Consider an Ising model. A state is described by giving the spin $s^x_x$ at each site $x$. On the set of states one has the Boltzmann prob. measure. The Green's functions are then the moments

$$< \prod_x s^x_x >$$

where $n^x = 0$, except for a finite number of $x$. This Ising situation can be generalized as follows. One has a field described by variables $\phi_x$ at each site $x$. To be specific, think of $\phi_x$ as a real scalar; (it could be a vector). Then we have a partition function

$$Z = \int e^{-\beta S(\phi)} d\phi$$

and Boltzmann measure

$$\frac{e^{-\beta S(\phi)}}{Z}$$

on the set of configurations. The Green's functions are the moments of the Boltzmann measure and can be obtained from the generating function

$$Z(J) = \int e^{\beta \sum_x J_x \phi_x - \beta S(\phi)} d\phi / \int e^{-\beta S(\phi)} d\phi$$

(Perhaps $\beta$ should be absorbed into $S$ and $J$.)

Consider now the grand partition function

$$Z = \sum \frac{z^n}{n!} \int e^{-\beta \sum \phi_x} d\phi_1 ... d\phi_n.$$
provided we define a configuration to be a finite sequence of points \( q_1, \ldots, q_n \) modulo changing the order. A better way to say this is a configuration is a positive divisor \( \sum q_i g_i \) on \( g \)-space.

Consequently we have a field \( n(g) \) over \( g \)-space taking the values 0, 1, 2, \ldots; with only finitely many \( n(g) > 0 \). A configuration is a set of \( n(g) \) and we have the Boltzmann measure on this set of configurations. It is clear now what the Green's functions are, from this viewpoint. They are expectation values of monomials in the variables \( n(g) \).

What is a generating function? Let's take a simple example where

\[
U_n (g_1, \ldots, g_n) = \sum_{i=1}^{n} f(g_i)
\]

Then

\[
\frac{z^n}{n!} \int e^{-\beta U_n} d g_1 \cdots d g_n = \frac{1}{n!} \left( \int e^{-\beta f(g)} d g \right)^n
\]

so to get a generating function it seems like we want to replace \( z \) by a function of \( g \). Thus our generating function is something like

\[
\tilde{z} = \sum_n \frac{1}{n!} \int z(g_1) \cdots z(g_n) e^{-\beta U_n (g_1, \ldots, g_n)} d g_1 \cdots d g_n
\]

Maybe we should put \( z(g) = e^{\beta \mu(g)} \) where \( \mu(g) \) = chemical potential at \( g \). Then assuming (6) we have

\[
\tilde{z} = \exp \left( \int e^{\beta \mu(g)} - \beta f(g) \, d g \right).
\]
Thus \( \tilde{Z} \) is a function of the variables \( \mu(x) \), i.e. a function of the function \( \mu \). Perhaps I should think of \( \tilde{Z} \) as a power series in the variables \( \tilde{Z}(x) = e^{\beta \mu(x)} \).

Now fix a site \( z_0 \) and consider the moments \( \langle n(z_0)^{x} \rangle \). In the independent particle model \((*)\) we see that formally

\[
\tilde{Z} = \exp \left( -\tilde{z}(z_0) e^{-\beta f(z_0)} + \text{stuff independent of } \tilde{z}(z_0) \right)
\]

since

\[
\langle n(z_0)^{x} \rangle = \frac{1}{\tilde{Z}} \left( \frac{\tilde{z}(z_0)}{\tilde{Z}(z_0)} \right)^{x} \tilde{Z}
\]

we should understand what sort of prob. distribution on \( N \) belonging to a partition function of the form

\[
e^{z \alpha + b} = e^{b} \sum_{n} \frac{e^{n \alpha}}{n!}
\]

Clearly we get a Poisson distribution

\[
p_{x} = e^{-z \alpha} \frac{(z \alpha)^{x}}{x!}
\]

so therefore we see that in the independent particle model \((*)\) the variable \( n(z_0) \) is governed by a Poisson distribution with mean \( e^{\beta (f(z_0) - f(z_{0}))} \).

This is like the formulas

\[
\langle n_{x} \rangle = e^{\beta \mu - \beta \bar{x}}
\]

\[
= \frac{1}{e^{\beta (\bar{x} - \mu)} + 1} \quad \text{Boze statistics} + \text{FD statistics}
\]

and suggests refining the latter to an understanding
of the actual probability distribution.

In the FD case \( n = 0 \) or 1 and the probabilities are

\[
\frac{p_0}{p_1} = \frac{p_0}{1-p_0} = e^{\beta(\langle x \rangle - \mu)}
\]

\[
p_0 = \frac{e^{\beta(\langle x \rangle - \mu)}}{e^{\beta(\langle x \rangle - \mu)} + 1}
\]

so the mean \( \langle x \rangle = p_1 \) completely specifies the distribution.

In the BE case the probabilities form a geometric series.

\[
p_n = \left( e^{-\beta(\langle x \rangle - \mu)} \right)^n / Z
\]

Thus \( p_n = r^n - r^{n+1} \) where \( r = e^{-\beta(\langle x \rangle - \mu)} \).

and \( \langle x \rangle = x + x^2 + 2(x^3 - x^2) + ... = x + x^2 + x^3 + ... = \frac{x}{1-r} \)
specifies the distribution, provided we assume it has this geometric form.

One of the ideas I really want to explore is the possibility of obtaining amplitudes for some quantum theory as integrals over discrete configurations.

---

Given a measure \( d\mu \) on \( R \) one has its moments

\[
c_n = \frac{\int x^n d\mu}{\int d\mu} = \frac{1}{Z} \frac{d^n Z}{dJ^n} \bigg|_{J=0}
\]

where \( Z(J) = \int e^{Jx} d\mu / (d\mu) \)

One also has reduced moments

\[
b_n = \frac{d^n}{dJ^n} \log Z \bigg|_{J=0}
\]
Thus \( b_0 = 0 \) \( \quad b_1 = c_1 \)

\[
b_2 = \frac{d^2}{dJ^2} \log Z \bigg|_{J=0} = \frac{d}{dJ} \left( Z^{-1} \frac{d^2}{dJ^2} \right) = Z^{-1} \frac{d^2}{dJ^2} - Z^{-1} \frac{d}{dJ} \frac{d}{dJ} \frac{d}{dJ} \left|_{J=0} \right.
\]

\[
\quad = c_2 - c_1^2 \quad \text{i.e.} \quad \langle x^2 \rangle - \langle x \rangle^2
\]

\[
b_3 = Z^{-1} \frac{d^3}{dJ^3} - 3Z^{-1} \frac{d^2}{dJ^2} \frac{d}{dJ} + 3Z^{-1} \frac{d}{dJ} \frac{d^2}{dJ^2} \frac{d}{dJ} Z^{-1} \frac{d^2}{dJ^2}
\]

\[
\quad = c_3 - 3c_1 c_2 + 2c_1^3
\]

