September 13, 1980

Plane wave solutions of Maxwell's equations. I want to use (ϕ, A) instead of (E, B), so the equations are

$$\begin{cases} \Box(\phi, A) = 4\pi(\rho, J) \quad (= 0 \text{ in vacuum}) \\ \dfrac{\partial\phi}{\partial t} + \nabla\cdot A = 0 \end{cases}$$

I want plane wave solutions $(\phi, A) = (\hat{\phi}, \hat{A})e^{i(kx - \omega t)}$ whence I have

$$\omega^2 = |k|^2$$

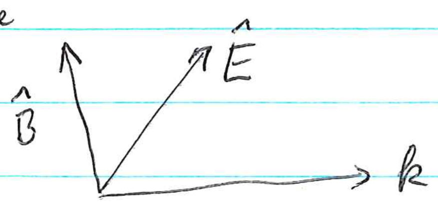
$$\omega\hat{\phi} = k\cdot\hat{A}$$

Recall that (ϕ, A) can be changed by $\left(-\frac{\partial f}{\partial t}, \nabla f\right)$ where $\Box f = 0$, without affecting the physics. This means that $(\hat{\phi}, \hat{A})$ is arbitrary up to adding $(\omega\hat{f}, k\hat{f})$ $= \hat{f}(\omega, k)$. In particular we get a unique choice for $(\hat{\phi}, \hat{A})$ provided we require $\hat{\phi} = 0$.

Therefore we see that a plane wave solution of [redacted] Maxwell's equations can be uniquely represented:

$$\hat{E} = i\,\omega\hat{A} \qquad \text{where } k\cdot\hat{A} = 0$$
$$\hat{B} = i\,k\times\hat{A} \qquad \omega = |k|$$

Picture



so that $k\times E = \omega B$

Note that $\hat{E}$ and $\hat{A}$ are in the same direction.

Next consider the space of solutions of Maxwell's equations, i.e. global solutions. In some sense the plane waves just described should be a basis.

(This has to be done carefully: suppose we consider the initial value problem. We are given E, B on $t=0$ such that $\nabla \cdot E = \nabla \cdot B = 0$. Expand in Fourier integrals

$$E(0,x) = \int E_k(0) e^{ikx} \frac{d^3k}{(2\pi)^3} \qquad k \cdot E_k = 0$$

$$B(0,x) = \int B_k(0) e^{ikx} \frac{d^3k}{(2\pi)^3} \qquad k \cdot B_k = 0$$

Then $E_k(t)$, $B_k(t)$ should satisfy the ordinary DE's

$$(*) \quad \begin{cases} ik \times E_k = -\dfrac{dB_k}{dt} \\[2ex] ik \times B_k = \dfrac{dE_k}{dt} \end{cases}$$

Since E_k, B_k lie in the 2-diml space of vectors $\perp k$ there is a unique solution of these ODE's with given values at $t=0$. Since

$$-k \times (k \times E_k) = -\frac{d^2 E_k}{dt^2}$$
$$\qquad \| $$
$$|k|^2 E_k$$

The solution has to be of the form

$$\begin{pmatrix} E_k \\ B_k \end{pmatrix}(t) = \begin{pmatrix} \hat{E}_k \\ \hat{B}_k \end{pmatrix} e^{-i\omega t} + \begin{pmatrix} \hat{E}'_k \\ \hat{B}'_k \end{pmatrix} e^{i\omega t}$$

where $\omega = |k|$.

For some reason, which I have to understand, the second solution, with $e^{i\omega t}$, shouldn't be there. It should be true that $E_k \perp B_k$ for EM waves with a given wave vector. [redacted] For (*) one has only that $E_k \cdot B_k$ is constant, and it comes from the fact that $\hat{E}_k$ and $\hat{B}'_k$ needn't be $\perp$.

What happens is that when we form

$$\begin{pmatrix} E_k \\ B_k \end{pmatrix}(t)\, e^{ikx} = \begin{pmatrix} \hat{E}_k \\ \hat{B}_k \end{pmatrix} e^{i(kx-\omega t)} + \begin{pmatrix} \hat{E}'_k \\ \hat{B}'_k \end{pmatrix} e^{i(kx+\omega t)}$$

and add it to its conjugate, the second term gets interpreted as the conjugate of a wave with dependence $e^{i(-kx-\omega t)}$, hence with wave vector $-k$. Conclusion: If E, B are resolved into components with given wave vectors k, these components are not the same as the components of the EM field with wave vector k.)

The correct way to think is this: For each wave vector k $(\neq 0)$, the set of EM fields with wave vector k is a 2-dimensional complex vector space (4 diml real vector space). It is the set of complex 3-vectors $\hat{A}$ such that $k \cdot \hat{A} = 0$.

Unfortunately this description is not Lorentz-invariant. Here is an invariant description. Start with the vector space of (t, x), i.e. space-time, and suppose given the dual vector $(\omega, k)^{\neq 0}$ with $\omega = |k|$. Then we consider other dual vectors (ϕ, A) perpendicular to (ω, k), i.e. $\omega\phi - k \cdot A = 0$, modulo multiples of (ω, k). So what we have done is to take an isotropic vector

(ω, k) and formed the space

$$(\omega, k)^{\perp} / \mathbb{R}\cdot(\omega, k)$$

which is 2 diml over $\mathbb{R}$. It is isomorphic to A in $\mathbb{R}^3$ with $k \cdot A = 0$, and hence carries a natural Euclidean structure. The space of plane EM fields with wave-frequency vector (ω, k) is its complexification.

September 14, 1980:

I want to quantize the EM field as a bunch of independent harmonic oscillators, one for each wave vector k and polarization. One has

$$A(x,t) = \mathrm{Re}\left(\sum_k{}' A_k e^{i(k\cdot x - \omega t)}\right) \qquad \omega = c|k|$$

where $A_k \in \mathbb{C}^3$ is such that $k \cdot A_k = 0$. This is a classical formula; to quantize it seems we need the energy in terms of the A_k.

For example take the simple harmonic oscillator

$$m\ddot{q} + Kq = 0 \qquad \text{solns:} \quad q = \mathrm{Re}\left(Ae^{-i\omega t}\right) \qquad \omega = \sqrt{\tfrac{K}{m}}$$

$$p = m\dot{q} = m\omega\, \mathrm{Im}\left(Ae^{-i\omega t}\right) \qquad m\omega^2 = K$$

so

$$H = \frac{p^2}{2m} + \frac{K}{2}q^2 = \frac{1}{2}\left(m\omega^2\, \mathrm{Im}^2 + K\, \mathrm{Re}^2\right) = \frac{1}{2}K|A|^2$$

When we quantize

$$q(t) = \frac{Ae^{-i\omega t} + \bar{A}e^{i\omega t}}{2}$$

$$p(t) = m\omega\,\frac{Ae^{-i\omega t} - \bar{A}e^{i\omega t}}{2i}$$

become operators such that

$$[p, q] = \frac{\hbar}{i}$$

$$\shortparallel$$

$$\frac{m\omega}{4i}\left([A,\bar{A}] - [\bar{A},A]\right) = \frac{m\omega}{2i}[A,\bar{A}]$$

$$\therefore \frac{m\omega}{2\hbar}[A,\bar{A}] = 1 \qquad \text{so} \qquad a = \sqrt{\frac{m\omega}{2\hbar}}\,A$$

$$\boxed{\begin{aligned} q &= \sqrt{\frac{2\hbar}{m\omega}}\ \frac{a+a^*}{2} \\ p &= \sqrt{2\hbar m\omega}\ \frac{a-a^*}{2i} \end{aligned}}$$

$$H = \frac{p^2}{2m} + \frac{\overset{m\omega^2}{K}}{2} q^2 = \hbar\omega\left[\left(\frac{a-a^*}{2i}\right)^2 + \left(\frac{a+a^*}{2}\right)^2\right]$$

$$= \hbar\omega\left(\frac{aa^* + a^*a}{2}\right) = \hbar\omega\left(a^*a + \frac{1}{2}\right) \qquad \text{checks.}$$

This calculation shows that there is a conversion factor between the amplitude A and the annihilation operator a which depends on the formula for the energy.

