May 10, 1980

The program is to understand an interacting fermi system. It should be a perturbation of the free system, so there should be a vacuum state, and 1-particle states indexed by a wave vector, and 2-particle states. What intrigues me is the nature of these one-particle states - somehow they are waves travelling thru the gas. It seems profitable to understand the analogue of phonons in a crystal.

Previous study of phonons is June 25-July 6, 1979. Picture a crystal as a lattice \( \Lambda \) in \( \mathbb{R}^3 \), the points of \( \Lambda \) denoting the equilibrium positions of the atoms. Let \( q_i \) be the displacement of the \( i \)-th atom and \( p_i \) its momentum. Then assuming small displacements

\[
H = \sum \frac{p_i^2}{2m} + \frac{1}{2} \sum_{i \neq j} q_i \cdot A_{ij} \cdot q_j
\]

where \( A_{ij} \) depends only on \( i-j \) in \( \Lambda \). The Einstein approximation consists in regarding the \( i \)-th atom as moving in the potential where all the other atoms are fixed at equilibrium. This means we retain only the diagonal terms \( A_{ii} \) which do not depend on \( i \), and so each atom is an independent 3-dim oscillators.

Let's derive the Einstein and Debye formulas for the specific heat of the crystal. Recall for a single oscillator

\[
Z = \sum_{n \geq 0} e^{-\beta (n+\frac{1}{2}) \hbar \omega} = \frac{e^{-\frac{1}{2} \beta \hbar \omega}}{1 - e^{-\beta \hbar \omega}} = \frac{1}{2 \sinh(\frac{\beta \hbar \omega}{2})}
\]

\[
U = -\frac{\partial}{\partial \beta} \log Z = \frac{\cosh(\frac{\beta \hbar \omega}{2})}{\sinh(\frac{\beta \hbar \omega}{2})} \hbar \omega = \hbar \omega \left[ \frac{e^{\beta \hbar \omega}}{1 - e^{\beta \hbar \omega}} + \frac{1}{2} \right]
\]

\[
= \frac{1}{2} \hbar \omega + \frac{\hbar \omega}{e^{\beta \hbar \omega} - 1}
\]

Thus for a crystal with vibration modes \( \omega \) we have

\[
U = \sum_{\alpha} \left( \frac{1}{2} \hbar \omega + \frac{\hbar \omega}{e^{\beta \hbar \omega} - 1} \right)
\]
In the Einstein approximation all the \( \omega_n \) are assumed to have the same value \( \omega \). This gives

\[
U = 3N \left( \frac{1}{2} \frac{\hbar \omega}{k_B T} + \frac{\hbar \omega}{e^{\beta \hbar \omega} - 1} \right)
\]

if the crystal has \( N \) atoms. To get the specific heat one can forget the ground energy and look at

\[
\frac{3N \hbar \omega}{e^{\beta \hbar \omega} - 1} \rightarrow \begin{cases} 
3N \hbar \omega e^{-\beta \hbar \omega} & \text{exponentially small as } T \to 0 \\
\frac{3N \hbar \omega}{\beta \hbar \omega} = \frac{3N}{\hbar} T & \text{as } T \to \infty.
\end{cases}
\]

The low temperature behavior is wrong—it should be \( \propto T^4 \), so as to get a specific heat \( \propto T^3 \).

In the Debye version one gets a better idea of the modes. First note that the modes \( \omega \) are described by pairs \((k, \lambda)\) where \( k \) is a wave vector in the Brillouin zone (better \( k \in \text{Hom}(\Lambda, S^1)\)) and \( \lambda \) describes the three possible polarizations. Debye's idea is to identify such modes with the modes of vibration of the crystal as an elastic body, and to take the frequency of the modes as calculated by elastic body theory. This has the effect of assuming the dispersion law

\[
\omega_k = s_\lambda |k|
\]

where \( s_\lambda \) is the speed for transverse or longitudinal waves.

To simplify suppose \( \omega_k = s|k| \). Instead of the Brillouin zone lets use a sphere \( |k| < k_0 \) so as to get the correct number of modes. Thus we want for a crystal of volume \( V \) with \( N \) atoms

\[
N = \left( \frac{4}{3} \right) \sum_{|k| < k_0} \frac{1}{2} = \frac{4}{3} V \int_{|k| < k_0} \frac{d^3k}{(2\pi)^3} = \frac{4}{3} V \left( \frac{4\pi}{(2\pi)^3} \right) \frac{k_0^3}{3}
\]
so \( h k_0 \) is like the fermi momentum. Then

\[
U = \sum_{k < k_0} \left( \frac{1}{2} h^2 k \right) - \frac{h k_1}{e^{\beta h k} - 1}
\]

\[
= V \cdot 3 \cdot \frac{4\pi}{(2\pi)^3} \int_0^{k_0} k^2 dk \left( \frac{1}{2} h^2 k + \frac{h k_1}{e^{\beta h k} - 1} \right)
\]

Ignoring the ground energy gives

\[
U = U_0 + V \cdot 3 \cdot \frac{4\pi}{(2\pi)^3} \int_0^{k_0} h^2 k^3 dk
\]

\[
U = U_0 + V \cdot 3 \cdot \frac{4\pi}{(2\pi)^3} \int_0^{k_0} k^3 dk \frac{h^2 k^3}{e^x} \text{d}x \frac{h s}{(\beta h s)^4}
\]

For high temperature

\[
U = U_0 + V \cdot 3 \cdot \frac{4\pi}{(2\pi)^3} \int_0^{k_0} h^2 k^3 dk
\]

\[
U = U_0 + V = 3kT
\]

For low temperature

\[
U - U_0 = \text{const.} \cdot T^4
\]

so the specific heat is \( \sim T^3 \). In practice it is put in the form \( \Theta_0 / T \) where \( \Theta_0 \) is the Debye temperature.

Finally, Planck's radiation law is the same thing because we can think of radiation as vibrations of the ether. Light waves are transverse vibrations of the ether; they have speed \( c \) so that the relation between frequency and wave number is

\[
\omega_k = c |k|
\]

These are 2 polarizations.