Examples:

1) Gaussian \( \mu = \frac{1}{\sqrt{2\pi}a} \)

\[
Z(J) = \int e^{Jx - \frac{x^2}{2a}} \frac{dx}{\sqrt{2\pi}a} = \pi^2 \quad \text{a} \quad \frac{x^2}{2}
\]

\[
\therefore \log Z(J) = a \frac{J^2}{2} \quad \text{so} \quad b_2 = a \quad \text{and} \quad \text{rest} \quad b_n = 0.
\]

2) Poisson \( \quad p_n = e^{-\lambda} \frac{\lambda^n}{n!} \quad \text{for} \quad n \in \mathbb{N} \)

\[
Z(J) = \sum e^{Jn} e^{-\lambda} \frac{\lambda^n}{n!} = e^{\lambda(e^{J-1})}
\]

\[
\log Z(J) = \lambda(e^{J-1}) = \sum_{n=1}^{\infty} \frac{\lambda^n}{n!} \quad \text{hence all} \quad b_n = \lambda
\]

3) FO with \( \quad p_1/p_0 = e^{-\beta \varepsilon} \). Then \( \quad p_0 = \frac{1}{1 + e^{-\beta \varepsilon}} \quad p_1 = \frac{e^{-\beta \varepsilon}}{1 + e^{-\beta \varepsilon}} \)

\[
Z(J) = \frac{1 + e^{J-\beta \varepsilon}}{1 + e^{-\beta \varepsilon}}
\]

4) BE with \( \quad p_n/p_{n-1} = e^{-\beta \varepsilon} \). Then

\[
Z(J) = \frac{\sum e^{Jn}(e^{-\beta \varepsilon})^n}{\sum(e^{-\beta \varepsilon})^n} = \frac{1 - e^{-\beta \varepsilon}}{1 - e^{J-\beta \varepsilon}}
\]

There doesn't seem to be a simple formula for the reduced moments.
We are trying to understand the Green's function belonging to the grand canonical partition function

\[
Z_g = \sum_{\mathbf{n}} \frac{z^n}{n!} \int e^{-\beta \mathbf{U} \mathbf{n}_g} d\mathbf{q}_1 \ldots d\mathbf{q}_n
\]

Yesterday we saw the possibility of viewing this as a sum over configurations, where a configuration is defined as a positive divisor in \( \mathbb{R} \)-space. Thus instead of a spin variable at each site we have an occupation number \( n(q) \) which is in \( \mathbb{N} \) and is a.e. 0.

Let us simplify by taking \( \mathbb{R} \)-space a finite set, say \( q \in \{1, \ldots, R\} \). Then because the weight of a configuration

\[
\begin{array}{ccc}
1, 1, \ldots, 1 & 2, \ldots, 2 & \ldots, R, \ldots, R \\
\hline
n_1 & n_2 & \ldots & n_R
\end{array}
\]

is

\[
\sum_{\mathbf{n}_g} \frac{1}{\prod n_g!} e^{-\beta \mathbf{U}(1, \ldots, 1, 2, \ldots, 2, \ldots, R, \ldots, R)}
\]

we can write

\[
Z_g = \sum_{n_1, \ldots, n_R} \frac{2^{\sum n_g}}{\prod n_g!} e^{-\beta \mathbf{U}(n_1, \ldots, n_R)}
\]

A Green's function in analogy with Ising models is a moment of a polynomial \( \prod n_g! \) in the variables \( n_g \) describing the configuration. We can get at these moments by introducing activity variables
\[ \tilde{Z}_{gr} = \sum_{n_1, \ldots, n_R} \frac{1}{\prod_{n} z_0^{n_0}} e^{-\beta \tilde{U}(n_1, \ldots, n_R)} \]

Then one has

\[ \langle \prod_n z_0^{n_0} \rangle = \frac{1}{\tilde{Z}_{gr}} \tilde{Z}_{gr} \bigg|_{z_0 = z} \]

Look at

\[ \langle n_1^{x_1} \rangle = \frac{1}{\tilde{Z}_{gr}} \left( \frac{\partial}{\partial z_1} \right)^{x_1} \tilde{Z}_{gr} \bigg|_{z_0 = z} \]

We can write

\[ \tilde{Z}_{gr} = \sum_n \frac{z_1^{n_1}}{n_1!} \sum_{n_2, \ldots, n_R} \frac{z_2^{n_2} \cdots z_R^{n_R}}{n_2! \cdots n_R!} e^{-\beta \tilde{U}(n_1, n_2, \ldots, n_R)} \]

Let's set \( z_2 = \ldots = z_R = z \) in the 2nd sum and observe that it is a partition function for configurations at the sites \( q = 2, \ldots, R \) with energy calculated as if \( n_i \) atoms are present at \( q = 1 \). Put

\[ Q(n_1, z) = \sum_{n_2, \ldots, n_R} \frac{z_1^{n_1} \cdots z_R^{n_R}}{n_2! \cdots n_R!} e^{-\beta \tilde{U}(n_1, n_2, \ldots, n_R)} \]

Then

\[ \tilde{Z}_{gr} \bigg|_{z_2 = \ldots = z_R = z} = \sum \frac{z_1^{n_1}}{n_1!} Q(n_1, z) \]

so

\[ \langle n_1^{x_1} \rangle = \frac{\sum (n_1)^{x_1} \frac{z_1^{n_1}}{n_1!} Q(n_1, z)}{\sum \frac{z_1^{n_1}}{n_1!} Q(n_1, z)} \bigg|_{z_1 = z} \]

The denominator is just \( \tilde{Z}_{gr} \).
So what we find is that the probability distribution for the variable \( n_i \) is given by

\[
P(n_i = m) = \frac{\sum \frac{z^{\sum n_i}}{\prod n_i!} e^{-\beta U(n_1, \ldots, n_g)}}{\sum \frac{z^{\sum n_i}}{\prod n_i!} e^{-\beta U(n_1, \ldots, n_g)}}
\]