The simplest way to get the conversion factor is as follows. Classically

$$H = \frac{1}{2}K|A|^2 = \hbar\omega|a|^2$$

so that

$$a = \sqrt{\frac{K}{2\hbar\omega}}\,A = \sqrt{\frac{m\omega}{2\hbar}}\,A$$

Consider now the EM field with $c=1$

$$A(t,x) = \mathrm{Re}\left(\sum_k A_k e^{i(kx-\omega_k t)}\right) \qquad \begin{aligned} \omega_k &= |k| \\ k\cdot A_k &= 0 \end{aligned}$$

The energy should be the sum of the energies for each mode.

$$E = -\frac{\partial A}{\partial t} = \mathrm{Re}\left(\sum_k i\omega_k A_k e^{i(kx-\omega_k t)}\right)$$

$$B = \nabla\times A = \mathrm{Re}\left(\sum_k i(k\times A_k) e^{i(kx-\omega_k t)}\right)$$

The energy is for the k-th mode

$$\frac{1}{4\pi}\int_V \left(\frac{1}{2}E^2 + \frac{1}{2}B^2\right)dx = \frac{1}{8\pi}\left(\frac{1}{2}\omega_k^2|A_k|^2 + \frac{1}{2}|k\times A_k|^2\right)V$$

$$= \frac{1}{8\pi}\,\omega_k^2|A_k|^2\, V$$

Thus the total energy in the box V is

$$V\left(\frac{1}{8\pi}\sum_k \omega_k^2 |A_k|^2\right) \qquad \omega_k^2 = |k|^2.$$

When we quantize we want

$$\frac{V}{8\pi}\omega_k^2 |A_k|^2 = \hbar\omega |a_k|^2$$

$$\text{or} \qquad A_k = \sqrt{\frac{8\pi\hbar}{V\omega_k}}\, a_k .$$

The basic operator formula is

$$\boxed{A(t,x) = \sum_{k,s}\sqrt{\frac{8\pi\hbar}{V\omega_k}}\;\frac{a_{k,s}e^{i(kx-\omega t)} + a_{k,s}^{*}e^{-i(kx-\omega t)}}{2}}$$

s sums over the 2 polarizations for k.

Let's check units in the Feynman system where Coulomb's law is

$$\text{Force} = \frac{q_1 q_2}{4\pi\varepsilon_0 r^2} \qquad \frac{\text{gr cm}}{\text{sec}^2} = \frac{\text{coul}^2}{[\varepsilon_0]\,\text{cm}^2}$$

The 4π becomes $\frac{1}{\varepsilon_0}$ so that we have

$$A_k = \sqrt{\frac{2\hbar}{V\varepsilon_0\omega_k}}\, a_k \qquad a_k \text{ dimensionless}$$

$$[A_k] = \left(\frac{\text{gr cm}^2/\text{sec}}{\frac{\text{coul}^2\text{sec}^2}{\text{gr}}\frac{1}{\text{sec}}}\right)^{1/2} = \frac{\text{gr cm}}{\text{coul sec}}$$

so that $[eA] = \frac{\text{gr cm}}{\text{sec.}}$ has the units of momentum.

I have forgotten the polarization. But we can include this by interpreting the sum as over wave vectors + polarization.

It seems the above is valid even when $c \neq 1$.

Review: $$A = \mathrm{Re}\left(\sum_k A_k e^{i(kx-\omega_k t)}\right) \qquad k\cdot A_k = 0, \quad \omega_k = c|k|.$$

$$E = -\frac{\partial A}{\partial t} = \mathrm{Re}\left(\sum_k \omega_k i A_k e^{i(kx-\omega_k t)}\right)$$

$$B = \nabla\times A = \mathrm{Re}\left(\sum_k k\times(iA_k)\, e^{i(kx-\omega_k t)}\right)$$

Energy $$H = \frac{\varepsilon_0}{2}\int_V \left(|E|^2 + c^2|B|^2\right)dx$$

Now if $t=0$
$$E = \sum_k \omega_k \frac{1}{2}\left(iA_k e^{ikx} + \overline{iA_k}\, e^{-ikx}\right)$$

$$= \sum_k \frac{\omega_k}{2}\left(iA_k + \overline{iA_{-k}}\right)e^{ikx}$$

$$B = \sum_k \frac{1}{2}\, k\times\left(iA_k - \overline{iA_{-k}}\right)e^{ikx}$$

so by Parseval.

$$H = \frac{\varepsilon_0}{2}\sum_k\left(\frac{\omega_k^2}{4}\left|iA_k + \overline{iA_{-k}}\right|^2 + \frac{c^2}{4}\left|k\times(iA_k - \overline{iA_{-k}})\right|^2\right)V$$

$$= \frac{\varepsilon_0 V}{2}\sum_k \frac{\omega_k^2}{4}\underbrace{\left(\left|A_k - \overline{A}_{-k}\right|^2 + \left|A_k + \overline{A}_{-k}\right|^2\right)}_{2|A_k|^2 + 2|\overline{A}_{-k}|^2}$$

$$\boxed{H = \frac{\varepsilon_0 V}{2}\sum_k \omega_k^2 |A_k|^2}$$

When quantized $$H = \sum_k \hbar\omega_k a_k^* a_k$$

Thus we want $$A_k = \sqrt{\frac{2\hbar\omega_k}{\varepsilon_0 V\omega_k^2}}\, a_k = \sqrt{\frac{2\hbar}{\varepsilon_0 V\omega_k}}\, a_k$$

and so as an operator in the Schroedinger picture

$$A(x) = \sum_{k\,[redacted]} \sqrt{\frac{2\hbar}{\varepsilon_0 V \omega_k}} \; \frac{e^{ikx} a_k + e^{-ikx} a_k^*}{2}$$

Let's put in the polarizations explicitly. A_k is a vector so that

$$A_k = [redacted]\; \varepsilon^{(1)}(k) A_{k1} + \varepsilon^{(2)}(k) A_{k2}$$

where $\varepsilon^{(\alpha)}(k)$ are orth. unit vectors also $\perp$ to k; in fact $(\varepsilon^{(1)}, \varepsilon^{(2)}, k/|k|)$ are usually taken to be right-handed. Then the formulas [redacted] are

$$\vec{A}(x) = \sum_{k,\alpha} \sqrt{\frac{[redacted]\,\hbar}{\varepsilon_0 V \omega_k 2}} \left(\vec{\varepsilon}^{(\alpha)}(k) e^{ikx} a_{k\alpha} + \vec{\varepsilon}^{(\alpha)}(k) e^{-ikx} a_{k\alpha}^* \right)$$

$$H = [redacted] \sum_{k,\alpha} \hbar\omega_k \, a_{k\alpha}^* a_{k\alpha}$$

The program is now to understand absorption and emission of photons by a charged particle. Suppose the free particle (e.g. electron) is described by $\frac{p^2}{2m}$. In the presence of the field A, the Hamiltonian becomes

$$\frac{(p - eA)^2}{2m} = \frac{p^2}{2m} \underbrace{- \frac{e}{2m}(p \cdot A + A \cdot p) + \frac{e^2}{2m} A^2}_{H_{int}}$$

One also has $\quad H_{int}^{(spin)} = -\frac{e\hbar}{2mc}(\vec{\sigma} \cdot c\vec{B})$

which we will ignore for the moment

The combined system consisting of the electron + EM field is described by the Hamiltonian

$$H = H_{el} + H_A + H_{int}$$

acting on the tensor product of the electron Hilbert space and the photon Hilbert space. The latter [redacted] is the boson Fock space with the operator $a_{k\alpha}, a^*_{k\alpha}$ and the former is the space of functions of x. One interpret[redacted]s $A(x)$ as a vector of three operators made up of things like $e^{ikx} a_{k\alpha}$ which is to be interpreted as $e^{ikx} \otimes a_{k\alpha}$. p is interpreted as $\frac{\hbar}{i}\nabla_x$. Because [redacted] $k \cdot \varepsilon^{(\alpha)}(k) = 0$ it follows that $p \cdot A = A \cdot p$.

Next we want to use this Hamiltonian to describe the processes of emission and absorption of photons by the electron. We work to first order in H_{int}, and we need the matrix elements of H_{int} between base states describing free particles. Suppose we start off with the electron in state A and only $n_{k\alpha}$ photons of type $k\alpha$ and we are interested in the absorption of 1 photon leaving the electron in the state B. The matrix element is

$$\langle B; n_{k\alpha}-1 \,|\, H_{int} \,|\, A, n_{k\alpha} \rangle$$

Only [redacted] the part of A involving $a_{k\alpha}$ occurs. Recall that for bosons

$$|n\rangle = \frac{(a^*)^n|0\rangle}{\sqrt{n!}}$$

so

$$\langle n-1 | a | n \rangle = \langle n-1 | \frac{n}{\sqrt{n!}} (a^*)^{n-1} | 0 \rangle = \sqrt{n}$$

similarly

$$\langle n+1 | a^* | n \rangle = \sqrt{n+1}$$

[redacted] As the $\frac{e^2A^2}{2m}$ part of Hint changes photon no by $0, \pm 2$

$$\langle B; n_{k\alpha}-1 | H_{int} | A; n_{k\alpha} \rangle = -\frac{e}{m}\sqrt{\frac{\hbar}{V\varepsilon_0 2\omega_k}} \langle B | (\varepsilon^{(\alpha)}(k)\cdot p) e^{ik\cdot x} | A \rangle \sqrt{n_{k\alpha}}$$

similarly for emission

$$\langle B; n_{k\alpha}+1 | H_{int} | A; n_{k\alpha} \rangle = -\frac{e}{m}\sqrt{\frac{\hbar}{V\varepsilon_0 2\omega_k}} \langle B | e^{-ik\cdot x} \varepsilon^{(\alpha)}(k)\cdot p | A \rangle \sqrt{n_{k\alpha}+1}$$

Next I want the transition probabilities.

