December 12, 1980

Recall that for a simple harmonic oscillator with a finite time perturbation

$$H = \frac{p^2}{2} + \frac{1}{2}\left(\omega^2 + \varepsilon(t)\right)\frac{q^2}{2}$$

one gets an S-operator

$$\langle e_{\bar{\mu}} | S | e_\lambda \rangle = \langle 0 | S | 0 \rangle \, e^{\frac{1}{2}(R\lambda^2 + 2T\lambda\mu + \tilde{R}\mu^2)}$$

where one has the asymptotic behavior

$$Re^{i\omega t} + e^{-i\omega t} \longleftrightarrow Te^{-i\omega t}$$

$$Te^{i\omega t} \longleftrightarrow e^{i\omega t} + \tilde{R}e^{-i\omega t}$$

for the solutions of the classical equation of motion

$$\left(\frac{d^2}{dt^2} + \omega^2 + \varepsilon\right) q = 0$$

(see p. 247).

Next I want to apply this to the infinite dimensional oscillator represented by the wave eqn.

$$\left[\partial_t^2 + (-\partial_x^2)\right]\phi = 0$$

and the perturbation given by the wave equation with potential:

(*) $$\left[\partial_t + (-\partial_x^2 + V)\right]\phi = 0$$

The idea is that because of the continuous spectrum of $(-\partial_x)^2$ we can deal with V independent of t.

Let us first examine the asymptotic behavior of the solutions of the classical equation of motion (*). Let's work on $x \geq 0$ with bdry condition $\partial_x \phi = 0$ at $x = 0$. Then a solution of (*) is of the form

$$\phi(x,t) = \int \frac{d\omega}{2\pi}\, u(x,\omega)\, e^{-i\omega t} f(\omega)$$

where $\quad (-\omega^2 - \partial_x + V)\, u_\omega(x) = 0 \;, \quad \partial_x u_\omega = 0 \quad$ at $x=0$

and

$$u_\omega(x) \sim e^{-i\omega x} + R(\omega) e^{i\omega x} \qquad x \to +\infty .$$

Notice that ϕ can be written

$$\phi(x,t) = \int_0^\infty \frac{d\omega}{2\pi} e^{-i\omega t}\, u(x,\omega) f(\omega) + \int_0^\infty \frac{d\omega}{2\pi} e^{i\omega t}\, u(x,-\omega) f(-\omega)$$

$$= e^{-iPt} \int_0^\infty \frac{d\omega}{2\pi} u_\omega\, f(\omega) + e^{iPt} \int_0^\infty \frac{d\omega}{2\pi} u_{-\omega} f(-\omega)$$

where $\quad P = +\sqrt{-\partial_x^2 + V}$.

Because of the Riemann-Lebesgue lemma, we know that $\phi(x,t)$ has the following asymptotic behavior

$$\phi(x,t) \sim \int \frac{d\omega}{2\pi} \left(e^{-i\omega x} + e^{i\omega x}\right) e^{-i\omega t} f(\omega) \qquad t \to -\infty$$

$$\sim \int \frac{d\omega}{2\pi} \left(\quad " \quad \right) e^{-i\omega t} R(\omega) f(\omega) \qquad t \to +\infty$$

where on the right are solutions of the "free" wave equation. Therefore if we use these asymptotics we have

$$e^{-iP_0 t} \int_0^\infty \frac{d\omega}{2\pi} u^0_\omega f(\omega) \quad \overset{\phi}{\longleftrightarrow} \quad e^{-iP_0 t} \int_0^\infty \frac{d\omega}{2\pi} u^0_\omega R(\omega) f(\omega)$$

$$+ e^{iP_0 t} \int_0^\infty \frac{d\omega}{2\pi} u^0_{-\omega} f(-\omega) \qquad\qquad + e^{iP_0 t} \int_0^\infty \frac{d\omega}{2\pi} u^0_{-\omega} R(-\omega) f(-\omega)$$

Hence we have

$$e^{-iP_0 t} \hat{f} \longleftrightarrow e^{-iP_0 t} \widehat{Rf}$$

$$e^{iP_0 t} \hat{g} \longleftrightarrow e^{-iP_0 t} \widehat{\bar{R}f}$$

where $\quad \hat{f} = \int \frac{d\omega}{2\pi} u^0_\omega f(\omega)$

Thus there is no reflection, only transmission, and the T operator is multiplication by $R(\omega)$.

It follows that the S-operator on the quantum level is essentially trivial, in the sense that there is no creation and annihilation of particles, only a phase shift.

But it would be interesting to ask about $\langle 0|S|0\rangle$. We have

$$\langle e_{\bar{\mu}}|S|e_{\lambda}\rangle = \langle 0|S|0\rangle e^{\mu^t T\lambda}$$

and also

$$|\langle 0|S|0\rangle|^2 = |\det T| = 1.$$

Consider the fermion analogue of the perturbed oscillator. One is given H_0 on $\mathcal{H}$, then H_0 is extended to Fock space $\Lambda(\mathcal{H})$ as a derivation. Assuming H_0 does not have 0 as eigenvalue, the ground state of H_0 on $\Lambda(\mathcal{H})$ is the element belonging to the subspace $\mathcal{H}^-$ of $\mathcal{H}$ on which $H_0 < 0$. Next suppose $H_0^{\text{on } \mathcal{H}}$ is perturbed:

$$H = H_0 + V(t)$$

where $V(t)$ has compact support. We want to compute the S-matrix of this perturbation on $\Lambda\mathcal{H}$.

Because H is extended as a derivation, the operators e^{-itH_0}, $U(t,t')$, $U_D(t,t')$ are the autos. of $\Lambda\mathcal{H}$ induced by the corresponding autos. of $\mathcal{H}$. Then the S-matrix on $\Lambda\mathcal{H}$ is the extension of the S-matrix on $\mathcal{H}$.

By analogy with boson case, we would like to write S in terms of creation + annihilation operators. The idea will be to work with particles and hole operators as follows. Let H_0 have the eigenvalues ω_α and $H_0 v_\alpha = \omega_\alpha v_\alpha$, and let a_α, a_α^* be the corresponding annihilation and creation operators. Let $b_\alpha = a_\alpha$ if $\omega_\alpha > 0$ and $c_\alpha = a_\alpha^*$ if $\omega_\alpha < 0$. Then $V(t)$ lies in the Lie algebra of operators spanned by $a_\alpha^* a_\beta$ and these fall into four types

$\begin{matrix} b^* & b \\ c & c^* \end{matrix}$

$b^* c^*$ $\qquad$ $b^* b$ $\qquad$ $c b^*$ $\qquad$ $c b$

Consequently we expect the S-matrix to appear in the form

$$S = \langle 0|S|0\rangle \, e^{\alpha b^*c^*} \, e^{\beta b^* b} \, e^{\gamma c c^*} \, e^{\delta c b}$$

Furthermore we know that $S: \mathcal{H}^+ \oplus \mathcal{H}^- \longrightarrow \mathcal{H}^+ \oplus \mathcal{H}^-$ breaks up into 4 pieces, [redacted] hence we expect these pieces will be essentially $\alpha, \beta, \gamma, \delta$.