Which should have been completely obvious from the beginning — namely you just take the part of the partition function belonging to these occupation numbers.
functional differentiation. Let us consider a functional $F(f)$, that is, a function of the function $f$. I want to work out the concept of Taylor series for $F$. First we need linear functionals. In good cases a linear functional is of the form

$$F(f) = \int f(x) g(x) \, dx$$

i.e. if we think of $f$ as an infinite-dimensional vector with the components $f(x)$, then $F(f)$ is the dot product of $f$ with the vector $g = (g(x))$. A quadratic functional is obtained by multiplying two linear functionals.

$$\int f(x) g_1(x) \, dx \cdot \int f(x) g_2(x) \, dx$$

$$= \int \int f(x_1) f(x_2) \, g_1(x_1) g_2(x_2) \, dx_1 \, dx_2$$

$$= \frac{1}{2!} \int \int f(x_1) f(x_2) \left[ g_1(x_1) g_2(x_2) + g_1(x_2) g_2(x_1) \right] \, dx_1 \, dx_2$$

$$g(x_1, x_2) \quad \text{where } g \text{ is symm.}$$

The Taylor series of $F$ can be expected to look like

$$F(f) = g_0 + \int g_1(x) f(x) \, dx_1 + \frac{1}{2!} \int g_2(x_1, x_2) f(x_1) f(x_2) \, dx_1 \, dx_2 + \cdots$$

and now what I want is to obtain the coefficient functions $g_n(x_1, \ldots, x_n)$ by the process of functional differentiation.

The idea is to define the functional derivative by
varying \( f \to f + \delta f \). Thus

\[
F(f + \delta f) = F(f) + \text{linear term} + O(\delta f^2)
\]

One can be precise by putting \( \delta f = \varepsilon \eta \) and then

\[
F(f + \varepsilon \eta) = F(f) + \frac{d}{d\varepsilon} F(f + \varepsilon \eta) \bigg|_{\varepsilon = 0} \cdot \varepsilon + O(\varepsilon^2)
\]

In good cases the linear term can be expressed

\[
\frac{d}{d\varepsilon} F(f + \varepsilon \eta) \bigg|_{\varepsilon = 0} = \int g(x) \eta(x) dx
\]

and one write, \( \eta = \nabla F(f) \)

so that

\[
F(f + \varepsilon \eta) = F(f) + \nabla F(f) \cdot \varepsilon \eta + O(\varepsilon^2)
\]

Another notation uses is that

\[
\nabla F(f)(x) = \frac{\delta F}{\delta f(x)}(f)
\]

The point is that this functional derivative depends on the measure \( dx \) used to define the dot product of functions.

It is clear that if \( \phi \) holds, then we can functionally differentiate

\[
\frac{\delta F}{\delta f(x)} = g_1(x) + \int g_2(x, x_1) f(x_1) dx_1 + \frac{1}{2!} \int g_3(x, x_1, x_2) f(x_1) f(x_2) dx_1 dx_2 + \ldots
\]
In effect
\[ \frac{1}{n!} \int \cdots \int g_n(x_1, \ldots, x_n) f(x_1) \cdots f(x_n) \, dx_1 \cdots dx_n \]
\[ = \frac{1}{n!} \int \cdots \int g_n(x_1, \ldots, x_n) \left[ \delta f(x_1) f(x_2) \cdots f(x_n) + f(x_1) \delta f(x_2) f(x_3) \cdots f(x_n) + \cdots \right] \, dx_1 \cdots dx_n + O(\delta f)^2 \]

By the symmetry of \( g_n \), this is
\[ \frac{1}{(n-1)!} \int g_n(x_1, \ldots, x_n) \delta f(x_1) f(x_2) \cdots f(x_n) \, dx_1 \cdots dx_n \]
so it works.

Therefore by repeated differentiation we obtain
\[ \frac{\partial^n}{\partial f(x_1) \cdots \partial f(x_n)} F(f) \bigg|_{f=0} = g_n(x_1, \ldots, x_n) \]

Consider next a grand partition \( F \).
\[ Z(\zeta) = \sum_n \frac{1}{n!} \int z(q_1) \cdots z(q_n) e^{-\beta U(q_1, \ldots, q_n)} \, dq_1 \cdots dq_n \]
which is evidently a functional of the activity function \( z(q) \). We get the usual grand partition function by setting \( z(q) = \text{the constant} \), however we can obviously ask questions about energy and density for an arbitrary activity level. Is there a particle density \( \rho(\zeta) \) function belonging to a general variable activity level \( z(q) \)?

Compute
\[ \frac{\partial Z}{\partial z(q)} = \sum_n \frac{1}{n!} \int e^{-\beta U(q_1, \ldots, q_n)} z(q_1) \cdots z(q_n) \, dq_1 \cdots dq_n \]
Is it possible for \( z^2 \frac{\delta}{\delta z} \) \( \log \tilde{Z} \) to be the particle density at \( \mathbf{q} \)?

\[
\int z^2 \frac{\delta}{\delta z} \log \tilde{Z} \, d\mathbf{q} = \sum_n \frac{1}{n!} \int e^{-\beta U(\mathbf{q}_1, \ldots, \mathbf{q}_n) / z(\mathbf{q}_1, \ldots, \mathbf{q}_n)} \, dz_1 \ldots dz_n
\]

\[
= \sum_n \frac{n}{n!} \int e^{-\beta U(\mathbf{q}_1, \ldots, \mathbf{q}_n) / z(\mathbf{q}_1, \ldots, \mathbf{q}_n)} \, dz_1 \ldots dz_n
\]

so when divided by \( \tilde{Z} \), you get \( \langle n \rangle \) the average number of particles.

Therefore

\[
\langle n(\mathbf{q}) \rangle = z(\mathbf{q}) \frac{\delta}{\delta z} \log \tilde{Z}
\]

is the particle density at the point \( \mathbf{q} \). When all \( z(\mathbf{q}) = z \) and \( U \) is translation invariant, it will be \( \mathbf{N} \) for all \( \mathbf{q} \).