The formula for the internal energy (leaving off the ground energy...
Which is infinite) is

\[ U = 2 \sum \frac{k^2 \omega \varepsilon}{\varepsilon^2 + \omega^2} \frac{k^2}{e^{\pi k} - 1} = V \cdot 2 \int \frac{\omega^3}{(2\pi)^3} \frac{e^{\pi k}}{e^{\pi k} - 1} \]

so

\[ \frac{U}{V} = \frac{8\pi \hbar c}{(2\pi)^3} \int_0^\infty \frac{\omega^3}{e^{\pi k} - 1} \]

Thus the energy distribution is

and the total energy is

\[ U = \frac{8\pi \hbar c}{(2\pi)^3} \int_0^\infty \frac{\omega^3}{e^{\pi k} - 1} \]

\[ \Gamma(4)\left(1 + \frac{1}{2^4} + \frac{1}{3^4} + \ldots\right) = \int_0^\infty \frac{x^4}{x} (e^x + e^{-2x} + \ldots) \]

\[ = 6 \cdot \frac{\pi^4}{90} = \frac{\pi^4}{15} \quad \text{hence} \]

\[ \text{total energy density} = \frac{8\pi}{c^3 (2\pi)^3 \hbar^3} \frac{\pi^4}{15} (kT)^4 = \frac{8\pi^5 k^4}{15^5 \hbar^3 c^3} \]

which is the Stefan-Boltzmann law.
May 17, 1980

Let's review spin waves as an example of an interacting Bose systems. First we have to understand the spin Hamiltonian. Let's recall the non-relativistic Pauli equation with external field; it gives the Hamiltonian

\[ H = e\Phi + \frac{1}{2m} (\sigma \cdot (p - eA))^2 \]

\[ = e\Phi + \frac{1}{2m} (p - eA)^2 - \frac{e\hbar}{2m} \sigma \cdot B \] (March 14, 1980, p. 686)

where the Pauli spin matrices \( \sigma_x = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \), \( \sigma_y = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \), \( \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \) have square 1, anti-commute, and have \( \sigma_x \sigma_y \sigma_z = i \). (Recall these matrices generate a group of order 16 which is an extension of a rank 2 2-torus by \( \mathbb{Z}_2 \); there are two dim. 2 irreducible reps, one where \( \sigma_x \sigma_y \sigma_z = i \) and the other with \( \sigma_x \sigma_y \sigma_z = -i \).

Thus due to the spin the electron has the extra energy

\[ \frac{e\hbar}{2m} \sigma \cdot B. \]

Here's the classical version: Suppose we have a small current loop

\[ \begin{array}{c}
\begin{array}{c}
\bullet
\end{array}
\end{array} \quad j = \text{current thru loop.} \]

uniform magnetic field \( B \). Let's find the torque \( \tau \) on the loop. The total force on the loop will be zero, since \( B \) is assumed not to change appreciably over the loop. Hence the torque will not depend on where the origin is taken (remember: \( \tau = r \times F \)), and so we take it in the center \( O \) of the loop. The force on the bottom is \( j(\hat{a} \times B) \) and its torque is \( -\frac{1}{2} x j(\hat{a} \times B) \). Thus

\[ \tau = j \left[ -\frac{1}{2} x (\hat{a} \times B) + \frac{\hat{a}}{2} x (b \times B) + \frac{b}{2} x (-\hat{a} \times B) + \frac{b}{2} x (b \times B) \right] \]

\[ = j \left[ a \times (b \times B) - b \times (a \times B) \right] = j (a \times b) \times B \] by Jacobi identity
Hence the torque on the loop is
\[ \tau = \mu \times B \quad \mu = j \times (a \times b) \]
\( \mu \) is called the magnetic moment of the loop.

Next compute the work done in turning the loop against the field.

\[ |\tau| = |\mu \times B| = \mu B \sin \theta \quad \text{directed into the paper} \]

\[ \text{Work} = \int_0^\pi \mu B \sin \theta \, d\theta = \left[ \mu B (-\cos \theta) \right]_0^\pi = 2\mu B \]

Thus the potential energy is \( \mu B \cos \theta = \mu \cdot B \) up to sign. The stable position is \( \theta = 0 \), hence we have

\[ \text{Magnetic energy of the current loop with moment } \mu = -\vec{\mu} \cdot \vec{B} \]

Next consider a charge \( e \), rotating circularly.

The angular momentum is \( J = mvr \). The magnetic moment is
\[ \mu = \frac{e}{2\pi} \cdot \frac{\pi r^2}{2} = \frac{e}{2m} mvr = \frac{e}{2m} J \]

When quantized \( J = \hbar j \), \( j = 1, 2, \ldots \) and so
\[ \mu = \frac{e\hbar}{2m} j. \]

Again we get the Bohr magneton \( \frac{e\hbar}{2m} \). (Note the Dirac theory says for an electron at rest, when \( j = \frac{1}{2} \), that \( \mu = \frac{e\hbar}{2m} \), which is twice the above. This has to do with the Landé g-factor, see Feynman).
The field of a dipole:

\[ \varphi = \frac{g}{|r|} + \frac{h}{|r-a|} = \frac{g}{|r|} \left( 1 - \frac{|r-a|}{|r|} \right) \approx \frac{g}{r^2} \left( a \cos \theta \right) \]

In the limit \( g \to 0 \), \( g^2 = \text{const.} \)

Next let's compute the field in a small current loop:

Recall Maxwell's equations:

\begin{align*}
\text{Divergence} & : \grad \cdot B = 0 \\
\text{Curl} & : \grad \times E = -\frac{\partial B}{\partial t}
\end{align*}

\[ \grad \cdot E = 4\pi j \quad \text{and} \quad \grad \times B = \mu_0 j + \frac{\partial E}{\partial t} \]

In the magnetostatic situation \( B = \grad \times A \) when \( \grad \cdot A = 0 \) must satisfy:

\[ c^2 \left( \grad \times B \right) = c^2 \left( \grad \times \left( \grad \times A \right) \right) = -c^2 \Delta A = 4\pi j \]

and

\[ c^2 A = \iiint \frac{j(r') \, dr'}{|r-r'|} \]

In other words, the vector potential belonging to a current element \( \int j \, d^3r \) is \( \frac{1}{c^2} \int \frac{j(r') \, dr'}{|r-r'|} \). Thus for the current loop above:

\[ \frac{c^2 \vec{A}}{I} = \left( \frac{\vec{b}}{|r-a|} + \frac{-\vec{a}}{|r-b|} + \frac{-\vec{b}}{|r+a|} + \frac{\vec{a}}{|r-b|} \right) \]

\[ = \frac{\vec{b}(a-r) - \vec{a}(b-r)}{r^3} = \frac{(\vec{a} \times \vec{b}) \times \vec{r}}{r^2} \]

Thus the vector potential for a small current loop is

\[ \vec{A} = \frac{1}{c^2} \frac{\vec{\mu} \times \vec{r}}{r^3} \quad \vec{\mu} = I(\vec{a} \times \vec{b}) \quad \text{magnetic moment} \]

Another way to write this is \( A = -\frac{1}{c^2} \vec{\mu} \times \grad \left( \frac{1}{r} \right) \). Thus

\[ B = \grad \times A = -\frac{1}{c^2} \grad \times \left( \vec{\mu} \times \grad \left( \frac{1}{r} \right) \right) \]

\[ = (\grad \cdot \grad \left( \frac{1}{r} \right)) \vec{\mu} - (\vec{\mu} \cdot \grad) \grad \left( \frac{1}{r} \right) \]
\[ \vec{B} = \frac{1}{c^2} \left( \mu \cdot \nabla \right) \vec{V} \left( \frac{1}{r} \right) = -\nabla \left( -\frac{1}{c^2} \mu \cdot \nabla \left( \frac{1}{r} \right) \right) \] 

\text{const curl vector field} \quad \text{dipole potential.}

Therefore one sees that the magnetic field of a current loop is the same as a magnetic dipole field.

Next deduce the energy of a magnetic dipole in a magnetic field \( \vec{B} \).

