ \ddot{u} + i \omega_o u = 0 \]

for outgoing waves, so

\[ M(\dot{u}_o^2 + \omega_o^2 u_o) = -\lambda \dot{u}_o \]

or

\[ \ddot{u}_o + \gamma \dot{u}_o + \omega_o^2 u_o = 0 \]

where \( \gamma = \frac{\lambda}{M} \)

The solutions are

\[ u_o = e^{-\frac{\gamma t}{2}} \text{Re} (A e^{-i\omega t}) \]

\[ \omega_1 = \sqrt{\omega_o^2 - \frac{\gamma^2}{4}} \]

In effect the roots of \( +\omega^2 + i\gamma \omega - \omega_o^2 = 0 \) are

\[ \omega = -\frac{i\gamma}{2} \pm \sqrt{\omega_o^2 - \frac{\gamma^2}{4}} \]

Thus we see that the energy, potential or kinetic, has the decay rate \( e^{-\gamma t} \). (Assume \( \gamma \ll \omega_o \).)
The problem is still to find a simple example of emission and absorption. Let's consider a simple oscillator coupled to a continuous family of oscillators. The Hamiltonian is

$$H = \omega_0 b^* b + \sum_{\alpha} \omega_{\alpha} a_{\alpha}^* a_{\alpha} + \sum_{\alpha} b^* (\delta_{\alpha} a_{\alpha}) + (a_{\alpha}^* \delta_{\alpha}) b.$$ 

Because this Hamiltonian is of degree 0, i.e., it commutes with the number operator $b^* b + \sum_{\alpha} a_{\alpha}^* a_{\alpha}$, it follows that its effect on states of many particles is determined by what it does to 1-particle states. More precisely, the Hilbert space is the symmetric algebra on the 1-particle space $\mathcal{H}$ which is spanned by the vectors $b^* |0\rangle$, $a_{\alpha}^* |0\rangle$. On this 1-particle space, time evolution is described by

$$i \frac{\partial}{\partial t} \psi = H \psi \quad \text{with} \quad H = \left( \begin{array}{cc} \omega_0 & \delta \\gamma & H_0 \end{array} \right),$$

where $H_0 = \sum_{\alpha} |\alpha\rangle \langle \alpha|$, $\delta = \sum_{\alpha} \delta_{\alpha} |\alpha\rangle \langle \alpha|$, and $\delta^* = \sum \delta_{\alpha}^* |\alpha\rangle \langle \alpha|$. Let

$$\Phi = b^* |0\rangle.$$ 

The first thing is to understand how $\Phi$ decays. Thus we want to understand $\langle \Phi | e^{-iHt} | \Phi \rangle$ as $t \to +\infty$. One has

$$\int_0^\infty dt e^{iWt} \langle \Phi | e^{-iHt} | \Phi \rangle = \langle \Phi | \frac{-1}{i(W - H)} | \Phi \rangle.$$ 

Hence

$$\langle \Phi | e^{-iHt} | \Phi \rangle = \int_{-\infty}^{\infty} \frac{dW}{2\pi} \langle \Phi | \frac{-1}{i(W + i\epsilon - H)} | \Phi \rangle e^{-iWt} \quad \text{for } \epsilon > 0.$$
Now we can compute $\frac{1}{W-H}$ by diagrams as in Weinberg's paper. One finds
\[ \langle \Phi | \frac{1}{W-H} | \Phi \rangle = \frac{1}{\Phi} + \frac{1}{\Phi} \langle g | \frac{1}{W-H_0} | g \rangle \frac{1}{W-H_0} + \ldots \]
So
\[ \langle \Phi | \frac{1}{W-H} | \Phi \rangle = \frac{1}{W-H_0} - \langle g | \frac{1}{W-H_0} | g \rangle \frac{1}{W-H_0} \]

Remark: We can split the space on which $H_0$ operates into the cyclic subspace spanned by $g$ and the orthogonal complement. We can forget about the latter in trying to understand $H$ from $H_0$, and so can suppose $g$ is a cyclic vector for $H_0$.