Here's an important point about the expansion:

\[
F(t) = g_0 + \int g_1(x_1) f(x_1) \, dx_1 + \frac{1}{2!} \int g_2(x_1, x_2) f(x_1) f(x_2) \, dx_1 \, dx_2 + \ldots
\]

\[
g_n(x_1, \ldots, x_n) = \frac{\delta^n}{\delta x_1 \ldots \delta x_n} \left. F \right|_{t=0}
\]

The point is that the values of \( g_n \) for equal values of the arguments \( x_1, \ldots, x_n \), say \( x_i = x'_i \), are not really well-defined. Hence there may be some sort of problem with

\[
z(\mathbf{q}_1) \frac{\delta}{\delta z(\mathbf{q}_1)} \quad z(\mathbf{q}_2) \frac{\delta}{\delta z(\mathbf{q}_2)}
\]
\[
\frac{\delta Z}{\delta z(x)} = \sum_n \frac{z(n)}{n!} \int e^{-\mu(x, y_1, \ldots, y_n)} Z(y_1) \cdots Z(y_n) \, dy_1 \cdots dy_n.
\]
This is a function of \( x \) and \( z \). Now vary \( z \)

\[
\delta \left( \frac{\delta z(x)}{\delta z(y)} \right) = \sum_n \frac{\delta z(x)}{n!} \int e^{-\mu(x, y_1, \ldots, y_n)} z(y_1) \cdots z(y_n) \, dy_1 \cdots dy_n
\]

\[
+ \sum_n \frac{\delta z(x)}{n!} \int e^{-\mu(x, y_1, \ldots, y_n)} \delta z(y_1) z(y_2) \cdots z(y_n) \, dy_1 \cdots dy_n.
\]

\[
\frac{\delta}{\delta z(y)} \left( \frac{\delta z(x)}{\delta z(x)} \right) = \delta(x-y) \sum_n \frac{1}{n!} \int e^{-\mu(x, y_1, \ldots, y_n)} z(y_1) \cdots z(y_n) \, dy_1 \cdots dy_n
\]

\[
+ \sum_n \frac{\delta z(x)}{n!} \int e^{-\mu(x, y, y_1, \ldots, y_n)} z(y) \cdots z(y_n) \, dy \cdots dy_n.
\]

Therefore we obtain the rule

\[
\frac{\delta}{\delta z(y)} (z(x)) = \delta(x-y)
\]

because

\[
z(x) = \int \delta(x-y) z(y) \, dy
\]

\[
\frac{\delta^2 Z}{\delta z(x) \delta z(y)} = \sum_n \frac{1}{n!} \int e^{-\mu(x, y_1, \ldots, y_n)} z(y_1) \cdots z(y_n) \, dy_1 \cdots dy_n.
\]

\[
\int dx dy \, z(x) z(y) \frac{\delta^2 Z}{\delta z(x) \delta z(y)} = \sum_{n=1}^{n+1+h} \frac{1}{(n+2)!} \int e^{-\mu(x, y_1, \ldots, y_{n+2})} z(y_1) \cdots z(y_{n+2}) \, dy_1 \cdots dy_{n+2}.
\]

Thus

\[
\frac{1}{Z} \int dx dy \, z(x) z(y) \frac{\delta^2 Z}{\delta z(x) \delta z(y)} = \frac{1}{Z} \sum \frac{n(n-1)}{n!} \int e^{-\mu(\delta y_1, \delta y_2)} z(y_1) \cdots z(y_n) \, dy_1 \cdots dy_n
\]

\[
= \langle n(n-1) \rangle
\]
which is consistent with
\[
\frac{1}{2} \int dx dy \ z(x) \frac{\delta}{\delta x} \left( z(y) \frac{\delta}{\delta y} \tilde{Z} \right) = \frac{1}{2} \int dx dy \ \delta(x-y) \frac{\delta^2 \tilde{Z}}{\delta x \delta y} + \delta(x-y) z(x) \frac{\delta \tilde{Z}}{\delta y} + \delta(x-y) z(y) \frac{\delta \tilde{Z}}{\delta x} \quad \tilde{Z}(x) \tilde{Z}(y)
\]

\[
\langle (n)(n-1) \rangle + \langle n \rangle = \langle n^2 \rangle.
\]

Hence we see that \( z(x) \frac{\delta}{\delta z(x)} \left( z(y) \frac{\delta}{\delta z(y)} \tilde{Z} \right) \) is not smooth although \( z(x) z(y) \frac{\delta^2 \tilde{Z}}{\delta z(x) \delta z(y)} \) is smooth. It seems that
\[
\frac{1}{2} \ z(x) \frac{\delta}{\delta z(x)} \left( z(y) \frac{\delta}{\delta z(y)} \tilde{Z} \right) = \langle n(x) n(y) \rangle
\]

has a valid interpretation. Thus
\[
\langle n(x) n(y) \rangle = \frac{1}{2} \ z(x) z(y) \frac{\delta^2 \tilde{Z}}{\delta z(x) \delta z(y)} + \delta(x-y) \frac{\delta \tilde{Z}}{\delta z(y)} \quad \text{smooth} \quad \langle n(y) \rangle
\]

as a distribution on the product x,y space. It seems therefore to be effectively meaningless to talk about the moments \( \langle n(x)^2 \rangle \)

Example: \( \mathcal{U}_n (\xi_1, \ldots, \xi_n) = \sum_{i=1}^{\nu} V(\xi_i) \)

Then
\[
\tilde{Z} = \exp \left( \int z(x) e^{-\beta V(x)} dx \right)
\]

\[
\langle n(x) \rangle = \frac{\delta}{\delta \tilde{Z}(x)} \log \tilde{Z} = e^{-\beta V(x)}
\]

\[
\frac{\delta}{\delta z(y)} \left( \frac{\delta \tilde{Z}}{\delta z(y)} \right) = \frac{\delta \tilde{Z}}{\delta z(y)} e^{-\beta V(y)} = \tilde{Z} e^{-\beta V(y)} - \beta V(x)
\]
\[ \langle n(x) n(y) \rangle = e^{-f(V(x) + V(y))} + \delta(x-y)e^{-fV(x)} \]

i.e.

\[ \langle n(x) n(y) \rangle = \langle n(x) \rangle \langle n(y) \rangle + \delta(x-y)\langle n(y) \rangle \]

which agrees with the following calculation for the Poisson distribution:

\[ \langle n^2 \rangle = \sum_{n>0} n^2 \frac{e^{-\lambda} \lambda^n}{n!} = \sum_{n>0} \frac{n(n-1)}{n!} e^{-\lambda} \lambda^n + \sum_{n>0} \frac{n}{n!} e^{-\lambda} \lambda^n \]

\[ = \lambda^2 + \lambda = \langle n \rangle^2 + \langle n \rangle \]
We consider a classical grand partition function with variable activity $z(q)$:

$$\tilde{Z}(z) = \sum_{n} \frac{1}{n!} \int \frac{dz_1...dz_n}{z_1^{(g_1)}...z_n^{(g_n)}} z^{(g_1)...z^{(g_n)}} e^{-\beta U(x_1,...,x_n)}$$

We can think of this as a partition function where there is one configuration for each positive divisor of $g$-space. Then a point in $g$ gives a function $n(g)$ on the set of configurations. We want to think of $n(g)$ as a random variable in the space of configurations, but this doesn't work because the probability of there being a particle exactly at $g$ is zero.