Idea: Can understand physics by thinking of a gas as a collections of **world** lines? Steady-state picture consists of a probability distribution on the set of configurations of points. When time is added these points become world lines (or curves). Thus the analogue of the particle density is the density-current 4 vector $(\rho, \vec{j})$. Is it possible that gauge fields occur naturally because they go with the particle current?

Return to transmission line as an analogue of the EM field:

$$\frac{\partial V}{\partial x} = -L\frac{\partial I}{\partial t} \qquad \frac{\partial I}{\partial x} = -C\frac{\partial V}{\partial t}$$

$$V = \mathrm{Re}\left(\sum_k \hat{V}_k\, e^{i(kx-\omega t)}\right)$$

$$I = \mathrm{Re}\left(\sum_k \frac{\hat{V}_k}{R}\,\frac{k}{|k|}\, e^{i(kx-\omega t)}\right)$$

is the general solution over $[0,\ell]$ with certain bdry conditions

Here $R = \sqrt{\frac{L}{C}}$ $\quad \omega = c|k| \quad c = \frac{1}{\sqrt{LC}}$.

The energy is the sum of the energy of the different modes

$$H = \int_0^\ell \left(\tfrac{1}{2}CV^2 + \tfrac{1}{2}LI^2\right)dx = \sum_k \left(\tfrac{1}{2}C\,\tfrac{1}{2}|\hat{V}_k|^2\ell + \tfrac{1}{2}L\,\tfrac{1}{2}\frac{|\hat{V}_k|^2}{R^2}\ell\right)$$

$$\frac{L}{R^2} = C$$

$$= \sum_k \frac{\ell}{2}C|\hat{V}_k|^2$$

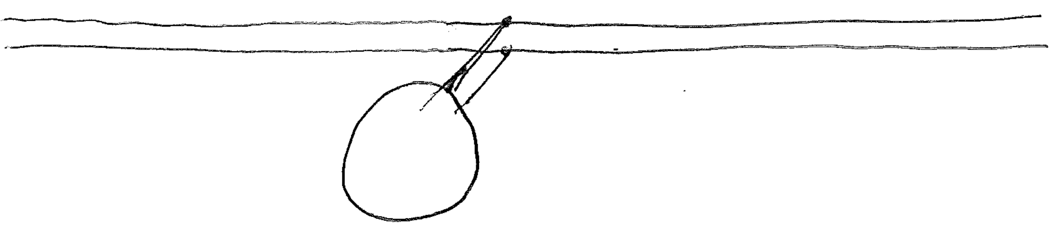
When we quantize $\quad \frac{\ell}{2}C|\hat{V}_k|^2 = \hbar\omega_k|a_k|^2$

$$\hat{V}_k = \sqrt{\frac{2\hbar\omega_k}{\ell C}}\, a_k$$

and so as operators

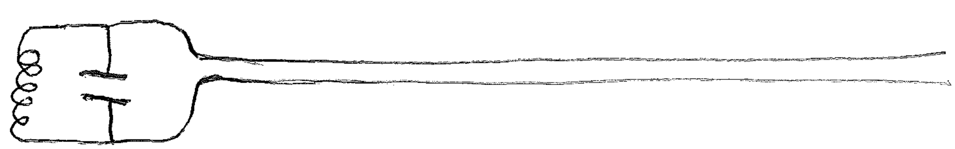
$$\begin{cases} V = \sum_k \sqrt{\dfrac{2\hbar\omega_k}{lC}}\ \dfrac{1}{2}\left(e^{i(kx-\omega t)}a_k + e^{-i(kx-\omega t)}a_k^*\right) \\[2ex] I = \sum_k \sqrt{\dfrac{2\hbar\omega_k}{lL}}\ \dfrac{k}{|k|}\ \dfrac{1}{2}\left(e^{i(kx-\omega t)}a_k + e^{-i(kx-\omega t)}a_k^*\right) \end{cases}$$

The next project is to couple the transmission line to a system with different energy levels so that we can see the process of absorption and emission. I already tried to do this according to the picture



This corresponds to adding the space of states of the line with the space of states of the addition on the classical level. When we quantize it's like taking the symmetric algebra of the classical state space, so we get the tensor product of the two quantum state spaces.

It seems desirable then to understand the quantum version of a tuned circuit coupled to a transmission line



September 19, 1980

Interaction of EM field with a charged particle is described by Hamiltonian: (no spin)

$$H = \frac{(p-eA)^2}{2m} + V(x) + \underbrace{H_{\text{EM fld.}}}_{\gamma}$$

$$= \underbrace{\frac{p^2}{2m} + V(x) + H_\gamma}_{H_0 = H_e + H_\gamma} + \underbrace{-\frac{e}{2m}(p\cdot A + A\cdot p) + \frac{e^2A^2}{2m}}_{H_{int}}$$

When A is interpreted [redacted] as an operator, one [redacted] takes the value of A at the position x of the electron.

We want the matrix elements of H_{int} between states consisting of an H_e-eigenstate of the particle and [redacted] a H_γ-eigenstate of H_γ. Let's consider just the process of absorption of a photon of type k, α and in which the electron has the transition $m \to n$. Then the matrix element is proportional to (neglect A^2 term)

$$\langle n | e^{ikx} \varepsilon^\alpha(k) \cdot p | m \rangle$$

Because [redacted] atomic dimensions are small one can assume $x \sim 0$, so this becomes

$$\varepsilon^\alpha(k) \cdot \langle n | \vec{p} | m \rangle \qquad [p, x] = \frac{\hbar}{i}$$

Finally $$\vec{p} = \left[\vec{x}, \frac{p^2}{2m_e}\right] \frac{m_e}{\hbar i} = \frac{m_e}{\hbar i}\Big[\vec{x}, \underbrace{\frac{p^2}{2m_e} + V(x)}_{H_e}\Big]$$

so

$$\langle n | \vec{p} | m \rangle = \frac{m_e}{\hbar i}(E_m - E_n) \langle n | \vec{x} | m \rangle$$

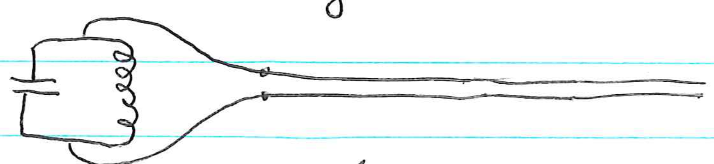
So the matrix element of the transition $m \to n$ caused by absorption of a photon of type $k\alpha$ is proportional to

$$\varepsilon^{\alpha}(k) \cdot \langle n | e\vec{x} | m \rangle$$

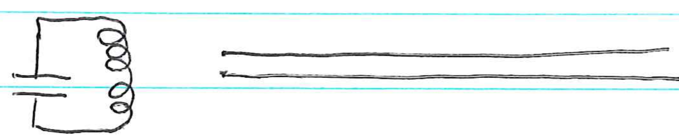
The vector $\langle n | e\vec{x} | m \rangle$ is called the <u>electric dipole moment of the $m \to n$ transition</u>.

Notice that making the approximation $e^{ikx} \sim 1$ amounts to forcing the field to interact only thru its value at $x=0$.

I tried to realize

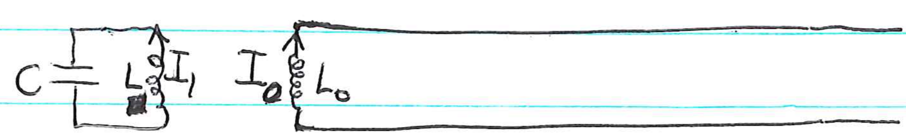


as a perturbation on the situation



but this doesn't work. The extra wires carry currents not in the space of states of the latter system.

The following seems to work

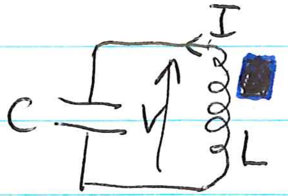


Assume the two inductances have a mutual inductance which can be varied by changing the separation. The interaction energy is a nice bilinear function

$$\frac{1}{2} M I_1 I_0$$

where M is the mutual inductance.