One has
\[ \langle g | \frac{1}{W-H_0} | g \rangle = \int \frac{d\mu(w)}{(W-w)} \]
for some measure on the line. Let's suppose
\[ d\mu(w) = |\omega|^2 dw \]
and put
\[ g(W) = \int \frac{|\omega|^2 dw}{W-w} = \langle g | \frac{1}{W-H_0} | g \rangle \]
for $W$ in the UHP. Define $g$ by analytic continuation from the UHP. From the formula
\[ \langle \Phi | e^{-iHt} | \Phi \rangle = \int \frac{dW}{2\pi i} e^{-iWt} \left( \frac{-1}{W-H_0-g(W)} \right) \]
and the fact that $e^{-i\omega t}$ decays in the LHP for $t \rightarrow \infty$, we see that the behavior of $\langle \bar{\Phi} | e^{-iHt} |\Phi \rangle$ is controlled by the singularities in the LHP nearest the real axis.

Suppose $\delta$ is small. Then $g(W)$ should be small so that near $\omega_0$ is a root $\tilde{\omega}_0$ of $W - \omega_0 - g(W)$, which can be found by iterating:

\[
W^{(0)} = \omega_0 \\
W^{(1)} = \omega_0 + g(\omega_0) \\
W^{(2)} = \omega_0 + g(\omega_0 + g(\omega_0)) = \omega_0 + g(\omega_0) + g'(\omega_0)g(\omega_0) + \ldots
\]

Then $W^{(n)} \rightarrow \tilde{\omega}_0$. Now $g(W) \in \text{LHP}$ when $W \in \text{UHP}$, in fact we know that on the real $W$ axis

\[
\text{Im}(g(W)) = \int |\delta W|^2 \, d\omega \, \text{Im}\left( \frac{-i\delta}{(W + i0^+ - \omega)} \right)
\]

\[
= \int |\delta W|^2 \left( -\pi \delta'(W - \omega) \right) \, d\omega
\]

\[
= -\pi |\delta W|^2
\]

Therefore $\tilde{\omega}_0$ is in the LHP. We can deform the contour $W$-plane

\[
\text{so as to get}
\]

\[
\langle \Phi | e^{-iHt} | \bar{\Phi} \rangle = \frac{1}{1 - g'(\tilde{\omega}_0)} e^{-i\tilde{\omega}_0 t} + \text{faster decaying exponentials}
\]
Thus it should be true that
\[ \langle \overline{\pi} | e^{-iHt} | \pi \rangle = (1 + \gamma'(\omega_0)) e^{-i\omega_0t + \gamma(\omega_0)t} (1 + o(\eta^2)) \]
\[ = e^{-i\omega_0t} (1 + \gamma'(\omega_0) \cdot -i\gamma(\omega_0)t + o(\eta^2)) \]

Let's check this by perturbation theory.
\[ \langle \overline{\pi} | e^{-iHt} | \pi \rangle = e^{-i\omega_0t} + (-i) \int_0^t dt_1 \int_0^{t_1} dt_2 e^{-i\omega_0(t-t_1)} \langle \gamma | e^{-iH_0(t-t_1)} | \gamma \rangle \]
\[ + (-i)^2 \int_0^t dt_1 \int_0^{t_1} dt_2 \int_0^{t_2} dt_3 \langle \gamma | e^{-iH_0(t-t_3)} | \gamma \rangle \]
\[ \times e^{-i\omega_0(t-t_2)} \]
\[ = e^{i\int_0^t d\omega |\gamma(\omega)|^2} \int_0^t dt_1 \int_0^{t_1} dt_2 \ e^{i\omega_0(t_1-t_2)} e^{-i\omega(t_1-t_2)} \]

Now,
\[ \int_0^t dt_1 \int_0^{t_1} dt_2 \ e^{i\omega(t-t_2)} = \int_0^t dt \ e^{i\omega t_1} \frac{e^{-i\omega t} - 1}{-i\omega} = \frac{1 + i\omega t - e^{i\omega t}}{\omega^2} \]
\[ \langle \overline{\pi} | e^{-iHt} | \pi \rangle = e^{-i\omega_0t} \left[ 1 - \int d\omega |\gamma(\omega)|^2 \frac{1 + i(\omega_0-\omega)t - e^{i(\omega_0-\omega)t}}{(\omega_0-\omega)^2} \right] \]