Our probability space looks as follows: $X \in g$-space

$$pt \downarrow X \downarrow (X/\Sigma_g) \downarrow \ldots \downarrow = SP(X)$$

Suppose we have a subset $A$ of $X$ with complement $\overline{A}$. Then

$$SP(X) = SP(A) \times SP(\overline{A})$$

and so we get a map

$$SP(X) \longrightarrow SP(A)$$

and hence an induced probability measure on $SP(A)$.

$\tilde{Z}$ can be written

$$\tilde{Z} = \sum_{m,n} \frac{1}{m! n!} \int_{A^m \times \overline{A}^n} \frac{dz_1...dz_m}{z_1^{(g_1)}...z_m^{(g_m)}} \int_{A^m \times \overline{A}^n} \frac{dz_1'...dz_n'}{z_1^{(g_1')...z_n^{(g_n')}}} e^{-\beta U(x_1,...,x_n)}$$

If I take $A$ to be a point, then $dg$ restricted to $A$
gives 0, hence all the terms with \( m \neq 0 \) vanish.

On the other hand, when we decompose

\[
SP_n(x) = \bigoplus_{m=0}^{n} SP_n(A) \times SP_{n-m}(\bar{A})
\]

the pieces with \( m > 0 \) have measure 0, so the
induced probability measure on \( SP(A) = \bigoplus \mathbb{N} \) is
the 0 measure at 0.

To get something interesting we must assume
\( A \) has positive measure in \( X \), say \( A \) is a nice region.

First note how the measure on \( SP(x) \) arises. On \( X \)
we have the measure

\[
\frac{1}{n!} \prod_{i=1}^{n} dz_i \prod_{i=1}^{n} \Pi(z_i) e^{-\mu(z_1, \ldots, z_n)}
\]

and we push it via \( X^n \to SP_n(x) \) to get a
measure \( d\mu_n \) on \( SP_n(x) \). Then we have measures on
each \( SP_n(x) \) and we have to normalize the sum measure
on \( SP(x) = \bigoplus SP_n(x) \) to be a prob. measure. This means

\[
\sum_n \int d\mu_n = Z.
\]

Now look what happens in the case of \( SP(A) \).
We have

\[
SP_n(x) = \bigoplus SP_n(A) \times SP_{n-m}(\bar{A})
\]

and the measure \( d\mu_n \) decomposes as follows. On
\( A^m \times \bar{A}^{n-m} \) we have

\[
\frac{1}{m!(n-m)!} \prod_{i=1}^{m} dz_i \prod_{i=1}^{m} \Pi(z_i) \prod_{i=1}^{n-m} \frac{dz_i'}{\Pi(z_i')} e^{-\mu(z, z')} \]

and pushing this via \( A^m \times \bar{A}^{n-m} \to SP_m(A) \times SP_{n-m}(\bar{A}) \) gives
a measure \( d\mu_{m,n-m} \) on the latter. Clearly

\[
d\mu = \bigoplus_{m=0}^{n} d\mu_{m,n-m}
\]
So now it is clear what kind of measure is obtained on $\text{SP}(A)$. On the piece $\text{SP}_m(A)$ we get
\[ \sum_{n=0}^{\infty} \text{d}\mu_{m,n} \text{ pushed via } \text{SP}_m(A) \otimes \text{SP}_n(A) \xrightarrow{\text{p.e.}} \text{SP}_m(A). \]
Thus we get
\[ \frac{1}{m!} \int \prod_{i=1}^{m} d\phi_i \prod_{i=1}^{m} \Pi(z_i) \sum_n \frac{1}{n!} \int \prod_{i=1}^{n} d\phi_i \prod_{i=1}^{n} \Pi(z_i) e^{-\beta U(\phi, \phi', \phi'')} \]

What this means is that we have a measure on $\text{SP}(A)$ based on a factor like $e^{-\beta U(\phi)}$ which is computed by summing over the possible configurations in $\overline{A}$. We could normalize and divide by the partition function for $\overline{A}$ if we want. Let's put:
\[ U_n(\phi, \phi') = U_n(\phi) + \tilde{U}_{mn}(\phi, \phi') + U_n(\phi'). \]
Then upon dividing by $\tilde{Z}$ for $\overline{A}$ we get
\[ \frac{1}{m!} \int d\phi \prod_{i=1}^{m} \Pi(z_i) e^{-\beta U_n(\phi)} \sum_n \frac{1}{n!} \int \prod_{i=1}^{n} d\phi' \prod_{i=1}^{n} \Pi(z_i) e^{-\beta U_n(\phi')} e^{-\beta \tilde{U}_{mn}(\phi, \phi')} \]
\[ \sum_n \frac{1}{n!} \int \prod_{i=1}^{n} d\phi' \prod_{i=1}^{n} \Pi(z_i) e^{-\beta U_n(\phi')} \]
average of $e^{-\beta \tilde{U}_{mn}(\phi, \phi')}$ over $\overline{A}$. 

So we see that the probability measure on $\text{SP}(A)$ is a grand measure associated to a weighting which involves averaging out over the possible configurations in $\overline{A}$.