The force on this dipole is

\[
m \vec{B} \left( \vec{r} + \frac{\vec{a}}{2} \right) - m \vec{B} \left( \vec{r} - \frac{\vec{a}}{2} \right) = (m \vec{a} \cdot \nabla) \vec{B}
\]

hence the work done in bring this dipole in from \( \infty \) is

\[
W = -\int_{\infty}^{r} (m \vec{a} \cdot \nabla) \vec{B} \cdot d\vec{s}
\]

\[\text{Note for a constant dipole} \quad \mu \vec{a} \]

Put \( \mu = m \vec{a} \) and use the identity

\[ \mu \times (\nabla \times B) = \nabla (\mu \cdot B) - (\mu \cdot B) \nabla \]

Some remarks are necessary. Here \( \mu \) is constant as far as \( \nabla \) is concerned, because \( \mu \) will depend on the path parameter, assuming the dipole rotates. Thus we have a situation like in

\[ \vec{V} \times (\nabla \times A) = \nabla (\mu \cdot A) - (\mu \cdot A) \nabla \]

encountered in deriving the Lagrangian for a charged particle.

Therefore

\[
W = -\int_{\infty}^{r} (\mu \cdot \nabla) B \cdot d\vec{s} = -\int_{\infty}^{r} \nabla (\mu \cdot B) d\vec{s} + \int_{\infty}^{r} \mu \times (\nabla \times B) \cdot d\vec{s}
\]

\[\mu \cdot B + \int_{\infty}^{r} \mu \times (\nabla \times B) \cdot d\vec{s} \]
so we get \( W = -\mu \cdot B \) provided the path is in a current-free region, or provided the path has \( \mu || ds \). This is consistent with thinking of the dipole as two opposite poles; where the field isn't conservative, one expects a closed curve will get work \( \neq 0 \), unless the two poles get treated the same, say \( \mu || ds \).

Suppose we have two dipoles with moments \( \mu_1, \mu_2 \) with the first at origin and the second at position \( r \). Then field \( B \) at \( r \) due to \( \mu_1 \) is

\[
B = \frac{1}{c^2} \nabla \left( \mu_1 \cdot \nabla \left( \frac{1}{r} \right) \right)
\]

and so the energy of the two dipoles is

\[
-\mu_2 \cdot B = -\frac{1}{c^2} \left( \mu_2 \cdot \nabla \right) \left( \mu_1 \cdot \nabla \right) \left( \frac{1}{r} \right)
\]

Now

\[
\frac{\partial}{\partial x_i} \frac{\partial}{\partial x_j} \left( \frac{1}{r} \right) = \frac{\partial}{\partial x_i} \left( - \frac{x_j}{r^3} \right) = \frac{3 x_i x_j}{r^5} - \frac{\delta_{ij}}{r^3}
\]

so

\[
\sum_{ij} \mu_i \mu_j \nabla^2 \left( \frac{1}{r} \right) = \frac{3 (\mu_1 \cdot \hat{r}) (\mu_2 \cdot \hat{r})}{r^3} - \frac{\mu_1 \cdot \mu_2}{r^3}
\]

hence

\[
\text{energy of two dipoles} = -\frac{1}{c^2} \frac{3 (\mu_1 \cdot \hat{r}) (\mu_2 \cdot \hat{r})}{r^3} - \frac{\mu_1 \cdot \mu_2}{r^3}
\]

This classical formula doesn't seem to imply that the energy is minimum when the spins line up.
May 20, 1980

Angular momentum: In two dimensions,

\[ J = \hat{\mathbf{r}} \times m\dot{\mathbf{r}} = \hat{\mathbf{r}} \times \hat{\mathbf{p}} \]

\[
\begin{vmatrix}
  \hat{i} & \hat{j} & \hat{k} \\
  x & y & z \\
  p_x & p_y & p_z \\
\end{vmatrix} = (x p_y - y p_x) \hat{k}
\]

Thus the angular momentum about the origin is the operator

\[ J = x p_y - y p_x = \hbar \left( x \frac{\partial}{\partial y} - y \frac{\partial}{\partial x} \right) \]

As

\[
\frac{\partial u}{\partial \theta} = \frac{\partial u}{\partial x} \frac{\partial x}{\partial \theta} + \frac{\partial u}{\partial y} \frac{\partial y}{\partial \theta} = \left( -\frac{y}{\sin \theta} + \frac{x}{\sin \theta} \right) u
\]

we see that in polar coordinates

\[ J = x p_y - y p_x = \frac{\hbar}{i} \frac{\partial}{\partial \theta} \]

Operating on functions on the plane, the eigenfunctions are \( e^{i n \theta} \) and the eigenvalues are \( n \hbar \), \( n \in \mathbb{Z} \).

In three dimensional angular momentum has 3 components

\[ J_x = y p_z - z p_y, \quad J_y = z p_x - x p_z, \quad J_z = x p_y - y p_x \]

satisfying

\[
\begin{bmatrix}
  J_x \\
  J_y \\
\end{bmatrix} = \begin{bmatrix}
  y p_z - z p_y \\
  z p_x - x p_z \\
\end{bmatrix} = \frac{\hbar}{i} \begin{bmatrix}
  p_x \\
  p_y \\
\end{bmatrix}
\]

\[ [J_x, J_y] = i \hbar J_z \quad \text{(cyclic perm.)} \]

\[ [J_x, J_y] = i \hbar J_z \quad \text{for others.} \]

Recall that the Lie algebra of \( SO_3 \) can be identified with \( \mathbb{R}^3 \) by assigning to \( \omega \in \mathbb{R}^3 \) the infinitesimal rotation

\[ \omega \in \mathbb{R}^3 \rightarrow \omega \times \mathbb{R}^3 \]
and that the bracket is given by the ordinary cross product of vectors. Hence a representation of Lie(SO₃) is given by operators (skew-adjoint) \( \Theta_x, \Theta_y, \Theta_z \) corresponding to the unit vectors \( i, j, k \) satisfying

\[
[\Theta_x, \Theta_y] = \Theta_z
\]

and cyclic perms.

Thus the angular momentum operators give a repn. of Lie(SO₃) if we put

\[
J_x = ik \Theta_x
\]

etc.

May 21, 1980:

The key example is furnished by the Pauli matrices

\[
\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}
\]

which anti-commute, have square 1, and satisfy \( \sigma_x \sigma_y = i \sigma_z \) + cyclic permutations. Hence

\[
[\sigma_x, \sigma_y] = 2i \sigma_z \quad \text{and cyclic perms.}
\]

which means we get a representation of the angular momentum operators provided we put

\[
J_x = \frac{\hbar}{2} \sigma_x
\]

etc.

