Spin waves (continued). Recall the Hilbert space is the exterior algebra with basis \( e_n \) where \( e_n \) is the spin assignment with spin down at \( n \) and spin up elsewhere. Better

\[
\mathcal{H} = \bigotimes_n \mathbb{C}^2
\]

This identification is not compatible with signs (see page 513)

and we think of \( \mathbb{C}^2 \) as \( \Lambda \mathcal{E} \) with \( 1 = |+> \) and \( 1 = |-> \)

generator of \( \Lambda \mathcal{E} \). The Hamiltonian is

\[
H = \sum_\sigma \frac{1}{2} (1 - P_\sigma^{ex})
\]

where \( \sigma \) runs over pairs of nearest neighbor sites.

I want to write \( H \) as \( H_0 + H' \) where \( H_0 \)

is a derivation of the exterior algebra extending \( \mathcal{H} \) on \( \mathcal{H}' \)

and where \( H' \) is a 2-particle interaction extended

canonically to several particles by summing over pairs. It's enough to do this for the operator

\[
\frac{1}{2} (1 - P_\sigma^{ex})
\]

Suppose \( \sigma = \{1, 2\} \). Then

\[
\mathcal{H} = \Lambda e_1, e_2 \otimes \Lambda e_{n+2}^3
\]

and the operator \( \frac{1}{2} (1 - P_\sigma^{ex}) \) is of the form \( A \otimes 1 \). So we need only compute \( A \) on \( \Lambda e_1, e_2 \).

\[
1 \rightarrow 0
\]

\[
e_1 \rightarrow \frac{1}{2} (e_1 - e_2)
\]

\[
e_2 \rightarrow \frac{1}{2} (e_2 - e_1)
\]

\[
e_1 e_2 \rightarrow 0
\]
Let $H_0$ be the derivation extending $A$ on $\Lambda^1$. Then

$$H_0(e_1 \wedge e_2) = \frac{1}{2}(e_1 \cdot e_2) e_2 + e_1 \cdot \frac{1}{2}(e_2 \cdot e_1) = e_1 e_2$$

so we have $A = H_0 + H'$ where

$$H_0 = \frac{1}{2}(a_1^* a_2^*) a_1 + \frac{1}{2}(a_2^* a_1^*) a_2 = \frac{1}{2}(a_1^* - a_2^*)(a_1 - a_2)$$

$$H' = -a_1^* a_2^* a_1 a_2$$

Hence the formula for the Heisenberg chain is

$$H = \sum_n \frac{1}{2}(a_n^* - a_{n+1}^*)(a_n - a_{n+1}) - a_n^* a_{n+1} a_n a_{n+1}$$

Let's now pass to the momentum representation. We suppose that our set of sites is $\mathbb{Z}/N\mathbb{Z}$ and we want to change from the orth. basis $e_n$ to the orth basis $u_k$

$$u_k = \frac{1}{\sqrt{N}} \sum_n e^{i k n} e_n$$

$$e_n = \frac{1}{\sqrt{N}} \sum_k e^{-i k n} u_k$$

$$\left\{ \begin{align*}
    a_n^* &= e(e_n) = \frac{1}{\sqrt{N}} \sum_k e^{-i k n} a_k^* \\
    a_n &= \frac{1}{\sqrt{N}} \sum_k e^{i k n} a_k
\end{align*} \right.$$
Now if the lattice spacing were a we would get

\[ H_0 = \sum_k (1 - \cos ak) a_k^* a_k \quad \text{if} \quad k \in \frac{2\pi}{Na} \mathbb{Z}/\frac{2\pi}{a} \mathbb{Z} \]

so the energy of the "particle" \( \psi_k \) of momentum \( k \) is

\[ \varepsilon_k = 1 - \cos ak \approx \frac{1}{2} a^2 k^2 \]

Consider \( H' = + \sum_n a_n^* a_{n+1} a_{n+1} a_n \)

\[ = + \frac{1}{2} \sum_{m,n} a_m^* a_n^* V(m-n) a_n a_m \]

where \( V(m-n) = \begin{cases} -1 & |m-n| = a \\ 0 & \text{otherwise} \end{cases} \)