But if we let \( \omega_0 \) be approached from the UHP
\[ -\int d\omega |\gamma(\omega)|^2 \frac{1 + i(\omega_0-\omega)t}{(\omega_0-\omega)^2} = -i \gamma(\omega_0)t + \gamma'(\omega_0) \]
and the term
\[ -\int d\omega |\gamma(\omega)|^2 \frac{-e^{i(\omega_0-\omega)t}}{(\omega_0-\omega)^2} \]

for \( \omega_0 \) above the real axis, can be evaluated by pushing the
the contour till it catches the singularities of \( |\tilde{\omega}|^2 \); hence it contributes faster decaying exponentials. Thus our conclusion is that the exact behavior is

\[
\langle \Phi | e^{-i H t} | \Phi \rangle = \frac{1}{1 - g^2(\tilde{\omega})} e^{-i\tilde{\omega}_0 t} + \text{faster decay terms}
\]

where \( \tilde{\omega}_0 \) satisfies:

\[
\tilde{\omega}_0 = \omega_0 + g(\tilde{\omega})
\]

Thus

\[
\text{Im}(\tilde{\omega}_0) = \text{Im} g(\tilde{\omega}) = -\pi |\tilde{\omega}_0|^2 + O(1/\tilde{\omega}_0^4)
\]

Next we could use some specific examples. Ultimately we want \( H_0 \) to have spectrum \( \geq 0 \) so \( g(\omega) \) should be an analytic function with a cut along the positive real axis. Unfortunately this puts a singularity on the real axis so there is no way to construct an example of the above type where \( \langle \Phi | e^{-i H_0 t} | \Phi \rangle \) has exponential decay and such that \( H_0 \) has spectrum \( \geq 0 \).

So let's look at some examples where the spectrum of \( H_0 \) is the whole line.
December 26, 1980

Let's return to what happens on the quantum mechanics level. We are working with a harmonic oscillator hence we know that there is a boson particle structure on the quantum states.

Review the formalism for an oscillator with a finite number of degrees of freedom. Say

\[ H = \frac{\dot{q}^2}{2m} + \frac{1}{2} m \omega^2 q^2 \]

where \( \omega^2 \) is a positive-definite matrix of which \( \omega \) is the positive square root. The classical equation of motion is

\[ \left( \frac{d^2}{dt^2} + \omega^2 \right) q = 0 \]

and it has solutions

\[ q = e^{-i\omega t} A + e^{i\omega t} B \quad B = \overline{A} \]

or better

\[ q = Re \left( e^{-i\omega t} A \right) \]

The energy of this solution is

\[ E = \frac{1}{2} |wA|^2 \]
December 27, 1980

Let's look at some examples of light strings terminated differently. Define the impedance to be \( Z = \frac{1+R}{1-R} \) where \( R \) is the reflection coefficient.

1) tied at \( x=0 \): Here \( R = -1 \)
2) free at \( x=0 \): \( R = 1 \).
3) weightless stretch of length \( b \) tied at \( x=-b \)

\[
\lambda \left( \partial_x u \right)_0 = \frac{u_0}{b} \quad \lambda s(1-R) = \frac{1}{b(1+R)}
\]

\[
\therefore Z = \frac{1+R}{1-R} = \lambda b s
\]

As \( s = -i\omega \) goes from \( 0 \) to \( -i\omega \),

\( Z \) goes from \( 0 \) to \( -i\infty \), \( \infty \)

\[
R = \frac{Z-1}{Z+1} \quad \text{goes from } -1 \text{ to } 1
\]

\[
\arg(R) \quad \text{goes from } -\pi \text{ to } 0
\]

4) mass \( M \) at \( x=0 \):

\[
M \ddot{u}_0 = \lambda \left( \partial_x u \right)_0 \Rightarrow M s^2 (1+R) = \lambda s (1-R)
\]

\[
\therefore Z = \frac{1+R}{1-R} = \frac{2s}{M s^2} = \frac{2}{M s}
\]

\( Z \) goes from \( i\infty \) to \( 0 \) so \( \arg(R) \) goes from \( 0 \) to \( \pi \)

5) mass \( M \) at \( x=0 \) + weightless stretch tied at \( x=-c \)

\[
M \ddot{u}_0 + \frac{1}{c} u_0 = \lambda \left( \partial_x u \right)_0
\]

\[
(M s^2 + \frac{1}{c})(1+R) = \lambda s (1-R)
\]

\[
Z = \frac{\lambda s}{M s^2 + \frac{1}{c}} \quad \text{goes from } 0 \text{ to } -i\omega, \text{then } i\omega \text{ to } 0.
\]
\[ M \dot{\theta} + \frac{1}{c} \dot{\theta} = \frac{u_0 - \dot{\theta}}{b} = \lambda (2 \pi \omega) \]

Since \[ u = Ae^{-\omega t} \]

\[ (M \dot{\theta}^2 + \frac{1}{c} \dot{\theta}) = \frac{1 + R - A}{b} = \lambda s (1 - R) \]

\[ (M \dot{\theta}^2 + \frac{1}{c} + \frac{1}{b}) = \frac{1}{b} (1 + R) \]

\[ (1 + R) \left[ 1 - \frac{1}{b (M \dot{\theta}^2 + \frac{1}{c} + \frac{1}{b})} \right] = b \lambda s (1 - R) \]

\[ Z = \frac{b \lambda s}{1 - \frac{1}{b (M \dot{\theta}^2 + \frac{1}{c} + \frac{1}{b})}} = \frac{b \lambda s (M \dot{\theta}^2 + \frac{1}{c} + \frac{1}{b})}{M \dot{\theta}^2 + \frac{1}{c}} \]

Thus \[ \omega R \] goes from \(-\pi\) to \(0\) to \(\pi\) to \(0\).

A point that emerges from these examples is that \( \arg(R) \) changes \( \pi \) in going from \( \omega = 0 \) to \( \omega = \infty \) where \( n \) is the number of quadratic terms in the energy in the form. Thus we know that \( \frac{3}{2} k_b T \) is the thermal energy one can store in the two strings and the \( \frac{1}{2} M \) kinetic energy of the mass \( M \) in example 6.)
Let's now interpret these "strings" as (generalized) oscillators. First look at example 6: A configuration consists of \( \phi = (v, u) \) where \( v \in \mathbb{R} \), \( u \) is real fun in \( C_c^\infty (\mathbb{R}_+ \mathbb{R}) \). The kinetic energy is \( \frac{1}{2} \| \phi \|^2 \) where

\[
\frac{1}{2} \| \phi \|^2 = \frac{1}{2} M v^2 + \frac{\lambda}{2} \int_0^\infty u^2 \, dx
\]

and the potential energy is

\[
\frac{1}{2} P(\phi) = \frac{1}{2c} v^2 + \frac{1}{2b} (u_0 - v)^2 + \frac{\lambda}{2} \int_0^\infty (\partial_x u)^2 \, dx
\]

We want to write

\[
P(\phi) = \langle \phi \mid L \phi \rangle
\]

where \( L \) is an operator and \( \langle \phi \mid \phi \rangle = \| \phi \|^2. \) So polarize \( P: \)

\[
\frac{1}{2} P(\phi, \phi) = \frac{1}{2c} v^2 + \frac{1}{2b} (u_0 - v)^2 + \frac{\lambda}{2} \int_0^\infty (\partial_x u) (\partial_x u) \, dx
\]

\[-\frac{\lambda}{2} u_0 (\partial_x u)_0 + \frac{\lambda}{2} \int_0^\infty u (-\partial_x^2 u) \, dx
\]

In order that this can be written \( \langle \phi \mid L \phi \rangle \) we must have that \( \phi = (v, u) \) makes the \( u_0 \) vanish, hence

\[
\frac{1}{b} (u_0 - v) = \lambda (\partial_x u)_0
\]

whence

\[
\frac{1}{2} P(\phi, \phi) = \frac{1}{2} M v \left[ (\frac{1}{c} + \frac{1}{b}) v - \frac{u_0}{b} \right] \frac{1}{M} + \frac{\lambda}{2} \int_0^\infty u (-\partial_x^2 u) \, dx.
\]

Thus

\[
L(v, u) = \left( (\frac{1}{c} + \frac{1}{b}) v - \frac{u_0}{b} \right) v + \frac{\lambda}{2} \partial_x^2 u
\]

The eigenfunctions of \( L \) with eigenvalue \( \omega^2 \) are given by solutions of

\[
\omega^2 u = \frac{1}{M} \left[ (\frac{1}{c} + \frac{1}{b}) v - \frac{u_0}{b} \right] \quad \frac{1}{b} (u_0 - v) = \lambda (\partial_x u)_0
\]

\[
\omega^2 u = -\partial_x^2 u
\]
and hence are of the form \( \phi_\omega = (v_\omega, u_\omega) \)

\[
\begin{align*}
  v_\omega &= e^{-i\omega x} + R_\omega e^{i\omega x} \\
\end{align*}
\]

\[
\nu = \Box A_\omega
\]

where

\[
(M(-\omega^2) + \frac{1}{c}) A = \frac{1 + R - A}{b} = \lambda(\nu)(1 - R)
\]