A more convenient basis is furnished by

\[
\sigma_2 = \begin{pmatrix} 0 & 1 \\ 0 & -1 \end{pmatrix} \quad \sigma_+ = \frac{1}{2}(\sigma_x + i \sigma_y) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}
\]

\[
\sigma_- = \frac{1}{2}(\sigma_x - i \sigma_y) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}
\]

which is the standard basis for \( sl_2 \), and satisfies

\[
[J_2, \sigma_+] = \pm 2 \sigma_+ \quad [\sigma_+, \sigma_-] = \sigma_z
\]
If \( V \) denotes the standard 2-dimensional representation of \( \text{SU}_2 \), the irreducible representations are \( S_n V \) for \( n \geq 0 \). On this space, \( \sigma \) has the eigenvalues \(-n, -n+2, \ldots, n-2, n\), hence it has maximal weight \( n \). The Casimir element is a multiple of

\[
\sigma_x^2 + \sigma_y^2 + \sigma_z^2 = (\sigma_+ + \sigma_-)^2 - (\sigma_+ - \sigma_-)^2 + \sigma_z^2
\]

\[
= 2(\sigma_+ \sigma_- + \sigma_- \sigma_+) + \sigma_z^2 = 4\sigma_+ \sigma_- + 2\sigma_z^2 + \sigma_z^2
\]

which has the value \( 2n + n^2 = (n+1)^2 - 1^2 \). It seems that \( \sigma_x^2 + \sigma_y^2 + \sigma_z^2 \) is the Casimir element.

Consider a system with two spin sites. It is described by the space \( V \otimes V \) with the operator \( \sigma \otimes \sigma \)

\[
= \sigma_x \otimes \sigma_x + \sigma_y \otimes \sigma_y + \sigma_z \otimes \sigma_z.
\]

The Casimir operator in \( V \otimes V \) is

\[
\sum_{i=1}^{3} (\sigma_i \otimes 1 + 1 \otimes \sigma_i)^2 = \sigma \otimes 1 + 2(\sigma \otimes \sigma) + 1 \otimes \sigma^2.
\]

\( V \otimes V \cong S_2 V \otimes S_0 V \), so Casimir has eigenvalues \( 8, 0 \) on \( V \otimes V \) and \( 3, 3 \) on \( V \) hence

\[
2, 2, -2/6
\]

\( \sigma \otimes \sigma \) has eigenvalues \( 1, 1, 1, -3 \) on \( V \otimes V \) as we've already seen (recall \( \sigma \otimes \sigma = 2 \sigma_+ - 1 \)).

More generally suppose we have two sites of spins \( \frac{m}{2} \) and \( \frac{n}{2} \). This system is described by \( S_m V \otimes S_n V \) with the operator \( \sigma \otimes \sigma \). One way to motivate this is as follows. If \( \psi \in S_m V \) is a unit vector, then the value of \( \sigma \) in the state \( \psi \) is the vector

\[
\langle \psi | \sigma | \psi \rangle
\]

so if \( \psi \in S_m V \), \( \phi \in S_n V \) are unit vectors, then the value of \( \sigma \otimes \sigma \) in the state \( \psi \otimes \phi \) is

\[
\langle \psi \otimes \phi | \sum_{a} \sigma_a \otimes \sigma_a | \psi \otimes \phi \rangle = \sum_{a} \langle \psi | \sigma_a | \psi \rangle \langle \phi | \sigma_a | \phi \rangle.
\]
\[ \langle \psi \mid \mathbf{S} \cdot \mathbf{S} \mid \psi \rangle \]

which agrees with the feeling that the interaction energy should be proportional to the dot product of the spins.

Actually why do we have to keep the space at a site irreducible, why not take the state space to be \( S(V) = \bigoplus_{n \geq 0} V_n \)? Whatever we take, because of the identity

\[ \sum \left( \sigma_a \bullet 1 + 1 \bullet \sigma_a \right)^2 = \sigma_a \bullet 1 + 2 \sum \sigma_a \bullet \sigma_a + 1 \bullet \sigma_a^2 \]

it follows that \( \sigma \times \sigma \) is essentially the primitive part of the Casimir operator.

Now suppose we have several sites \( i \in I \), and let \( \mathbf{S}^i = \bigotimes (\sigma^i_a) \) be the spin operators belonging to the \( i \)-th site space extended by the identity in the other site spaces. The interaction energy is

\[ \frac{1}{2} \sum_{i \neq j} g_{ij} \mathbf{S}^i \cdot \mathbf{S}^j \]

where \( g_{ij} \) gives the coupling between the \( i \)-th and \( j \)-th sites. Suppose all \( g_{ij} = 1 \). Then

\[ \frac{1}{2} \sum_{i \neq j} \mathbf{S}^i \cdot \mathbf{S}^j = \frac{1}{2} \sum_i \left( \sum_{j \neq i} \sigma^i_a \sigma^j_a - \sum \sigma^i_a \sigma^i_a \right) \]

\[ = \frac{1}{2} \sum_i \left( \sum_{a} \sigma^i_a \right)^2 - \frac{1}{2} \sum_i \left( \sigma^i_a \right)^2 \]

and \( \sum_{a} \sigma^i_a \) is just \( \sigma^i \) on the big tensor product. Thus again the interaction is the non-primitive part of Casimir's

Digression: Compute Casimir on \( S(V) \). We have
that \( \sigma_x \), etc., are derivations of \( S(V) \), hence putting
\[
a = a^\uparrow, \quad b = a^\downarrow, \quad \sigma_x = a^* b + b^* a, \quad \sigma_x = a^* b
\]
\[
\sigma_y = \sum_i \langle i | a^i \rangle a^i_j, \quad \sigma_y = i(b^*a - a^*b)
\]
\[
\sigma_z = a^*a - b^*b
\]
Then Casimir is
\[
4 \sigma_+ \sigma_- + 2 \sigma_z \sigma_z = 4b^*a a^*b + 2(a^*a - b^*b)
\]
\[
+ (a^*a - b^*b)^2
\]
\[
= 4b^*b(a^*a + 1) + 2(a^*a - b^*b) + (a^*a)^2 - 2 a^*a b^*b + (b^*b)^2
\]
\[
= 2(a^*a + b^*b) + (a^*a + b^*b)^2 = 2N + N^2
\]
where \( N \) is the number operator.

Recall the electron gas, where one has an exterior algebra on a vector space with basis \( | \lambda > \), where \( \lambda = \uparrow \) or \( \downarrow \).

This exterior algebra is
\[
\Lambda(\bigoplus V) = \bigotimes \overset{\Lambda(V)}{x} \bigotimes 4 \text{ diml.}
\]

For each \( x \) one has the spin density operator \( \psi(x) \rightarrow \psi(x) \)
which involves projecting on the \( x \)-th component and using
the operator \( x^\downarrow \rightarrow \psi \) on \( \Lambda(V) \). Here
\[
\psi = (a^\uparrow a^\downarrow) \quad \psi^* = (a^\downarrow a^\uparrow)
\]
\[
\psi^x \sigma_x \psi = (a^\downarrow a^\uparrow)(0 \quad 1)(a^\downarrow a^\uparrow) = a^\uparrow a^\downarrow + a^\downarrow a^\uparrow
\]
Thus \( \psi^x \sigma_x \psi \) is just the derivation of \( \Lambda V \) belonging to \( \sigma_x \)
on \( \Lambda V \); similarly for \( \sigma_y, \sigma_z \).
Here's how we can handle the spin wave situation: Think of an electron gas with the electrons held in place by the atoms. At each site one can have either no electrons, or one electron with either ↑ or ↓ spin, or two electrons with opposite spin. In the first and last cases the net spin at the site is zero.

Let denote the sites by \( i \in I \) and let \( a_i, b_i \) be the destruction operators for ↑ and ↓ states, so that

\[
\sigma_x^i = a_i^* b_i + b_i^* a_i \\
\sigma_y^i = i(b_i^* a_i - a_i^* b_i) \\
\sigma_z^i = a_i^* a_i - b_i^* b_i
\]

Then the interaction between \( i \)-th and \( j \)-th sites is

\[
\sigma_x^i \sigma_x^j + \sigma_y^i \sigma_y^j + \sigma_z^i \sigma_z^j = (a_i^* b_i + b_i^* a_i)(a_j^* b_j + b_j^* a_j) \\
- (b_i^* a_i - a_i^* b_i)(b_j^* a_j - a_j^* b_j) \\
+ (a_i^* a_i - b_i^* b_i)(a_j^* a_j - b_j^* b_j)
\]

\[
= a_i^* b_i b_j^* a_j + b_i^* a_i a_j^* b_j + \sigma_z^i \sigma_z^j \\
+ a_i^* b_i b_j^* a_j + b_i^* a_i a_j^* b_j
\]

\[
= 2(\sigma_x^i \sigma_x^j + \sigma_y^i \sigma_y^j) + \sigma_z^i \sigma_z^j
\]

which is some fourth order combination of creation and annihilation operators. Perhaps you can describe this interaction by diagrams.
May 22, 1980

Let's consider a "box" into which we can put particles which can assume two states: either spin up or spin down. Suppose there are $N$ particles and that they are distinguishable, so that we $2^N$ possible states. Let's assume a standard form interaction energy:

$$E = -g \sum_{i \neq j} \sigma_i \sigma_j - B \sum_i \sigma_i$$

The parameter $g$ is going to play the role of $\frac{1}{2}$ volume of the box, so that if the volume is small $g$ is large.

Let's put

$$\sigma_i = 2s_i - 1$$

so that $s_i = 0 \Rightarrow \sigma_i = -1$ and $s_i = 1 \Rightarrow \sigma_i = 1$

and write $E$ in terms of the $s_i$.

$$\frac{1}{2} \sum_{i \neq j} \sigma_i \sigma_j = \frac{1}{2} \left[ \left( \sum_i \sigma_i \right)^2 - \sum_i (\sigma_i)^2 \right] = \frac{1}{2} \left[ \left( \sum_i s_i \right)^2 - N \right]$$

$$= \frac{1}{2} \left[ (2 \sum s_i - N)^2 - N \right]$$

hence

$$- E = \frac{g}{2} \left[ \left( \sum s_i - N \right)^2 - N \right] + B \left[ 2 \sum s_i^2 - N \right]$$

$$= 2g (\sum s_i)^2 + (2B + 2gN) \underbrace{\sum \sum s_i}_{b}$$

so if we put $n = \sum s_i$ the partition function is

$$Z = \sum_{n=0}^{N} \binom{N}{n} e^{(2g+2gN>b-n)}$$
What we want to do is to understand this for large $N$. Ideally what we want to do is take the low-density case ($g \to 0$) and increase the density until there is a phase transition (= singularity of some sort).
May 23, 1980  (Alice is 18)

I'm considering putting $N$ particles with either spin up or down into a "box". The particles are distinguishable so the partition function is

$$Z = \sum_{n=0}^{N} \frac{(N)}{n!} e^{-\beta E_n}$$

where $n$ is the number of up spins, and the energy of a configuration with $n$ spins up is quadratic in $n$

$$-E_n = 2gn^2 + bn$$

What we would like to do is to evaluate this partition function, and then understand how it varies with $N, \beta, g, b$.

Example: suppose $g = 0$. Then

$$Z = \sum_{n=0}^{N} \frac{(N)}{n!} e^{\beta bn} = (1 + e^{\beta b})^N$$

so

$$\frac{1}{N} \log Z = \log (1 + e^{\beta b})$$

and

$$\frac{\langle n \rangle}{N} = \frac{1}{\beta \partial b} \log Z = \frac{e^{\beta b}}{1 + e^{\beta b}}$$

leading to the curves:

![Graph showing $\langle n \rangle / N$ vs. $b$]

Recall that $-E_n = 2gn^2 + n(2b-2gN)$ so that if $b=0$, one has $-E_n = 2g(n-N)$ so $E_n$ is maximum when $n = \frac{N}{2}$ and minimum when $n=0$ or $N$. Thus the effect of the interaction is to weight the center region $n = \frac{N}{2}$.
less than the ends. The binomial factor \( N \) weights the center region heavily. So what I expect as \( q \) increases is a smooth change up to a point where the binomial factor is overcome, at which point we should have a phase transition.

Unfortunately we don't seem to be able to compute the partition function exactly, hence we resort to approximations.

\[
\log \left( \frac{N^N}{N!} \right) = N \log N - N + \frac{1}{2} \log N + \frac{1}{2} \log 2\pi - p N (\log p + \log N) + p N - \frac{1}{2} (\log p + \log N) - q N (\log q + \log N) + q N - \frac{1}{2} (\log q + \log N) - \frac{1}{2} \log 2\pi
\]

\[
= -N(p \log p + q \log q) - \frac{1}{2} \log N - \frac{1}{2} \log (2\pi p q)
\]

where \( q = 1 - p \). Let set \( x = \frac{p}{N} = \frac{q}{N} \). Then our first approx is:

\[
Z = \sum_{n=0}^{N} \left( \frac{N!}{n!} \right) e^{\beta (2g n^2 + b n)} = N \int_0^1 dx \: e^{-N(x \log x + (1-x) \log (1-x)) + \beta (2g N^2 x^2 + b N x)} \frac{1}{\sqrt{2\pi N(x(1-x))}}
\]

(This approximation ought to be good for large \( N \) provided one avoids weighting \( x = 0, 1 \) with the other factor). The next approximation is to use Laplace's method on the integral, since the large factor \( N \) appears in the exponent.

\[
f(x) = x \log x + (1-x) \log (1-x) - \beta 2 g N x^2 - \beta b x
\]

\[
f'(x) = \log x + x \frac{1}{x} - \log (1-x) + (1-x) \frac{-1}{1-x} - 2 \beta 2 g N x - \beta b
\]

\[
f''(x) = \frac{x}{x(1-x)} - 2 \beta 2 g N
\]
It's clear at this stage that if I want to get anywhere I want to let $N \to \infty$ with $gN$ constant. I am thinking of $g$ as $\frac{1}{\text{Vol}}$, hence this amounts to a infinite volume limit with fixed density.

Recall we are interested in $\lim_{N} \frac{1}{N} \log Z$. By Laplace we have

$$\lim_{N} \frac{1}{N} \log Z = -f(\bar{x})$$

where $\bar{x}$ is chosen to be the minimum point of $f$, i.e.

$$f'(\bar{x}) = 0$$

since $f''(x) = \frac{1}{x(1-x)} - 4\beta gN \geq 4 - 4\beta gN$, as long as $\beta gN < 1$ there is a unique minimum point, if it exists.