Then

\[ H' = + \frac{1}{2} \sum_{m,n} \left( \frac{1}{V} \sum_{k_1} e^{-ik_1 n} a_{k_1}^* \right) \left( \frac{1}{V} \sum_{k_2} e^{-ik_2 n} a_{k_2}^* \right) \times V(m-n) \left( \frac{1}{V} \sum_{k_4} e^{ik_4 n} a_{k_4}^* \right) \left( \frac{1}{V} \sum_{k_3} e^{ik_3 n} a_{k_3}^* \right) \]

\[ = + \frac{1}{2Na^2} \sum_{k_1, k_4} a_{k_1}^* a_{k_2}^* a_{k_4} a_{k_3} \sum_{m,n} i^{m+n}(k_3 - k_1) + i^m(k_4 - k_2) V(m-n) \frac{1}{N} \]

\[ \frac{1}{N} \sum_{m,n} e^{im(n)(k_3 - k_1) + in(k_4 - k_2)} V(m) \]

\[ = \delta(k_3 + k_4 - k_1 - k_2) \sum_m e^{im(k_3 - k_1)} V(m) \]

Picture:

- \( k_1 \)
- \( k_2 \)
- \( k_3 \)
- \( k_4 \)
Thus
\[ H^\prime = \frac{1}{2N} \sum_{k_1, k_2, g} a^*_{k_1-g} a^*_{k_2+g} \hat{V}(g) a_{k_2} a_{k_1} \]

In the situation considered above
\[ \hat{V}(g) = -(e^{iag} + e^{-iag}) = -2\cos(ag) \]

Notice that in the expression
\[ H^\prime = \frac{1}{2} \sum_{m,n} a^*_m a^*_n \hat{V}(m-n) a_n a_m \]

The value \( V(0) \) doesn't appear, and hence a constant can be added to \( \hat{V}(g) \) without affecting the expression at the top of this page for \( H^\prime \).

Also we can pass to a continuum limit if we redefine
\[ \hat{V}(g) = a \sum_n e^{-img} V(n) \rightarrow \int e^{-ixg} V(x) dx \]

Then we have
\[ H^\prime = \frac{1}{2Na} \sum_{k_1, k_2, g} a^*_{k_1-g} a^*_{k_2+g} \hat{V}(g) a_{k_2} a_{k_1} \]

It seems that for the Heisenberg chain this limit is useless, because then
\[ \hat{V}(g) = -a(e^{iag} + e^{-iag} + \text{const.}) = 2a(1 - \cos(ag)) \]
\[ \text{take } a = -2 \quad \propto \quad a^2 \]
and I want to divide $H$ by $a^2$ in order that $H_0$ have a good limit as $a \to 0$.

A more serious problem is that the identification of $\mathbb{H}$ with an exterior algebra is faulty. The idea was to use an ordering of the sites in order to go from

$$e_{i_1} \ldots e_{i_p} \leftrightarrow e_{i_1} \ldots e_{i_p} \quad \text{if} \quad i_1 < \ldots < i_p$$

But this gets in trouble with the Hamiltonian. Recall we wanted

$$\frac{1}{2} (1 - P_{mn}^{\text{ex}}) \leftrightarrow \frac{1}{2} (a_m^* a_n)^* (a_m^* a_n) - a_m^* a_n a_n a_m$$

Take $m = 1$, $n = 3$ and see what happens to $e_1 e_2$

$$\frac{1}{2} (1 - P_{13}^{\text{ex}}) e_1 e_2 = \frac{1}{2} e_1 e_2 - \frac{1}{2} e_3 e_2 \leftrightarrow \frac{1}{2} (e_1 e_2 - e_2 e_3)$$

$$(\frac{1}{2} (a_1 - a_3)^* (a_1 - a_3) - a_1^* a_3 a_3 a_1) (e_1^* e_2) = \frac{1}{2} (a_1^* - a_3^*) e_2$$

$$= \frac{1}{2} (e_1 e_2 - e_3 e_2) = \frac{1}{2} (e_1 e_2 + e_2 e_3)$$

But maybe this problem doesn't occur with the linear chain. There doesn't seem to be any trouble when the set of sites is $\mathbb{Z}$, but clearly the same problem arises in the periodic case for $P_{1N}^{\text{ex}}$.

Anyway, it seems to be more important to understand

$$H = \sum_k \varepsilon_k a_k^* a_k + \frac{1}{2V} \sum_{k, k' \neq 0} a_k^* a_{k'}^* \hat{V}(k) a_{k'} a_k$$

Here $\hat{V}(k)$ is irrelevant.
Schrödinger's anti-commuting c-numbers.

Let's consider $\mathcal{H} = \Lambda V$ with creation and annihilation operators $a^*_n = e(e_n)$, $a_n = i(e^*_n)$ defined by some orthonormal basis $e_n$ of $V$. Consider $H_0 = \sum \omega_n a^*_n a_n$. The problem will be to compute expectation values of products of the operators $a_n(t)$, $a^*_n(t)$ at different times. In particular, I want to see if there is some analogue of the time-ordered products encountered before.

