April 22, 1980.

Program: Understand density fluctuations in an electron gas by using an effective potential to be determined consistently. Suppose $\varphi(xt)$ denotes the electric potential. We assume the density $n(xt)$ of the gas can be computed well enough by regarding the electrons as moving independently in the field φ. Then we get, if φ is small, and we work to first order:

$$\delta n(xt) = \int \mathcal{P}(x-x', t-t')\, e\varphi(x't')$$

where $\mathcal{P}$ is a so-called retarded polarization propagator:

$$\mathcal{P}(x-x', t-t') = \frac{1}{i}\langle[\hat{n}(xt), \hat{n}(x't')]\rangle\,\Theta(t-t') \qquad \hat{n}(x) = \psi^*(x)\psi(x)$$

Taking Fourier transform we get one relation

$$\boxed{\delta n(q\omega) = \mathcal{P}(q,\omega)\, e\varphi(q\omega).}$$

The other relation comes from Maxwell's equations:

$$\nabla\cdot B = 0 \qquad \nabla\times E = -\frac{\partial B}{\partial t}$$

$$\nabla\cdot E = 4\pi\rho \qquad c^2\,\nabla\times B = \frac{\partial E}{\partial t} + 4\pi j \qquad \frac{\partial\rho}{\partial t} + \nabla\cdot\vec{j} = 0$$

which after F.T. become

$$iq\cdot B = 0 \qquad iq\times E = +i\omega B$$

$$iq\cdot E = 4\pi\rho \qquad c^2(iq\times B) = -i\omega E + 4\pi\vec{j}$$

I should have added

$$\nabla\times A = B \qquad E = -\nabla\varphi - \frac{\partial A}{\partial t}.$$

We assume $A = 0$ whence $B = 0$ and from

$$iq\times E = 0$$

we see that E is longitudinal. In fact

$$E = -\nabla\varphi = -iq\varphi$$

The current $\vec{j}$ is exactly what's needed to kill the displacement current $\frac{\partial E}{\partial t}$.

Because of the positive background

$$\rho = e(n - n_0) = e\,\delta n$$

and so the other equation for φ and δn is $-\Delta\varphi = 4\pi e\,\delta n$
or

$$\boxed{q^2 \varphi(q\omega) = 4\pi e\, \delta n(q\omega)}$$

$$= 4\pi e\; e\, P(q\omega)\, \varphi(q\omega)$$

leading to the condition

$$\boxed{1 = \frac{4\pi e^2}{q^2} P(q\omega)}$$

for density fluctuations.

Let now compute $P(q\omega)$. Recall

$$\hat{n}(x) = \psi^*(x)\psi(x) = \sum_{k',k}{}' \frac{1}{V} e^{i(-k'+k)x} a^*_{k'} a_k$$

$k' = k+q$
$-k'+k = -q$
sign trouble

$$= \frac{1}{V}\sum_q{}' e^{-iqx} n_q \qquad n_q = \sum_k{}' a^*_{k+q} a_k$$

$$P(xt, x't') = \frac{1}{i}\langle [\hat{n}(xt), \hat{n}(x't')]\rangle\, \theta(t-t')$$

$$= \frac{1}{V^2}\sum_{q,q'}{}' e^{i(qx+q'x')} \underbrace{\langle [\hat{n}_q(t), \hat{n}_{q'}(t')]\rangle\, \theta(t-t')\frac{1}{i}}$$

$$\sum_{k,k'} e^{i(\varepsilon_{k+q}-\varepsilon_k)t + i(\varepsilon_{k'+q'}-\varepsilon_{k'})t'}\, \theta(t-t')\frac{1}{i}$$

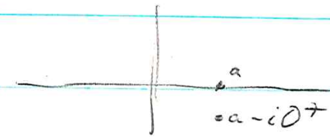
$$\times \underbrace{\langle [a^*_{k+q} a_k, a^*_{k'+q'} a_{k'}]\rangle}$$

$$\delta_{k+q,k'}\,\delta_{k,k'+q'}\left(n_{k+q} - n_k\right) \qquad = \langle a^*_{k+q} a_{k'}\rangle \delta_{k,k'+q'} - \langle a^*_{k'+q'} a_k\rangle \delta_{k+q,k'}$$

Now $k' = k+q$, $k = k'+q'$ $\Rightarrow$ $q+q'=0$ and

$$P(xt,x't') = \frac{1}{V}\sum_q e^{iq(x-x')}\,\frac{1}{V}\sum_k{}' e^{i(\varepsilon_{k+q}-\varepsilon_k)(t-t')}\,\theta(t-t')\frac{1}{i}\left(n_{k+q}-n_k\right)$$

Next use that $\quad \frac{1}{i}e^{-iat}\theta(t) = \int \frac{d\omega}{2\pi}\frac{e^{-i\omega t}}{\omega - a + i0^+}$



and then

$$P(xt,x't') = \frac{1}{V}\sum_q e^{iq(x-x')}\int\frac{d\omega}{2\pi}e^{-i\omega t}\,\frac{1}{V}\sum_k \frac{n_{k+q}-n_k}{\omega + (\varepsilon_{k+q}-\varepsilon_k)+i0^+}$$

Now let $V\to\infty$ to get

$$P(xt,x't') = \int\frac{d^3q\,d\omega}{2\pi}e^{i(qx-\omega t)}\underbrace{\int\frac{d^3k}{(2\pi)^3}\frac{n_{k+q}-n_k}{\omega+(\varepsilon_{k+q}-\varepsilon_k)+i0^+}}_{P(q,\omega)}$$

Next use $\quad n_{k+q}-n_k = n_{k+q}(1-n_k) - n_k(1-n_{k+q})$ and observe

$$\int\frac{d^3k}{(2\pi)^3}\frac{n_k(1-n_{k+q})}{\omega+(\varepsilon_{k+q}-\varepsilon_k)+i0^+} \overset{k\mapsto -k-q}{=} \int\frac{d^3k}{(2\pi)^3}\frac{n_{-k-q}(1-n_{-k})}{\omega+(\varepsilon_{-k}-\varepsilon_{-k-q})+i0^+}$$

$$= \int\frac{d^3k}{(2\pi)^3}\frac{n_{k+q}(1-n_k)}{\omega-(\varepsilon_{k+q}-\varepsilon_k)+i0^+}$$

$n_k = \langle a_k^\dagger a_k\rangle = \begin{cases}1 & |k|<k_F\\ 0 & |k|>k_F\end{cases}$

Hence

$$P(q,\omega) = \int\frac{d^3k}{(2\pi)^3}n_{k+q}(1-n_k)\left[\frac{1}{\omega+(\varepsilon_{k+q}-\varepsilon_k)+i0^+} - \frac{1}{\omega-(\varepsilon_{k+q}-\varepsilon_k)+i0^+}\right]$$

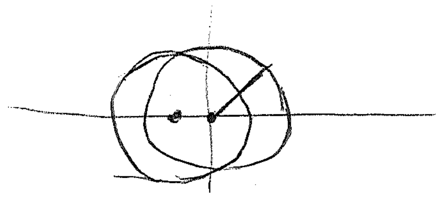
$$= \int_{\substack{|k+q|<k_F\\|k|>k_F}}\frac{d^3k}{(2\pi)^3}\left[\qquad ''\qquad\right]$$

better

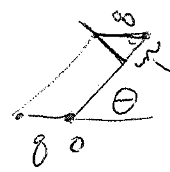
$$P(q,\omega) = \int_{\substack{|k|<k_F\\|k+q|>k_F}}\frac{d^3k}{(2\pi)^3}\left[\frac{1}{\omega-(\varepsilon_{k+q}-\varepsilon_k)+i0^+} - \frac{1}{\omega+(\varepsilon_{k+q}-\varepsilon_k)+i0^+}\right]$$