Let's set this up carefully. First suppose $M=0$ and discuss first the tank circuit.



$$V = -L\frac{dI}{dt} \qquad V = \mathrm{Re}(\hat{V}e^{-i\omega t})$$

$$I = C\frac{dV}{dt} \qquad I = \mathrm{Re}(\hat{I}e^{-i\omega t})$$

$$\hat{V} = -L(-i\omega)\hat{I} \qquad \omega = \frac{1}{\sqrt{LC}}$$

$$\hat{I} = -Ci\omega\,\hat{V}$$

Better:
$$V = \mathrm{Re}(Ae^{-i\omega t})$$

$$I = -C\omega\,\mathrm{Re}(Aie^{-i\omega t}) = C\omega\,\mathrm{Im}(Ae^{-i\omega t})$$

$$H = \tfrac{1}{2}CV^2 + \tfrac{1}{2}LI^2 = \tfrac{1}{2}C|A|^2$$

Thus $\frac{1}{2}C|A|^2 = \hbar\omega|a|^2$ so $A = \sqrt{\frac{2\hbar\omega}{C}}\,a$

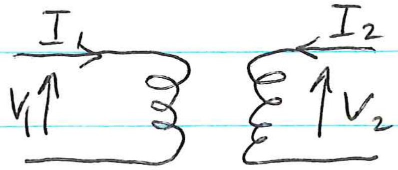
when quantized. Hence as operators

$$V = \sqrt{\frac{\hbar\omega}{C2}}(a + a^*)$$

$$I = C\omega\sqrt{\frac{2\hbar\omega}{C}}\left(\frac{a - a^*}{2i}\right)$$

September 20, 1980

Consider a transformer:



$$V_1 = L_1 \frac{\partial I_1}{\partial t} + M \frac{\partial I_2}{\partial t}$$

$$V_2 = M \frac{\partial I_1}{\partial t} + L_2 \frac{\partial I_2}{\partial t}$$

$$\begin{pmatrix} V_1 \\ V_2 \end{pmatrix} = \begin{pmatrix} L_1 & M \\ M & L_2 \end{pmatrix} \frac{\partial}{\partial t} \begin{pmatrix} I_1 \\ I_2 \end{pmatrix}$$

$$\text{power in} = (I_1 \; I_2) \begin{pmatrix} V_1 \\ V_2 \end{pmatrix} = \frac{\partial}{\partial t} \Big(\underbrace{\tfrac{1}{2} L_1 I_1^2 + M I_1 I_2 + \tfrac{1}{2} L_2 I_2^2}_{\text{internal energy}} \Big)$$

$$\hat{V}_1 = L_1 s \hat{I}_1 + Ms \hat{I}_2 \qquad \hat{I}_2 = \frac{1}{Ms}\hat{V}_1 - \frac{L_1}{M}\hat{I}_1$$

$$\hat{V}_2 = Ms \hat{I}_1 + L_2 s \hat{I}_2 \qquad \hat{V}_2 = Ms\hat{I}_1 + \frac{L_2}{M}\hat{V}_1 - s\frac{L_1 L_2}{M}\hat{I}_1$$

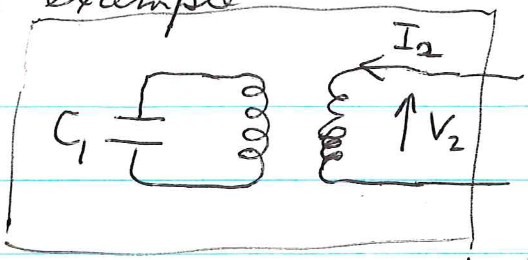
$$\begin{pmatrix} \hat{V}_2 \\ +\hat{I}_2 \end{pmatrix} = \begin{pmatrix} \frac{L_2}{M} & -s\left(M - \frac{L_1 L_2}{M}\right) \\ \frac{1}{sM} & +\frac{L_1}{M} \end{pmatrix} \begin{pmatrix} \hat{V}_1 \\ -\hat{I}_1 \end{pmatrix}$$

$$\begin{pmatrix} \hat{V}_2 \\ \hat{I}_2 \end{pmatrix} = \underbrace{\begin{pmatrix} \frac{L_2}{M} & s\left(\frac{L_1 L_2}{M} - M\right) \\ \frac{1}{sM} & \frac{L_1}{M} \end{pmatrix}}_{\det = 1} \begin{pmatrix} \hat{V}_1 \\ -\hat{I}_1 \end{pmatrix}$$

So if we use this transformer to connect up an impedance Z_1, we get the new impedance

$$Z_2(s) = \begin{pmatrix} L_2 & s(L_1 L_2 - M^2) \\ \frac{1}{s} & L_1 \end{pmatrix} Z_1(s)$$

For example



$$Z_1(s) = \frac{1}{C_1 s}$$

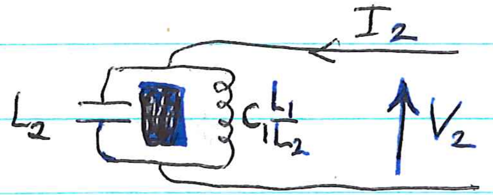
$$Z_2(s) = \frac{L_2\left(\frac{1}{C_1 s}\right) + s(L_1 L_2 - M^2)}{\frac{1}{s}\frac{1}{C_1 s} + L_1} = L_2 s - \frac{M^2 s^2}{\frac{1}{C_1 s} + L_1 s}$$

In general it is known that $M^2 \leq L_1 L_2$ with equality for an "ideal" transformer. I am ultimately interested in letting M go from $M = 0$ where there is no coupling and $Z_2(s) = L_2 s$ to $M = \sqrt{L_1 L_2}$ where

$$Z_2(s) = \frac{\frac{L_2}{C_1}}{\frac{1}{C_1 s} + L_1 s} \qquad \text{if } M^2 = L_1 L_2$$

$$= \frac{1}{\frac{1}{L_2 s} + \frac{C_1 L_1}{L_2} s}$$

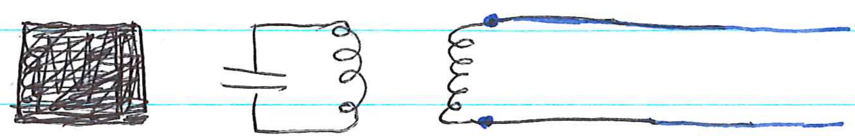
which is the same as



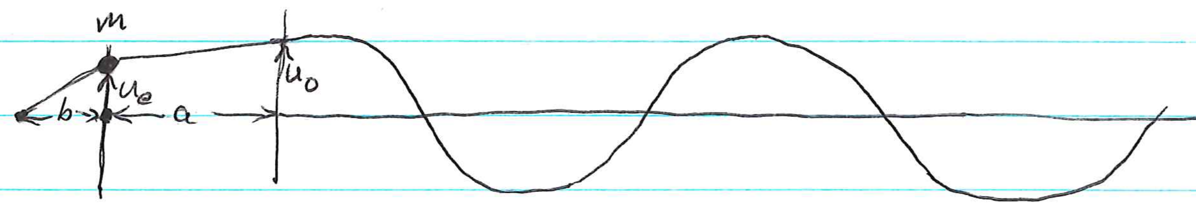
Perhaps this extreme case $M = \sqrt{L_1 L_2}$ is degenerate in some sense, and so should be avoided.

September 21, 1980

Review: I am trying to find a simple model of absorption and emission of photons. The idea was to use a transmission line coupled to a tuned circuit



Another example is to take a string of unit density for $x \geq 0$ and attach its end to a mass point.



This is an example of a (Krein) string. The energy is

$$\frac{1}{2} m \dot{u}_e^2 + \frac{1}{2}\frac{1}{b} u_e^2 + \frac{1}{2}\frac{1}{a}(u_0 - u_e)^2 + \int \left[\frac{1}{2}\left(\frac{\partial u}{\partial t}\right)^2 + \frac{1}{2}\left(\frac{\partial u}{\partial x}\right)^2 \right] dx$$

Both of these examples when quantized lead to the following situation: One has single oscillator described by $H_e = \nu b^* b$ and a continuous family of oscillators given by $H_\gamma = \sum \omega_k a_k^* a_k$. The interaction between these is described by a bilinear Hamiltonian

$$H_{int} = \lambda \sum{}' (b + b^*)(a_k + a_k^*) f_k$$

The combined system is therefore a generalized oscillator and its quantum Hilbert space is the symmetric algebra on its classical phase space (with natural complex structure)

The interesting point is that for $\lambda \neq 0$ the single oscillator disappears in the following sense. We can decompose the combined system into normal modes. For each $\omega > 0$ there is exactly one mode of vibration for $\lambda \neq 0$, however for $\lambda = 0$ and [redacted] ω = the frequency ν of the oscillator we get two modes. Thus the single oscillator vibrating by itself no longer occurs as a stationary state for the coupled system. The spectrum of the coupled system is absolutely continuous for $\lambda \neq 0$ and as $\lambda \to 0$ it acquires point spectrum at $\omega = \nu$.