Let's first review the boson situation. We have $H_0 = \omega a^* a$ and we form $H = \omega a^* a + J a + \tilde{J} a^*$ where $J$, $\tilde{J}$ are functions of $t$ with compact support. Then let $U(t,t')$ be the propagator for

$$\frac{\partial \psi}{\partial t} = -H \psi$$

and let's compute $\text{tr} U(\beta,0)$ by Schrödinger's method.

$$8 U(\beta,0) = -\int_0^\beta dt \ U(\beta,t) \left\{ 8 J a + 8 \tilde{J} a^* \right\} U(t,0)$$

$$\frac{d}{dt} U(\beta,t) \bigg|_{t=0} = U(\beta,t) [H, a] U(t,0)$$

$$\left( \frac{d}{dt} + \omega \right) \text{tr} \left( U(\beta,t) a U(t,0) \right) = -\tilde{J} \text{tr} \left( U(\beta,0) \right)$$

$$\left( \frac{d}{dt} - \omega \right) \text{tr} \left( U(\beta,t) a^* U(t,0) \right) = \tilde{J} \text{tr} \left( U(\beta,0) \right)$$

$$\frac{\text{tr} \left( U(\beta,t) a U(t,0) \right)}{\text{tr} U(\beta,0)} = -\left( \frac{d}{dt} + \omega \right)^{-1} \tilde{J}$$
\[
\frac{\text{tr} \left( U(\beta,t) a^* U(\psi,0) \right)}{\text{tr} U(\beta,0)} = - \left( -\frac{d}{dt} + \omega \right)^{-1} J
\]
where the inverses \((+-\frac{d}{dt}+\omega)^{-1}\) are computed using periodic boundary conditions on \([0,\beta]\). Thus

\[
\frac{\partial \text{tr} U(\beta,0)}{\text{tr} U(\beta,0)} = \int_0^\beta \left( \partial \tilde{J} \left( \frac{d}{dt} + \omega \right)^{-1} J + \partial \tilde{J} \left( -\frac{d}{dt} + \omega \right)^{-1} J \right) dt
\]

\[
eq \int_0^\beta \left( \partial \tilde{J} \left( \frac{d}{dt} + \omega \right)^{-1} \tilde{J} + \tilde{J} \left( \frac{d}{dt} + \omega \right)^{-1} \partial \tilde{J} \right) dt
\]

Thus integrating we get

\[
\frac{\text{tr} U(\beta,0)}{\text{tr} U(\beta,0)} = \exp \left[ \int_0^\beta \left( \tilde{J} \left( \frac{d}{dt} + \omega \right)^{-1} \tilde{J} \right) dt \right]
\]

Notice this isn't symmetric in \(J, \tilde{J}\). As a check you should let \(\beta \to +\infty\) and replace the lower limit 0 by \(\beta' \to -\infty\). Then

\[
\text{tr} U(\beta,\beta') \approx \langle 0 \mid U(\beta,\beta') \mid 0 \rangle \quad \beta \gg 0 \gg \beta'.
\]

Also \((+-\frac{d}{dt}+\omega)^{-1}\) has the kernel

\[
\begin{cases} 
  e^{-\omega(t-t')} & t > t' \\
  0 & t < t'
\end{cases}
\]

so we get

\[
\langle 0 \mid U(\beta,\beta') \mid 0 \rangle = \exp \int_{t>t'} \int_{t'>t} \tilde{J}(t) e^{-\omega(t-t')} \tilde{J}(t') dt dt'
\]

What kinds of Green's functions arise? Use Dyson's formula

\[
U(\beta,0) = U_0(\beta,0) \ast \int_0^\beta U_0(\beta, t_1) H_+(t_1) U_0(t_1,0) + \cdots
\]
Then
\[
\frac{\text{tr } U(\beta, 0)}{\text{tr } U_0(\beta, 0)} = \sum_{m, n} (-1)^{m+n} \int \frac{dt_1 \ldots dt_m dt_1' \ldots dt_n'}{m! n!} x \tilde{T}(t_1) \ldots \tilde{T}(t_m) \tilde{T}(t_1') \ldots \tilde{T}(t_n') \nonumber 
\]
\[
\times \langle T [a(t_1) \ldots a(t_m) a^*(t_1') \ldots a^*(t_n')] \rangle \nonumber 
\]
where \( a(t) = e^{tH_0} a e^{-tH_0} \) and \( \langle A \rangle = \frac{\text{tr } (e^{-\beta H_0} A)}{\text{tr } (e^{-\beta H_0})} \)

So it's clear the Green's functions are just
\[
\langle T [a(t_1) \ldots a(t_m) a^*(t_1') \ldots a^*(t_n')] \rangle 
\]
and one has Wick's theorem evaluating them.