On the other hand we should think of the integral defining $Z$ as a kind of partition function, hence the minimum $f$ value $\bar{x}$ is actually the average $\langle x \rangle = \lim \langle x \rangle = \frac{1}{N}$. Hence

$$f'(\bar{x}) = \log\left(\frac{\bar{x}}{1-\bar{x}}\right) - 4\beta gN \bar{x} - \beta b = 0$$

is the equation defining the fraction $\bar{x}$ of up spins in terms of the parameters $gN, \beta, b$. This is reminiscent of the Weiss theory.

Let's change variables to $2\bar{x} - 1 = \bar{s}$, so that $\bar{s}$ is the average spin. Then $\bar{x} = \frac{1}{2} (1 + \bar{s})$, $1-\bar{x} = \frac{1}{2} (1 - \bar{s})$ and the equation becomes

$$\log\left(\frac{1+\bar{s}}{1-\bar{s}}\right) = 4\beta gN \frac{1}{2} (1+\bar{s}) + \beta (2\bar{b}-2gN)$$

$$\frac{1}{2\beta} \log\left(\frac{1+\bar{s}}{1-\bar{s}}\right) = (gN)\bar{s} + B \quad \sigma$$

$$\bar{s} = \tanh \beta (gN)\bar{s} + B$$
Here's the way to plot this. Start with the inverse \( \tanh(\bar{s}) \) graph:

\[
\bar{s} = \tanh(\beta B) \quad \text{or} \quad B = \frac{1}{\beta} \cdot \frac{1}{2} \log\left(\frac{1+\bar{s}}{1-\bar{s}}\right)
\]

For small \( \bar{s} \), this is \( \bar{s} \approx \beta B \) so it has slope \( \beta \). Now consider

\[
B = \frac{1}{2\beta} \log\left(\frac{1+\bar{s}}{1-\bar{s}}\right) - (gN)\bar{s}
\]

which will distort the curve.

For small \( \bar{s} \), \( B \approx \frac{1}{\beta} \cdot \frac{1}{1-\bar{s}} - gN\bar{s} \) so the curve has slope

\[
\frac{1}{1-gN} = \frac{\beta}{1-\beta gN}
\]

and the phase transition occurs when \( \beta (gN) = 1 \).

Review the Mean theory. Let the spins be

\[
\text{Assume that the effect of the other spins on the } i\text{-th site can be replaced by their average. Thus the } i\text{-th site has two states } s_i = \pm 1
\]

with energy

\[
E = -B s_i - \sum_{j \neq i} g s_j s_i = -s_i \left( B + (N-1)g \right)
\]

replaced by \( \bar{s} \approx Ng \bar{s} \).

Using this to compute \( \langle s_i \rangle = \bar{s} \), we get

\[
\bar{s} = \frac{e^{\beta (B + Ng \bar{s})}}{e^{\beta (B + Ng \bar{s})} + e^{-\beta (B + Ng \bar{s})}} = \tanh(\beta (B + Ng \bar{s}))
\]
which is the same as before.

So we now have a model we might be able to understand completely, and which agrees with the Weiss theory above the critical temperature.
Review: I'm considering an Ising model with $N$ spins $s_1, ..., s_N$ each coupled to the others the same way. The energy is

$$E(s) = -B \sum s_i - g \frac{1}{2} \sum_{i < j} s_i s_j = -B \langle \sum s_i \rangle - g \frac{1}{2} \left( \langle \sum s_i \rangle^2 - N \right)$$

Write this in terms of $\bar{s} = \frac{1}{N} \sum s_i$ the average spin

$$E(\bar{s}) = -N \left( B \bar{s} - g \frac{N}{2} \bar{s}^2 - \frac{N}{2} \right)$$

We are going to let $N \to \infty$ in such a way that $gN$ is constant. (I think of $g$ as proportional to $\frac{1}{\sqrt{V}}$, think of particles spreading out all over the box; then $gN = \text{constant}$ is the same as constant density.)

The partition function is

$$Z = \sum_{n=0}^{N} \binom{N}{n} e^{\beta E_n}$$

where $n$ is the number of up spins: $\sum s_i = 2n - N$. Recall

$$\log \binom{N}{n} = -N \left( x \log x + (1-x) \log (1-x) \right) - \frac{1}{2} \log (N2\pi x(1-x))$$

by Stirling's formula, where $x = \frac{n}{N}$. Now

$$\bar{s} = \frac{1}{N} \sum s_i = 2x - 1 \quad x = \frac{1}{2} (1+\bar{s}) \quad 1-x = \frac{1}{2} (1-\bar{s})$$

$$x \log x + (1-x) \log (1-x) = \frac{1}{2} (1+\bar{s}) \log (1+\bar{s}) + \frac{1}{2} (1-\bar{s}) \log (1-\bar{s}) - \log 2$$

Thus

$$Z \approx \sum e^{-N \left( \frac{1}{2} (1+\bar{s}) \log (1+\bar{s}) + \frac{1}{2} (1-\bar{s}) \log (1-\bar{s}) - \beta B \bar{s} - \beta g \frac{\bar{s}^2}{2} \right) + \beta g \bar{s}}$$

$$\times \sqrt{\frac{N2\pi}{e} \frac{1-\bar{s}^2}{4}}$$

We can ignore constants independent of spin $\bar{s}$, and so we get
\[ Z = C \int \frac{d\tilde{s}}{\sqrt{1 - \tilde{s}^2}} \ e^{-N f(\tilde{s})} \quad \text{C depends on N, b} \]

where

\[ f(\tilde{s}) = \frac{1}{2} (1 + \tilde{s}) \log (1 + \tilde{s}) + \frac{1}{2} (1 - \tilde{s}) \log (1 - \tilde{s}) - (b \beta \tilde{s}) \tilde{s} - (b \beta N) \tilde{s}^{-2} \]

Let's graph. Put

\[ h(s) = \frac{1}{2} (1 + s) \log (1 + s) + \frac{1}{2} (1 - s) \log (1 - s) = \frac{s^2}{2} + \frac{s^4}{12} + O(s^6) \]

\[ h'(s) = \frac{1}{2} \log \left( \frac{1 + s}{1 - s} \right) = s + \frac{s^3}{3} + O(s^5) \]

\[ h''(s) = \frac{1}{2} \left[ \frac{1}{1 + s} + \frac{1}{1 - s} \right] = \frac{1}{1 - s^2} = 1 + s^2 + O(s^3) \]

\[ h \to \log 2 = 0.7 \quad \text{as} \quad s \to \pm 1 \]

Now we are interested in the function

\[ f(s) = h(s) - bs - \frac{1 + \varepsilon}{2} s^2 \approx -bs - \frac{\varepsilon}{2} s^2 + \frac{s^4}{12} + O(s^6) \]

where we have put \( b = \beta B \), \( 1 + \varepsilon = \beta gN \).