Let's consider the simple case of independent particles
\[ U_n(\phi) = \sum_{i=1}^{n} V(\phi_i). \]
Then
\[ Z_x = \exp \{ \int \varepsilon(q) e^{-\beta V(q)} dq \} \]
\[ \tilde{Z}_A = \exp \{ \int_A \varepsilon(q) e^{-\beta V(q)} dq \} \]

So it's clear that we get the probability measure on \( SP(A) \) with partition function
\[ \tilde{Z}_A = \exp \{ \int_A \varepsilon(q) e^{-\beta V(q)} dq \}. \]

Let's look at this as \( A \) shrinks to a point \( q_0 \):
\[ \int_A \varepsilon(q) e^{-\beta V(q)} dq \sim \text{vol}(A) \cdot \varepsilon(q_0) e^{-\beta V(q_0)} \]

Look at the function \( n_A = \text{number of particles on } A \). This is a random variable on the probability space \( SP(A) \), and we have that its distribution is a Poisson distribution
\[ p_n = \frac{e^{-\lambda} \lambda^n}{n!} \]

where \( \lambda = \int_A \varepsilon(q) e^{-\beta V(q)} dq \sim \text{vol}(A) \cdot \text{const.} \). Thus
\[ \langle n_A^2 \rangle = \lambda^2 + \lambda \]

and although we \( \text{can make sense of} \)
\[ \lim_{A \to q_0} \frac{\langle n_A \rangle}{\text{vol}(A)} \]

as density at \( q_0 \)

it is not possible to make sense of
\[ \lim_{A \to q_0} \frac{\langle n_A^2 \rangle}{\text{vol}(A)^2} \]

Therefore we have \( \langle n(q) \rangle \) defined \( \checkmark \), but not \( \langle n(q)^2 \rangle \).
although it seems that $\langle n(x_1) \ldots n(x_n) \rangle$
can be interpreted as distributions on $X^n$.

Let $f(x)$ be a smooth function on $X$. Then
the quantities

$$\int \langle n(x) \rangle f(x) \, dx$$
$$\int \langle n(x)n(y) \rangle f(x)f(y) \, dx \, dy \quad \text{etc.}$$

make sense, which suggests that they are moments of a random variable

$$\int n(x)f(x) \, dx$$
on $SP(X)$. For example $n_A = \int n(x)X_A(x) \, dx$ might be true. Perhaps the value of $\int n(x)f(x) \, dx$ at a
configuration $(q_1, \ldots, q_n)$ should be $\sum_{j=1}^n f(q_j)$. Maybe

$$n(x) = \sum_{i} \delta(x-q_i) \quad \text{at the configuration} \ (q_1, \ldots, q_n).$$

Therefore $n(x)$ is random-variable-valued distribution
in the sense that for any test function $f(x)$ the
express $\int n(x)f(x) \, dx$ is a random-variable. Compare
with $\phi(x)$ in field theory being an operator-valued distribution
in the sense that $\int \phi(x)f(x) \, dx$ is an operator.
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\[ S^p(x) = pt + x + x^2/\Sigma_z \]

On \( S_{n'}(x) \) we have the measure \( \frac{1}{n!} \, dx_1 \cdots dx_n \); this means we take this measure on \( X^n \) and push it to \( S_{n'}(x) \), so that \( \frac{1}{n!} \int f(x_1, \ldots, x_n) \, dx_1 \cdots dx_n \) is the integral of a function on \( S_{n'}(x) \). Now given a function \( f(x) \) on \( X \) we can extend it to a "1-particle" function on \( S_{n'}(x) \) given by

\[ \tilde{f}(x_1, \ldots, x_n) = \sum_{i=1}^{n} f(x_i) \]

For example if \( f = \delta \) i.e. \( f(x) = \delta(x-q) \), then we get the gadget \( n(q) \)

\[ n(q)(x_1, \ldots, x_n) = \sum \delta(x_i - q) \]

We can think of \( n(q) \) as a random variable with distribution values (distributions on \( X \), or as a distribution on \( X \) with values in random variables on \( S_{n'}(x) \). In the latter interpretation, given a test \( f(x) \), \( f(x) \) we can convert it to get

\[ \int f(x) \, n(x) \, dx \]

which is a function on \( S_{n'}(x) \), namely the function \( \tilde{f} \) above. Thus

\[ \int f(x) \, n(x) \, dx = \tilde{f} \quad \text{because} \quad n(x) = \delta_x \]

\( \tilde{f} \) is a real valued function on \( S_{n'}(x) \) and hence it gives rise to a distribution on \( \mathbb{R} \). Let's compute its char. fn.

\[ \int e^{i \tilde{f}} \, dx = \sum_{n} \frac{1}{n!} \int e^{i f(x_1) + \ldots + i f(x_n)} \, dx_1 \cdots dx_n \]
\[ = \exp \left\{ \int e^{Jf(x)} \, dx \right\} \]

Now \[ \int \, d\mu = \sum_{n=0}^{\infty} \frac{1}{n!} V^n = e^V, \quad V = \int \, dx. \]

hence the characteristic function of \( f \) as a random variable on the probability space \( X \) is

\[ W = \exp \left\{ \int (e^{Jf(x)} - 1) \, dx \right\} \]

There's no way I can see how to let \( f \to 0 \)

in this expression.

Look at the reduced moments

\[ b_n = \frac{d^n}{dJ^n} \log(W) \bigg|_{J=0} \]

\[ \log W = \int (e^{Jf(x)} - 1) \, dx = J \int f(x) \, dx + \frac{J^2}{2} \left( \int f(x) \, dx \right)^2 \]

Thus \[ b_n = \int f^n(x) \, dx \quad n \geq 1 \]

Let's return to the problem of the Green's fun.

belonging to the grand partition fun.

\[ \tilde{Z} = \sum_{n=1}^{\infty} \frac{1}{n!} \int d^nx \prod \tilde{z}(\xi_i) \, e^{-\beta u(x)} \]

Then \[ \frac{\partial \tilde{Z}}{\partial z(x)} = \sum_{n=1}^{\infty} \frac{1}{n!} \int \prod \tilde{z}(\xi_i) \, e^{-\beta u(x)} \]

and similarly for higher derivatives. So there is no problem with defining the functions.
and these will be nice smooth funs. on $X^n$. If we use the identity

$$\frac{\delta}{\delta z(x)} z(y) = \delta(x-y)$$

then we can make sense of

$$\frac{1}{\prod_{i=1}^{n} \left( z(x_i) \frac{\delta}{\delta z(x_i)} \right)^2}$$

as a distribution on $X^n$. It should be possible to justify the formula

$$\left< h(x_1) \ldots h(x_n) \right> = \frac{1}{2} \frac{1}{\prod_{i=1}^{n} \left( z(x_i) \frac{\delta}{\delta z(x_i)} \right)^2}$$

as distributions on $X^n$. How?