**Question:** Does there exist a path integral description for \( \text{tr } (U(\beta, 0)) \)?

**Possible approach:** \( Z(J, \bar{J}) = \text{tr } (U(\beta, 0) J, \bar{J}) \) is a function of the pair \( J, \bar{J} \) which we might try to represent as the Fourier (or Laplace) transform of a measure on the dual space. So the problem becomes describing this measure. A first problem is whether \( \bar{J} \) should be required to be \( \bar{J} \).

We can work backwards from the formula
\[
\int e^{-\frac{1}{2} x^* A x + J^* x} (dx) = \frac{e^{\frac{1}{2} J \bar{A} \bar{J}}}{(\det A)^{1/2}} 
\]
which is valid for \( \text{Re}(A) > 0 \). So look at
\[
\frac{\text{tr } (U(\beta, 0))}{\text{tr } (U_0(\beta, 0))} = \exp \int J \left( \frac{\partial}{\partial \psi} + \psi \right)^{-1} \bar{J}. 
\]
Is the quadratic function
\[ T, \tilde{T} \rightarrow \int T (\frac{d}{dt} + \omega)^{-1} \tilde{T} \]
positive-definite? Obviously not since it changes sign when \( \tilde{T} \) does. Note that if \( \tilde{T} = T \), then
\[ \int T (\frac{d}{dt} + \omega)^{-1} T = \int (\frac{d}{dt} + \omega)^{-1} T \cdot T \]
but it's simpler to work with Fourier series:
\[
T(t) = \sum_{n e^{\frac{2\pi i n}{N}}} J_n e^{int} 
\]
\[ \bar{T} = \sum_{n} \bar{J_n} e^{-int} \]
\[ (\frac{d}{dt} + \omega)^{-1} \bar{T} = \sum_{n} \frac{\bar{J_n} e^{-int}}{-in + \omega} \]
\[ \int T (\frac{d}{dt} + \omega)^{-1} \bar{T} = \sum_{n} \frac{|J_n|^2}{-in + \omega} \]

So we can conclude that there should be a Gaussian measure on the dual space to the set of all \( T, \tilde{T} = \bar{T} \) periodic on \([0, \beta]\). This space is the space of periodic functions (really distributions) at \( t \).

Recall the formula
\[ \int e^{-a|z|^2 + \bar{T} \cdot \bar{T}} \, dzd\bar{z} = e^{\frac{|T|^2}{a}} \cdot \text{const} \]
Consequently for the form
\[ \sum_{n} \frac{|J_{n}|^2}{-in + \omega} \]
we are going to want to use something like
\[ (*) \quad \int \Delta a \Delta \bar{a} \quad e^{-\frac{1}{2} \left( a \frac{d\bar{a}}{dt} + \omega \bar{a} \right)^2 + T a + \bar{T} \bar{a}} \quad dt \]

Let's evaluate by Fourier series:
\[ J = \sum J_{n} e^{int} \quad a = \sum a_{n} e^{int} \]

\[ \int \left( a \frac{d\bar{a}}{dt} + \omega \bar{a} \right) dt = \sum a_{n} (-in) \bar{a}_{n} + \omega a_{n} \bar{a}_{n} \]

\[ \int T a \quad dt = \sum_{n} J_{n} a_{n} \]

and so
\[ (*) = \int \Delta a \Delta \bar{a} \quad e^{-\sum_{n} \frac{1}{2} (in+\omega)|a_{n}|^2 + T_{n} a_{n} + \bar{T}_{n} \bar{a}_{n}} \]

\[ = \text{const.} \quad e^{\frac{1}{2} \sum \frac{|J_{n}|^2}{-in + \omega}} \]

It seems like we do get a path-integral representation for the Schwinger generating function
\[ \frac{\text{tr} \ U(\beta, 0)}{\text{tr} \ U_{0}(\beta, 0)} \]