\[ f'(s) = -b - 3s + \frac{s^3}{3} + O(s^5) \]

\[ f''(s) = \frac{1}{1 - s^2} - (1 + \varepsilon) \approx -\varepsilon + s^2 + O(s^4) \]

If \( \varepsilon < 0 \), then \( f(s) \) is concave upward, and so at least for
small $b$ it has a unique minimum near zero. The point is that for large $N$, the partition function can be evaluated by steepest descent. One should find that

$$\lim_{N \to \infty} \frac{1}{N} \log Z = -f(s)$$

where $\overline{s}$ is the minimum point for $f$. More generally if there are more than one minimum points one should take the sum of the $f$ values. Now we want to work around $b, \varepsilon = 0$. For $\varepsilon$ slightly $> 0$ and $b = 0$ we get

$$f(s) = -\frac{\varepsilon s^2}{2} + \frac{s^4}{12}$$

$$f'(s) = -\varepsilon s + \frac{s^3}{3} = 0 \quad \text{when} \quad s = \pm \sqrt[3]{3\varepsilon}, 0$$

Now if $b \neq 0$ the two minima points should separate in $f$-value, so there should be only one minimum point, which is consistent with the Lee-Yang result that there is no phase transition with $b \neq 0$.

Let us fix $\beta, B, gN$. Actually we might as well put $gN = 1$. I know that for large $N$ the Stirling approximation for $(N)_n$ is good provided $n/N$ stays away from the ends. Look carefully at the exponent.

$$f(s) = \frac{1}{2} ((1+s) \log (1+s) + \frac{1}{2} (1-s) \log (1-s) - \beta Bs - \frac{\beta s^2}{2}$$
Then \( f'(s) = \frac{1}{2} \log \left( \frac{1+s}{1-s} \right) - \beta B - \beta s \) goes from \(-\infty\) to \(\infty\) as \(s\) goes from \(-1\) to \(1\). So there is at least one zero.

The graph of \(f(s)\) looks as follows (when \(B < 0\))

\[
\begin{align*}
\log 2 & - \beta B - \frac{B}{2} \\
\log 2 + \beta B - \frac{B}{2} & \\
-1 & \quad \text{slope} \quad -\beta B \quad 1
\end{align*}
\]

Also \(f''(s) = \frac{1}{1-s^2} - \beta\). If \(\beta < 1\), the graph is concave upward and can have only one minimum point necessarily \(< 0\), (assuming \(B < 0\)). Otherwise you have two inflection points and possibly a local minimum for \(s > 0\) as drawn. In any case it's clear that we have a unique minimum point for \(B \neq 0\) no matter what \(B, \beta\) are. (Notice that even for \(\beta > 1\) the graph is obtained by shearing the case \(B = 0\):

Since the minimum point occurs inside \(-1 < s < 1\), we can be confident that the Stirling approximation as well as replacing sum by integral are valid as \(N \to \infty\) and we get

\[
\frac{1}{N} \log 2 \to f(\bar{s}) \quad \text{where} \quad \bar{s} \quad \text{is the unique minimum point for} \quad B \neq 0.
\]

When \(B = 0\) we have two symmetrical minimum points but they have the same \(f\) value.

It's clear now that this model gives the Weiss theory on
The internal energy is
\[
\frac{U}{N} = -BSs - \frac{s^2}{2}
\]
because the basic probability distribution peaks at the value of \( s \) given by the equation of state.

Note that \( \frac{U}{N} = 0 \) for \( \beta < 1 \) when \( B = 0 \). If \( \beta > 1 \) and \( B = 0 \) we need to find \( -\frac{s^2}{2} \), where \( s \) is the minimum point. This minimum point is the spontaneous magnetization and is found by solving

\[
\frac{1}{2\beta} \log \left( \frac{1+s}{1-s} \right) - s = 0
\]

\[
\Rightarrow \quad \frac{1}{\beta} \left( s + \frac{s^2}{3} \right) \quad \frac{1+\frac{s^2}{3}}{3} = \beta \quad s = \sqrt{3(\beta-1)}
\]

Thus for \( \beta > 1 \) but close to 1 we have

spontaneous magnetization \( \propto \sqrt{3(\beta-1)} \)

internal energy \( \frac{U}{N} \propto -\frac{3}{2}(\beta-1) \)
May 26, 1980

We should go over the previous calculations for the symmetric Ising model with the idea that \( N \) is finite, but that we are calculating asymptotic expansions as \( N \to \infty \). If we think of \( N \) being infinite, then because \( gN = 1 \), there is no coupling between the spins. To be precise, we have for a given choice of \( \beta, H \) (\( H \) being the external field, denoted previously by \( B \)) a probability measure on \( \{ \pm 1 \}^N \) and hence for any subset \( i \subset i \subset i \subset i \) of spins we have an induced probability measure on \( \{ \pm 1 \}^{\tilde{N}} \). If we let \( N \to \infty \), keeping \( \tilde{N} \) fixed, then it's clear that on \( \{ \pm 1 \}^{\tilde{N}} \) we get the product measure for the measure on \( \{ \pm 1 \}^N \) giving the correct average spin \( S \) determined by the Curie–Weiss law

\[
\frac{1}{2\beta} \log \left( \frac{1 + S}{1 - S} \right) - S = H \quad \text{with suitable modifications for } \beta > 1.
\]

In terms of \( \beta, H \). What we want to do is to compute the induced measure on \( \{ \pm 1 \}^{\tilde{N}} \) as an asymptotic expansion in \( N \).

Since the approximations (Stirling, \( \Sigma i \to \int \) give \( \frac{1}{N} \) errors, we want to express the partition function

\[
\sum_n (N^n) e^{-\beta E_n}
\]
directly as an integral. Recall

\[
-E_n = H \sum_i s_i + \frac{\beta}{2} \sum_{i \neq j} s_i s_j
\]

\[
= H \sum_i s_i + \frac{\beta}{2} \left[ (\sum_i s_i)^2 - (\sum_i s_i^2) \right]
\]

\[
= g \frac{\sigma^2}{2} + (H - \frac{g\beta}{2}) \sigma - \frac{N_g}{2} \quad \sigma = \sum s_i = 2n - N
\]

\[
-\beta E_n = \beta g \frac{\sigma^2}{2} + (\beta H - \beta \frac{g\beta}{2}) \sigma - \beta \frac{N_g}{2}
\]
\[
\frac{1}{\sqrt{2\pi a}} \int_{-\infty}^{\infty} e^{-\frac{1}{2a}k^2 + bk + \sigma k} \, dk = \frac{1}{\sqrt{2\pi a}} \int_{-\infty}^{\infty} e^{-\frac{1}{2a}((k-(b+\sigma)a)^2 + (b+\sigma)^2 a/2) \, dk
\]

\[
= e^{\frac{a}{2}(b+\sigma)^2} = e^{\frac{a}{2}\sigma^2 + ab\sigma + \frac{a^2}{2}} - \frac{1}{2a} (k-ab)^2
\]

\[
e^{\frac{a}{2}\sigma^2 + (ab)\sigma} = \frac{1}{\sqrt{2\pi a}} \int_{-\infty}^{\infty} e^{-\frac{1}{2a}k^2 + bk - \frac{a}{2}b^2 + k\sigma} \, dk
\]

\[
\sum (N) e^{-\frac{1}{2}\sigma^2 + (\beta H-\frac{\beta}{2})k} = \frac{1}{\sqrt{2\pi \beta g}} \int_{-\infty}^{\infty} e^{-\frac{1}{2\beta g} (k-\beta H+\frac{\beta}{2})^2} \sum (N) e^{k\nu} \, dk
\]

\[
(e^k + e^{-k})^N = e^{-kN}(1+e^{2k})^N = \sum (N) e^{k(2n-N)}
\]

Therefore

\[
z = \sum (N) e^{-\beta E_n} = \frac{1}{\sqrt{2\pi \beta g}} \int_{-\infty}^{\infty} e^{-\frac{1}{2\beta g}(k-\beta H+\frac{\beta}{2})^2 + N\log(e^k + e^{-k})} \, dk
\]

and this is an exact formula. Now we want to put \( g = \frac{1}{N} \) and evaluate this asymptotically in \( N \). The exponent is

\[
-\frac{N}{2\beta} (k-\beta H+\frac{\beta}{2N})^2 + N\log(e^k + e^{-k})
\]

and its critical points are given by

\[
-\frac{1}{\beta} (k-\beta H+\frac{\beta}{2N}) + \frac{e^k - e^{-k}}{e^k + e^{-k}} = 0
\]

If we put \( S = \frac{e^k - e^{-k}}{e^k + e^{-k}} \), then \( k = \frac{1}{2} \log(1+S) \) and the equation becomes

\[
S + H - \frac{1}{2N} = \frac{k}{\beta} = \frac{1}{2\beta} \log \left( \frac{1+S}{1-S} \right)
\]

which is the Weiss equation, at least when \( N \rightarrow \infty \).
It doesn't seem to be possible to get the approximate integral for $Z$ from the exact one by the substitution $s = \frac{e^k - e^{-k}}{e^k + e^{-k}}$.

The quantity $\frac{hB}{kT}$ seems to be interpretable as the magnetic field $B$. Let's go back to the energy

$$-E(s) = H \sum_i s_i + \frac{g}{2} (\sum_i s_i^2 - s_i)$$

Then $-\frac{\partial E}{\partial s_i} = H + g \sum_{j \neq i} s_j$ is something like the magnetic field $B_i$ at the $i$-th site. The only thing that's mysterious is the $\frac{g}{2}$, in fact it's absurd to differentiate with respect to something that only takes the values $\pm 1$. The correct thing if you are thinking of continuous spins is

$$-E(s) = H \sum_i s_i + \frac{g}{2} (\sum_i s_i^2 - s_i^2)$$

which gives $B_i = -\frac{\partial E}{\partial s_i} = H + g \sum_{j \neq i} s_j$. This is confusing, but somehow what's involved is whether you think of $s_i$ as contributing to the field at the $i$-th site. Let's tentatively define

$$B = H + g \sum_{j \neq i} s_j - \frac{g}{2} = H + gNs - \frac{g}{2}$$

where $s = \frac{1}{N} \sum_i s_i$ is the average spin. Then the Weiss equation becomes

$$\begin{cases}
B = H + (Ng) s - \frac{g}{2} \\
S = \frac{e^{\beta B} - e^{-\beta B}}{e^{\beta B} + e^{-\beta B}}
\end{cases}$$

$$Z = \frac{1}{2N} \sum(N)e^{-\beta E_n} = \sqrt{\frac{2}{2\pi g}} \int_{-\infty}^{\infty} e^{-\frac{1}{2g} (B - H + \frac{g}{2})^2 + N \log \frac{e^{\beta B} + e^{-\beta B}}{2}} dB$$
May 27, 1980

It is necessary to correct yesterday's error. The energy is

\[-E(s) = H \sum s_i + g \frac{1}{2} \sum s_i s_j\]

\[= H \sum s_i + g \frac{1}{2} \left( \sum_i s_i s_i - \sum_{i \neq j} s_i s_j \right)\]

\[= H \sum s_i + g \frac{1}{2} \left( (\sum s_i)^2 - N \right)\]

\[= H \sigma + g \frac{1}{2} \sigma^2 - \text{const} \quad \text{where} \quad \sigma = \sum s_i = 2n - N\]

However, adding a constant to the energy doesn't affect the predictions, so put

\[-E(s) = H \sigma + g \frac{1}{2} \sigma^2\]

\[-\beta E(s) = \beta H \sigma + \frac{g}{2} \sigma^2\]

Then we use

\[e^{\frac{a}{2} \sigma^2} = \int e^{-\frac{k^2}{2a} + \sigma k - \frac{a}{2} \sigma^2 + \frac{a}{2} \sigma^2} \frac{dk}{\sqrt{2\pi a}}\]

\[e^{\frac{a}{2} \sigma^2 + b \sigma} = \int e^{-\frac{(k-b)^2}{2a} + \sigma k} \frac{dk}{\sqrt{2\pi a}}\]

\[e^{\frac{b}{2} \sigma^2 + \beta H \sigma} = \int e^{-\frac{(k-\beta H)^2}{2b} + \sigma k} \frac{dk}{\sqrt{2\pi b}}\]

\[Z = \sum_{n=0}^{N} \frac{n^N}{N!} e^{\frac{b}{2} \sigma^2 + \beta H \sigma} = \int \sum_{n=0}^{N} \frac{(n^N)}{N!} e^{k n - N} \frac{dk}{\sqrt{2\pi b}}\]

\[(\cosh k)^N = \left( \frac{e^k + e^{-k}}{2} \right)^N = 2^{-N} (e^k - e^{-k})^N (1 + e^{2k})^N\]
Thus
\[ Z = \sum_{n=0}^{N} \frac{1}{2^N} \binom{N}{n} e^{\frac{k^2}{2} \sigma^2 + \beta H_0} = \int_{-\infty}^{\infty} e^{-\frac{1}{2\beta g} (k - \beta H)^2 + N \beta g \log \cosh k} \frac{dk}{\sqrt{2\pi \beta g}} \]

and this gets evaluated by steepest descent. The maximum point for the integrand occurs at

\[ -\frac{1}{\beta g} (k - \beta H) + N \frac{e^k - e^{-k}}{e^k + e^{-k}} = 0 \]

Thus if I put \( B = \frac{k}{\beta} \) I get the equations

\[
\begin{align*}
B &= H + N g s \\
S &= \frac{e^{\beta B} - e^{-\beta B}}{e^{\beta B} + e^{-\beta B}}
\end{align*}
\]

of the Weiss theory. \( B \) is the average magnetic field at any of the sites. Normally \( B \) the field \( B_i \) at the \( i \)-th site would be

\[
B_i = H + g \sum_{j \neq i} S_j \implies B = \langle B_i \rangle = H + (N-1)gs
\]

which means that the theory "wants" a self-energy of \( -\frac{g^2}{2} s_i^2 \) of the \( i \)-th site with itself, so it seems.