Here's how to determine α. Note that

$$S|0\rangle = \langle 0|S|0\rangle \, e^{\alpha b^* c^*} |0\rangle$$

Now if the eigenvectors of H_0 in increasing order of eigenvalue are $v_1, \ldots, v_p, \ldots, v_n$ with $\omega_1 < \ldots < \omega_p < 0 < \omega_{p+1} < \ldots < \omega_n$, then

$$|0\rangle = v_1 \wedge \ldots \wedge v_p$$

$$S|0\rangle = (Sv_1) \wedge (Sv_2) \wedge \ldots \wedge (Sv_p)$$

[redacted] Assume that we are in the nice case where [redacted] $S\mathcal{H}^- \cap \mathcal{H}^+ = 0$. Then $S\mathcal{H}^-$ is the graph of a linear map T from $\mathcal{H}^-$ to $\mathcal{H}^+$, and so we can find an orthonormal basis $e_1, \ldots, e_p$ for $\mathcal{H}^-$; $e_{p+1}, \ldots, e_n$ for $\mathcal{H}^+$ so that

$$Te_i = \lambda_i e_{p+i} \qquad \lambda_1 \geq \lambda_2 \geq \ldots \, .$$

Then

$$S(e_i) = \frac{e_i + \lambda_i e_{p+i}}{\sqrt{1+\lambda_i^2}}$$

and

$$S|0\rangle = \prod_{i=1}^{p} \frac{1}{\sqrt{1+\lambda_i^2}} \underbrace{(e_1 + \lambda_1 e_{p+1}) \wedge \ldots \wedge (e_p + \lambda_p e_{2p})}_{e^{\lambda_1 a_1^* a_{p+1} + \ldots + \lambda_p a_p^* a_{2p}} (e_1 \wedge \ldots \wedge e_p)}$$

for the e_i basis

But it is even simpler than this, maybe:

$$e^{\alpha b^* c^*}|0\rangle = e^{\sum \alpha_{ij} a_i^* a_j}(v_1 \wedge \cdots \wedge v_p) \qquad j \le p,\ i \ge p$$

[redacted]

$$= \left(e^{\sum \alpha_{i1} a_i^* a_1} e_1\right) \wedge \cdots \wedge \left(e^{\sum \alpha_{ip} a_i^* a_p} e_p\right)$$

$$= \left(e_1 + \sum \alpha_{i1} e_i\right) \wedge \cdots \wedge \left(e_p + \sum \alpha_{ip} e_i\right)$$

The point therefore seems to be that [redacted] any element of $\Lambda^p(\mathcal{H})$ belonging a subspace [redacted] complementary to $\mathcal{H}^+$ is uniquely represented in the form

$$e^{\alpha b^* c^*}|0\rangle.$$

In fact this is clear $\alpha b^* c^* = \sum_{\substack{j \le p \\ i > p}} \alpha_{ij} a_i^* a_j$ is a nilpotent (square zero) operator on $\mathcal{H}$ carrying $\mathcal{H}^-$ to $\mathcal{H}^+$.

It seems clear then that writing S in the form

$$S = \langle 0|S|0\rangle\, e^{\alpha b^* c^*}\, e^{(\log \beta) b^* b}\, e^{(\log \gamma) c c^*}\, e^{\delta c b}$$

$$e^{\alpha a_+^* a_-}\, e^{(\log \beta) a_+^* a_+}\, e^{(\log \gamma) a_-^* a_-}\, e^{\delta a_-^* a_+}$$

[redacted] corresponds to factoring $S : \mathcal{H}^- \oplus \mathcal{H}^+ \longrightarrow \mathcal{H}^- \oplus \mathcal{H}^+$ in the form

$$S = \begin{pmatrix} 1 & 0 \\ \alpha & 1 \end{pmatrix} \begin{pmatrix} \beta & 0 \\ 0 & \gamma \end{pmatrix} \begin{pmatrix} 1 & \delta \\ 0 & 1 \end{pmatrix}$$

December 15, 1980

Yesterday we looked at $H = H_0 + V$ on $\mathcal{H}$ extended to $\Lambda\mathcal{H}$. Then S on $\Lambda\mathcal{H}$ is the automorphism extending [redacted]

$$S = T\left\{e^{-i\int dt\, V_D(t)}\right\} \qquad \text{on } \mathcal{H}$$

The perturbation will create pairs when

$$S : \mathcal{H}^- \oplus \mathcal{H}^+ \longrightarrow \mathcal{H}^- \oplus \mathcal{H}^+$$

does not preserve this decomposition, because $S|0\rangle$ is the element of $\Lambda\mathcal{H}$ corresponding to the subspace $S(\mathcal{H}^-)$.

So now let us consider an infinite dimensional situation, say where H_0 is the Dirac operator and V is the perturbation represented by a weak external time-independent EM field. Then we know the scattering operator S exists on $\mathcal{H}$ and that it commutes with H_0. In particular S has to preserve the decomposition into positive and negative eigenspaces. Thus S [redacted] cannot create pairs. So what happens is that S separately moves the particles and holes around.

Consider two oscillators described by

$$H_0 = \omega_1 a_1^* a_1 + \omega_2 a_2^* a_2$$

and a coupling between them, say

$$H_{int} = \varepsilon\, q_1 q_2 \qquad q_i = \frac{a_i + a_i^*}{\sqrt{2\omega_i}}$$

In practice ω_1, ω_2 are large relative to $\omega_1 - \omega_2$, hence the terms involving $a_1^* a_2^*$, $a_1 a_2$ in H_{int} have rapidly changing phase in the Dirac picture. So if one were to compute

$$U_D(t, t') = 1 - i \int_{t'}^{t} dt_1\, \varepsilon\, q_1(t_1) q_2(t_1)$$

to first order these terms would be small. Thus one make the "rotating wave" approximation and drops them.