However, the "measure" on the space of paths is a Gaussian with imaginary part.
Yesterday we computed $\text{tr} \ (U(\beta,0))$ for $H = \omega a^*a + J_a + J_{a^*}$ using Schwinger's variational method. However a simpler approach uses

$$e^{+\beta H_0} U(\beta,0) = T e^{-\int_0^\beta (J_a + \bar{J}_{a^*})(t) \, dt}$$

where $T$ is time-ordering. Also one uses

$$e^{A} e^{B} = e^{[A,B]} e^{B} e^{A}$$

when $[A,B]$ commutes with $A, B$. Then one has

$$e^{\gamma a} e^{\gamma a^*} = e^{\gamma a} e^{\gamma a^*} e^{\gamma a}$$

Since

$$a(t) = e^{+\beta H_0} a \ e^{-\beta H_0} = e^{-\omega t} a$$

and

$$a^*(t) = e^{\omega t} a^*$$

We therefore can evaluate the time-ordering by showing $T(t) a^*(t) \, dt$ thru the previous $a$-part

$$e^{+\beta H_0} U(\beta,0) = \int_0^\beta \int_0^\beta \ e^{\int_0^\beta -\omega t \, dt} e^{\int_0^\beta J(t) \, dt} a \ e^{-\int_0^\beta (J(t) e^{-\omega t} \, dt) a^*}$$

What we have to do is multiply by $e^{-\beta H_0} = e^{-\beta \omega a^* a}$ and take the trace. Somehow this doesn't look easy. Here are two ways of doing it:

Put $\langle A \rangle = \text{tr} (e^{+\beta H_0} A) / \text{tr} (e^{-\beta H_0})$. Then
\[
\langle e^{z' a^*} e^{z a} \rangle = \sum_{m,n} \frac{(y^*)^m (y)^n}{m! n!} \langle a^m a^n \rangle
\]

Wick's thm says
\[
\langle a^m a^n \rangle = \begin{cases} 
0 & m \neq n \\
n! \langle a^* a \rangle^n & \text{if } m = n.
\end{cases}
\]

and
\[
\langle a^* a \rangle = \frac{\sum n e^{-\beta \omega n}}{\sum e^{-\beta \omega}} = -\frac{d}{d(\beta \omega)} \log \frac{1}{1-e^{-\beta \omega}} = \frac{e^{-\beta \omega}}{1-e^{-\beta \omega}}
\]

Therefore
\[
\langle e^{z' a^*} e^{z a} \rangle = e^{e^{-\beta \omega} z' z}
\]

Use holomorphic representation: Want trace of
\[
f(z) \xrightarrow{e^{z a}} f(z + \alpha) \xrightarrow{e^{z' a^*}} e^{z' z} f(z + \alpha) \xrightarrow{e^{-\beta \omega z a^*}} e^{e^{-\beta \omega z} z'} f(e^{-\beta \omega} z + \alpha)
\]

Let's compute the trace using as basis the powers of \(z - \alpha\) where \(\alpha\) is the fixed point for \(z \mapsto e^{-\beta \omega} z + \alpha\), i.e.
\[
\alpha = \frac{\theta}{1-e^{-\beta \omega}}
\]

Multiplication by \(e^{z' e^{-\beta \omega} z}\) relative to this basis is triangular with diagonal entries equal to
\[
e^{e^{-\beta \omega} \alpha} = e^{\frac{e^{-\beta \omega}}{1-e^{-\beta \omega}} z' z}
\]

The operator \(f(z) \mapsto f(e^{-\beta \omega} z + \alpha)\) relative to this basis is diagonal with eigenvalues \(e^{-\beta \omega n}\) for \((z - \alpha)^n\), hence one sees that
\[
(\eta^* \{ a, a^* \} - \{ \eta a^* \eta, a \} \eta + a^*(\eta \{ a, a^* \} - \{ \eta a^* \eta, a \})
\]

= \eta \tilde{\eta}

commutes with other operators.

Thus we can use the \( e^A e^B = e^{[A,B]} e^A e^B \) formula to evaluate the time-ordered product. We get

\[
e^{\beta H_0} U(\beta,0) = e^{\frac{\beta}{2} \int_{t_1}^{t_2} \tilde{\eta}(t) e^{-\beta t} dt} \eta(t_1) e^{-\beta t_2} \eta(t_2) - (\int_0^\beta \tilde{\eta}(t) e^{-\beta t} dt) a
\]

Now

\[
\langle e^{-a^* \tilde{\eta}} e^{-\eta a} \rangle = \langle (1 - a^* \tilde{\eta})(1 - \eta a) \rangle
\]

= \langle 1 + \tilde{\eta}a + a\tilde{\eta} + \eta a^* a \eta \rangle

= 1 + \tilde{\eta} \langle a^* a \rangle \eta

\[
\langle a^* a \rangle = \frac{e^{-\beta \omega}}{1 + e^{-\beta \omega}}
\]