So it appears that from the scattering data, i.e. the reflection coefficient [redacted] $R(\omega)$, we should be able to see the frequency ν approximately (the width of the spectral line should be related to the lifetime of the excited states). [redacted] The size of λ should be important in determining the lifetimes and width of the lines.

---

damped harmonic oscillator

$$\ddot{x} + \gamma \dot{x} + \omega_0^2 x = \mathrm{Re}(e^{-i\omega t})$$

$$x = \mathrm{Re}(A e^{-i\omega t})$$

$$A = \frac{1}{(\omega_0^2 - \omega^2) - i\gamma\omega}$$

Compute rate of energy dissipation. $F\dot{x}$ = rate at which work is done by external force

$$F\dot{x} = \mathrm{Re}\,[redacted](e^{-i\omega t})\ \mathrm{Re}(A e^{-i\omega t})$$

$-i\omega$

The time average is

$$\langle F\dot{x}\rangle = \left\langle \frac{(e^{-i\omega t}+e^{i\omega t})(-i\omega A e^{-i\omega t} + i\omega \bar{A} e^{i\omega t})}{4} \right\rangle$$

$$= \frac{1}{4}\left(i\omega \bar{A} - i\omega A\right) = \frac{\omega}{2}\,\frac{A-\bar{A}}{2i} = \frac{\omega}{2}\,\mathrm{Im}\,A$$

Thus

$$\text{power loss} = \frac{\omega}{2}\,\mathrm{Im}\,A = \frac{\omega}{2}\,\frac{\gamma\omega}{(\omega^2-\omega_0^2)^2+\gamma^2\omega^2}$$

Another derivation

$$F\dot{x} = \ddot{x}\dot{x} + \gamma\dot{x}^2 + \omega_0^2 x\dot{x}$$

$$= \frac{d}{dt}\left(\frac{\dot{x}^2}{2} + \frac{\omega_0^2 x^2}{2}\right) + \gamma\dot{x}^2$$

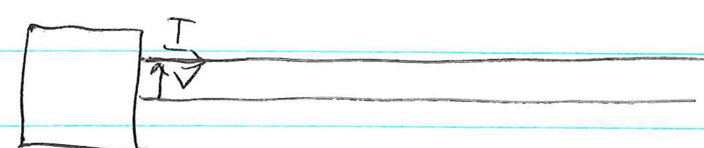
$$\therefore \quad \langle F\dot{x}\rangle = \gamma\langle \dot{x}^2\rangle = \gamma\left\langle \mathrm{Re}\left(-i\omega A e^{-i\omega t}\right)\right\rangle$$

$$= \gamma\omega^2\,\frac{|A|^2}{2}$$

September 22, 1980

Review: I am trying to understand absorption and emission of radiation by finding a simple example which illustrates the phenomena. My idea is to a transmission line (or semi-infinite string) and couple it weakly with a simple harmonic oscillator. What I found was that the Hilbert space describing the coupled system is the boson Fock space for the terminated string. The simple oscillator states are no longer stationary for the coupled system. I believe the terminated string is completely described by the scattering data (= reflection coefficient). Therefore it ought to be possible to "see" the stationary states of the simple oscillator in the scattering data.

Consider a transmission line terminated by an impedance Z.



$$V = \mathrm{Re}\left(\hat{V}\left(e^{-ikx} + R(k)e^{ikx}\right)e^{-ikt}\right) \qquad k>0.$$

$$I = \mathrm{Re}\left(\hat{V}\left(-e^{-ikx} + R(k)e^{ikx}\right)e^{-ikt}\right)$$

Then

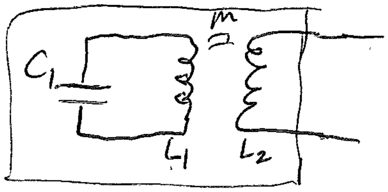
$$Z(k) = \left.\frac{\hat{V}}{-\hat{I}}\right|_{x=0} = \frac{R+1}{-R+1} \qquad \text{or} \qquad R(k) = \frac{Z-1}{Z+1}$$

I want the simplest possible Z. It should look act a damped harmonic oscillator if the line is replaced by a resistance of 1. Thus the equation

$$Z(s) = -1 \qquad s = -i\omega$$

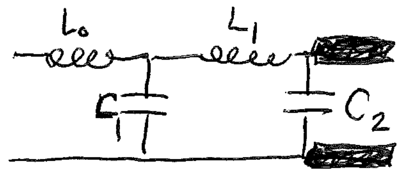
should have two conjugate complex roots $-\frac{\gamma}{2} \pm i\omega_0$ with γ positive and very small. (The Q, defined to be $\frac{\omega_0}{\gamma}$ should be large.)

(Digression: The port

(*) 

has impedance
$$Z(s) = L_2 s - \frac{M^2 s^2}{\frac{1}{C_1 s} + L_1 s}$$

and so leads to a cubic equation: $Z(s) = -1$. What's nice about this circuit is that M is a natural coupling constant. Note that the port

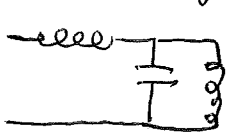


has the
$$Z = L_0 s + \cfrac{1}{C_1 s + \cfrac{1}{L_1 s + \cfrac{1}{C_2 s}}}$$

Hence the port (*) is equivalent to one of this form. In fact the formulas are

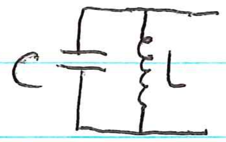
$$L_2 s - \frac{M^2 s^2}{\frac{1}{C_1 s} + L_1 s} = \left(L_2 - \frac{M^2}{L_1}\right) s + \cfrac{1}{\frac{L_1^2 C_1}{M} s + \cfrac{1}{\frac{M^2}{L_1} s}}$$

so that we get an equivalent circuit



This appears too complicated)

Consider the obvious candidate



$$Z = \frac{1}{\frac{1}{Ls} + Cs} = \frac{Ls}{LCs^2+1}$$

The equation

$$Z(s) = -1 \qquad \text{gives}$$

$$LCs^2 + Ls + 1 = 0$$

$$\gamma = \frac{1}{C}$$

or

$$s = \frac{-L \pm \sqrt{L^2 - 4LC}}{2LC} \qquad \omega_0 = \frac{1}{\sqrt{LC}}$$

$$= -\frac{1}{2C} \pm i \frac{1}{\sqrt{LC}} \left(1 - \frac{L}{4C}\right)^{1/2}$$

I want to work with $\omega_0 = \frac{1}{\sqrt{LC}} = 1$ whence

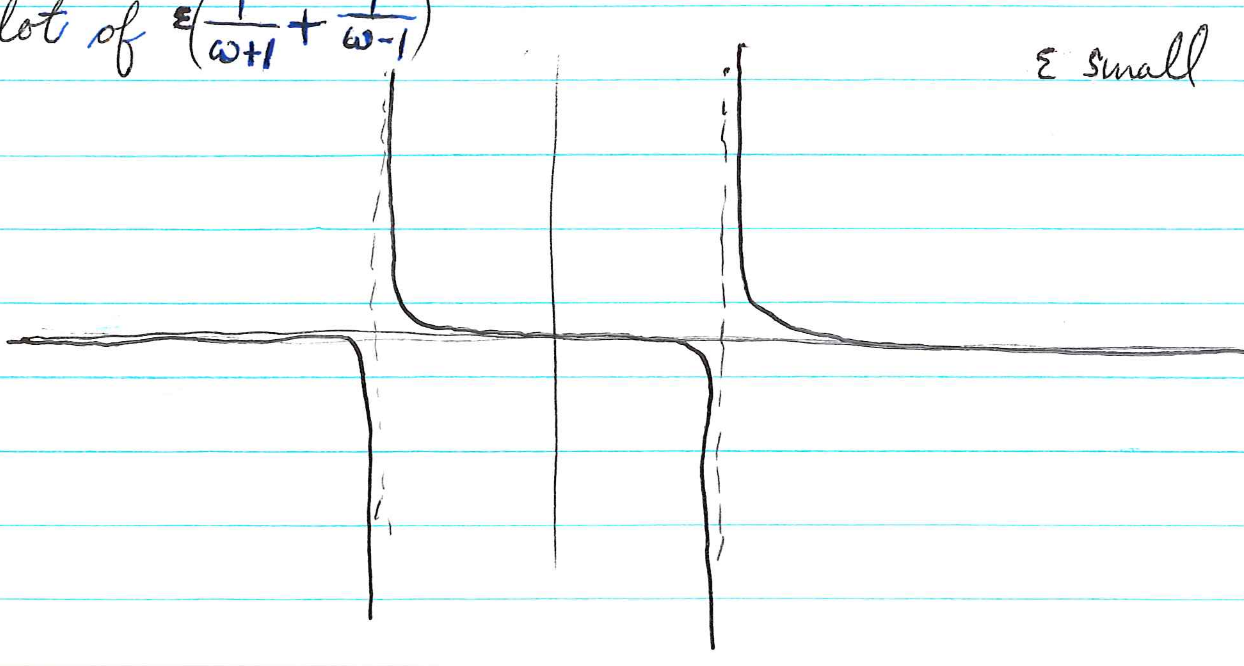
$$Q = \frac{\omega_0}{\gamma} = C = \frac{1}{L}$$

is assumed to be very large and L is very small.