We want the asymptotic behavior of $P(q,\omega)$ as $q \to 0$.
This will give an integral over a hemisphere.

$$\int \frac{d^3k}{(2\pi)^3} \; \frac{2(\varepsilon_{q+k} - \varepsilon_k)}{\omega^2}$$



I need the thickness of the shell



$$d^3k = 2\pi k_F^2 \sin\theta \underbrace{dr}_{q\cos\theta} d\theta$$

Also $\quad \varepsilon_{q+k} - \varepsilon_k = \frac{1}{2m}(2q\cdot k + q^2) = \frac{1}{2m}(2q\,k_F \cos\theta)$.

$$P(q,\omega) \approx \frac{1}{(2\pi)^3} \int_0^{\pi/2} 2\pi\, k_F^2 \sin\theta \; q\cos\theta \, d\theta \; \frac{1}{\cancel{2}m} \frac{\cancel{2}(2q\, k_F \cos\theta)}{\omega^2}$$

$$= \frac{1}{(2\pi)^3} \frac{4\pi\, k_F^3 q^2}{m\omega^2} \underbrace{\int_0^{\pi/2} \cos^2\theta \sin\theta \, d\theta}_{\left[-\frac{\cos^3\theta}{3}\right]_0^{\pi/2} = \frac{1}{3}}$$

$$P(q,\omega) \approx \underbrace{\frac{1}{(2\pi)^3} \frac{4\pi}{3} k_F^3}_{n_0} \frac{q^2}{m\omega^2}$$

$$\boxed{P(q,\omega) \sim \frac{n_0 q^2}{m\,\omega^2} \quad \text{as } q \to 0}$$

So the equation $\quad 1 = \frac{4\pi e^2}{q^2} P(q,\omega) \quad$ becomes

$$1 = \frac{4\pi e^2}{\cancel{q^2}} \frac{n_0 \cancel{q^2}}{m\omega^2} \quad \text{or}$$

$$\omega^2 = \frac{4\pi e^2 n_0}{m} \quad \text{so} \quad \omega = \omega_{pl} \quad \text{the plasma frequency}$$

There's a simplification as follows:

$$P(q,\omega) = \int \frac{d^3k}{(2\pi)^3} \frac{n_{k+q} - n_k}{\omega + (\varepsilon_{k+q} - \varepsilon_k) + i0^+}$$

Take $\displaystyle\int \frac{d^3k}{(2\pi)^3} \frac{n_{k+q}}{\omega + (\varepsilon_{k+q} - \varepsilon_k) + i0^+} = \int \frac{d^3k}{(2\pi)^3} \frac{n_{-k}}{\omega + (\varepsilon_{+k} - \varepsilon_{+k+q}) + i0^+}$

giving

$$P(q,\omega) = \int \frac{d^3k}{(2\pi)^3}\, n_k \left[\frac{1}{\omega - (\varepsilon_{k+q} - \varepsilon_k) + i0^+} - \frac{1}{\omega + (\varepsilon_{k+q} - \varepsilon_k) + i0^+} \right]$$

$$= \int_{|k|<k_F} \frac{d^3k}{(2\pi)^3} \left[\qquad \text{''} \qquad \right]$$

which leads directly to the estimate

$$P(q,\omega) \approx \int_{|k|<k_F} \frac{d^3k}{(2\pi)^3} \frac{k(2k\cdot q + q^2)}{\omega^2 k m} = \frac{1}{(2\pi)^3} \frac{4}{3}\pi k_F^3 \frac{q^2}{m\omega^2}$$

Thus it seems simpler to integrate over the fermi disk rather than the shell.

April 26, 1980

Have fun - do Bohr theory of hydrogen atom. $Z>0, e<0$ $e=-Z$.

$$L = \frac{m}{2}\left(\dot{r}^2 + (r\dot\theta)^2\right) \quad \frac{Ze}{r^2} \qquad P.E. = \frac{Ze}{r} < 0$$

$$p_\theta = \frac{\partial L}{\partial \dot\theta} = mr^2\dot\theta \qquad \text{angular momentum}$$

$$\dot p_\theta = \frac{\partial L}{\partial \theta} = 0 \qquad \text{so } p_\theta = \text{constant}$$

quantization condition: action: $\oint p_\theta\, d\theta = 2\pi p_\theta = nh \quad n=1,2,..$

so $p_\theta = n\hbar \quad n = 1, 2, \cdots$

Also $p_r = m\dot r$ $\quad \dot p_r = \frac{\partial L}{\partial r} = mr\dot\theta^2 + \frac{Ze}{r^2} = 0$ for circular orbit.

For circular orbit $\quad K.E. = \frac{m}{2}r^2\dot\theta^2 = \frac{r}{2}(mr\dot\theta^2) = -\frac{Ze}{2r} = -\frac{1}{2}P.E.$

and $\quad E = K.E. + P.E. = \frac{1}{2}P.E. = \frac{Ze}{2r}$. Also

$$mr\dot\theta^2 = -\frac{Ze}{r} \Rightarrow p_\theta^2 = m^2r^4\dot\theta^2 = m r^3\left(-\frac{Ze}{r^2}\right) = -\frac{mZe}{r}$$

$$\text{so } \frac{1}{r} = \frac{-mZe}{p_\theta^2}$$

and so for a circular orbit of angular momentum p_θ we get

$$E = \frac{-mZ^2e^2}{2p_\theta^2} = \frac{-me^4}{2\hbar^2 n^2} \qquad \text{and } r = \frac{n^2\hbar^2}{me^2}$$

For $n=1$, we get $\quad E_0 = \frac{me^4}{2\hbar^2}$ = the energy to remove the electron (this is called the Rydberg = 13.6 electron volts)

and $\quad r_0 = \frac{\hbar^2}{me^2}$ = Bohr radius ($\approx .529$ angstroms)

April 27, 1980

Recall from p. 728 that TF approximation leads to the static linear response:

$$\delta n(x) = \gamma \cdot \varphi(x)$$

$$\gamma = \frac{1}{(2\pi)^3} 4\pi \, m k_F (-e)$$

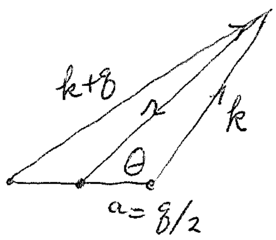
Here's an exact derivation of this approximation. We know the exact linear response is given on the F.T. level by

$$\delta n(q) = P(q,0)\, e\, \varphi(q)$$

where

$$P(q,0) = \int_{\substack{|k|<k_F \\ |k+q|>k_F}} \frac{d^3k}{(2\pi)^3} \frac{-2}{\varepsilon_{k+q}-\varepsilon_k} \qquad \text{(see p. 730)}$$

So what I want to do is to compute $P(q,0)$ for small q. This is going to be like a dipole computation; we use spherical coordinates with center at $\vec{q}/2$ and z-axis pointing in the $\vec{q}$-direction.