So we get

\[
\frac{\text{tr} \ U(\beta,0)}{\text{tr} \ U_0(\beta,0)} = \langle e^{\beta H_0} U(\beta,0) \rangle = e^{\frac{\beta}{2} \int_{t_1}^{t_2} \tilde{\eta}(t) e^{-\beta t} dt} \langle (1 + \frac{\beta}{1 + e^{-\beta \omega}} \tilde{\eta} \rangle
\]

where \( \gamma' = \int_0^\beta \tilde{\eta}(t) e^{\omega t} dt \quad \gamma = \int_0^\beta \eta(t) e^{-\omega t} dt \)

Recall that \( \eta(t), \tilde{\eta}(t) \) are elements of degree 1 of some Grassman algebra, hence the same is true for \( \gamma' \) and \( \gamma \), so that we have \( \gamma'^2 = \gamma^2 = 0 \) and so I can write

\[
\langle e^{-a^* \tilde{\eta}} e^{-\eta a} \rangle = 1 + \tilde{\eta} \gamma' e^{-\beta \omega} \frac{1 + e^{-\beta \omega}}{1 + e^{-\beta \omega}} = e^{-\frac{\beta \omega}{1 + e^{-\beta \omega}}} \tilde{\eta} \gamma'
\]

It follows that we have
\[ \text{tr} \left( e^{-\beta \omega a^* a} e^{\beta \omega a a^*} \right) = \frac{1}{1-e^{-\beta \omega}} e^{\frac{\omega^2}{1-e^{-\beta \omega}}} \]

leading to the same formula for \( \langle e^{\beta \omega a^* a} \rangle \).

It follows from the above formulas that

\[
\frac{\text{tr} \; U(\beta, 0)}{\text{tr} \; U_0(\beta, 0)} = \langle e^{\beta H_0} U(\beta, 0) \rangle = e^0 \int_0^{\beta} J(t_2) G(t_1, t_2) \tilde{J}(t_2)
\]

where

\[
G(t_1, t_2) = e^{-\omega (t_1-t_2)} \Theta(t_1-t_2) + e^{-\omega (t_1-t_2)} \frac{e^{-\beta \omega}}{1-e^{-\beta \omega}}
\]

Notice that

\[
\left( \frac{d}{dt} + \omega \right) G(t_1, t_2) = \delta(t_1-t_2)
\]

and

\[
G(\beta, t_2) = e^{-\omega (\beta-t_2)} \left[ 1 + \frac{e^{-\beta \omega}}{1-e^{-\beta \omega}} \right]
\]

\[
G(0, t_2) = e^{\omega t_2} \left[ \frac{e^{-\beta \omega}}{1-e^{-\beta \omega}} \right]
\]

so \( G \) is the Green's function for \( (\frac{d}{dt} + \omega)^{-1} \) with periodic boundary conditions.

---

Next let's go over the fermion case. We introduce "sources" \( \eta, \tilde{\eta} \) which are elements in a Grassman algebra, and consider

\[ H = \omega a^* a + \eta a + a^* \tilde{\eta} \]

where \( \{a, a^*\} = 1 \), \( \{a, \eta\} = \{a^* a^*\} = 0 \). Consider

\[ e^{\beta H_0} U(\beta, 0) = T e^{-\int_0^\beta \eta(t) e^{-\omega t} a \; dt} e^{-\int_0^\beta \tilde{\eta}(t) e^{\omega t} a^* dt} \]

and then note that

\[ [\eta a, a^* \tilde{\eta}] = [\eta a, a^* \tilde{\eta}] + a^* [\eta a, \tilde{\eta}] \]
\[
\begin{align*}
\frac{\text{tr } U(\beta,0)}{\text{tr } U_0(\beta,0)} &= e^{t_1 t_2} \int \eta(t_1) e^{-\omega(t_1 t_2)} \tilde{\eta}(t_2) dt_2 - \int \eta(t_1) e^{-\omega(t_1 t_2)} \eta(t_2) \frac{e^{-\beta \omega}}{1 + e^{-\beta \omega}} dt_2 \\
&= e \int \eta(t_1) G(t_1 t_2) \tilde{\eta}(t_2)
\end{align*}
\]

where

\[
G(t_1, t_2) = e^{-\omega(t_1 t_2)} \left\{ \Theta(t_1, t_2) - \frac{e^{-\beta \omega}}{1 + e^{-\beta \omega}} \right\}
\]