Look at $Z(\omega) = +iL \dfrac{\omega}{\omega^2 - 1}$

$$= iL \, \frac{1}{2} \left[\frac{1}{\omega+1} + \frac{1}{\omega-1} \right]$$

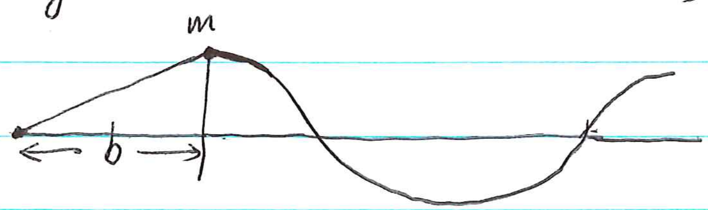
Plot of $\varepsilon\left(\frac{1}{\omega+1} + \frac{1}{\omega-1}\right)$



Therefore the impedance $Z(\omega)$ is close to zero everywhere on $0<\omega<\infty$ except near $\omega_0 = 1$.

The problem is now as follows: For each $\omega>0$ we get a mode and the collection of these modes describes the coupled system + oscillator. For [redacted] ω not too close to ω_0 it looks like the line is short-circuited, but as $\omega \to \omega_0$ it changes rapidly to an open-circuit-like termination. [redacted] In what way is the coupled system [redacted] to be regarded as a perturbation of the line + oscillator? How can I see absorption of photons?

String model. tension $=1$, $\rho = 1$ for $x>0$.



Equations of motion: $\quad \dfrac{\partial^2 u}{\partial t^2} = \dfrac{\partial^2 u}{\partial x^2} \quad$ for $x>0$

$$m\ddot{u}_0 = -\frac{1}{b}u_0 + \left.\frac{\partial u}{\partial x}\right|_{x=0}$$

Frequency ω:

$$u(x,t) = \mathrm{Re}\left(\left(e^{-i\omega x} + R e^{i\omega x}\right)e^{-i\omega t}\right)$$

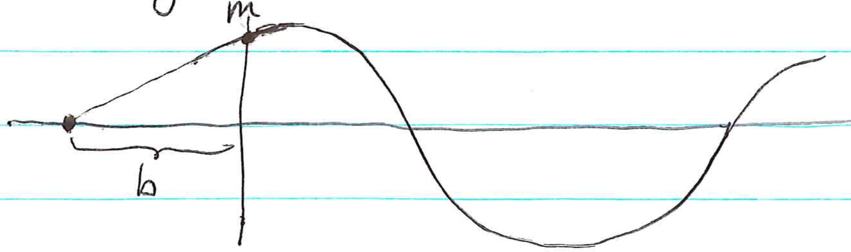
$$ms^2(1+R) = -\frac{1}{b}(1+R) + (+1-R)s$$

or

$$\boxed{ms^2 + \frac{1}{b} = s\,\frac{1-R}{1+R}}$$

September 24, 1980

String model



Equations of motion:
$$\begin{cases} m\ddot{u}_0 = -\frac{1}{b}u_0 + \left.\frac{\partial u}{\partial x}\right|_{x=0} \\ \ddot{u} = \partial_x^2 u \qquad x>0. \end{cases}$$

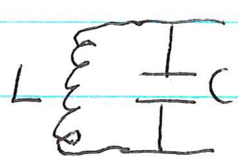
modes of freq. ω : $\quad u(x,t) = \mathrm{Re}\left(\left(e^{-i\omega x} + R(\omega)e^{i\omega x}\right)e^{-i\omega t}\right)$

$$m(-i\omega)^2(1+R) = -\frac{1}{b}(1+R) + \left((-i\omega) + R\cdot(i\omega)\right)$$

$$ms^2 + \frac{1}{b} = s\,\frac{1-R}{1+R} \qquad s = -i\omega$$

$$ms + \frac{1}{bs} = \frac{1-R}{1+R}$$

Recall that for a transmission line
$$V(x,t) = \left(e^{-i\omega x} + Re^{i\omega x}\right)e^{-i\omega t}$$
$$I(x,t) = \left(-e^{-i\omega x} + Re^{i\omega x}\right)e^{-i\omega t}$$
so that when coupled to an impedance $Z(\omega)$ we get
$$Z(\omega) = -\frac{1+R}{-1+R} = \frac{1+R}{1-R}$$



For tuned circuit $\quad Z(s) = \dfrac{1}{Cs + \frac{1}{Ls}} = \dfrac{1+R}{1-R}$

hence we see that we have the dictionary:
$$\boxed{\begin{array}{c} m \longleftrightarrow C \\ b \longleftrightarrow L \\ u(x,t) \longleftrightarrow V(x,t) \end{array}}$$

What is the high Q situation? For the tuned circuit the decay modes are given by

$$\frac{1}{Cs+\frac{1}{Ls}} = -1 \quad \text{or} \quad Cs + 1 + \frac{1}{Ls} = 0$$

$$\text{or} \quad s^2 + \frac{1}{C}s + \frac{1}{LC} = 0$$

Thus

$$Q = \frac{\omega_0}{\gamma} = \frac{1}{\sqrt{LC}} \Big/ \frac{1}{C} = \sqrt{\frac{C}{L}}$$

so $\quad$ high Q $\Longleftrightarrow$ C large $\quad$ L small

$\qquad\qquad\qquad\qquad$ m large, b small

So high Q means the mass m is large and the spring constant is also large.

How much energy is in the [redacted] tank at the frequency ω?

Suppose we have the mode

$$u(x,t) = Re\left[\left(e^{-i\omega x} + Re^{i\omega x}\right)e^{-i\omega t}\right].$$

Then the mass at $x=0$ moves via

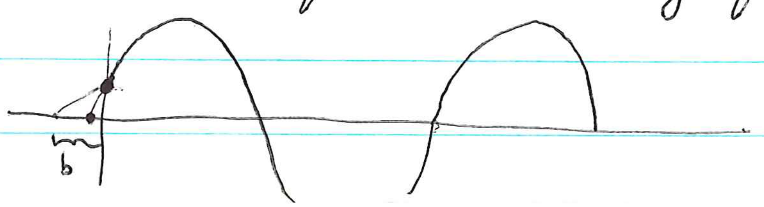
$$u(0,t) = Re\left((1+R)e^{-i\omega t}\right)$$

hence it has amplitude $|1+R|$. [redacted] Since

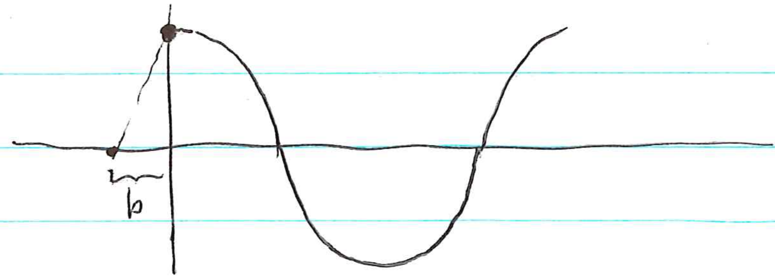
$$\frac{1-R}{1+R} = Cs + \frac{1}{Ls} \qquad \frac{1+R}{1-R} = \frac{\overset{\text{small}}{Ls}}{LCs^2+1}$$