Then the sphere $|k| = k_F$ and the sphere $|k+q| = k_F$ intersect when $\theta = \pi/2$. We put

$$r_1 = |k| = \sqrt{r^2+a^2-2ar\cos\theta}$$

$$r_2 = |k+q| = \sqrt{r^2+a^2+2ar\cos\theta}$$

Then

$$\varepsilon_{k+q} - \varepsilon_k = \frac{1}{2m}(r_2^2 - r_1^2) = \frac{1}{2m} 4ar\cos\theta \qquad \text{and so}$$

$$P(q,0) = \frac{-2 \cdot 2m}{(2\pi)^3} 2\pi \int_0^{\pi/2} \sin\theta \, d\theta \int_{r_2=k_F}^{r_1=k_F} r^2 dr \frac{1}{4ar\cos\theta}$$

$$\int_{r_2=k_F}^{r_1=k_F} r\,dr = \frac{r^2}{2}\Bigg]_{r^2+a^2+2ar\cos\theta=k_F^2}^{r^2+a^2-2ar\cos\theta=k_F^2}$$

For $\frac{a}{k_F}$ small we can use the linear approximations

$$r_1 = r - a\cos\theta \qquad r = k_F + a\cos\theta \quad \text{when } r_1 = k_F$$
$$r_2 = r + a\cos\theta \qquad r = k_F - a\cos\theta \quad \text{when } r_2 = k_F$$

to get the above integral $= \frac{1}{2}\left[(k_F + a\cos\theta)^2 - (k_F - a\cos\theta)^2\right]$

$$= 2ak_F\cos\theta.$$

Thus

$$P(q,0) \approx \frac{-2m\cdot 4\pi}{(2\pi)^3}\underbrace{\int_0^{\pi/2}\sin\theta\,d\theta}_{1}\ \frac{2\cancel{a}k_F\cancel{\cos\theta}}{4\cancel{a}\cancel{\cos\theta}}$$

$$= -\frac{4\pi}{(2\pi)^3}\,mk_F$$

leading to

$$\delta n(q) \approx \frac{4\pi}{(2\pi)^3}(-emk_F)\,\varphi(q)$$

for small q which is the TF approximation.

The exact evaluation of $P(q,0)$ can apparently be done as follows.

$$r^2+a^2-2ar\cos\theta = k_F^2 \implies r^2 - 2ar\cos\theta + (a^2-k_F^2) = 0$$

$$r = a\cos\theta \pm\sqrt{a^2\cos^2\theta - a^2 + k_F^2}$$

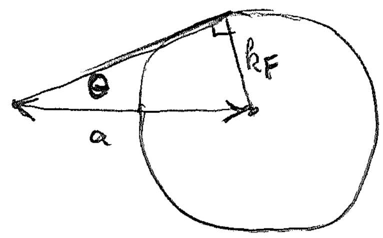
$$r^+ = a\cos\theta + \sqrt{k_F^2 - a^2\sin^2\theta} \qquad \text{upper limit}$$

$$r^- = -a\cos\theta + \sqrt{} \qquad \text{lower limit}$$

$$\frac{1}{2}\left((r^+)^2 - (r^-)^2\right) = 2a\cos\theta\sqrt{k_F^2 - a^2\sin^2\theta}$$

$$P(q,0) = -\frac{4\pi}{(2\pi)^3}\,m\int_0^{\pi/2}\sin\theta\,\sqrt{k_F^2 - a^2\sin^2\theta}\,d\theta \qquad \text{valid for } a \lesssim k_F.$$

and probably when $a > k_F$ the same formula holds except that θ goes from 0 to $\sin^{-1}\left(\frac{k_F}{a}\right)$



April 30, 1980

Review calculation of $\mathcal{P}^R(q,\omega)$: This describes density fluctuations in a non-interacting fermi gas:

$$\hat{n}(x) = \psi^*(x)\,\psi(x) \qquad \psi(x)=\frac{1}{\sqrt{V}}\sum_k e^{ikx} a_k$$

$$\psi^*(x) = \frac{1}{\sqrt{V}}\sum_{k'} e^{-ik'x} a^*_{k'}$$

$$= \frac{1}{V}\sum_{k',k}{}' e^{-ik'x+ikx}\, a^*_{k'} a_k$$

$$= \frac{1}{V}\sum_q{}' e^{-iqx} \sum_k{}' a^*_{k+q} a_k$$

$$\mathcal{P}^R(xt,x't') = \frac{1}{i}\langle[\hat{n}(xt),\hat{n}(x't')]\rangle\,\theta(t-t')$$

$$= \frac{1}{V^2}\sum_{\substack{q',k'\\ q,k}} e^{-iqx-iq'x'}\langle[(a^*_{k+q}a_k)(t),(a^*_{k'+q'}a_{k'})(t')]\rangle \frac{1}{i}\theta(t-t')$$

$$= \frac{1}{V^2}\sum_{q,k,q',k'} e^{-iqx-iq'x'}\, e^{i(\varepsilon_{k+q}-\varepsilon_k)t + i(\varepsilon_{k'+q'}-\varepsilon_{k'})t'} \underbrace{\langle[a^*_{k+q}a_k, a^*_{k'+q'}a_{k'}]\rangle}\frac{1}{i}\theta(t-t')$$

$$= \delta_{k+q,k'}\,\delta_{k,k'+q'}\Big(\underbrace{\langle a^*_{k+q}\, a_{\overset{k'}{\overset{\shortparallel}{k+q}}}\rangle}_{f_{\underset{k'}{k+q}}} - \underbrace{\langle a^*_k a_{\underset{k'+q'}{k}}\rangle}_{f_{\underset{k'+q'}{k}}}\Big)$$

$$\begin{array}{l} k'=k+q \\ k'+q'=k \end{array} \Rightarrow q'=-q$$

$$= \frac{1}{V^2}\sum_{k',q'} e^{iq'(x-x')-i(\varepsilon_{k'+q'}-\varepsilon_{k'})(t-t')}(f_{k'}-f_{k'+q'})\,\frac{1}{i}\theta(t-t').$$

$\therefore$

$$\mathcal{P}(xt,x't') = \frac{1}{V^2}\sum_{q,k} e^{iq(x-x')-i(\varepsilon_{k+q}-\varepsilon_k)t}(f_k-f_{k+q})\,\frac{1}{i}\theta(t-t')$$

$$= \frac{1}{V}\sum_q{}' e^{iq(x-x')}\int\frac{d\omega}{2\pi}e^{-i\omega(t-t')}\underbrace{\left\{\frac{1}{V}\sum_k{}'\frac{f_k-f_{k+q}}{\omega-(\varepsilon_{k+q}-\varepsilon_k)+i0^+}\right\}}_{\mathcal{P}(q,\omega)}$$

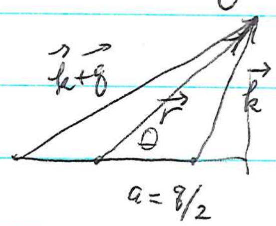
Using $f_k - f_{k+q} = f_k(1-f_{k+q}) - f_{k+q}(1-f_k)$ we can get

$$\mathcal{P}(q,\omega) = \frac{1}{V}\sum_k{}' f_k(1-f_{k+q})\left\{\frac{1}{\omega-(\varepsilon_{k+q}-\varepsilon_k)+i0^+} - \frac{1}{\omega+(\varepsilon_{k+q}-\varepsilon_k)+i0^+}\right\}$$

and finally letting $V \to \infty$ we get

$$P(q,\omega) = \int_{\substack{|k|<k_F \\ |k+q|>k_F}} \frac{d^3k}{(2\pi)^3} \left\{ \frac{1}{\omega - (\varepsilon_{k+q} - \varepsilon_k) + i0^+} - \frac{1}{\omega + (\varepsilon_{k+q} - \varepsilon_k) + i0^+} \right\}$$

To evaluate this integral we use spherical coords. with axis in the q-direction and origin in the center of q.