Then

\[
G(\beta, t_2) = e^{-\omega(\beta t_2)} \left\{ 1 - \frac{e^{-\beta \omega}}{1 + e^{-\beta \omega}} \right\} = \frac{e^{-\beta \omega + \omega t_2}}{1 + e^{-\beta \omega}}
\]

\[
G(0, t_2) = e^{\omega t_2} \left\{ -\frac{e^{-\beta \omega}}{1 + e^{-\beta \omega}} \right\} = -G(\beta, t_2)
\]

so that \( G \) is anti-periodic:

\[
G = \left( \frac{d}{dt} + \omega \right)^{-1} \quad \text{an anti-periodic function.}
\]

We can check things a bit

\[
\begin{align*}
G(t_1, t_2) &= \langle T[ a(t_1) a^*(t_2) ] \rangle = e^{-\omega(t_1 t_2)} \int \langle a a^* \rangle_{t_1 < t_2} e^{-\beta \omega} \frac{e^{-\beta \omega}}{1 + e^{-\beta \omega}} dt_2 \\
&= e^{-\omega(t_1 t_2)} \left\{ \Theta(t_1, t_2) - \frac{e^{-\beta \omega}}{1 + e^{-\beta \omega}} \right\}
\end{align*}
\]

Notice also that

\[
e^{\eta(t_1)} G(t_1, t_2) \tilde{\eta}(t_2) = \overline{\int_{(t_1, t_2)}} e^{\eta(t_1)} G(t_1, t_2) \tilde{\eta}(t_2) dt_1 dt_2
\]

\[
= \overline{\int_{(t_1, t_2)}} (1 + \eta(t_1) G(t_1, t_2) \tilde{\eta}(t_2)) dt_1 dt_2
\]
Schwinger derivation of \( \pi \) on 522 goes as follows:

\[
\delta \text{tr}(U(\beta,0)) = -\int_0^\beta \text{tr} \left( U(\beta,t) (\eta(t)a + a^* \tilde{\eta}(t)) U(t,0) \right) dt
\]

or

\[
\delta \log \text{tr}(U(\beta,0)) = -\int_0^\beta (\eta(t) \langle a(t) \rangle + \langle a^*(t) \rangle \tilde{\eta}(t)) dt
\]

Again one shows

\[
\left( \frac{d}{dt} + \omega \right) \langle a(t) \rangle = -\tilde{\eta}(t)
\]

\[
\left( \frac{d}{dt} - \omega \right) \langle a^*(t) \rangle = \eta(t)
\]

so that

\[
\langle a(t) \rangle = -\left( \frac{d}{dt} + \omega \right)^{-1} \tilde{\eta}(t)
\]

But the mistake to avoid goes as follows:

\[
\langle a(t) \rangle = \text{tr}(U(\beta,t)a U(t,0)) / \text{tr}(U(\beta,0))
\]

This looks periodic in \( \pi \) because one would think

\[
\text{tr} (a U(\beta,0)) = \text{tr} (U(\beta,0) a)
\]

However the trace picks up only the even part of \( a U(\beta,0) \), and hence depends on the odd part of \( U(\beta,0) \), hence there is a minus sign. Anyway the conclusion is that \( \langle a(t) \rangle \) is anti-periodic, so we get the anti-periodic Green's function, and the rest of the argument proceeds without change.

Two questions: What is the analogue of the path integral in the fermion case? How do things look when we Fourier transform \( \eta, \tilde{\eta} \)?
January 6, 1980

Yesterday we considered $A[V, V = C e, a = -i(e^*)$
$a^* = e(e), H_0 = w a^* a$. We computed the generating
function for the Green's functions and found it to be

$$e \int \eta G \eta$$

To fix the ideas suppose we want the finite temperature
situation: The generating function is

$$\frac{\text{tr} \left( U(\beta, 0) \right)}{\text{tr} \left( e^{\beta H_0} \right)} = e \int \eta G \eta$$

where $G$ is the kernel for $(\frac{d}{dt} + w)^{-1}$ with anti-
periodic boundary conditions.

$$e \int \eta G \eta = \sum_n \frac{1}{n!} \int dt_1 dt_1' \ldots dt_n dt_n' \left( \eta(t_1) G(t_1, t_1') \eta(t_1') \right) \ldots (\ldots)$$

$$= \sum_n \frac{1}{n!} \int dt_1 \ldots dt_n \eta(t_1) \ldots \eta(t_n) \eta(t_1') \ldots \eta(t_n') \prod G(t_i, t_i')$$

$$= \sum_n \frac{1}{n! n!} \int \prod \frac{\sum \text{sgn}(\sigma)}{\lambda} G(t_{\sigma(i)}, t_{\sigma(i)'}$$

$$= \sum_n \frac{1}{n! n!} \int dt_1 \ldots dt_n \eta(t_1) \ldots \eta(t_n) \det G(t_i, t_i')$$