[redacted]

we know that $R \sim -1$ for ω not too close to ω_0, and that $R = 1$ at $\omega = \omega_0$. So the picture of the mode for ω away from ω_0 is



and for $\omega = b\omega_0$ it is



Let's compute the energy density for the mode

$$(*)\quad \begin{cases} V = \operatorname{Re}\left(A\left(e^{-i\omega x} + Re^{i\omega x}\right)e^{-i\omega t}\right) \\ I = \operatorname{Re}\left(A\left(-e^{-i\omega x} + Re^{i\omega x}\right)e^{-i\omega t}\right) \end{cases}$$

Suppose $R = -e^{-2i\delta(\omega)}$; the normal state of affairs is when $R = -1$. [redacted]

$$V = \operatorname{Re}\left((-A)\left(e^{i\omega x - i\delta} - e^{-i\omega x + i\delta}\right)e^{-i\delta}e^{-i\omega t}\right)$$

$$= \operatorname{Re}\left(\left(-2iAe^{-i\delta}\right)\sin(\omega x - \delta)\, e^{-i\omega t}\right)$$

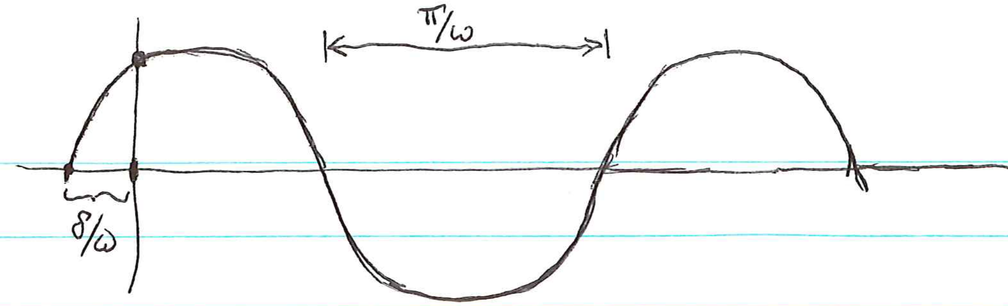
$$\boxed{\begin{aligned} V &= 2|A| \sin(\omega x - \delta)\sin(\omega t - \phi) \\ I &= 2|A| \cos(\omega x - \delta)\cos(\omega t - \phi) \end{aligned}}$$

The energy density averaged over time is

$$\left\langle \tfrac{1}{2}\left(V^2 + I^2\right)\right\rangle = \tfrac{1}{2}\left(4|A|^2\left(\sin^2(\omega x - \delta) + \cos^2(\omega x - \delta)\right)\tfrac{1}{2}\right)$$

$$= |A|^2$$

Notice that δ/ω can be intepreted as the imaginary point where the standing wave seems to start:



The energy density for the string situation is slightly different:

$$\left\langle \frac{1}{2}\left(\frac{\partial u}{\partial t}\right)^2 + \frac{1}{2}\left(\frac{\partial u}{\partial x}\right)^2 \right\rangle = \text{[redacted]}\ \omega^2 |A|^2$$

Here I think of $u(x,t) = V(x,t)$ as given on the previous pages. Let's compute oscillators energy

L C $\uparrow V$

$$V = \mathrm{Re}\,(1+R)e^{-i\omega t} = V_C$$

$$\left\langle \tfrac{1}{2} C V_C^2 \right\rangle = \tfrac{1}{4} C |1+R|^2$$

$$I_L = \mathrm{Re}\left(\frac{1+R}{Ls} e^{-i\omega t}\right)$$ so [redacted]

[redacted] $$\left\langle \tfrac{1}{2} L I_L^2 \right\rangle = \frac{1}{2} L \frac{|1+R|^2}{L^2\omega^2}\frac{1}{2} = \frac{1}{4}\frac{1}{L\omega^2}|1+R|^2$$

Thus

$$\langle H_{tank} \rangle = \frac{1}{4}\left(C + \frac{1}{L\omega^2}\right)|1+R|^2$$

$$\approx \frac{1}{2} C |1+R|^2 \quad \text{for} \quad \omega \approx \omega_0 = \frac{1}{\sqrt{LC}}$$

On the other hand if we compute mechanically

$$H_m = \frac{1}{2}\frac{1}{b}\ \text{[redacted]}\ u_0^2 + \frac{1}{2} m \dot{u}_0^2$$

$$u_0 = \mathrm{Re}\left((1+R)e^{-i\omega t}\right)$$

$$\dot{u}_0 = \omega\, \mathrm{Im}\left((1+R)e^{-i\omega t}\right)$$

$$\therefore \langle H_m \rangle = \frac{1}{2}\frac{1}{b}\frac{|1+R|^2}{2} + \frac{1}{2} m\omega^2 \frac{|1+R|^2}{2}$$

$$\langle H_m \rangle = \frac{1}{4}\left(m\omega^2 + \frac{1}{b}\right)|1+R|^2$$

Thus in general one has

$$\langle H_m \rangle = \omega^2 \langle H_{tank} \rangle$$

which means that mechanical energy is ω^2 times electrical energy. We can also write this

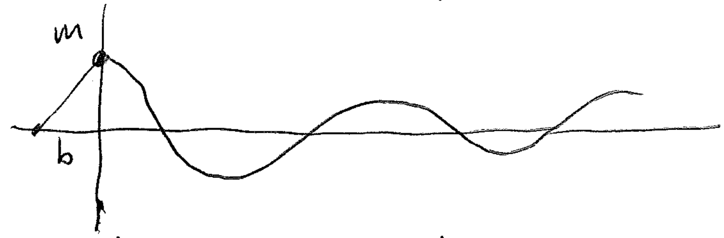
$$\langle H_m \rangle = \frac{1}{4} m\left(\omega^2 + \omega_0^2\right)|1+R|^2$$

Recall that $R(\omega) \approx -1$ for ω not close to ω_0 and that $R(\omega_0) = 1$. Thus we see that around resonance we get energy in the tank circuit.

Still, what does this have to do with emission and absorption of photons?

September 26, 1980

Recall the model for emission + absorption of photons:



The equations of motion are

$$\begin{cases} m\ddot{u}_0 = -\frac{1}{b}u_0 + \left.\frac{\partial u}{\partial x}\right|_{x=0} \\ \frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} \qquad x>0 \end{cases}$$

The normal modes are

$$(*) \qquad u(x,t) = A\left(-e^{-i\omega x} + R(\omega)\, e^{i\omega x}\right)e^{-i\omega t}$$

where if $s = -i\omega$ we have

$$ms + \frac{1}{bs} = \frac{1+R}{1-R}$$

or

$$\boxed{R = \frac{1-Z}{1+Z} \qquad Z = \frac{1}{ms+\frac{1}{bs}}}$$

I have changed the sign of R from yesterday so that in the high Q-case $R \approx 1$ for $\omega \neq \omega_0$ and $R \approx -1$ at $\omega = \omega_0$. This means

$$R = e^{-2i\delta}$$

where δ (or $-\delta$) is the phase shift.

The energy in the mode $(*)$ for the mass m is

$$\langle H_m \rangle = \tfrac{1}{2}m\left(\frac{\omega^2+\omega_0^2}{2}\right)|A|^2\,|1-R|^2$$

It seems that a simpler model results if we take the string to be discrete with masses [redacted] a located at $x \in \mathbb{N}a$. The equations of motion are

$$(*) \quad \begin{cases} \ddot{u}_0 + \omega_0^2 u_0 = \dfrac{1}{m} \dfrac{u_a - u_0}{a} \\ \ddot{u}_x = \dfrac{1}{a^2}\left(u_{x+a} - 2u_x + u_{x-a}\right) \end{cases}$$

The dispersion relation for waves on the line is

$$\omega^2 = \frac{1}{a^2}(2 - 2\cos ka) = \frac{4\sin^2(ka/2)}{a^2}$$

or

$$\omega = \left| \frac{\sin(ka/2)}{a/2} \right|$$

Notice that $\frac{1}{m}$ plays the role of the coupling constant. [redacted]

The goal is to understand the quantized version of the model. The virtue of [redacted] the discrete model is that the Hamiltonian structure of the equations $(*)$ is clear. One has the coordinates u_x $x \in \mathbb{N}a$ and the momenta $m\dot{u}_0$, a[redacted] $\dot{u}_x$ $x > 0$.

Let's begin with the free case $\frac{1}{m} = 0$. Notice that this is not a terminated line & oscillator not coupled together, but rather it's the case where the end is so heavy that the line doesn't affect it.

Unfortunately there seems to be difficulty quantizing the case where the end is infinitely heavy.