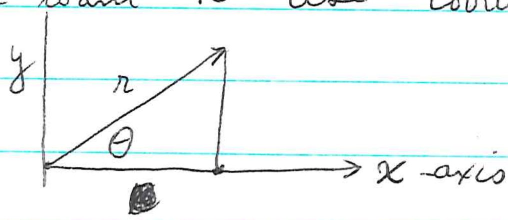
Perhaps cylindrical coordinates are even better. I want

$$\varepsilon_{k+q} - \varepsilon_k = \frac{1}{2m}\left[\left(r+\frac{q}{2}\right)^2 - \left(r - \frac{q}{2}\right)^2\right]$$

$$= \frac{1}{2m} 2\vec{r}\cdot\vec{q} = \frac{|q|}{m} \underbrace{r\cos\theta}_{x}$$

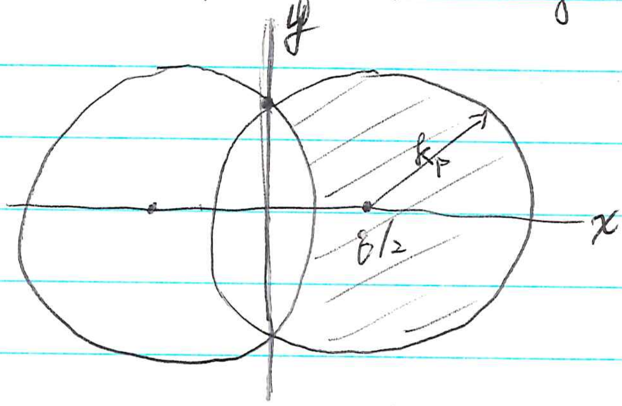
Thus you want to use coordinates x (instead of z)

y = distance from $\vec{q}$ axis

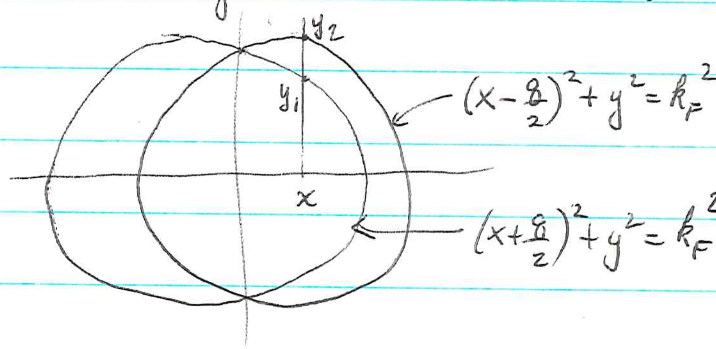


So $d^3k = 2\pi y\, dy\, dx$

I need limits for the region:



Actually we want $k_F \gg q$ so the region should be drawn



When we do $\displaystyle\int_{y_1}^{y_2} y\,dy = \frac{1}{2}(y_2^2 - y_1^2) = \frac{1}{2}\left(\left(k_F^2 - \left(x-\frac{q}{2}\right)^2\right) - \left(k_F^2 - \left(x+\frac{q}{2}\right)^2\right)\right)$

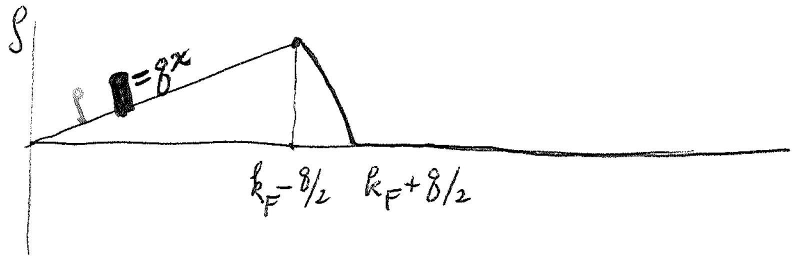
$$= \begin{cases} qx & \text{valid for } 0 \le x \le k_F - \frac{q}{2} \\ \frac{1}{2}\left(k_F^2 - \left(x-\frac{q}{2}\right)^2\right) & k_F - \frac{q}{2} \le x \le k_F + \frac{q}{2} \\ 0 & x \ge k_F + q/2 \end{cases}$$

So we get

$$P(q,\omega) = \frac{1}{(2\pi)^3}\int 2\pi\, dx \left(\int_{y_1(x)}^{y_2(x)} y\,dy\right)\left(\frac{1}{\omega - \frac{q}{m}x + i0^+} - \frac{1}{\omega + \frac{q}{m}x + i0^+}\right)$$

$$= \frac{1}{(2\pi)^2}\int_0^\infty dx\, \rho(x)\left(\frac{1}{\omega - \frac{q}{m}x + i0^+} - \frac{1}{\omega + \frac{q}{m}x + i0^+}\right)$$

where $\rho(x) = \int_{y_1(x)}^{y_2(x)} y\,dy$ has the picture



Recall that we computed the speed of sound waves in a fermi gas [redacted] classically and got

$$C = \sqrt{\frac{p_0 \gamma}{m n_0}} = \frac{1}{\sqrt{3}} \frac{k_F}{m}$$

Let's do the computation in dimension d. One has

$$n_0 = \frac{N}{V} = \int_{|k|<k_F} \frac{d^d k}{(2\pi)^d} = \frac{vol(S^{d-1})}{(2\pi)^d} \frac{k_F^d}{d}$$

E_0 = ground energy

$$\mathcal{E}_0 = \frac{E_0}{V} = \int_{|k|<k_F} \frac{d^d k}{(2\pi)^d} \underbrace{\varepsilon_k}_{\frac{\hbar^2 k^2}{2m}} = \frac{vol(S^{d-1})}{(2\pi)^d} \frac{\hbar^2}{2m} \frac{k_F^{d+2}}{d+2}$$

$$\therefore \quad \frac{\mathcal{E}_0}{n_0} = \frac{E_0}{N} = \frac{\hbar^2}{2m} k_F^2 \frac{d}{d+2} = \frac{d}{d+2} \varepsilon_F$$

To compute pressure one computes [redacted] $-\frac{\partial}{\partial V} E_0$ as N is held fixed. The number of energy levels doesn't change, but ε_k varies as L^{-2} where $V = L^d$ so $\varepsilon_k \sim V^{-2/d}$ and so $E_0 \sim V^{-2/d}$. Another way to see this is that for fixed N,

$$V \sim k_F^{-d} \qquad E_0 \sim V k_F^{d+2} \sim V^1 V^{-\frac{d+2}{d}} = V^{-2/d}.$$