Example where \( b = 0 \):

\[
\|u\| = Mu_0^2 + \lambda \int_0^\infty u^2 \, dx
\]

\[
P(v) = \frac{1}{c} u_0^2 + \lambda \int_0^\infty (\partial_x u)^2 \, dx
\]

\[
P(\overline{v}, u) = \frac{1}{c} \overline{u}_0 u_0 - \lambda \overline{u}_0 (\partial_x u)_0 + \lambda \int_0^\infty \overline{u} (\partial_x u^2) \, dx
\]

\[
\langle \overline{u} | L u \rangle = Mu_0 (L u)_0 + \lambda \int_0^\infty \overline{u} L u \, dx
\]

leads to

\[
L u = -\partial_x^2 u
\]

\[
M(Lu)_0 = \frac{1}{c} u_0 - \lambda (\partial_x u)_0
\]

Thus for \( u \) to be in the domain of \( L \) we must have

\[
M(\partial_x^2 u)_0 + \frac{1}{c} u_0 = \lambda (\partial_x u)_0
\]

The eigenfunctions are

\[
u_\omega = e^{-i\omega x} + R_\omega e^{i\omega x}
\]

where

\[
\frac{1 + R}{1 - R} = \frac{\lambda s}{Ms^2 + \frac{1}{c}}
\]

Now let's fix a length \( l \) and tie the string at \( x = l \). We then get eigenvalues \( \omega_\alpha \) by solving
\[ R_{\omega} + e^{-2i\omega t} = 0 \]

and corresponding eigenfunctions \( u_n = u_n \). Assuming these eigenfunctions are complete we know that

\[ \frac{u_n}{\|u_n\|} \]

is an orthonormal basis for functions on \([0, l]\) with the norm \( \|f\|^2 = M|f_0|^2 + \lambda \int_0^l |f|^2 \, dx \). Hence we should have an expansion formula

\[ f = \sum_n \frac{u_n}{\|u_n\|^2} \langle u_n | f \rangle \]

Now let \( l \to \infty \). For large \( l \), the variation of \( R_{\omega} \) can be neglected so that the number of eigenvalues \( \lambda \) in a range \( \delta \omega \) is given by \( d\lambda = \frac{\delta \omega}{\pi} \).

Also

\[ \|u_n\|^2 = M|u_n(0)|^2 + \lambda \int_0^l |u_n|^2 \, dx \approx 2\frac{\lambda}{\delta} \frac{l}{\pi} = 2\lambda \delta \]

Thus the expansion formula becomes

\[ f(x) = \int \frac{d\omega}{2\lambda \pi} u_{\omega}(x) \langle u_{\omega} | f \rangle \]

For example, take the free line: \( u_{\omega}(x) = 2 \cos(\omega x) \)

\[ f(x) = \int \frac{d\omega}{\lambda \pi} \cos \omega x \cdot \left( 2\lambda \int_0^\infty \cos(\omega x') f(x') \, dx' \right) \]

which checks.

Another check is as follows. Take again and let

\[ f(x) = \begin{cases} 0 & x > 0 \\ 1 & x = 0 \end{cases} \]

Then

\[ \langle u_\omega | f \rangle = M(1 + R_{\omega}) \]

and so we should.
have the formula
\[ f(x) = \int_{-\infty}^{\infty} \frac{M \, d\omega}{2 \pi} \left( \frac{e^{-i\omega x} + R_\omega e^{i\omega x}}{1 + R_\omega} \right) \]
\[ \frac{e^{-i\omega x} + e^{i\omega x} + R_\omega e^{i\omega x} + R_\omega e^{-i\omega x}}{1 + R_\omega} \]
\[ = \int_{-\infty}^{\infty} \frac{M \, d\omega}{2 \pi} (1 + R_\omega) e^{i\omega x} \]

Now, \( R_\omega \) is analytic in the UHP, so this will vanish for \( x > 0 \). If \( x = 0 \) then use
\[ \frac{1 + R}{2} = \frac{1 + R}{1 - R} = \frac{\lambda s}{M s^2 + \lambda s + M \omega^2} \approx \frac{\lambda}{Ms} \]

\[ f(0) = \int \frac{M}{2 \pi} \frac{2\pi \, d\omega}{M(-i\omega)} = \int_{-\pi}^{\pi} \frac{Re^{i\theta} \, d\theta}{i - Re^{i\theta}} = 1 \]

which checks.