I guess that one can conclude from this that

$$\langle T \left[ a(t_1) \ldots a(t_n) a^*(t_1') \ldots a^*(t_n') \right] \rangle = \det G(t_i, t_i')$$

where $G(t_i, t_i') = \langle T \left[ a(t) a^*(t') \right] \rangle$. 
The Pfaffian: Let $A = [a_{ij}]$ be a skew-symmetric $n \times n$ matrix with $n$ even. In the exterior algebra with generators $\eta_1, \ldots, \eta_n$ we can form $\omega = \frac{1}{n!} \sum a_{ij} \eta_i \wedge \eta_j$. Then

$$\omega^{n/2} = \text{Pfaff}(A) \cdot \eta_1 \wedge \cdots \wedge \eta_n$$

Why should this be true? First of all, the above construction should be invariant under the action of $SO(n)$. Second, for the matrix

$$A = \begin{pmatrix}
0 & a_1 & \cdots & a_n \\
-\alpha_1 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
-\alpha_n & 0 & \cdots & 0
\end{pmatrix}
$$

it gives $\omega = \Sigma \lambda_j \eta_{j-1} \eta_j$, hence $\text{Pfaff}(A) = \lambda_1 \cdots \lambda_{n/2}$.

These characterize the Pfaffian.

Notice that the Pfaffian is given as a sum over the $(2n-1)!!$ ways of partitioning $1, 2, \ldots, n$ into pairs.

---

I want now to do the Fourier transform.

We have this generating function

$$Z(\eta, \tilde{\eta}) = e^{\int \eta \tilde{\eta}}$$

and we've seen that $G(t, t')$ is anti-periodic, in fact

$$G(t, t') = \sum_{k} \frac{e^{ik(t-t')}}{i k + \omega} e^{\frac{2\pi}{\beta} (\xi + \frac{1}{2})}$$

So we should regard $\eta, \tilde{\eta}$ as anti-periodic on $[0, \beta]$. Suppose
\[ \eta = \frac{1}{\beta} \sum \eta_k e^{ikt} \quad \tilde{\eta} = \frac{1}{\sqrt{\beta}} \sum \tilde{\eta}_k e^{ikt} \]

Then
\[ \mathcal{A} \tilde{\eta} = \frac{1}{\sqrt{\beta}} \sum \frac{1}{ik + \omega} \tilde{\eta}_k e^{ikt} \]

and
\[ \mathcal{S} \eta \mathcal{G} \tilde{\eta} = \sum_{k} \frac{\eta_{k} \tilde{\eta}_{k}}{ik + \omega}. \]

Here \( \eta_k, \tilde{\eta}_k \) are anti-commuting variables.

What are we after? I eventually want to be able to handle Green's functions for interacting fermions, and I want to get the perturbation expansion from a kind of path integral. The path integral should look like
\[ \int \text{something} \quad e^{- i S} \quad e^{- i \int a^{\dagger} \tilde{\eta} \, dt} \quad Da(t) \quad Da^{\dagger}(t) \]

And hence
\[ \int \text{something} \quad Da \quad Da^{\dagger} \quad a(t_1) \cdots a(t_m) \quad a^{\dagger}(t'_1) \cdots a^{\dagger}(t'_n) \]

should be a Green's function like the one obtained. The "something" brings in the Green's function.

Let's generalize slightly so that instead of having variables \( a, a^{\dagger} \) we have something that goes with the Pfaffian, namely, a set of anti-commuting variables \( \eta_1, \eta_2, \ldots, \eta_n \). These are like the variables \( \xi_i \). I need a quadratic form to go with them, i.e. take
\[ \omega = \frac{1}{2} \sum_{ij} a_{ij} \eta_i \eta_j \]

where \( [a_{ij}] \) is a skew-symmetric matrix. Now there has
to be somehow to make sense out of
\[ \int e^{\omega} \eta_1 \cdots \eta_f \]
so that Wick's theorem holds. The obvious thing to try is take cap product with \( \eta_1 \ldots \eta_n \). Then
\[ \int e^{\omega} = \int \frac{\omega^{n/2}}{(n/2)!} = \text{Pfaff}(A) \]
Recall that \( \text{Pf}(A) = (\det A)^{1/2} \) which looks good in comparison with
\[ \int e^{-\frac{1}{2}x^tAx} \, dx = \text{const} \, (\det A)^{-1/2} \]