Thus

$$p_{\text{[redacted]}} = -\frac{\partial}{\partial V} E_0 = \frac{2}{d} \frac{E_0}{V} \qquad \text{or} \qquad p_{\text{[redacted]}} V = \frac{2}{d} E_0$$

$$\boxed{p_{\text{[redacted]}} = \frac{2}{d} \frac{E_0}{V}}$$

Equation of state gives $p \sim k_F^{d+2} \sim n_0^{\frac{d+2}{d}}$ so $\gamma = \frac{d+2}{d}$ ($= \frac{5}{3}$ when $d = 3$). Finally

$$c^2 = \frac{p_0 \gamma}{m n_0} = \frac{\frac{2}{d} \frac{E_0}{V} \frac{d+2}{\cancel{d}}}{m \frac{N}{V}} = \frac{2}{dm} \frac{d}{d+2} \underbrace{\varepsilon_F}_{\frac{k_F^2}{2m}} \frac{\cancel{d+2}}{\cancel{d}} = \frac{\hbar^2 k_F^2}{dm^2}$$

yielding $\boxed{c = \frac{1}{\sqrt{d}} \frac{\hbar k_F}{m} \quad \text{speed of sound in a d-diml. fermi gas}}$

One of the things I would like to do is to reconcile this classical calculation with the exact quantum calculation of density fluctuations. Let recall what these look like. Let Φ be the ground state of the gas and let Ψ be a state nearby. We suppose Ψ has the form

$$\Psi = \Phi + \sum_{\substack{|k| < k_F \\ |l| > k_F}} A(l,k) \; a_l^* a_k \Phi$$

(Somehow this is what seems to be a first order variation of Φ. Notice it is a variation tangent to the decomposable vectors in $\Lambda^N \mathcal{H}$, not a general tangent vector to Φ in Λ^N.)

When the gas is in the state Φ at $t=0$, the density change at xt is given by

$$\delta n(xt) = \langle \Psi | \hat{n}(xt) | \Psi \rangle - \langle \Phi | \hat{n}(xt) | \Phi \rangle = \langle \Phi | \hat{n}(xt) | \Psi - \Phi \rangle + \text{c.c.}$$

$$= \frac{1}{V} \sum_{\substack{q, q' \\ l, k}} e^{-iq'x + iqx} \langle \Phi | a_{q'}^*(t) a_q(t) a_l^* a_k | \Phi \rangle A(l,k) + \text{c.c.}$$

$$= \frac{1}{V} \sum_{q, q'} e^{i(q-q')x - i(\varepsilon_q - \varepsilon_{q'})t} A(l,k) \underbrace{\langle \Phi | a_{q'}^* a_q a_l^* a_k | \Phi \rangle}_{\text{non-zero only for } q = l,\ q' = k} + \text{c.c.}$$

$$= \frac{1}{V} \sum_{\substack{|l| > k_F \\ |k| < k_F}} e^{i(l-k)x - i(\varepsilon_l - \varepsilon_k)t} A(l,k) + \text{c.c.}$$

What this says is that the density fluctuation $\delta n(xt)$ is made up out of waves

$$e^{i(l-k)x - i(\varepsilon_l - \varepsilon_k)t}$$

where $|l| > k_F$ and $|k| < k_F$. Now we have seen that such waves don't have a dispersion relation giving the frequency $\varepsilon_l - \varepsilon_k$ as a function of the wave number $|l-k|$, so the question is how to get the speed $c = \frac{1}{\sqrt{d}} \frac{\hbar k_F}{m}$. A possible idea is to let $\hbar \to 0$, keeping the fermi velocity $\frac{\hbar k_F}{m}$ constant, and to do a stationary phase calculation.

It is first necessary to put in $\hbar$ in the preceding density fluctuation calculation. Simply replace time-dependence $\varepsilon_q t \longmapsto \varepsilon_q t/\hbar$. Since

$$\varepsilon_{q+k/2} - \varepsilon_{q-k/2} = \frac{\hbar^2}{2m}(2q\cdot k)$$

we have that the part of $\delta n(xt)$ with wave number q is

$$e^{iqx} \int_{\substack{|k+q/2| > k_F \\ |k-q/2| < k_F}} e^{-i\frac{\hbar}{m} k\cdot q\, t} A(k+q/2, k-q/2)$$

which we can rescale to

$$e^{iqx} \int_{\substack{\left|\frac{k}{k_F} - \frac{q}{2k_F}\right| < 1 \\ \left|\frac{k}{k_F} + \frac{q}{2k_F}\right| > 1}} e^{-iq\cdot\frac{k}{k_F}\underbrace{\left(\frac{\hbar k_F}{m}\right)}_{v_F = \text{Fermi velocity}} t} A\left(k_F\left(\frac{k}{k_F} + \frac{q}{2k_F}\right), k_F\left(\frac{k}{k_F} - \frac{q}{2k_F}\right)\right)$$

Unfortunately as $\hbar \to 0$ and $k_F \to \infty$, the exponent doesn't change, so we can't do stationary phase.

Summary: I still haven't found a way to explain or explain away sound waves of speed $c = \frac{1}{\sqrt{3}} v_F$ in a Fermi gas. Stationary phase doesn't work.

Possibly, the density fluctuations you consider involve transverse waves as well as longitudinal waves and these account for the different frequencies. Maybe a fermi gas resembles an ~~[illegible]~~ elastic solid.

May 7, 1980

Specific heat of the fermi gas. Recall as $V \to \infty$

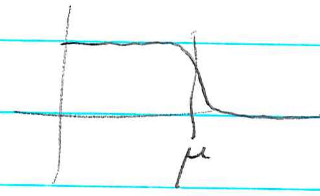
$$\frac{N}{V} = \int \frac{d^3k}{(2\pi)^3} f_k \qquad \frac{E}{V} = \int \frac{d^3k}{(2\pi)^3} \varepsilon_k f_k$$

where $$f_k = \frac{e^{\beta(\mu-\varepsilon_k)}}{1+e^{\beta(\mu-\varepsilon_k)}}$$ is the fermi function.

Since $d^3k = 4\pi k^2 dk$ in the above integrals, we want to evaluate

$$(*) \qquad \int_0^\infty \varepsilon^n d\varepsilon \, \frac{e^{\beta(\mu-\varepsilon)}}{1+e^{\beta(\mu-\varepsilon)}} \qquad \text{for large } \beta$$

Here $\mu > 0$ and the fermi function has graph



so if one integrates by parts

$$\int_0^\infty \frac{\varepsilon^{n+1}}{n+1} \left\{ -\frac{d}{d\varepsilon}\left(\frac{e^{\beta(\mu-\varepsilon)}}{1+e^{\beta(\mu-\varepsilon)}} \right) \right\} d\varepsilon$$

the function in brackets approaches a δ-function as $\beta \to \infty$. Hence the integral $(*)$ should have an asymptotic expansion in powers of $\frac{1}{\beta}$ involving derivatives of ε^n at μ.