What we have with the grand configuration space $SP(x)$ is an analogue of the set of classical configurations $\phi$ which occur in the Feynman amplitude formula. In this example, there is no problem with the existence of the measure $D\phi$, because we have the measure $d\mu = \frac{\delta}{\delta z(x_i)} d\mu$ on $SP(x)$. So the formula

$$Z(J) = \int e^{-S(\phi) + \int J\phi dx} D\phi$$

de field theory becomes

$$Z(J) = \int e^{-\beta \tilde{u} + \int Jn dx} d\mu$$
\[ Z(J) = \sum_{n \geq 0} \frac{1}{n!} \int d^n \phi \, e^{-\beta U(\phi)} + \sum \frac{\delta^2 Z(J)}{\delta J(\phi)} \]

Consistent with the formula

\[ \int J(x) n(x) \, dx = \sum \frac{\delta n}{\delta J(\phi_j)} \quad \text{at} \quad \phi = (\phi_1, \ldots, \phi_n) \]

At least formally, the Green's functions (= moments of the \( n(x) \)) will be given by

\[ \langle n(x_1) \cdots n(x_k) \rangle = \left( \frac{\delta^n Z(J)}{\delta J(x_1) \cdots \delta J(x_k)} \right) \]

\[ = \frac{\int n(x_1) \cdots n(x_k) \, e^{-\beta \hat{U}} + \int \text{Ind} \, d\mu}{\int e^{-\beta \hat{U}} + \int \text{Ind} \, d\mu} \]

Let's try:

\[ \int n(x) \, e^{-\beta \hat{U}} + \int \text{Ind} \, d\mu = \sum \frac{1}{n!} \int d^n \phi \, \delta^2 U(\phi) \, e^{-\beta U(\phi)} + \sum \frac{\delta^2 Z}{\delta J(\phi)} \]

\[ = \sum \frac{1}{(n-1)!} \int d^{n-1} \phi \, \delta^2 U(\phi) \, e^{-\beta U(\phi)} + J(x) + \sum \frac{\delta^2 Z}{\delta J(x)} \]

\[ \int n(x) n(y) \, e^{-\beta \hat{U}} + \int \text{Ind} \, d\mu = \sum \frac{1}{n!} \int d^n \phi \, \delta^n U(\phi) \delta(\phi) \, e^{-\beta U + \sum \delta J(\phi)} \]

Break up: \[ \sum \frac{\delta(n-\delta_i)}{\delta J(\phi_i)} \frac{\delta y - \delta_j}{\delta J(\phi_j)} = \sum_{i \neq j} \delta + \sum_{j = j} \delta_{i = j} \]
The \( i \neq j \) part is

\[
\sum \frac{1}{n!} n(n-1) \int d^{n-2} \delta \ e^{\beta \mu_{0}(x,y) + J(x) + J(y) + \sum J_{(j)}}
\]

\[= \mathcal{Z}(y,x) \frac{\delta^2 \mathcal{Z}}{\delta \mathcal{Z}(x) \delta \mathcal{Z}(y)} \quad \text{where} \quad \mathcal{Z}(x) = e^{J(x)}
\]

The \( i = j \) part is

\[
\sum \frac{1}{n!} \int \, \delta(x-y) \delta(y-z) \int d^{n-2} \delta \ e^{-\beta \mu_{0}(x,y) + J(x) + \sum J_{(j)}}
\]

The point is that

\[
\int \delta(x-y) \delta(y-z) F(z) = \delta(y-x) F(x)
\]

using the fundamental property of \( \int \delta(x-y) f(y) \) (?).

Hence the \( i = j \) part becomes

\[
\delta(y-x) \sum \frac{1}{n!} \int d^{n-2} \delta \ e^{-\beta \mu_{0}(x,y) + J(x) + \sum J_{(j)}}
\]

\[= \delta(y-x) \frac{\delta \mathcal{Z}}{\delta \mathcal{Z}(x)}
\]

Hence we end up with the familiar formula.

\[
\frac{\int n(x) n(y) \ e^{-\beta \mu + \sum J_{(x)}} \ d\mu}{\int e^{-\beta \mu + \sum J_{(x)}} \ d\mu} = \frac{1}{\mathcal{Z}(x) \mathcal{Z}(y) \frac{\delta^2 \mathcal{Z}}{\delta \mathcal{Z}(x) \delta \mathcal{Z}(y)} + \delta(x-y) \mathcal{Z}(x) \frac{\delta \mathcal{Z}}{\delta \mathcal{Z}(x)}}
\]

So now it is clear that we can define distributions by

\[
\langle n(x_1) \cdots n(x_k) \rangle = \frac{1}{2} \frac{\delta^{(k)} \mathcal{Z}(J)}{\delta \mathcal{Z}(x) \delta \mathcal{Z}(k)}
\]
and that in principle we can compute them in terms of pieces belonging to the different strata of $SP_e(X)$.

Suppose we have a two particle function, i.e. $F(x_1, x_2)$ defined on $SP_2(X)$. Then we can extend it to $SP(X)$ by

$$
\tilde{F}(\tilde{g}_1, \ldots, \tilde{g}_n) = \frac{1}{2} \sum_{i \neq j} F(\tilde{g}_i, \tilde{g}_j),
$$

and average it over the grand canonical ensemble,

$$
\frac{1}{Z} \int F e^{-\beta \tilde{u}} + \sum_{n} \frac{n(n-1)}{n!} \int d^n \tilde{g}_1 F(\tilde{g}_1, \tilde{g}_2) e^{-\beta \tilde{u}(\tilde{g}_1) + \tilde{J}(\tilde{g}_1)} / Z
$$

$$
= \frac{1}{2} \sum_{n} \frac{n(n-1)}{n!} \int d^n \tilde{g}_1 F(\tilde{g}_1, \tilde{g}_2) e^{-\beta \tilde{u}(\tilde{g}_1) + \tilde{J}(\tilde{g}_1)} / Z
$$

$$
= \frac{1}{2} \sum_{n} \frac{1}{n!} \int dxdy d^n \tilde{g}_1 F(x, y) e^{-\beta U(x, y, g) + \tilde{J}(x) + \tilde{J}(y) + 2 \tilde{J}(\tilde{g}_1)} / Z
$$

$$
= \frac{1}{2} \int dxdy F(x, y) \langle n(x)n(y) \rangle
$$

This means the diagonal part is omitted.