So we consider $H_{int} = \varepsilon(a_1^* a_2 + a_2^* a_1)$. The classical equations of motion are

$$\dot{a}_1 = i[H, a_1] = i(-\omega_1 a_1 - \varepsilon a_2)$$
$$\dot{a}_2 = i(-\omega_2 a_2 - \varepsilon a_1)$$

or simply

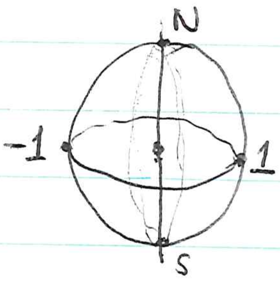
(☆) $$i \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}^{\cdot} = \begin{pmatrix} \omega_1 & \varepsilon \\ \varepsilon & \omega_2 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$$

Notice that $H = \omega_1 a_1^* a_1 + \omega_2 a_2^* a_2 + \varepsilon(a_1^* a_2 + a_2^* a_1)$ preserves "particle number". Hence [redacted] the quantum situation is the symmetric algebra on the 1-particle space which is the same for both H_0 and H. The 1-particle space is described by the "classical" equations (☆) which happens to be a Schroedinger equation for a 2-state system.

Think of the states of a 2-particle system as corresponding to points on the Riemann sphere, with $\binom{1}{0} \leftrightarrow$ north pole $\binom{0}{1} \leftrightarrow$ south pole. If $\varepsilon = 0$, these points stay fixed and

$$\frac{a_1}{a_2}(t) = \frac{e^{-i\omega_1 t} a_1}{e^{-i\omega_2 t} a_2} = e^{i(\omega_2-\omega_1)t}\left(\frac{a_1}{a_2}\right)$$

rotates with angular frequency $\omega_2 - \omega_1$.



Suppose $\omega_1 = \omega_2$. The eigenvalues of $\begin{pmatrix} \omega_1 & \varepsilon \\ \varepsilon & \omega_1 \end{pmatrix}$ are $\omega_1 \pm \varepsilon$ and the eigenvectors are $\binom{1}{-1}$ for $\omega_1 - \varepsilon$, $\binom{1}{1}$ for $\omega_1 + \varepsilon$

Thus the perturbed system is described by [redacted] rotation [redacted] leaving 1, −1 fixed. [redacted] The state $\binom{1}{0}$ evolves as

$$\psi(t) = e^{-i\omega t}\begin{pmatrix} \cos \varepsilon t \\ -i \sin \varepsilon t \end{pmatrix}$$

and alternates between N and S. It's a mixture of the N, S states. It seems the the probability splitting

$$\cos^2 \varepsilon t + \sin^2 \varepsilon t = 1$$

is the [redacted] relative height, or level, between N and S.

On the other hand if $\varepsilon \ll |\omega_1 - \omega_2|$, then the eigenstates for the perturbed system are very close to those of the unperturbed system. Thus the perturbed system is described by rotation about an axis very close to NS. So one sees that the states N, S are nearly invariant in this case.

These two cases exhibit the transitions $N \leftrightarrow S$ when $\omega_1 = \omega_2$, and the impossibility of such [redacted] transitions when $\omega_1 \neq \omega_2$ or more precisely $|\varepsilon| \ll |\omega_1 - \omega_2|$.

Review Nyquist's relation: Suppose we have a transmission line described by

$$\frac{\partial I}{\partial x} = -C_0 \frac{\partial V}{\partial t} \qquad \frac{\partial V}{\partial x} = -L_0 \frac{\partial I}{\partial t} .$$

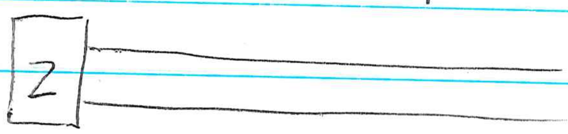
The travelling waves are $\begin{pmatrix} V \\ I \end{pmatrix} = \begin{pmatrix} \hat{V} \\ \hat{I} \end{pmatrix} e^{i(kx-\omega t)}$

with $\quad k\hat{I} = C_0 \omega \hat{V} \qquad k\hat{V} = L_0 \hat{I} \omega \quad$ or

$$\frac{\hat{V}}{\hat{I}} = \frac{k}{C_0\omega} = \frac{L_0 \omega}{k} \qquad \text{speed} = \frac{\omega}{k} = \frac{1}{\sqrt{L_0 C_0}}$$

$$R_0 = \text{imped.} = \sqrt{\frac{L_0}{C_0}}$$

Suppose the transmission line is terminated by a circuit at $x=0$ with impedance $Z(\omega)$:



The normal mode are

$$V(x,t) = \mathrm{Re}\left(A\left(e^{-ikx} + S(\omega)e^{ikx}\right)e^{-i\omega t}\right)$$

$$I(x,t) = \frac{1}{R_0}\mathrm{Re}\left(A\left(-e^{-ikx} + S(\omega)e^{ikx}\right)e^{-i\omega t}\right)$$

$$\frac{k}{\omega} = \sqrt{L_0 C_0}$$

where the reflection coefficient $S(\omega)$ satisfies

$$+Z(\omega) = R_0 \frac{S+1}{S-1}$$

The time-averaged energy density for this mode is

$$\left\langle \tfrac{1}{2} C_0 V^2 + \tfrac{1}{2} L_0 I^2 \right\rangle = \frac{1}{2} C_0 \Big(\frac{1}{2}|A|^2 |e^{-ikx} + Se^{ikx}|^2$$

$$+ \frac{1}{2}|A|^2 |-e^{-ikx} + Se^{ikx}|^2 \Big)$$

$$= C_0 |A|^2$$

assuming the circuit is lossless, so that $|S| = 1$.

Fix a length ℓ of the line and terminate it say by a short circuit. Then the normal modes are given by frequencies ω satisfying

$$e^{-ik\ell} + S(\omega)e^{ik\ell} = 0 \quad \text{or} \quad S(\omega) + e^{-2ik\ell} = 0$$

If ℓ is large and we want the density of modes around a fixed frequency, the variation of $S(\omega)$ can be neglected, and so the modes are distributed for

$$k\ell = n\pi + \underset{\text{small change}}{\delta(\omega)} \qquad n \in \mathbb{Z}.$$

hence the number in the range dk or $d\omega$ is

$$dn = \frac{\ell}{\pi}\, dk = \frac{\ell}{\pi}\sqrt{L_0 C_0}\, d\omega.$$

Let's [redacted] now suppose the system is at the temperature T. Then the situation is described by the Maxwell-Boltzmann distribution in phase space. Since we have an oscillator, this means each amplitude A_ω is a Gaussian variable such that the average energy of the mode is kT: The energy is $C_0\ell\langle|A_\omega|^2\rangle$ + the energy inside the [redacted] circuit. If ℓ is large then approximately we have

$$C_0 \ell \langle |A_\omega|^2 \rangle = kT$$

Now

$$V\big|_{x=0} = \sum_\omega \mathrm{Re}\left(A_\omega \left(1 + S(\omega)\right)\right)$$