Consider

$$\int_{-\infty}^{\infty} g(\varepsilon) \frac{e^{-\beta\varepsilon}}{1+e^{-\beta\varepsilon}} d\varepsilon$$

where g vanishes for $\varepsilon \ll 0$ and doesn't grow too fast as $\varepsilon \to +\infty$. Then

$$\int_{-\infty}^{0} g(\varepsilon) \frac{e^{-\beta\varepsilon}}{1+e^{-\beta\varepsilon}} d\varepsilon = \int_{-\infty}^{0} g(\varepsilon) \left[\frac{-1}{1+e^{-\beta\varepsilon}} + 1 \right] d\varepsilon$$

$$= \int_{-\infty}^{0} g(\varepsilon) d\varepsilon + \int_{+\infty}^{0} g(-\varepsilon) \frac{+d\varepsilon}{1+e^{+\beta\varepsilon}} = \int_{-\infty}^{0} g(\varepsilon) d\varepsilon - \int_0^\infty g(\varepsilon) \frac{e^{-\beta\varepsilon}}{1+e^{-\beta\varepsilon}} d\varepsilon$$

so we have

$$\int_{-\infty}^{\infty} g(\varepsilon)\frac{e^{-\beta\varepsilon}}{1+e^{-\beta\varepsilon}}d\varepsilon = \int_{-\infty}^{0} g(\varepsilon)d\varepsilon + \int_0^\infty [g(\varepsilon)-g(-\varepsilon)]\frac{e^{-\beta\varepsilon}}{1+e^{-\beta\varepsilon}}d\varepsilon$$

$$= \int_{-\infty}^{0} g(\varepsilon)d\varepsilon + \sum_{n=0}^{\infty}\frac{g^{(n)}(0)-(-1)^n g^{(n)}(0)}{\cancel{n!}}\underbrace{\int_0^\infty \varepsilon^{n+1}\left[e^{-\beta\varepsilon}-e^{-2\beta\varepsilon}+\cdots\right]\frac{d\varepsilon}{\varepsilon}}_{\frac{\cancel{\Gamma(n+1)}}{\beta^{n+1}}\left[1-\frac{1}{2^{n+1}}+\frac{1}{3^{n+1}}-\cdots\right]}$$

$$\therefore \boxed{\int_{-\infty}^{\infty} g(\varepsilon)\frac{e^{-\beta\varepsilon}}{1+e^{-\beta\varepsilon}}d\varepsilon = \int_{-\infty}^{0} g(\varepsilon)d\varepsilon + \frac{2g'(0)}{\beta^2}\left(1-\frac{1}{2^2}+\frac{1}{3^2}-\cdots\right) + \frac{2g'''(0)}{\beta^4}\left(1-\frac{1}{2^4}+\frac{1}{3^4}-\cdots\right)+\cdots}$$

$$\frac{N}{V} = \frac{4\pi}{(2\pi)^3}\int k^2\,dk\,f_k \qquad \varepsilon = \frac{k^2}{2m} \qquad k = (2m\varepsilon)^{1/2}$$

$$d\varepsilon = \frac{k\,dk}{m}$$

$$\int_0^\infty k^{n+1}\,dk\,f_k = m\int k^n\frac{k\,dk}{m}f_k = m(2m)^{n/2}\int_0^\infty \varepsilon^{n/2}\frac{e^{\beta(\mu-\varepsilon)}}{1+e^{\beta(\mu-\varepsilon)}}d\varepsilon$$

$$= m(2m)^{n/2}\left\{\frac{\mu^{n/2+1}}{\frac{n}{2}+1} + \frac{2}{\beta^2}\,\frac{n}{2}\mu^{\frac{n}{2}-1}\left(1-\frac{1}{2^2}+\frac{1}{3^2}-\cdots\right)+\cdots\right\}$$

Here $g(\varepsilon) = (\mu+\varepsilon)^{n/2}$ $\quad g'(0) = \frac{n}{2}\mu^{n/2-1}$. Recall

$$\frac{\pi^2}{6} = 1+\frac{1}{2^2}+\frac{1}{3^2}+\cdots \qquad \frac{\pi^2}{8} = 1+\frac{1}{3^2}+\frac{1}{5^2}+\cdots$$

$$2\,\frac{1}{4}\,\frac{\pi^2}{6} = 2\left(\frac{1}{2^2}+\frac{1}{4^2}+\frac{1}{6^2}+\cdots\right) \qquad \frac{\pi^2}{24} = \frac{1}{2^2}+\frac{1}{4^2}+\cdots$$

$$\frac{\pi^2}{12} = 1-\frac{1}{2^2}+\frac{1}{3^2}-\frac{1}{4^2}+\cdots \qquad \pi^2\underbrace{\left(\frac{1}{8}-\frac{1}{24}\right)}_{\frac{1}{12}} = 1-\frac{1}{2^2}+\frac{1}{3^2}-\cdots$$

So therefore

$$\int_0^\infty k^{n+1}\,dk\, f_k = m(2m)^{n/2}\left\{\frac{\mu^{\frac{n}{2}+1}}{\frac{n}{2}+1} + \frac{\cancel{2}}{\beta^2}\frac{n}{\cancel{2}}\mu^{\frac{n}{2}-1}\frac{\pi^2}{12} + O\left(\frac{1}{\beta^4}\right)\right\}$$

$$\boxed{\begin{aligned}\frac{N}{V} &= \frac{4\pi}{(2\pi)^3}\int_0^\infty k^2 dk\, f_k = \frac{4\pi}{(2\pi)^3} m(2m)^{1/2}\left\{\frac{2}{3}\mu^{3/2} + \frac{\pi^2}{12}\mu^{-1/2}\frac{1}{\beta^2} + O\left(\frac{1}{\beta^2}\right)\right\}\\ \frac{E}{V} &= \frac{4\pi}{(2\pi)^3}\int_0^\infty k^2 dk \frac{k^2}{2m} f_k = \frac{4\pi}{(2\pi)^3}\frac{m}{\cancel{2m}}(2m)^{1/2}\left\{\frac{2}{5}\mu^{5/2} + \frac{\pi^2}{12}3\mu^{1/2}\frac{1}{\beta^2} + O\left(\frac{1}{\beta^2}\right)\right\}\end{aligned}}$$

To compute the specific heat at constant V we let T, μ vary so that N doesn't change.

$$\text{const } d\left(\frac{N}{V}\right) = \left(\mu^{1/2} - \frac{1}{2}\frac{\pi^2}{12}\mu^{-3/2}K^2T^2\right)d\mu + \left(\frac{\pi^2}{12}\mu^{-1/2}K^2 2T\right)dT = 0$$

or

$$\frac{d\mu}{dT} = -\frac{\frac{\pi^2}{12}\mu^{-1/2}K^2 2T}{\mu^{1/2} - \frac{1}{2}\frac{\pi^2}{12}\mu^{-3/2}K^2T^2} = \underbrace{-\left(\frac{\pi^2}{12}\right)2K^2\frac{T}{\mu}}\quad + O(T^3)$$

Then the specific heat per unit volume is

$$\frac{d}{dT}\left(\frac{E}{V}\right) = \frac{4\pi}{(2\pi)^3} m\sqrt{2m}\left\{\left(\mu^{3/2} + \frac{\pi^2}{12}\frac{1}{2}3\mu^{-1/2}K^2T^2\right)\frac{d\mu}{dT} + \frac{\pi^2}{12}3\mu^{1/2}K^2 2T\right\} + O(T^3)$$

$$= \frac{4\pi}{(2\pi)^3} m\sqrt{2m}\left\{6\frac{\pi^2}{12}K^2\mu^{1/2}T - 2\frac{\pi^2}{12}K^2\mu^{1/2}T + O(T^3)\right\}$$

$$\boxed{\text{specific heat} = \frac{4\pi}{(2\pi)^3} m\sqrt{2m}\; 3\cdot\frac{4}{2}\frac{\pi^2}{12}K^2\frac{2}{3}\mu^{3/2}\frac{T}{\mu} + O(T^3)}$$