Let's put the equations of motion in Hamiltonian form. The kinetic energy + potential energy are

$$KE = \frac{1}{2} m \dot{u}_0^2 + \frac{1}{2} a \dot{u}_a^2 + \frac{1}{2} a \dot{u}_{2a}^2 + \cdots$$

so $\quad p_0 = m\dot{u}_0 \qquad p_{na} = a\dot{u}_{na} \qquad n > 0$

$$PE = \frac{1}{2b} u_0^2 + \frac{1}{2a}(u_a - u_0)^2 + \frac{1}{2a}(u_{2a} - u_a)^2 + \cdots$$

$$= \frac{1}{2}\left(\frac{1}{b} + \frac{1}{a}\right) u_0^2 - \frac{u_0 u_a}{a} + \frac{u_a^2}{a} - \frac{u_a u_{2a}}{a} + \frac{u_{2a}^2}{a} - \cdots$$

Thus the Hamiltonian is

$$H = \frac{p_0^2}{2m} + \frac{p_a^2}{2a} + \frac{p_{2a}^2}{2a} + \cdots$$

$$+ \frac{1}{2}\left(\frac{1}{b} + \frac{1}{a}\right) u_0^2 - \frac{u_0 u_a}{a} + \frac{u_a^2}{a} - \cdots$$

and the equations of motion are

$$\dot{p}_0 = -\left(\frac{1}{b} + \frac{1}{a}\right) u_0 + \frac{u_a}{a} \qquad \dot{u}_0 = \frac{p_0}{m}$$

$$\dot{p}_a = \frac{u_0}{a} - \frac{2u_a}{a} + \frac{u_{2a}}{a} \qquad \dot{u}_a = \frac{p_a}{a}$$

as they should be.

Now make the canonical transformation

$p_0 \mapsto \sqrt{m}\, P, \quad u_0 \longmapsto \dfrac{Q}{\sqrt{m}}$ and we obtain

$$\boxed{\begin{aligned} H = {} & \frac{P^2}{2} + \frac{p_a^2}{2a} + \frac{p_{2a}^2}{2a} + \cdots\cdots \\ & + \frac{1}{2}\left(\frac{1}{bm} + \frac{1}{am}\right) Q^2 - \frac{Q u_a}{\sqrt{m}\, a} + \frac{u_a^2}{a} - \frac{u_a u_{2a}}{a} + \cdots \end{aligned}}$$

From this formula we see that as $m \to \infty$, $bm = \omega_0^2$

we get the Hamiltonian

$$H = \frac{P^2}{2} + \frac{\omega_0^2}{2}Q^2 + \left[\left(\frac{p_a^2}{2a} + \frac{p_{2a}^2}{2a} + \cdots\cdots\right) + \left(\frac{u_a^2}{2a} + \frac{(u_a - u_{2a})^2}{2a} + \cdots\right)\right]$$

representing a simple harm. oscillator of freq. ω_0 running independently of the string, and string is tied down at $x=0$. I can visualize this as follows. Let's fix a range of energy which is available, e.g. suppose that we work around the temperature T and hence [redacted] our oscillator has energy around [redacted]

$$\frac{1}{2}\hbar\omega_0 + \frac{\hbar\omega_0}{e^{\hbar\omega_0/kT} - 1} \approx kT \qquad \text{if } \hbar\omega_0 \ll kT.$$

Then if m is very large, the amplitude A of the oscillator has to be small, since the energy is

$$\frac{1}{2} m\, \omega_0^2 A^2$$

Thus the end of the string is essentially fixed for large m.

<u>Idea</u>: Unbounded operators A tend to be best understood as correspondences

$$\begin{tikzcd} & \mathcal{D}_A \arrow[dl] \arrow[dr] & \\ \mathcal{H} & & \mathcal{H} \end{tikzcd}$$

[redacted] Would it be possible to formulate QM, especially expectation values, transition probabilities in terms of traces of suitable correspondences?

<u>September 27, 1980</u>

The Hamiltonian of interest is $H = H_0 + H_{int}$ where

$$H_0 = \frac{1}{2}P^2 + \frac{1}{2}\omega_0^2 Q^2 + \frac{1}{2a}p_a^2 + \frac{1}{2a}p_{2a}^2 + \cdots\cdots$$
$$+ \frac{1}{2a}u_a^2 + \frac{1}{2a}(u_{2a}-u_a)^2 + \cdots\cdots$$

$$H_{int} = \lambda Q u_a$$

H_0 represents a simple harmonic oscillator together with the discrete string of masses a at the points $x = a, 2a, \cdots\cdots$ with $u_0 = 0$.

To quantize this system we introduce creation + ann. operators. I now have a nice way to do this by computing the energy of the modes.

$$Q = \mathrm{Re}\,(Ae^{-i\omega_0 t})$$
$$P = \dot{Q} = \omega_0\,\mathrm{Im}\,(Ae^{-i\omega_0 t})$$

$$H = \frac{1}{2}P^2 + \frac{1}{2}\omega_0^2 Q^2 = \frac{1}{2}\omega_0^2 |A|^2$$

When quantized $H = \omega_0 a_0^* a_0$ so that we want to put

$$A = \sqrt{\frac{2}{\omega_0}}\, a$$

and hence

$$\boxed{Q = \frac{1}{\sqrt{2\omega_0}}(a_0 + a_0^*) \qquad P = \sqrt{2\omega_0}\left(\frac{a_0 - a_0^*}{2i}\right)}$$

For the string we have the modes

$$u_x = \mathrm{Re}\left(A_k \sin(kx)\, e^{-i\omega_k t}\right)$$

where $\omega_k = \left|\dfrac{\sin ka/2}{a/2}\right|$. Let's work with a fixed length L of the string with $N = \frac{L}{a}$ masses, and suppose it is clamped at the end $x=L$; so $u_L=0$. Then

$$k = \frac{\pi}{L} n \qquad 1 \le n < \frac{L}{a}$$

We have

$$\dot{u}_x = \omega_k \, \mathrm{Im}\left(A_k \sin(kx) e^{-i\omega_k t}\right).$$

Notice that the kinetic energy is sinusoidal in time, and also the potential energy, so taking time average of either gives half the energy of the mode.

$$\therefore \text{energy of mode} = 2 \, \frac{1}{2} a \omega_k^2 |A_k|^2 \underbrace{\sum_x \sin^2(kx)}_{N/2} \underbrace{\frac{1}{2}}_{\substack{\text{time average} \\ \text{of imag. part.}}}$$

$$= \frac{1}{4} L \omega_k^2 |A_k|^2$$

Consequently we want $A_k = \frac{2}{\sqrt{L\omega_k}} a_k$ and so when quantized

$$\boxed{u_x = \sum_k \frac{1}{\sqrt{L\omega_k}} \sin kx \, (a_k + a_k^*)}$$

Check: $$a\dot{u}_x = \sum_k \frac{a}{\sqrt{L\omega_k}} \sin kx \, (-i\omega_k)(a_k - a_k^*)$$

$$[a\dot{u}_x, u_{x'}] = \sum_{k,k'} \frac{a}{\sqrt{L\omega_k}} \frac{1}{\sqrt{L\omega_{k'}}} \sin kx \sin k'x' \, (-i\omega_k) \underbrace{[a_k - a_k^*, a_{k'} + a_{k'}^*]}_{2\delta_{kk'}}$$

$$= \frac{1}{i} \sum_k \underbrace{\frac{2a}{L}}_{2/N} \sin(kx)\sin(kx') = \frac{1}{i}\delta_{xx'}$$

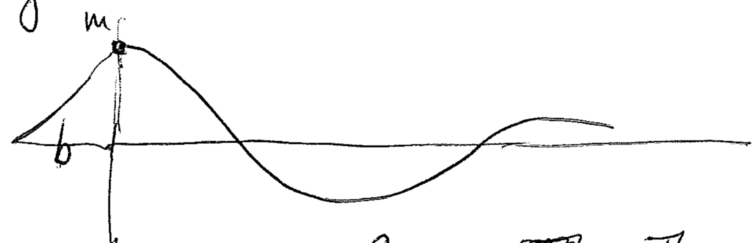
So now we have quantized the system:

$$H_0 = \omega_0 a_0^* a_0 + \sum_k \omega_k a_k^* a_k \qquad k = \frac{\pi}{L} n \qquad 1 \le n < \frac{L}{a}$$

$$H_{int} = \lambda Q u_a \qquad Q = \frac{1}{\sqrt{2\omega_0}}(a_0 + a_0^*)$$

$$u_a = \sum_k \frac{1}{\sqrt{L\omega_k}} \sin(ka)\,(a_k + a_k^*)$$

Is it possible to quantize the continuous string with weight at end:



in a similar way? The problem is how to let $a \to 0$ in the discrete Hamiltonian at the top of p 165. Recall that

$$\omega_0^2 = \left(\frac{1}{bm} + \frac{1}{am}\right) \qquad \text{(see 163)}$$