Thus

$$
\langle n(x)n(y) \rangle = \frac{1}{2} \frac{\delta(x-x') \delta(z-y) \delta^2 Z}{\delta x \delta z y}
$$

Let's denote this by $g^{(2)}(x, y)$, or simply $g(x)$ assuming it depends only on $|x-y|$. Then we get

$$
\langle U \rangle = \frac{1}{2} \int dxdy U(x) g(x)
$$

$$
= \frac{W}{2} \int 4\pi m^2 dr U(r) g(r)
$$
Green's function for grand partition function

\[ Z(z) = \sum_{n!} \int d^n g \, e^{-\beta U_n(g)} \prod_{i=1}^{n} \, z(g_i) \]

\[ = \int d\mu \, e^{-\beta A} + \int J dx \quad Z \mu = e^{J(x)} \]

\[ G_1(x) = \langle n(x) \rangle = z(x) \frac{\partial}{\partial z(x)} \log Z \bigg|_{\text{all } z(x) = z} \]

\[ = z \sum z^n \frac{z^n}{n!} \int d^n g \, e^{-\beta U_n(x, g)} \]

\[ = \sum \frac{z^n}{n!} \int d^n g \, e^{-\beta U_n(g)} \]

\[ G_2(x, y) = z(x) z(y) \frac{\partial^2 Z}{\partial z(x) \partial z(y)} \bigg|_{\text{all } z(x) = \pm} \]

\[ = \frac{2z^2}{n!} \sum \frac{z^n}{n!} \int d^n g \, e^{-\beta U_{n+2}(g, y, g)} \]

\[ = \frac{1}{n!} \sum \frac{z^n}{n!} \int d^n g \, e^{-\beta U_n(g)} \]

To now I want to assume $z$ adjusted so that the dominant term method can be used. First note that under the assumption of translation invariance for the energy, we

\[ \int d^n g \, e^{-\beta U_n(x, g)} = \frac{1}{\sqrt{Z_n}} \int d^n g \, e^{-\beta U_n(g)} \]

so consequently

\[ G_1(x) = \sum_n \frac{z^n}{n!} \frac{1}{\sqrt{Z_n}} Z_n / \sum \frac{z^n}{n!} Z_n \]
\[ \frac{V}{N} \]

assuming that the series \( \sum \frac{z^n}{k!} Z_n \)

has dominant term at \( n = N \).

Similarly, in the expression for \( G_2(x, y) \)

if the dominant term in the partition function

occurs at \( n = N \), then we expect the same is true

for the numerator in

\[ G_2(x, y) \approx \frac{z^N}{(N-2)!} \frac{z^N}{N!} \int d^n \phi e^{-\beta U_N(x, y, \phi)} \]

\[ = \frac{z^N}{(N-2)!} \frac{z^N}{N!} \int d^n \phi e^{-\beta U_N(\phi)} \]

Better approaches. Write partition function in the

form

\[ Z_i = \sum \frac{1}{n!} \int d^n \phi e^{-\beta U_N(\phi)} \prod_{i=1}^{n} z(\phi_i) \]

and define the Green's functions to be

\[ G_n(x_1, \ldots, x_n) = \frac{1}{Z} \frac{\delta^n Z}{\delta \phi(x_1) \cdots \delta \phi(x_n)} \]

and the connected Green's functions by

\[ G_n(x_1, \ldots, x_n) = \frac{\delta^n}{\delta \phi(x_1) \cdots \delta \phi(x_n)} \log Z \]

For an ideal gas all \( U_N(\phi) = 0 \), so

\[ Z = \sum \frac{1}{n!} \int d^n \phi \prod_{i=1}^{n} z(\phi_i) = \exp \left\{ \int z(\phi) d\phi \right\} \]
hence \( G_n^c = 0 \) for \( n \geq 2 \)
\( G_1^c(x) = 1 \) for all \( x \)

Also \( G_2(x_1, x_2) = \frac{1}{2} \sum_n \frac{1}{n!} \int d^4q \left( \sum_{n=0}^\infty \frac{1}{n!} \int d^4q \left( e^{-\beta U_{n+2}(x,y,\theta)} TR_2(\theta) \right) \right) \)

\[ = 1 \]

and more generally \( G_n(x_1, x_n) = 1 \) for all \( n \) in this notation.

Now we want to express these Green's funs. in terms of diagrams. Recall \( U_n(\theta) = \frac{1}{2} \sum_i U(\tau_i, j) \) and

\[ e^{-\beta U_n(\theta)} = \prod_{i < j} (1 + f_{ij}) \]

\[ f_{ij} = e^{-\beta U(\tau_i, j)} - 1 \]

The numerator for \( G_2(x_1, x_2) \) represents a sum over diagrams with two fixed vertices which get labelled by \( x_1, x_2 \).

\[ n = 0 \]

\[ n = 1 \]

We can simplify a little bit by taking out the factor \( e^{-\beta U(x, y)} \) common to all the terms of \( G_2(x, y) \). This means we may assume \( x, y \) are not connected by an edge. Hence

\[ n = 0 \]

\[ n = 1 \]
Next suppose we use the connected diagram decomposition. Then we get two types of diagrams:

- $x,y$ in same component:
- $x,y$ in diff. components:

When we divide by $Z$ we cancel the components not containing $x$ or $y$ and we find

\[ G^c_2(x,y) = G^c_2(x,y) + G^c_1(x) G^c_1(y) \]

So now it's clear that

\[ G^c_2(x,y) = \ldots + (\ldots + \ldots + \ldots) + \ldots \]

Let's compute carefully:

\[ G_1(x) = \frac{\delta Z}{2\delta Z(x)} = \frac{1}{2} \sum_{n=1}^{\infty} \frac{1}{n!} \int d^{n-1} \theta e^{-\beta U_n(x,\theta)} \]

\[ = \frac{1}{2} \frac{\langle n \rangle}{V} = \frac{S}{Z} \]

According to page 961 we have

\[ \frac{S}{Z} = 1 + F_2 Z + F_3 \frac{Z^2}{2!} + \ldots \]

\[ = \ldots + (\ldots + \ldots + \ldots) \]

\[ = 1 + aZ + (3a^2 + b) \frac{Z^2}{2} + \ldots \]
where 

\[ a = \int f(n^1) \, d^2 \beta = \frac{1}{\nu} \int f_{12} \, d\beta_1 d\beta_2 \]

\[ b = \int f_{12} f_{13} f_{23} \, d\beta_2 d\beta_3 \quad \text{where} \quad \beta_1 = 0. \]