It's probably better to write this in terms of N/V. Notice that $\mu(T) = \mu_0 + O(T^2)$. So we get

$$\boxed{\frac{d}{dT}\left(\frac{E}{V}\right) = \frac{N}{V}\left(\frac{\pi^2}{2}K^2\right)\frac{T}{\mu_0} + O(T^3)}$$

Sackur-Tetrode formula: First review entropy for the grand canonical ensemble.

$$Z_{gr} = \sum_\alpha e^{-\beta(E_\alpha - \mu N_\alpha)}$$

function of β, V, μ.

$$U = -\frac{\partial}{\partial\beta}\log Z_{gr} = \sum (E_\alpha - \mu N_\alpha)\, prob_\alpha = E - \mu N.$$

$$p = \left\langle -\frac{\partial E_\alpha}{\partial V} \right\rangle = \frac{1}{\beta}\frac{\partial}{\partial V}\log Z_{gr}$$

$$N = \langle N_\alpha \rangle = \frac{1}{\beta}\frac{\partial}{\partial\mu}\log Z_{gr}$$

Now if system changes slightly the internal energy change dU is a sum

$$dU = \underset{\text{heat in}}{dQ} - \underset{\text{work done}}{p\,dV} - \underset{\text{like changing potential energy } d\mu \text{ for each particle}}{N\,d\mu}$$

or

$$dQ = dU + p\,dV + N\,d\mu$$

Thus

$$\beta\,dQ = \beta\,dU + \underbrace{\beta p\,dV + \beta N\,d\mu}_{\frac{\partial}{\partial V}(\log Z_{gr})\,dV + \frac{\partial}{\partial\mu}(\log Z_{gr})}$$

$$= \beta\,dU + d(\log Z_{gr}) - \underbrace{\frac{\partial}{\partial\beta}(\log Z_{gr})\,d\beta}_{U\,d\beta}$$

$$= d(\beta U + \log Z_{gr})$$

Thus if $\boxed{\frac{S}{k} = \beta U + \log Z_{gr}}$ we have $\frac{dQ}{T} = dS$

Technically, S is defined this way up to an additive constant, but the third law wants S to be zero at absolute zero. Actually for a quantum system we have an exact probabalistic formula for S

$$\frac{S}{k} = -\sum_\alpha pr_\alpha \log pr_\alpha = \langle \beta(E_\alpha - \mu N_\alpha) + \log Z \rangle = \beta U + \log Z.$$

so really is defined absolutely by this formula.

Next take the fermi gas. Better let's first consider an ideal monatomic gas with grand partition function computed in the standard way

$$Z_{gr} = \sum_{N=0}^{\infty} z^N \frac{1}{N!}(Z_1)^N \qquad z = e^{\beta\mu}$$

(Somehow dividing by $N!$ is a quantum effect).

$$Z_1 = \int e^{-\beta \frac{p^2}{2m}} \frac{d^3p\, d^3x}{(2\pi\hbar)^3} = V \cdot \left(\frac{m}{\beta\, 2\pi\hbar^2}\right)^{3/2}$$

So

$$\log Z_{gr} = z Z_1 = e^{\beta\mu} V \left(\frac{m}{\beta\, 2\pi\hbar^2}\right)^{3/2}$$

$$\boxed{\cancel{U = -\frac{\partial}{\partial\beta}\log Z_{gr} = E - \mu N}}$$

$$N = \frac{1}{\beta}\frac{\partial}{\partial\mu}\log Z_{gr} = \log Z_{gr} \qquad e^{\beta\mu} = \frac{N}{V}\left(\frac{2\pi\hbar^2\beta}{m}\right)^{3/2}$$

$$p = \frac{1}{\beta}\frac{\partial}{\partial V}\log Z_{gr} = \frac{1}{\beta}\frac{\log Z_{gr}}{V} = \frac{N}{\beta V} \qquad \therefore\ pV = NkT.$$

$$U = -\frac{\partial}{\partial\beta}\log Z_{gr} = -\mu N + \frac{3}{2}\frac{N}{\beta}$$

Thus

$$\frac{S}{k} = \beta U + \log Z_{gr} = \frac{3}{2}N - \beta\mu N + N = N\left(\frac{5}{2} - \beta\mu\right)$$

so

$$S = kN\left(\frac{5}{2} - \log \frac{N}{V}\left(\frac{2\pi\hbar^2\beta}{m}\right)^{3/2}\right)$$

$$= Nk\left(\frac{5}{2} + \log \frac{V}{N}\left(\frac{2\pi m kT}{h^2}\right)^{3/2}\right)$$

which is the Sackur-Tetrode formula.

Next take the fermi gas

$$\log Z_{gr} = \sum_{k} \log\left(1 + e^{\beta(\mu - \varepsilon_k)}\right) \sim V \int \frac{d^3k}{(2\pi)^3} \log\left(1 + e^{\beta(\mu - \varepsilon_k)}\right)$$

$$N = \frac{1}{\beta}\frac{\partial}{\partial\mu}\log Z_{gr} \doteq V\int\frac{d^3k}{(2\pi)^3}\,\frac{ze^{-\beta\varepsilon_k}}{1+ze^{-\beta\varepsilon_k}}$$

$$P = \underbrace{\frac{1}{\beta}\frac{\partial}{\partial V}\log Z_{gr}}_{\doteq\,\frac{1}{\beta}\frac{\log Z_{gr}}{V}} = \sum_k \underbrace{\left(-\frac{\partial}{\partial V}\varepsilon_k\right)}_{\frac{2}{3}\frac{\varepsilon_k}{V}}\,\frac{ze^{-\beta\varepsilon_k}}{1+ze^{-\beta\varepsilon_k}} \doteq \frac{2}{3}\underbrace{\int\frac{d^3k}{(2\pi)^3}\,\varepsilon_k\,\frac{ze^{-\beta\varepsilon_k}}{1+ze^{-\beta\varepsilon_k}}}_{E/V}$$

Suppose that $\mu<0$, i.e. that $z<1$. Then we can use the geometric series to get

$$\frac{N}{V} \doteq \frac{4\pi}{(2\pi)^3}\int_0^\infty k^2dk\left(ze^{-\frac{\beta\hbar^2}{2m}k^2} - z^2e^{-2\frac{\beta\hbar^2}{2m}k^2} + \cdots\right)$$

$$\frac{4\pi}{(2\pi)^3}\int_0^\infty \underbrace{\left(\frac{2m\varepsilon}{\hbar^2}\right)^{1/2}}_{k}\,\underbrace{d\varepsilon\,\frac{m}{(\hbar)^2}}_{\frac{\hbar^2k\,dk}{m}}\left(ze^{-\beta\varepsilon} - ze^{-2\beta\varepsilon} + \cdots\right)$$

$$\frac{4\pi}{(2\pi)^3}\,\frac{\sqrt{2}\,m^{3/2}}{\hbar^3}\,\frac{\overset{\frac{1}{2}\sqrt{\pi}}{\cancel{\Gamma(3/2)}}}{\beta^{3/2}}\left(z - \frac{z^2}{2^{3/2}} + \cdots\right)$$

$$\begin{cases} \dfrac{N}{V} = \dfrac{m^{3/2}}{(2\pi)^{3/2}\hbar^3}\,\beta^{-3/2}\left(z - \dfrac{z^2}{2^{3/2}} + \dfrac{z^3}{3^{3/2}} - \cdots\right) \\[2ex] \dfrac{E}{V} = \dfrac{m^{3/2}}{(2\pi)^{3/2}\hbar^3}\,\dfrac{3}{2}\beta^{-5/2}\left(z - \dfrac{z^2}{2^{5/2}} + \dfrac{z^3}{3^{5/2}} - \cdots\right) \end{cases}$$