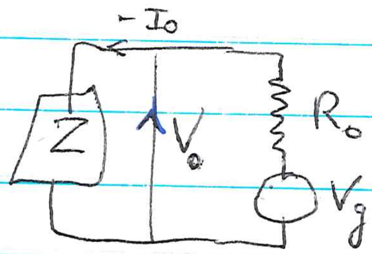
so

$$\langle V_0^2 \rangle = \sum \frac{1}{2} \underbrace{\langle |A_\omega|^2 \rangle}_{\sim kT/C_0\ell} \left|1 + S(\omega)\right|^2 \underbrace{dn}_{\frac{\ell}{\pi}\sqrt{L_0C_0}\,d\omega}$$

so

$$\langle V_0^2 \rangle = \int_0^\infty \frac{d\omega}{\pi} \cdot \frac{1}{2} kTR_0 \cdot |1+S(\omega)|^2$$

On the other hand we can think of the line as a resistor R_0 plus noise generator



whence
$$\hat{V}_0 = \frac{Z}{R_0+Z} \hat{V}_g$$ [redacted]

[redacted]

$$\frac{Z}{R_0+Z} = \frac{\frac{S+1}{S-1}}{1+\frac{S+1}{S-1}} = \frac{S+1}{2S}$$

$$\left|\frac{Z}{R_0+Z}\right|^2 = \left|\frac{S+1}{2}\right|^2 \qquad \text{since } |S| = 1.$$

$$\therefore \quad \langle V_0^2 \rangle = \int_0^\infty \frac{d\omega}{\pi} \underbrace{2kTR_0 \left|\frac{S+1}{2}\right|^2}_{|\hat{V}_0(\omega)|^2}$$

and so one concludes* that

$$\langle V_g^2 \rangle = \int_0^\infty \frac{d\omega}{\pi} 2kTR_0$$

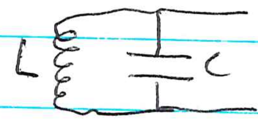
which is the Nyquist relation. [redacted]

[redacted]

The above is sloppy in many respects at the end. First of all one has used the Wiener-Khinchin theorem to get

$$\langle V_0^2\rangle = \int_0^\infty \frac{d\omega}{\pi} |\hat{V}_0(\omega)|^2$$

and secondly one argues that one can take various impedances Z. For example Wannier takes Z to be a tuned circuit



with a sharp resonance frequency. By equipartition one knows that

$$\frac{1}{2} C\langle V_0^2\rangle = \frac{1}{2} kT,$$

so by computing $\left|\frac{Z}{R_0+Z}\right|^2$ one can check the Nyquist formula at the resonant frequency

Let's go over this in detail. For the tuned circuit

$$Z(\omega) = \frac{1}{Cs + \frac{1}{Ls}} \qquad s = -i\omega$$

so

$$\frac{S+1}{2S} \simeq \frac{Z}{R_0+Z} = \frac{1}{R_0\left(Cs + \frac{1}{Ls}\right) + 1} = \frac{\frac{L}{R_0}s}{LCs^2 + \frac{Ls}{R_0} + 1}$$

Thus $$\langle V_0^2\rangle = 2kTR_0 \int_0^\infty \frac{d\omega}{\pi} \left|\frac{s}{R_0C\left(s^2 + \frac{1}{R_0C}s + \frac{1}{LC}\right)}\right|^2$$

$$= \frac{2kTR_0}{(R_0C)^2} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \frac{\omega^2}{(\omega^2 - \omega_0^2)^2 + \gamma^2\omega^2} \qquad \omega_0 = \frac{1}{\sqrt{LC}}$$

$$\gamma = \frac{1}{R_0C}$$

The integral can be evaluated by residues and yields $\frac{1}{2\gamma}$. Thus

$$\langle V_0^2 \rangle = \frac{2kTR_0}{(R_0C)^2} \frac{R_0C}{2} = \frac{kT}{C}$$

so that $\quad \frac{1}{2}kT = \frac{1}{2}C\langle V_0^2\rangle$ as required.

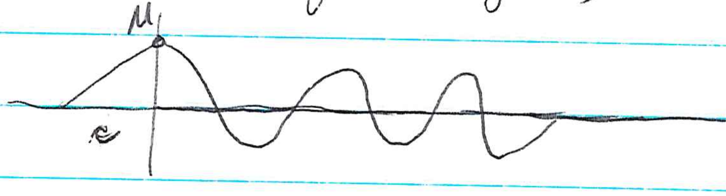
Let's go over why this has to hold. We are calculating $\langle V_0^2 \rangle$ by the Maxwell-Boltzmann distribution. The phase space involved has points given by the pair of functions $\begin{pmatrix} V(x) \\ I(x) \end{pmatrix}$, together with the current thru the inductance, call it I_L. Then on this vector space we have the energy

$$H: \quad \frac{1}{2}LI_L^2 + \frac{1}{2}CV_0^2 + \int\left\{\frac{1}{2}C_0V(x)^2 + \frac{1}{2}L_0I(x)^2\right\}dx$$

Therefore on this vector space goes the Gaussian measure $e^{-\beta H}$, and hence $\quad \langle \frac{1}{2}LI_L^2\rangle = \langle\frac{1}{2}CV_0^2\rangle = \frac{1}{2}kT$.

December 22, 1980

Consider a line of density λ, tension λ connected to an oscillator



$$\lambda \partial_t^2 u = \lambda \partial_x^2 u \qquad x > 0$$

$$M(\ddot{u}_0 + \omega_0^2 u_0) = \lambda (\partial_x u)_0$$

Suppose the string is tied down at $x = l$, where l is very large. The motion is of the form

$$u(x,t) = \sum_\omega \mathrm{Re}\left(A_\omega \left(e^{-i\omega x} + S_{(\omega)} e^{-i\omega x}\right) e^{-i\omega t}\right)$$

where ω runs over solutions > 0 of

$$e^{-i\omega l} + S(\omega) e^{i\omega l} = 0$$

and $S(\omega)$ is determined by

$$M(s^2 + \omega_0^2)(1 + S) = \lambda s (1 - S) \qquad s = -i\omega$$

$$\frac{1+S}{1-S} = \frac{\lambda s}{M(s^2 + \omega_0^2)}$$

The energy density along the string is

$$\frac{1}{2} \lambda \dot{u}^2 + \frac{1}{2} \lambda (\partial_x u)^2$$

For the ω-th mode, the time-averaged energy density is

$$\lambda |A_\omega|^2 \omega^2$$

so proceeding as in the Nyquist problem

$$\langle u_0^2 \rangle = \int_0^\infty \frac{d\omega}{\pi} \frac{2kT}{\lambda \omega^2} \left| \frac{1 + S_\omega}{2} \right|^2 .$$

Here $\frac{1+S}{2} =$ [redacted] $\frac{Z}{Z+1} = \frac{\lambda s}{M(s^2+\omega_0^2)+\lambda s}$

and a residue [redacted] calculation again gives

$$\frac{1}{2} M\omega_0^2 \langle u_0^2 \rangle = \frac{1}{2} kT$$

as required by equi-partition.

It seems to be possible to understand emission classically as follows. Consider the string as an absorber, so that only outgoing waves are allowed. Then we have a damped harmonic oscillator. In effect

$$\partial_x u = -\dot{u} \qquad \text{for outgoing waves, so}$$

$$M(\ddot{u}_0 + \omega_0^2 u_0) = -\lambda \dot{u}_0$$

or

$$\ddot{u}_0 + \gamma \dot{u}_0 + \omega_0^2 u_0 = 0 \qquad \gamma = \frac{\lambda}{M}$$

The solutions are

where [redacted]

$$\boxed{u_0 = e^{-\frac{\gamma}{2}t} \, Re\left(Ae^{-i\omega_1 t}\right) \qquad \omega_1 = \sqrt{\omega_0^2 - \frac{\gamma^2}{4}}}$$

In effect the roots of $+\omega^2 + i\gamma\omega - \omega_0^2 = 0$ are

$$\omega = -\frac{i\gamma}{2} \pm \sqrt{\omega_0^2 - \frac{\gamma^2}{4}}.$$

Thus we see that the energy, potential or kinetic, has[redacted] the decay rate $e^{-\gamma t}$. (Assume $\gamma \ll \omega_0$).

December 25, 1980

The problem is still to find a simple example of emission and absorption. Let's consider a simple oscillator coupled to a continuous family of oscillators. The Hamiltonian is

$$\mathbb{H} = \underbrace{\omega_0 b^* b + \sum_\alpha \omega_\alpha a_\alpha^* a_\alpha}_{H_0} + \sum_\alpha b^*(\gamma_\alpha a_\alpha) + (a_\alpha^* \bar{\gamma}_\alpha) b .$$

Because this Hamiltonian is of degree 0, i.e. it commutes with the number operator $b^*b + \sum_\alpha a_\alpha^* a_\alpha$, it follows that its effect on states of many particles is determined by what it does to 1-particle states. More precisely the Hilbert space is the symmetric algebra on the 1-particle space $\mathcal{H}$ which is spanned by the vectors $b^*|0\rangle$, $a_\alpha^*|0\rangle$. On this 1-particle space, time evolution is described by

$$i \frac{\partial}{\partial t} \psi = \mathbb{H} \psi \qquad \mathbb{H} = \left(\begin{array}{c|c} \omega_0 & \gamma^* \\ \hline \gamma & H_0 \end{array} \right)$$

where $H_0 = \sum_\alpha |\alpha\rangle \omega_\alpha \langle\alpha|$, $\quad \gamma^* = \sum_\alpha \bar{\gamma}_\alpha \langle\alpha|$.

Let

$$\Phi = b^* |0\rangle .$$

The first thing is to understand how Φ decays. Thus we want to understand

$$\langle \Phi | e^{-i\mathbb{H}t} | \Phi \rangle$$

as $t \to +\infty$. One has

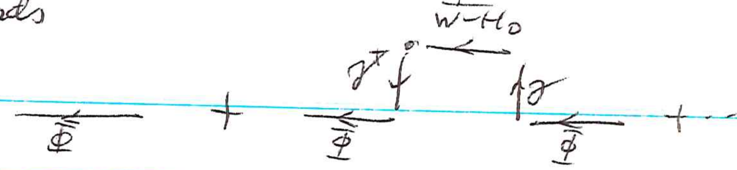
$$\int_0^\infty dt\, e^{+iWt} \langle \Phi | e^{-i\mathbb{H}t} | \Phi \rangle = \langle \Phi | \frac{-1}{i(W_{+i0^+} - \mathbb{H})} | \Phi \rangle .$$

hence

$$\langle \Phi | e^{-i\mathbb{H}t} | \Phi \rangle = \int_{-\infty}^{\infty} \frac{dW}{2\pi} \langle \Phi | \frac{-1}{i(W + i0^+ - \mathbb{H})} | \Phi \rangle e^{-iWt} \quad \text{for } t > 0.$$

Now we can compute $\frac{1}{W-H}$ by diagrams as in Weinberg's paper. One finds

$$\langle\Phi|\frac{1}{W-H}|\Phi\rangle =$$



$$= \frac{1}{W-\omega_0} + \frac{1}{W-\omega_0}\langle\gamma|\frac{1}{W-H_0}|\gamma\rangle\frac{1}{W-\omega_0} + \cdots$$

so

$$\boxed{\langle\Phi|\frac{1}{W-H}|\Phi\rangle = \frac{1}{W-\omega_0-\langle\gamma|\frac{1}{W-H_0}|\gamma\rangle}}$$

Remark [redacted]: We can split the space on which H_0 [redacted] operates into the cyclic subspace spanned by γ and the orthogonal complement. We can forget about the latter in trying to understand H from H_0, and so can suppose γ is a cyclic vector for H_0.

[redacted] One has

$$\langle\gamma|\frac{1}{W-H_0}|\gamma\rangle = \int [redacted] \frac{d\mu(\omega)}{(W-\omega)}$$

for some measure on the line. Let's suppose

$$d\mu(\omega) = |\gamma_\omega|^2 d\omega$$

and put

$$g(W) = \int \frac{|\gamma_\omega|^2 d\omega}{W-\omega} = \langle\gamma|\frac{1}{W-H_0}|\gamma\rangle.$$

for W in the UHP. Define g by analytic continuation from the UHP. From the formula

$$\langle\Phi|e^{-iHt}|\Phi\rangle = \int\frac{dW}{2\pi i}e^{-iWt}\left(\frac{(-1)}{W-\omega_0-g(W)}\right) \qquad t>0$$

and the fact that e^{-iWt} decays in the LHP for $t>0$, we see that the behavior of $\langle \Phi | e^{-iHt} | \Phi \rangle$ is controlled by the singularities in the LHP nearest the real axis.

Suppose γ is small. Then $g(W)$ should be small so that near ω_0 is a root $\tilde{\omega}_0$ of $W-\omega_0-g(W)$, which can be found by iterating: $W = \omega_0 + g(W)$

$$W^{(0)} = \omega_0$$

$$W^{(1)} = \omega_0 + g(\omega_0)$$

$$W^{(2)} = \omega_0 + g(\omega_0 + g(\omega_0)) = \omega_0 + g(\omega_0) + g'(\omega_0)g(\omega_0) + \dots$$

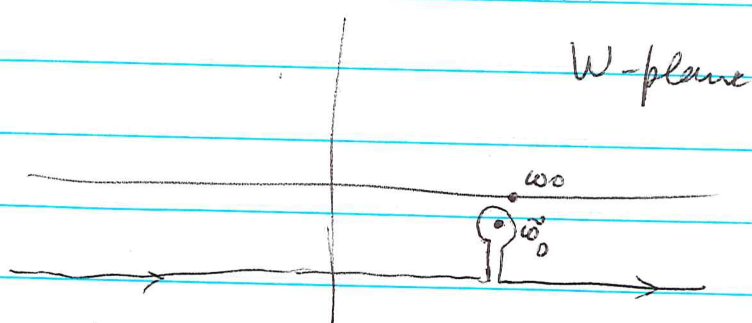
Then $W^{(n)} \longrightarrow \tilde{\omega}_0$. Now $g(W) \in \text{LHP}$ when $W \in \text{UHP}$, in fact we know that on the real W-axis

$$\text{Im}(g(W)) = \int |\gamma_\omega|^2 \, d\omega \; \text{Im}\underbrace{\left(\frac{1}{W+i0^+-\omega}\right)}_{\frac{-i\varepsilon}{(W-\omega)^2+\varepsilon^2}}$$

$$= \int |\gamma_\omega|^2 \left(-\pi \delta(W-\omega)\right) d\omega$$

$$= -\pi |\gamma_W|^2$$

Therefore $\tilde{\omega}_0$ is in the LHP. We can deform the contour



so as to get

$$\langle \Phi | e^{-iHt} | \Phi \rangle = \frac{1}{1-g'(\tilde{\omega}_0)} e^{-i\tilde{\omega}_0 t} + \text{faster decaying exponentials}$$

Thus it should be true that

$$\langle\Phi|e^{-iHt}|\Phi\rangle \simeq \left(1+g'(\omega_0)\right) e^{-i(\omega_0+g(\omega_0))t}\left(1+O(|\gamma|^4)\right)$$

$$= e^{-i\omega_0 t}\left(1+g'(\omega_0) - i g(\omega_0)t + O(|\gamma|^4)\right)$$

Let's check this by perturbation theory.

$$\langle\Phi|e^{-iHt}|\Phi\rangle = e^{-i\omega_0 t} + (-i)\int 0$$

$$+ (-i)^2 \underbrace{\int_0^t dt_1\int_0^{t_1} dt_2\, e^{-i\omega_0(t-t_1)}\langle\gamma|e^{-iH_0(t_1-t_2)}|\gamma\rangle \times e^{-i\omega_0(t_2)}} + \dots$$

$$= e^{-i\omega_0 t}\int d\omega\, |\gamma_\omega|^2 \int_0^t dt_1\int_0^{t_1}dt_2\, e^{i\omega_0(t_1-t_2)}\, e^{-i\omega(t_1-t_2)}$$

Now

$$\int_0^t dt_1\int_0^{t_1}dt_2\, e^{i\alpha(t_1-t_2)} = \int_0^t dt_1\, e^{i\alpha t_1}\,\frac{e^{-i\alpha t_1}-1}{-i\alpha} = \int_0^t dt\,\frac{1-e^{i\alpha t_1}}{-i\alpha}$$

$$= \frac{t}{-i\alpha}+\frac{e^{i\alpha t}-1}{(i\alpha)^2} = \frac{1+i\alpha t - e^{i\alpha t}}{\alpha^2}$$

$$\therefore\ \langle\Phi|e^{-iHt}|\Phi\rangle = e^{-i\omega_0 t}\left\{1 - \int d\omega\,|\gamma_\omega|^2\,\frac{1+i(\omega_0-\omega)t - e^{i(\omega_0-\omega)t}}{(\omega_0-\omega)^2} + O(|\gamma|^4)\right\}$$

But if we let ω_0 be approached from the UHP

$$-\int d\omega\,|\gamma_\omega|^2\,\frac{1+i(\omega_0-\omega)t}{(\omega_0-\omega)^2} = -i\,g(\omega_0)t + g'(\omega_0)$$

and the term

$$-\int d\omega\,|\gamma_\omega|^2\,\frac{-e^{i(\omega_0-\omega)t}}{(\omega_0-\omega)^2},$$

for ω_0 above the real axis, can be evaluated by pushing the

ω-contour down till it catches the singularities of $|\gamma_\omega|^2$; hence it contributes faster decaying exponentials.

So our conclusion is that the exact behavior is

$$\langle \Phi | e^{-iHt} | \Phi \rangle = \frac{1}{1-g'(\tilde{\omega}_0)} e^{-i\tilde{\omega}_0 t} + \text{faster decay terms}$$

where $\tilde{\omega}_0$ satisfies:

$$\tilde{\omega}_0 = \omega_0 + g(\tilde{\omega}_0)$$

Thus

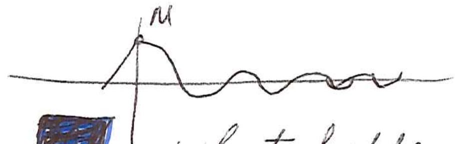
$$\operatorname{Im}(\tilde{\omega}_0) = \operatorname{Im} g(\tilde{\omega}_0) = -\pi |\gamma_{\omega_0}|^2 + O(|\gamma|^4)$$

Next we could use some specific examples. Ultimately we want H_0 to have spectrum ≥ 0 so $g(W)$ is to be an analytic function with a cut along the positive real axis. Unfortunately this puts a singularity on the real axis. So there is no way to construct an example of the above type where $\langle \Phi | e^{-iHt} | \Phi \rangle$ has exponential decay and such that H_0 has spectrum ≥ 0.

So let's look at some examples where the spectrum of H_0 is the whole line.

December 26, 1980

Let's return to



and try to understand what happens on the quantum mechanics level. We are working with a harmonic oscillator hence we know that there is a boson particle structure on the quantum states.

Review the formalism for an oscillator with a finite number of degrees of freedom. Say

$$H = \frac{p^2}{2} + \frac{1}{2} q \cdot \omega^2 q$$

where ω^2 is a positive-definite matrix of which ω is the positive square root. The classical equation of motion is

$$\left(\frac{d^2}{dt^2} + \omega^2\right) q = 0$$

and it has solutions

$$q = e^{-i\omega t} A + e^{i\omega t} B \qquad B = \bar{A}.$$

or better

$$q = Re\left(e^{-i\omega t} A\right).$$

The energy of this solution is

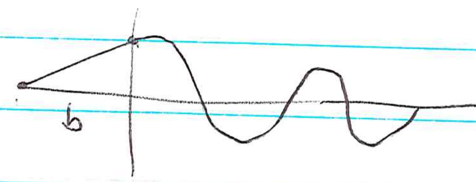
$$E = \frac{1}{2}|\omega A|^2$$

Lets look at some examples of light strings terminated differently. Define the impedance to be $Z = \frac{1+R}{1-R}$ where R is the reflection coefficient.

1) tied at $x=0$: Here $R = -1$

2) free at $x=0$: $R = 1$.

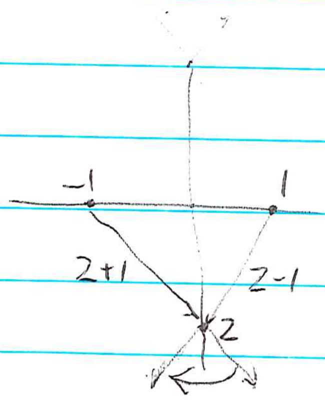
3) weightless stretch of length b tied at $x = -b$



$$\lambda(\partial_x u)_0 = \frac{u_0}{b} \qquad \lambda s(1-R) = \frac{1}{b}(1+R)$$

$\therefore Z = \lambda b s$. As $s = -i\omega$ goes from 0 to $-i\infty$

Z goes from 0 to $-i\infty$, so



$R = \frac{Z-1}{Z+1}$ goes from -1 to 1

$\arg(R)$ " " $-\pi$ to 0

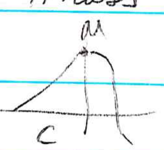
4) mass M at $x=0$:

$$M\ddot{u}_0 = \lambda(\partial_x u)_0 \Rightarrow Ms^2(1+R) = \lambda s(1-R)$$

$$\Rightarrow Z = \frac{1+R}{1-R} = \frac{\lambda s}{Ms^2} = \frac{\lambda}{Ms}$$

Z goes from $i\infty$ to 0 so $\arg(R)$ goes from 0 to π

5) mass M at $x=0$ + weightless stretch tied at $-c$.



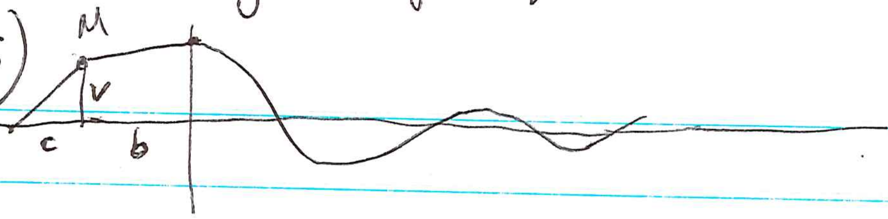
$$M\ddot{u}_0 + \frac{1}{c}u_0 = \lambda(\partial_x u)_0$$

$$\left(Ms^2 + \frac{1}{c}\right)(1+R) = \lambda s(1-R)$$

$Z = \frac{\lambda s}{Ms^2 + \frac{1}{c}}$ goes from 0 to $-i\infty$, then $i\infty$ to 0.

so $\arg R$ goes from $-\pi$ to π.

6)



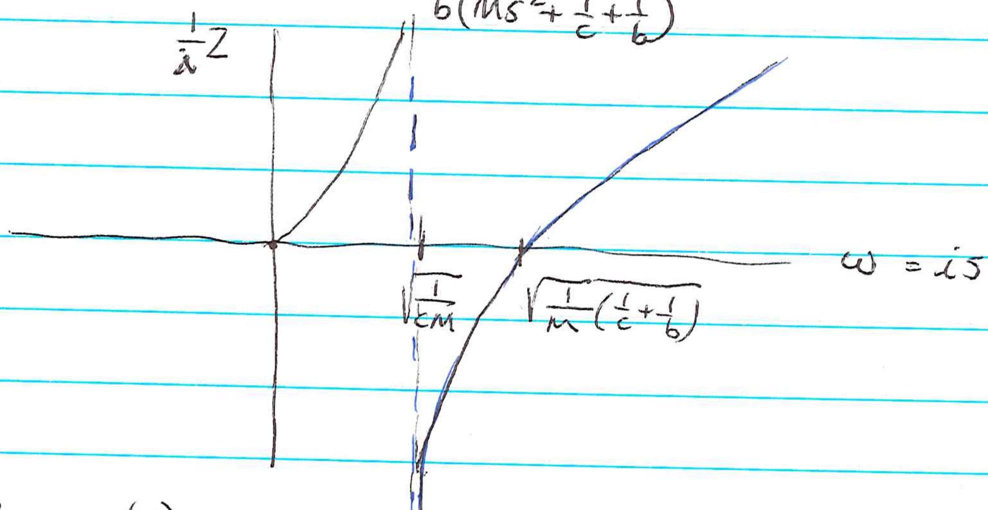
$$M\ddot{v} + \frac{1}{c}v = \frac{u_0 - v}{b} = \lambda (\partial_x u)_0 \qquad \text{say } v = Ae^{-i\omega t}$$

$$\left(Ms^2 + \frac{1}{c}\right)A = \frac{1+R-A}{b} = \lambda s(1-R)$$

$$\left(Ms^2 + \frac{1}{c} + \frac{1}{b}\right)A = \frac{1}{b}(1+R)$$

$$(1+R)\left[1 - \frac{1}{b\left(Ms^2 + \frac{1}{c} + \frac{1}{b}\right)}\right] = b\lambda s(1-R)$$

$$\therefore \quad Z = \frac{b\lambda s}{1 - \frac{1}{b\left(Ms^2+\frac{1}{c}+\frac{1}{b}\right)}} = \frac{b\lambda s\left(Ms^2 + \frac{1}{c} + \frac{1}{b}\right)}{Ms^2 + \frac{1}{c}}$$



Thus $\arg(R)$ goes from $-\pi$ to 0 to π to 0.

A point that emerges from these examples is that $\arg(R)$ changes $n\pi$ in going from $\omega = 0$ to $\omega = \infty$ where n is the number of quadratic terms in the energy in the port. Thus we know that $\frac{3}{2}k_B T$ is the thermal energy one can store in the two strings and the [redacted] kinetic energy of the mass M in example 6).

Let's now interpret these "strings" as (generalized) oscillators. First look at example 6): A configuration consists of $\phi = (v, u)$ where $v \in \mathbb{R}$, u is real fn in $C_0^\infty(\mathbb{R}_{\geq 0})$. The kinetic energy is $\frac{1}{2}\|\dot\phi\|^2$ where

$$\frac{1}{2}\|\phi\|^2 = \frac{1}{2}Mv^2 + \frac{\lambda}{2}\int_0^\infty u^2\,dx$$

and the potential energy is

$$\frac{1}{2}P(\phi) = \frac{1}{2c}v^2 + \frac{1}{2b}(u_0 - v)^2 + \frac{\lambda}{2}\int_0^\infty (\partial_x u)^2\,dx$$

We want to write

$$P(\phi) = \langle \phi | L\phi \rangle$$

where L is an operator and $\langle\phi|\phi\rangle = \|\phi\|^2$. So polarize P:

$$\frac{1}{2}P(\tilde\phi, \phi) = \frac{1}{2c}\tilde v v + \frac{1}{2b}(\tilde u_0 - \tilde v)(u_0 - v) + \underbrace{\frac{\lambda}{2}\int_0^\infty (\partial_x \tilde u)(\partial_x u)\,dx}_{-\frac{\lambda}{2}\tilde u_0(\partial_x u)_0 + \frac{\lambda}{2}\int_0^\infty \tilde u(-\partial_x^2 u)\,dx}$$

In order that this can be written $\langle\tilde\phi|L\phi\rangle$ we must have that $\phi = (v, u)$ makes the $\tilde u_0$ vanish, hence

$$\frac{1}{b}(u_0 - v) = \lambda(\partial_x u)_0$$

whence

$$\frac{1}{2}P(\tilde\phi, \phi) = \frac{1}{2}M\,\tilde v\left[\left(\frac{1}{c} + \frac{1}{b}\right)v - \frac{1}{b}u_0\right]\frac{1}{M} + \frac{\lambda}{2}\int_0^\infty \tilde u(-\partial_x^2 u)\,dx.$$

Thus

$$L(v, u) = \left(\frac{1}{M}\left[\left(\frac{1}{c} + \frac{1}{b}\right)v - \frac{u_0}{b}\right], -\partial_x^2 u\right)$$

The eigenfunctions of L with eigenvalue ω^2 are given by solutions of

$$\omega^2 v = \frac{1}{M}\left[\left(\frac{1}{c} + \frac{1}{b}\right)v - \frac{u_0}{b}\right]$$

$$\omega^2 u = -\partial_x^2 u \qquad \frac{1}{b}(u_0 - v) = \lambda(\partial_x u)_0$$

and hence are of the form $\phi_\omega = (v_\omega, u_\omega)$

$$u_\omega = e^{-i\omega x} + R_\omega e^{i\omega x}$$
$$v = A_\omega$$

where

$$\left(M(-\omega^2) + \frac{1}{c}\right) A = \frac{1+R-A}{b} = \lambda(-i\omega)(1-R)$$

Example where $b = 0$:

$$\|u\|^2 = M u_0^2 + \lambda \int_0^\infty u^2 \, dx$$

$$P(u) = \frac{1}{c} u_0^2 + \lambda \int_0^\infty (\partial_x u)^2 \, dx$$

$$P(\tilde{u}, u) = \frac{1}{c} \tilde{u}_0 u_0 - \lambda \tilde{u}_0 (\partial_x u)_0 + \lambda \int_0^\infty \tilde{u} (-\partial_x^2 u) \, dx$$

$$\| $$

$$\langle \tilde{u} | Lu \rangle = M \tilde{u}_0 (Lu)_0 + \lambda \int_0^\infty u \, Lu \, dx$$

leads to

$$Lu = -\partial_x^2 u$$

$$M(Lu)_0 = \frac{1}{c} u_0 - \lambda (\partial_x u)_0$$

Thus for u to be in the domain of L we must have

$$M(\partial_x^2 u)_0 + \frac{1}{c} u_0 = \lambda (\partial_x u)_0$$

The eigenfunctions are

$$u_\omega = e^{-i\omega x} + R_\omega e^{i\omega x}$$

where

$$\frac{1+R}{1-R} = \frac{\lambda s}{M s^2 + \frac{1}{c}}$$

Now let's fix a length l and tie the string at $x = l$. We then get eigenvalues ω_n by solving

$$R_\omega + e^{-2i\omega l} = 0$$

and corresponding eigenfunctions $u_n = u_{\omega_n}$. Assuming these eigenfns. are complete we know that

$$\frac{u_n}{\|u_n\|}$$

is an orthonormal basis for functions on $[0, l]$ with the norm $\|f\|^2 = M|f_0|^2 + \lambda \int_0^l |f|^2 dx$. Hence we should have an expansion formula

$$f = \sum_n u_n \frac{1}{\|u_n\|^2} \langle u_n | f \rangle$$

Now let $l \to \infty$. For large l, the variation of R_ω can be neglected so that the number of eigenvalues dn in a range $d\omega$ is given by $dn = \frac{l}{\pi} d\omega$.

Also $$\|u_\omega\|^2 = M|u_\omega(0)|^2 + \lambda \int_0^l \underbrace{|u_\omega|}_{2\cos(\omega x - \delta)}{}^2 dx \sim \lambda \cdot 4 \frac{l}{2} = 2\lambda l$$

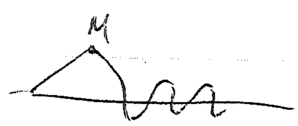
Thus the expansion formula becomes

$$f(x) = \int_0^\infty \frac{d\omega}{2\lambda\pi} u_\omega(x) \cdot \langle u_\omega | f \rangle$$

For example, take the free line: $u_\omega(x) = 2\cos(\omega x)$

$$f(x) = \int_0^\infty \frac{d\omega}{\lambda\pi} \cos \omega x \cdot \left(2\lambda \int_0^\infty \cos(\omega x') f(x') dx' \right)$$

which checks.

Another check is as follows. Take 

again and let $$f(x) = \begin{cases} 0 & x > 0 \\ 1 & x = 0 \end{cases}$$

Then $\langle u_\omega | f \rangle = M \overline{(1 + R_\omega)}$ and so we should have the formula

$$f(x) = \int_0^\infty \frac{M\,d\omega}{2\lambda\pi} \underbrace{\left(e^{-i\omega x} + R_\omega e^{i\omega x}\right)\left(1+\bar{R}_\omega\right)}_{e^{-i\omega x} + e^{i\omega x} + R_\omega e^{i\omega x} + \bar{R}_\omega e^{-i\omega x}}$$

$$= \int_{-\infty}^{\infty} \frac{M\,d\omega}{\lambda\, 2\pi} \left(1+R_\omega\right) e^{i\omega x}$$

Now R_ω is analytic in the UHP, so this will vanish for $x>0$. If $x=0$ then use

$$\frac{1+R}{2} = \frac{\frac{1+R}{1-R}}{1+\frac{1+R}{1-R}} = \frac{\lambda s}{Ms^2 + \lambda s + M\omega_0^2} \sim \frac{\lambda}{Ms}$$

$$f(0) = \int \frac{\cancel{M}}{\cancel{\lambda}2\pi} \frac{\cancel{2\lambda}\,d\omega}{\cancel{M}(-i\omega)} = \int_\pi^0 \frac{1}{\pi} \frac{\cancel{Re^{i\theta}}\,i\,d\theta}{-i\,\cancel{Re^{i\theta}}} = 1$$

which checks.