Consider these formulas with very small z, i.e. as $\frac{N}{V}\beta^{3/2} \to 0$, which will be the case if $\frac{N}{V}\to 0$, T fixed or $\frac{N}{V}$ fixed and $T\to\infty$. Then we get

$$P = \frac{2}{3}\frac{E}{V} = \frac{1}{\beta}\frac{N}{V} \qquad \text{which is ideal gas behavior.}$$

The entropy in this approximation is found as follows.

$$U = -\frac{\partial}{\partial\beta}\log Z_{gr} = E - \mu N$$

so $$S = k_B(\beta U + \log Z_{gr}) = k_B\left(\beta E - \beta\mu N + \tfrac{2}{3}\beta E\right)$$

$$= k_B N\left(\frac{5}{3}\frac{\beta E}{N} - \beta\mu\right) \qquad -\beta\mu = -\log z = \log\left(\frac{V}{N}\left(\frac{m}{2\pi\hbar^2\beta}\right)^{3/2}\right)$$

$$S = k_B N\left(\frac{5}{2} + \log\left(\frac{V}{N}\left(\frac{m}{2\pi\hbar^2\beta}\right)^{3/2}\right)\right)$$

Sackur-Tetrode formula

The same computation results if one uses the Bose gas in the small z approximation.

May 9, 1980

Interacting fermi system: Let's begin with the free Hamiltonian $H_0 = \frac{p^2}{2m}$ for single particles in a box of volume $V = L^3$. Orthonormal basis of H_0 eigenfunctions: $u_k = V^{-1/2} e^{ikx}$ where $k \in \frac{2\pi}{L}\mathbb{Z}^3$. Then we form Fock space $\mathcal{F} = \Lambda \mathcal{H}$, $\mathcal{H} = L^2(\text{box})$ and extend H_0 to $\hat{H}_0 = \sum' \varepsilon_k a_k^* a_k$, $\varepsilon_k = \frac{\hbar^2 k^2}{2m}$ on Fock space. Vectors in $\Lambda^N \mathcal{H}$ describe N independent fermions in the box. As usual $\Lambda^N \mathcal{H}$ is identified with the subspace of skew-symmetric elements in $\mathcal{H}^{\otimes N}$, i.e. with skew-symmetric wave functions $f(r_1, r_2, \ldots, r_N)$.

Next we consider an interaction between the particles given by a potential $V(r, r')$ such that $V(r', r) = V(r, r')$. This gives on $\mathcal{H}^{\otimes N}$ the operator of multiplication by

$$\hat{V} = \frac{1}{2} \sum_{i \neq j} V(r_i, r_j)$$

this is unnecessary if $\hat{V}$ defined in this way.

Recall how this is expressed as an operator on Fock space. The operators $\psi(x)$, $\psi^*(x)$ on $\mathcal{F}$ are interior multiplication by $\langle x|$ and exterior multiplication by $|x\rangle$ respectively:

$$\psi(x) = \sum_k V^{-1/2} e^{ikx} a_k \quad \Big(= a \text{ of } \sum_k{}' \langle x|u_k\rangle\langle u_k| \Big)$$

$$\psi^*(x) = \sum_k{}' V^{-1/2} e^{-ikx} a_k^*$$

On an element f of $\mathcal{H}$, $|f(x)|^2 = \langle f|\psi^*(x)\psi(x)|f\rangle$ is the probability density for the particle to be at x. If $f(r_1, \ldots, r_N) \in \mathcal{H}^{\otimes N}$, then the probability density of the gas at x is the sum

$$\sum_{i=1}^{N} \int |f(r_1, \ldots, \underset{\substack{\uparrow \\ i}}{x}, \ldots, r_N)|^2 \, dr_1 \cdots \widehat{dr_i} \cdots dr_N$$

of the probabilities of finding each particle at x. This can be written as $\langle f | \underbrace{\sum_{j=1}^{N} \delta(x-r_j)}_{\hat\rho(x)} | f \rangle$. Here $\hat\rho(x)$ denotes the operator of multiplying by $\sum \delta(x-r_j)$ on $\mathcal{H}^{\otimes N}$. When restricted to $\Lambda^N\mathcal{H}$ we see $\hat\rho(x)$ is the derivation extending multiplication by $\delta(x-r)$ on $\mathcal{H}$. Thus

$$\hat\rho(x) = \psi^*(x)\psi(x).$$

Now $\frac{1}{2}\int dx\,dx' \; \psi^*(x)\psi(x)\, V(x,x')\, \psi^*(x')\psi(x')$

is multiplication by $\frac{1}{2}\int dx\,dx' \sum_i \delta(x-r_i)\, V(x,x') \sum_j \delta(x'-r_j)$ on $\Lambda^N\mathcal{H}$.

$$= \frac{1}{2}\sum_{i,j} V(r_i, r_j).$$

So

$$\frac{1}{2}\int dx\,dx' \; \psi^*(x)\psi^*(x')\, V(x,x')\, \psi(x')\psi(x)$$

$[\psi(x), \psi^*(x')\psi(x)]$
$= \{\psi(x), \psi^*(x')\}\psi(x)$

$$= \frac{1}{2}\int dx\,dx' \left[\psi^*(x)\psi(x)\, V(x,x')\, \psi^*(x')\psi(x') - \psi^*(x)\,\delta(x'-x)\, V(x,x')\psi(x')\right]$$

$$= \frac{1}{2}\sum_{i,j} V(r_i,r_j) - \underbrace{\frac{1}{2}\int dx\; \psi^*(x)\, V(x,x)\, \psi(x)}_{-\frac{1}{2}\sum V(r_i,r_i)}$$

Thus

$$\boxed{\frac{1}{2}\int dx\,dx' \; \psi^*(x)\psi^*(x')\, V(x,x')\,\psi(x')\psi(x) = \text{mult by } \frac{1}{2}\sum_{i\neq j} V(r_i,r_j)}$$

Call this $\hat V$. Changing to momentum notation we get

$$\hat V = \frac{1}{2} V^{-2} \sum a^*_{k_1} a^*_{k_2} a_{k_3} a_{k_4} \underbrace{\int dx\,dx' \; e^{-ik_1x - ik_2x' + ik_3x' + ik_4x}\, V(x,x')}$$

Suppose $V(x,x') = V(x-x')$

$$\int dx\,dx' \; e^{i(-k_1+k_4)\underset{\downarrow\, x+x'}{x} + i(-k_2+k_3)x'}\; V(\underset{\downarrow\, x+x'}{x-x'})$$

Put $\quad V(q) = \int dx\, e^{-iqx} V(x) \qquad q = k_1 - k_4$

Then we get

$$\hat{V} = \frac{1}{2\,\mathrm{Vol}} \sum_{q,p,k} V(q)\, a^*_{k+q}\, a^*_{p-q}\, a_p\, a_k$$

the terms correspond to diagrams: