February 23, 1979

Consider [redacted] a coupled harmonic oscillator with Hamiltonian

$$H = \frac{1}{2}\Big(\sum_i p_i^2 + \sum_{i,j} q_i V_{ij} q_j \Big)$$

The matrix V is supposed pos. definite, hence $V = \gamma^2$ where $\gamma > 0$. Then introduce annihilation and creation operators:

$$a_i = \frac{1}{\sqrt{2}}\Big(\sum_j \gamma_{ij} q_j + i p_i \Big)$$

$$a_i^* = \frac{1}{\sqrt{2}}\Big(\sum_j \gamma_{ij} q_j - i p_i \Big)$$

and you have (using [redacted] the representation $q_i = x_i$, $p_i = \frac{1}{i}\frac{\partial}{\partial x_i}$)

$$[a_i, a_j^*] = \frac{1}{2}\Big[(\gamma x)_i + \frac{\partial}{\partial x_i} \,,\, (\gamma x)_j - \frac{\partial}{\partial x_j} \Big]$$

$$= \frac{1}{2}\big(\gamma_{ji} + \gamma_{ij} \big) = \gamma_{ij}$$

not the usual comm. relations.

$$[a_i, a_j] = \frac{1}{2}\big(\gamma_{ji} - \gamma_{ij} \big) = 0$$

$$[a_i^*, a_j^*] = 0$$

(The formula for a_i comes from the fact that

$$\frac{\partial}{\partial x_i} e^{-\frac{1}{2} x\cdot\gamma x} = e^{-\frac{1}{2} x\cdot\gamma x}\Big(-\frac{1}{2}\Big)\Big(\sum \gamma_{ij} x_j \cdot 2\Big) = -(\gamma x)_i \, e^{-\frac{1}{2} x\cdot\gamma x}$$

so $\frac{\partial}{\partial x_i} + (\gamma x)_i$ kills the "vacuum" state.)

Also we have

$$a_i^* a_i = \frac{1}{2}\left(-\frac{\partial}{\partial x_i} + (\gamma x)_i\right)\left(\frac{\partial}{\partial x_i} + (\gamma x)_i\right)$$

$$= \frac{1}{2}\left(-\frac{\partial^2}{\partial x_i^2} + (\gamma x)_i^2 - \gamma_{ii}\right)$$

so

$$\sum a_i^* a_i = H - \frac{1}{2}\operatorname{tr}(\gamma)$$

Next we consider the field equation

$$\ddot{\phi} = -(-\Delta + m^2)\phi$$

which comes from the Hamiltonian

$$H(\phi) =$$

$$= \frac{1}{2}\int dx\,\{\dot{\phi}(x)^2 + |\nabla_x \phi(x)|^2 + m^2\phi(x)^2\}$$

$$= \frac{1}{2}\{p\cdot p + q\cdot(-\Delta + m^2)q\}(\phi)$$

where $p_x(\phi) = \dot{\phi}(x)$, $q_x(\phi) = \phi(x)$. We can think of this as a continuous system of coupled harmonic oscillators. To quantize we introduce the positive square root $\gamma = \sqrt{-\Delta + m^2}$ which is given by the kernel

$$\gamma(x, x') = \int \frac{d\xi}{2\pi}\, e^{-i\xi(x-x')}\sqrt{\xi^2 + m^2}\,.$$

Let q_x, p_x become now operators satisfying the Heisenberg relations

$$[q_x, p_{x'}] = i\,\delta(x - x')$$

We could define ann.+creation ops in analogy

with the formulas on p. 612 $\left(a_x = \frac{1}{\sqrt{2}}[(\gamma q)_x + i p_x] \right)$
but it seems better to put

$$(*) \qquad \begin{aligned} a_x &= \frac{1}{\sqrt{2}}\left(q_x + i(\Gamma p)_x \right) \\ a_x^* &= \frac{1}{\sqrt{2}}\left(q_x - i(\Gamma p)_x \right) \end{aligned}$$

where $(\Gamma p)_x = \int dx' \, \Gamma(x,x') p_{x'}$ and Γ is the operator inverse to γ:

$$\Gamma(x,x') = \int \frac{d\xi}{2\pi} \frac{e^{-i\xi(x-x')}}{\sqrt{\xi^2+m^2}}$$

It follows that

$$[a_x, a_{x'}^*] = [q_x, (\Gamma p)_{x'}] \frac{1}{i}$$

$$= \int dx'' \, \Gamma(x',x'') \underbrace{[q_x, p_{x''}] \frac{1}{i}}_{\delta(x,x'')} = \Gamma(x,x')$$

and

$$\boxed{\cancel{a_x^* a_x = \frac{1}{2}\left\{ q_x^2 + \underbrace{i\, q_x (\Gamma p)_x - i(\Gamma p)_x q_x}_{-\Gamma(x,x)} + (\Gamma p)_x^2 \right\}}}$$

$$(\gamma a)_x^* (\gamma a)_x = \frac{1}{2}\left((\gamma q)_x - i p_x \right)\left((\gamma q)_x + i p_x \right)$$

$$= \frac{1}{2}\left\{ p_x^2 + (\gamma q)_x^2 + \int dx' \, \gamma(x,x')[q_{x'}, i p_x] \right\}$$

$$= \frac{1}{2}\left\{ p_x^2 + (\gamma q)_x^2 - \gamma(x,x) \right\}$$

Thus $$\int dx \, (\gamma a)_x^* (\gamma a)_x = H - \underbrace{\frac{1}{2}\int dx \, \gamma(x,x)}_{\text{infinite constant}}$$

Finally solving (*) we get

$$q_x = \frac{1}{\sqrt{2}}\left(a_x + a_x^*\right)$$

or
$$\phi(x) = \phi^+(x) + \phi^-(x) \quad \text{where}$$

$$\phi^+(x) = \frac{1}{2}\left(q_x + i(\Gamma p)_x\right)$$ is the destruction operator at the point x, and $\phi^-(x) = \phi^+(x)^*$ is the creation operator. We have

$$[\phi^+(x), \phi^-(x')] = \frac{1}{2}[a_x, a_{x'}^*] = \frac{1}{2}\Gamma(x,x')$$
$$= \int \frac{d\xi}{2\pi} \frac{e^{-i\xi(x-x')}}{2\sqrt{\xi^2+m^2}}.$$

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$$H = \frac{1}{2}\{p\cdot p + q\cdot\gamma^2 q\} = \frac{1}{2}\{-\Delta + \sum_i (\gamma x)_i^2\}$$

$$\frac{\partial}{\partial x_i} e^{-\frac{1}{2}x\cdot\gamma x} = -(\gamma x)_i\, e^{-\frac{1}{2}x\cdot\gamma x}$$

hence $e^{-\frac{1}{2}x\cdot\gamma x}$ is killed by the operators

$$b_i = \frac{1}{\sqrt{2}}\left(\frac{\partial}{\partial x_i} + (\gamma x)_i\right) = \frac{1}{\sqrt{2}}\left((\gamma q)_i + ip_i\right)$$

which satisfy

$$[b_i, b_j] = [b_i^*, b_j^*] = 0 \quad \text{and}$$

$$[b_i, b_j^*] = [(\gamma q)_i, -ip_j] = \sum_{i'}[\gamma_{ii'}q_{i'}, -ip_j] = \gamma_{ij}$$

Let Γ be the inverse matrix of γ and put

$$a_i = (\Gamma b)_i = \frac{1}{\sqrt{2}}\left(q_i + i(\Gamma p)_i\right)$$

Then we have $[a_i, a_j] = [a_i^*, a_j^*] = 0$, and

$$[a_i, b_j^*] = [(\Gamma b)_i, b_j^*] = \sum_k \Gamma_{ik}\gamma_{kj} = \delta_{ij}.$$

Since

$$b_i^* b_i = \frac{1}{2}\left(\frac{-\partial}{\partial x_i} + (\gamma x)_i\right)\left(\frac{\partial}{\partial x_i} + (\gamma x)_i\right)$$

$$= \frac{1}{2}\left\{-\frac{\partial^2}{\partial x_i^2} + (\gamma x)_i^2 - \gamma_{ii}\right)$$

we have

$$H = \sum b_i^* b_i + \frac{1}{2}\mathrm{tr}(\gamma)$$

$$= \sum \gamma_{ij} b_i^* a_j + \frac{1}{2}\mathrm{tr}(\gamma)$$

If I think of the Hilbert space as having the basis (not orthonormal) $\qquad (b^*)^\alpha u_0 \quad , \qquad u_0 = e^{-\frac{1}{2}x.\gamma x} ,$

then [redacted] a_j acts like differentiating with respect to the "variable b_j^*". Thus the number of particles operator is

$$N = \sum b_i^* a_i$$

Next project will be to consider the simplest oscillator with a small perturbation:

$$H = \tfrac{1}{2}\{p^2 + q^2\} \text{ [redacted] } - \tfrac{\varepsilon}{2} q^2$$

where $\varepsilon = \varepsilon(t)$ will be of compact support and small. The problem is to compute the scattering matrix. What this means is the following. We have this very nice description of the Hilbert space $L^2(\mathbb{R})$ under the action of $H_0 = \frac{1}{2}(p^2+q^2)$ in terms of the basis $(a^*)^n u_0$

$$H_0\,(a^*)^n u_0 = \left(a^*a + \tfrac{1}{2}\right)(a^*)^n u_0$$
$$= \left(n + \tfrac{1}{2}\right)(a^*)^n u_0$$

This means that time evolution of a state vector is given by

$$e^{-itH_0}(a^*)^n u_0 = e^{-i(n+\frac{1}{2})t}(a^*)^n u_0$$

If we leave off the $\frac{1}{2}$ which is consistent with the viewpoint that states are lines in the Hilbert space then in degree n we multiply by $(e^{-it})^n$; standard $\mathbb{G}_m$ action. Under the [redacted] time evolution described by the full Hamiltonian H a vector looks like $e^{-int}(a^*)^n u_0$ for $t \ll 0$ and then it evolves into a linear combination of these gadgets for $t \gg 0$. The matrix describing this process is the S-matrix.

We work in the interaction picture. This means that to solve the Schroedinger equation

$$\frac{\partial \psi}{\partial t} = \frac{1}{i} H \psi = \frac{1}{i}(H_0 - V)\psi$$

we write it $\frac{\partial \psi}{\partial t} + iH_0\psi = iV\psi$ or

$$\frac{\partial}{\partial t}\left(e^{iH_0 t}\psi\right) = \left(e^{iH_0 t} iV e^{-iH_0 t}\right) e^{iH_0 t}\psi$$

or

$$\frac{\partial}{\partial t}\psi_D = H_I(t)\,\psi_D$$

where $\psi_D(t) = e^{iH_0 t}\psi(t)$ is the Dirac state vector and

$$H_I(t) = e^{iH_0 t}\, iV e^{-iH_0 t}$$

is the interaction Hamiltonian. In the situation at hand $e^{iH_0 t}$ is multiplication by e^{int} in degree n. Now

$$V = \frac{\varepsilon}{2} q^2 \qquad a = \frac{1}{\sqrt{2}}(q + ip)$$

$$a^* = \frac{1}{\sqrt{2}}(q - ip)$$

$$q = \frac{a + a^*}{\sqrt{2}}$$

$$V = \frac{\varepsilon}{4}(a + a^*)^2 = \frac{\varepsilon}{4}\left(a^2 + aa^* + a^*a + a^{*2}\right)$$

Now $\quad e^{iH_0 t} a e^{-iH_0 t} (a^*)^n u_0 = e^{iH_0 t} a (a^*)^n u_0 \, e^{-int}$

$$= e^{-it} a \cdot (a^*)^n u_0$$

so

$$e^{iH_0 t} a e^{-iH_0 t} = e^{-it} a$$

$$e^{iH_0 t} a^* e^{-iH_0 t} = e^{+it} a^*$$

so therefore the interaction Hamiltonian becomes

$$H_I(t) = \frac{i\,\varepsilon(t)}{4}\Big(e^{-2it} a^2 + \underbrace{aa^*}_{a^*a+1} + a^*a + e^{2it} a^{*2}\Big)$$

Now we know ψ_D satisfies the integral equation

$$\psi_D(t) = \psi_D(-\infty) + \int_{-\infty}^{t} dt_1 H_I(t_1)\psi_D(t_1)$$

which leads to the perturbation expansion for the S-matrix:

$$S = I + \int_{-\infty}^{\infty} dt_1\, H_I(t_1) + \int_{-\infty}^{\infty} dt_1 \int_{-\infty}^{t_1} dt_2\, H_I(t_1) H_I(t_2) + \cdots\cdots$$

$$= I + \int_{-\infty}^{\infty} dt_1\, H_I(t_1) + \frac{1}{2!}\int_{-\infty}^{\infty} dt_1 \int_{-\infty}^{\infty} dt_2\, P\{H_I(t_1) H_I(t_2)\} + \cdots\cdots$$

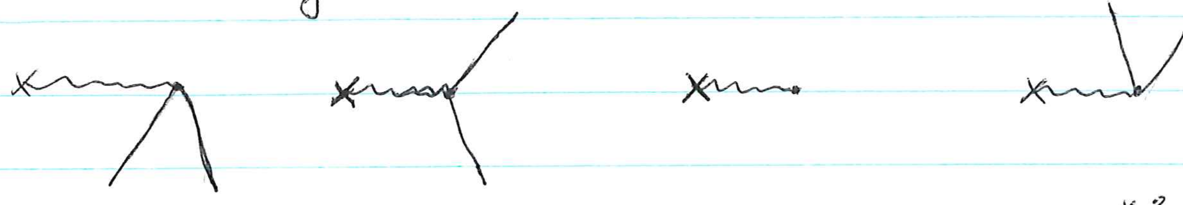
Let's analyze the first few terms in the S-matrix.

$$S^{(1)} = \int dt_1\, H_I(t_1) = \frac{i}{4}\Big[\hat{\varepsilon}(-2)\, a^2 + \hat{\varepsilon}(0)(2a^*a + 1) + \hat{\varepsilon}(2)\, a^{*2}\Big]$$

The vacuum-vacuum amplitude to first order is

$$1 + \langle 0|S^{(1)}|0\rangle = 1 + \frac{i}{4}\hat{\varepsilon}(0).$$

In $S^{(1)}$ one sees various processes. The a^2 term sees destruction of 2 particles, that is pair annihilation; the a^*a sees destruction followed by creation, that is, scattering. The four first order processes are described by the diagrams



a^2 $\qquad$ a^*a $\qquad$ 1 $\qquad$ a^{*2}

Next let's compute the second order contribution to the S-matrix. Put $\zeta_1 = e^{-2it_1}$, $\zeta_2 = e^{-2it_2}$

$$\left(\zeta_1 a^2 + 2a^*a + 1 + \bar\zeta_1 a^{*2}\right)\left(\zeta_2 a^2 + 2a^*a + 1 + \bar\zeta_2 a^{*2}\right)$$

$$= \zeta_1\zeta_2 a^4 + 2\zeta_1 \underbrace{a^2a^*a}_{(a^*a^2+2a)} + \zeta_1 a^2 + \zeta_1\bar\zeta_2 \underbrace{a^2a^{*2}}_{(a^{*2}a^2+4a^*a+2)}$$
$$+ 2\zeta_2 a^*a^3 + 4a^*\underbrace{aa^*}_{(a^*a+1)}a + 2a^*a + 2\bar\zeta_2 a^*\underbrace{aa^{*2}}_{(a^{*2}a+2a^*)}$$
$$+ \zeta_2 a^2 + 2a^*a + 1 + \bar\zeta_2 a^{*2}$$
$$+ \bar\zeta_1\zeta_2 a^{*2}a + 2\bar\zeta_1 a^{*3}a + \bar\zeta_1 a^{*2} + \bar\zeta_1\bar\zeta_2 a^{*4}$$

$$= \begin{cases} a^4(\zeta_1\zeta_2) + a^*a^3(2\zeta_1 + 2\zeta_2) + a^{*2}a^2(\zeta_1\bar\zeta_2 + 4 + \bar\zeta_1\zeta_2) \\ + a^{*3}a(2\bar\zeta_2 + 2\bar\zeta_1) + a^{*4}(\bar\zeta_1\bar\zeta_2) \end{cases}$$

$$+ \begin{cases} a^2(4\zeta_1 + \zeta_1 + \zeta_2) + a^*a(4\zeta_1\bar\zeta_2 + 4 + 2 + 2) \\ \qquad + a^{*2}(4\bar\zeta_2 + \bar\zeta_2 + \bar\zeta_1) \end{cases}$$

$$+ \Big\{ (2\zeta_1\bar\zeta_2 + 1)$$

so

$$\langle 0|S^{(2)}|0\rangle = \left(\frac{i}{4}\right)^2 \int_{-\infty}^{\infty} dt_1 \int_{-\infty}^{t_1} dt_2 \; \varepsilon(t_1)\varepsilon(t_2)\left\{2e^{-2it_1}e^{2it_2}+1\right\}$$

$$= \left(\frac{i}{4}\right)^2 \frac{1}{2!} \int_{-\infty}^{\infty} dt_1 \int_{-\infty}^{\infty} dt_2 \; \varepsilon(t_1)\varepsilon(t_2)\left\{2e^{-2i|t_1-t_2|}+1\right\}.$$

Now before analyzing this let's see if we can compute the third order correction to the vacuum-vacuum amplitude

$$(\zeta_1 a^2 + \cancel{2a^*a} + 1 + \cancel{\bar{\zeta}_1 a^{*2}})(\zeta_2 a^2 + 2a^*a + 1 + \bar{\zeta}_2 a^{*2})(\cancel{\zeta_3 a^2} + \cancel{2a^*a} + 1 + \bar{\zeta}_3 a^{*2})$$

$$\equiv (\cancel{\zeta_1 a^2}+1)(\zeta_2 a^2)(\cancel{1}+\bar{\zeta}_3 a^{*2}) \qquad \equiv \qquad \zeta_2\bar{\zeta}_3\, 2$$

$$+ (\zeta_1 a^2 + \cancel{1})(2a^*a)(\cancel{1}+\bar{\zeta}_3 a^{*2}) \qquad\qquad 8\zeta_1\bar{\zeta}_3$$

$$+ (\zeta_1 a^2 + 1)\;\; 1 \;\;(1+\bar{\zeta}_3 a^{*2}) \qquad\qquad +1+2\zeta_1\bar{\zeta}_3$$

$$+ (\zeta_1 a^2 \cancel{+1})(\bar{\zeta}_2 a^{*2})(1+\cancel{\bar{\zeta}_3 a^{*2}}) \qquad\qquad +2\zeta_1\bar{\zeta}_2$$

because $2a^2a^*a\,a^{*2} \equiv 2(a^*a^2+2a)(a^{*2}a+2a^*) \equiv 8aa^* \equiv 8.$

This looks messy:

$$\langle 0|S^{(3)}|0\rangle = \left(\frac{i}{4}\right)^3 \int_{-\infty}^{\infty} \varepsilon(t_1)\,dt_1 \int_{-\infty}^{t_1} \varepsilon(t_2)\,dt_2 \int_{-\infty}^{t_2} \varepsilon(t_3)\,dt_3 \left\{1+10e^{-2i(t_1-t_3)}+2e^{-2i(t_2-t_3)} + 2e^{-2i(t_1-t_2)}\right\}$$

One can simplify the above as follows. Recall that

$$H_I(t) = \frac{i\varepsilon(t)}{4}\left(e^{-2it}a^2 + 2a^*a + e^{2it}a^{*2}\right) + \frac{i\varepsilon(t)}{4}$$

Just as we dropped the $\frac{1}{2}$ from $H_0 = a^*a + \frac{1}{2}$, as the scalar $\frac{i\varepsilon(t)}{4}$ gives rise to the scalar factor $\exp\left\{\frac{i}{4}\int_{-\infty}^{\infty}\varepsilon(t)\,dt\right\}$

in the scattering matrix, we might as well simplify things by working with

$$H_I(t) = \frac{i\varepsilon(t)}{4}\left(e^{-2it}a^2 + 2a^*a + e^{2it}a^{*2}\right)$$

The last factor is $N\{(e^{-it}a + e^{it}a^*)^2\}$ where N [redacted] arranges any polynomial in a, a^* into normal product form.

Wick's theorem will tell how to compute the time-ordered product

$$P\{H_I(t_1)H_I(t_2)\}$$

in terms of normal products and contraction terms. For example

$$P(e^{-it_1}a \cdot e^{it_2}a^*) = \begin{cases} e^{-it_1+it_2}(a^*a+1) & \text{if } t_1 > t_2 \\ e^{-it_1+it_2}(a^*a) & \text{if } t_1 < t_2 \end{cases}$$

[redacted] hence

$$P(e^{-it_1}a \cdot e^{it_2}a^*) = N(e^{-it_1}a \cdot e^{it_2}a^*) + \begin{cases} e^{-i(t_1-t_2)} & t_1 > t_2 \\ 0 & t_1 < t_2 \end{cases}$$

To use Wick's thm. we need to calculate

$$P\left((\underbrace{e^{-it_1}}_{\lambda_1}a + e^{it_1}a^*)\cdot(\underbrace{e^{-it_2}}_{\lambda_2}a + e^{it_2}a^*)\right)$$

If $t_1 > t_2$ it is

$$\lambda_1\lambda_2 a^2 + \lambda_1\bar{\lambda}_2(a^*a+1) + \bar{\lambda}_1\lambda_2 a^*a + \bar{\lambda}_1\bar{\lambda}_2 a^{*2}$$

If $t_1 < t_2$ it is $(\lambda_2 a + \bar{\lambda}_2 a^*)(\lambda_1 a + \bar{\lambda}_1 a^*) =$

[redacted] $$\lambda_1\lambda_2 a^2 + \bar{\lambda}_2\lambda_1 a^*a + \bar{\lambda}_1\lambda_2(a^*a+1) + \bar{\lambda}_1\bar{\lambda}_2 a^{*2}$$

and so

$$P\{(e^{-it_1}a + e^{it_1}a^*)\cdot(e^{-it_2}a + e^{it_2}a^*)\} = N\{\quad\cdot\quad\}$$

$$+ \begin{cases} e^{-i(t_1-t_2)} & t_1 > t_2 \\ e^{it_1 - it_2} & t_1 < t_2. \end{cases}$$

The simplest way to write the contraction term is

$$k(t_1-t_2) = e^{-i|t_1-t_2|}.$$

For the moment put $\theta(t) = e^{-it}a + e^{it}a^*$. Then

left out const.

$$P\{H_I(t_1)\cdot H_I(t_2)\} = P\{N(\theta(t_1)^2)\cdot N(\theta(t_2)^2)\}$$

$$= N(\theta(t_1)^2\theta(t_2)^2) + 4k(t_1-t_2)\,N(\theta(t_1)\theta(t_2))$$

$$+ 2k(t_1-t_2)^2$$

should be true by Wick's theorem. Similarly

$$P\{N(\theta_{t_1}^2)N(\theta_{t_2}^2)N(\theta_{t_3}^2)\} = N(\theta_{t_1}^2\theta_{t_2}^2\theta_{t_3}^2)$$

$$+ 4k(t_1-t_2)N(\theta_{t_1}\theta_{t_2}\theta_{t_3}^2) + 4k(t_1-t_3)N(\theta_{t_1}\theta_{t_2}^2\theta_{t_3}) + 4k(t_2-t_3)N(\theta_{t_1}^2\theta_{t_2}\theta_{t_3})$$

$$+ \Bigg\{$$

$$+ 4k(t_1-t_2)k(t_2-t_3)k(t_3-t_2) + 4k(t_1-t_3)k(t_3-t_2)k(t_3-t_1)$$

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$$H_0 = \frac{1}{2}(p^2+q^2) = a^*a + \frac{1}{2} \qquad \text{where} \qquad a = \frac{1}{\sqrt{2}}(q+ip)$$

$$[a, a^*] = 1. \qquad\qquad a^* = \frac{1}{\sqrt{2}}(q-ip)$$

The ground state u_0 or $|0\rangle$ is characterized by

$$a|0\rangle = 0 \qquad\qquad \| |0\rangle \| = 1.$$

The other eigenvectors for H_0 are

$$|n\rangle = \frac{1}{\sqrt{n!}} a^{*n}|0\rangle$$

and time evolution is given by

$$e^{-iH_0 t}|n\rangle = e^{-i(n+\frac{1}{2})t}|n\rangle$$

Now I consider a perturbation

$$H = \frac{1}{2}(p^2+q^2) - \frac{\varepsilon(t)}{2}q^2$$

where $\varepsilon \in C_0^\infty(\mathbb{R})$. Since $q = \frac{1}{\sqrt{2}}(a+a^*)$,

$$q^2 = \frac{1}{2}(a+a^*)^2 = \frac{1}{2}(a^2 + \underbrace{aa^*}_{a^*a+1} + a^*a + a^{*2})$$

$$= \frac{1}{2}\left(a^2 + 2a^*a + a^{*2} + 1\right)$$

one has

$$H = a^*a + \frac{1}{2} - \frac{\varepsilon(t)}{4} - \frac{\varepsilon}{4}\underbrace{(a^2 + 2a^*a + a^{*2})}_{N((a+a^*)^2)}$$

Schroedinger's equation is

$$i\frac{\partial\psi}{\partial t} = H\psi.$$

Because ε has compact support, ψ [redacted] links solutions of the free Schroedinger equation for $t \ll 0$ and $t \gg 0$, and hence gives a scattering matrix.

Another point is that adding a real scalar like ε to the Hamiltonian produces a multiplication factor for the time evolution. For example the solution of

$$i \frac{\partial \psi}{\partial t} = \left(H_0 - \varepsilon(t)/4\right) \psi$$

is

$$\psi(t) = e^{-iH_0 t} e^{i \int_0^t \varepsilon(t')\,dt'/4} \psi(0)$$

So I will simplify [redacted] by taking out the scalar [redacted] $\frac{1}{2} - \frac{\varepsilon(t)}{4}$ and considering the Hamiltonian

$$H = \underbrace{a^* a}_{H_0} - \underbrace{\frac{\varepsilon}{4} N((a+a^*)^2)}_{V}$$

We know this leads to

$$S = I + [redacted]\; i \int_{-\infty}^{\infty} dt_1\, H_I(t_1) + \frac{(i)^2}{2!} \int_{-\infty}^{\infty} dt_1 \int_{-\infty}^{\infty} dt_2\, P\{H_I(t_1) H_I(t_2)\} + \cdots$$

where

$$H_I(t) = e^{iH_0 t} V e^{-iH_0 t}$$

$$= \frac{\varepsilon(t)}{4} N\left(\left(e^{+it} a + e^{-it} a^*\right)^2\right)$$

Here we use

$$e^{iH_0 t} a e^{-iH_0 t} |n\rangle = e^{iH_0 t} a e^{-int} |n\rangle$$

$$= e^{-int} e^{+i(n-1)t} a|n\rangle$$

should be e^{-it}

$$= e^{it} a |n\rangle$$

Let's put [redacted] $q_t = \frac{1}{\sqrt{2}}\left(e^{it}a + e^{-it}a^*\right)$

$$= e^{iH_0 t}\, q\, e^{-iH_0 t}$$

Then
$$H_I(t) = \frac{\varepsilon(t)}{2} N\left(q_t^2\right)$$

Now we are going to want to use Wick's theorem to calculate the time-ordered products. If $t_1 > t_2$

$$P\{q_{t_1} q_{t_2}\} = \frac{1}{2}\left(e^{it_1}a + e^{-it_1}a^*\right)\left(e^{it_2}a + e^{-it_2}a^*\right)$$
$$= N(q_{t_1} q_{t_2}) + \frac{1}{2}e^{i(t_1 - t_2)}$$

But if $t_1 < t_2$, then the wrong order factor is $e^{i(t_2 - t_1)} a a^*$, so the general formula is

$$\boxed{P\{q_{t_1} q_{t_2}\} = N\{q_{t_1} q_{t_2}\} + \underbrace{\frac{1}{2}e^{i|t_1 - t_2|}}_{k(t_1 - t_2)}}$$

[redacted] Wick's thm. gives

$$P\{N(q_{t_1}^2)\, N(q_{t_2}^2)\} = N(q_{t_1}^2 q_{t_2}^2) + 4\, k(t_1 - t_2)\, N(q_{t_1} q_{t_2})$$
$$+ 2\, k(t_1 - t_2)^2$$

Check carefully the vacuum-vacuum term.

$t_1 > t_2$

$$\frac{1}{4}\left(a_1^2 + 2a_1^* a_1 + a_1^{*2}\right)\left(a_2^2 + 2a_2^* a_2 + a_2^{*2}\right) \equiv \frac{1}{4} a_1^2 a_2^{*2} = e^{2i(t_1 - t_2)} \frac{a^2 a^{*2}}{4}$$
$$\equiv \frac{e^{2i(t_1 - t_2)}}{4}\, 2$$

seems OK.

third order. If $t_1 > t_2 > t_3$, and we want vacuum term

$$\frac{1}{8}\left(a_1^2 + \cancel{2a_1^* a_1} + \cancel{a_1^{*2}}\right)\left(a_2^2 + 2a_2^* a_2 + a_2^{*2}\right)\left(\cancel{a_3^2} + \cancel{2a_3^* a_3} + a_3^{*2}\right)$$

$$\equiv \frac{1}{8}\left(2\, a_1^2 a_2^* a_2\, a_3^{*2}\right) = \frac{1}{4} e^{2i(t_1 - t_3)} \underbrace{\left(a^2 a^* a\, a^{*2}\right)}_{a^2 a^* a^* 2 \;=\; 4}$$

$$\equiv e^{2i(t_1 - t_3)}$$

But $\quad P\{N(q_{t_1}^2)\, N(q_{t_2}^2)\, N(q_{t_3}^2)\}$

$$\equiv 4 \quad k(t_1, t_2)\, k(t_2, t_3)\, k(t_3, t_1)$$
$$+\, 4\, k(t_1, t_3)\, k(t_2, t_3)\, k(t_1, t_2)$$
$$= e^{i\{|t_1 - t_2| + |t_2 - t_3| + |t_1 - t_3|\}}$$
$$= e^{2i(t_{max} - t_{min})}$$

It appears that it perhaps wasn't a good idea to remove the $\varepsilon(t)$ scalar from the Hamiltonian. What would happen if one were to go back to

$$H_I(t) = \frac{\varepsilon(t)}{2} \cdot q_t^2$$

where $\quad q_t^2 = P\{q_t\, q_t\} = N\{q_t^2\} + \overset{k(t,t)}{\frac{1}{2}} \qquad ?$

We should compute the vacuum-vacuum amplitudes carefully.

$$\langle 0 | S^{(1)} | 0 \rangle = i \int_{-\infty}^{\infty} dt_1\, \frac{\varepsilon(t_1)}{2} \langle 0 | N(q_{t_1}^2) + \tfrac{1}{2} | 0 \rangle = \frac{i}{4} \int_{-\infty}^{\infty} dt_1\, \varepsilon(t_1)$$

Using Witt's thm. $\quad \langle 0 | P\{q_{t_1}^2 q_{t_2}^2\} | 0 \rangle =$

 $\frac{1}{4}$

 $+ \; k(t_1,t_2)^2$

 $+ \; k(t_1,t_2)^2$

$$\left.\begin{matrix} \frac{1}{4} \\ + k(t_1,t_2)^2 \\ + k(t_1,t_2)^2 \end{matrix}\right\} = \frac{1}{4}\left(1+2e^{2i|t_1-t_2|}\right)$$

So $\langle 0|S^{(2)}|0\rangle = \frac{1}{2!}\left(\frac{i}{4}\right)^2 \int dt_1 \int dt_2 \;\; \varepsilon(t_1)\varepsilon(t_2)\left(1+2e^{2i|t_1-t_2|}\right)$

Next $\langle 0|P\{g_{t_1}^2 g_{t_2}^2 g_{t_3}^2\}|0\rangle =$ sum of terms which we can classify according to $\sigma \in \Sigma_3$

$$= k(t_1,t_1)\,k(t_2,t_2)\,k(t_3,t_3)$$

$$+ \; k(t_1,t_1)\; 2k(t_2,t_3)^2 \; + k(t_2,t_2)\; 2k(t_1,t_3)^2 + \; k(t_3,t_3)2k(t_1,t_2)^2$$

$$+ \;\; 4\,k(t_1,t_2)\,k(t_2,t_3)\,k(t_1,t_3) + 4\,k(t_1,t_3)\,k(t_2,t_3)\,k(t_1,t_2).$$

$$= \frac{1}{8} + \frac{e^{2i|t_2-t_3|} + e^{2i|t_1-t_3|^2} + e^{2i|t_1-t_2|^2}}{4} + e^{2i(t_{max}-t_{min})}$$

$$\langle 0|S^{(3)}|0\rangle = \frac{1}{3!}\left(\frac{i}{4}\right)^3 \int dt_1 \int dt_2 \int dt_3 \;\; \varepsilon(t_1)\varepsilon(t_2)\varepsilon(t_3)\Big\{ 1 + 2e^{2i|t_1-t_2|} + 2e^{2i|t_2-t_3|} + 2e^{2i|t_1-t_3|} + 8e^{2i(t_{max}-t_{min})}\Big\}$$

Now what I want to do is to organize $\langle 0|S|0\rangle$ into $1/\det(1-K)$. Recall

$$\frac{1}{\det(1-K)} = e^{\sum_{m=1}^{\infty}\frac{1}{m}tr(K^m)} = \left(1+\tau_1+\frac{\tau_1^2}{2}+\frac{\tau_1^3}{3!}\right)\left(1+\frac{\tau_2}{2}\right)\left(1+\frac{\tau_3}{3}\right)$$

$$= 1 + \tau_1 + \left(\frac{\tau_1^2}{2}+\frac{\tau_2}{2}\right) + \left(\frac{\tau_1^3}{3!}+\frac{\tau_1\tau_2}{2}+\frac{\tau_3}{3}\right) + \cdots$$

Try $\quad \tau_1 = \int dt \, \frac{i\varepsilon(t)}{4}$

$$\tau_2 = \int dt_1 \int dt_2 \, \frac{i\varepsilon}{4}(t_1) \frac{i\varepsilon}{4}(t_2) \; 2e^{2i|t_1 - t_2|}$$

$$\tau_3 = \int dt_1 \int dt_2 \int dt_3 \frac{i\varepsilon}{4}(t_1) \frac{i\varepsilon}{4}(t_2) \frac{i\varepsilon}{4}(t_3) \; 4e^{2i|t_{max} - t_{min}|}$$

This gives the right formula for $\langle 0|S|0\rangle$. But if we try to interpret τ_k as $\mathrm{tr}(K^k)$ we seem to be off by a factor of 2. Thus suppose we put

$$K(t_1, t_2) = \frac{i\varepsilon(t_2)\, e^{i|t_1 - t_2|}}{2}$$

we get

$$\mathrm{tr}\, K = \frac{i}{2}\int \varepsilon \, dt = 2\tau_1$$

$$\mathrm{tr}\, K^2 = \int dt_1 \int dt_2 \, \frac{i\varepsilon(t_2)}{2} e^{i|t_1 - t_2|} \frac{i\varepsilon(t_1)}{2} e^{i|t_2 - t_1|} = 2\tau_2$$

$$\mathrm{tr}\, K^3 = 2\tau_3$$

Consequently it appears that

$$(*) \qquad \langle 0|S|0\rangle = \left(\frac{1}{\det(1-K)}\right)^{1/2}$$

~~Here's a possible explanation for this~~

Due to the error on p. 625 all exponentials $e^{i|t_1 - t_2|}$ should be $e^{-i|t_1 - t_2|}$.

Here's a possible way of seeing $(*)$ formally. Look at the new contributions to $\langle 0|P\{q_{t_1}^2 \cdots q_{t_n}^2\}|0\rangle$.

According to Wick's thm. we have to [redacted] make n contractions. The first q_{t_1} must be contracted against another q_{t_i}, then [redacted] the other q_{t_i} contracts somewhere else; this gives one a [redacted] cycle ending with the second q_{t_1}. Suppose the cycle is an n-cycle; then there are 2^{n-1} contractions belonging to this cycle, and there are $(n-1)!$ different n-cycles. It seems [redacted] each contraction gives the factor

$$k(t,t') = \frac{1}{2}e^{-i|t-t'|}.$$

Hence the contribution to $\langle 0|S^{(n)}|0\rangle$ coming from the n-cycles is

$$\frac{1}{n!}(n-1)!\,2^{n-1}\,\mathrm{tr}\left\{\left(\frac{i\varepsilon(t)}{2}\frac{1}{2}e^{-i|t-t'|}\right)^n\right\}$$

$$= \frac{1}{2}\,\frac{1}{n}\,\mathrm{tr}\left\{\left(\frac{i\varepsilon(t)}{2}e^{-i|t-t'|}\right)^n\right\}.$$

This expression contains the mysterious $\frac{1}{2}$.

Next point is to look at the classical DE described by the Hamiltonian

$$H = \frac{1}{2}(p^2+q^2) - \frac{\varepsilon}{2}(q^2)$$

$$\dot{q} = \frac{\partial H}{\partial p} = p \qquad \dot{p} = -\frac{\partial}{\partial q} = -(1-\varepsilon)q$$

$$\therefore \quad \ddot{q} = (-1+\varepsilon(t))q \qquad \text{or} \qquad \ddot{q}+q = \varepsilon(t)q$$

Among the Green's functions for $\left(\frac{d^2}{dt^2}+1\right)$ [redacted] are

$$\frac{e^{i|t|}}{2i} \quad \text{and} \quad \frac{e^{-i|t|}}{-2i} = \frac{i}{2}e^{-i|t|}.$$

The latter one has positive frequencies for pos. time and negative frequency for negative time. So if we use it on

$$\ddot{q} + q = \varepsilon(t)\, q$$

we get the integral equation

$$q(t) = q_0(t) + \int_{-\infty}^{\infty} \frac{i}{2} e^{-i|t-t'|}\, \varepsilon(t')\, q(t')\, dt'$$

whose determinant $\det(1-K)$ has the kernel

$$K = \frac{i}{2} e^{-i|t-t'|}\, \varepsilon(t')$$

encountered above.

So our next project maybe should be to understand Schwinger's transformation functions well-enough to check the result: $\det(1-K)^{-1/2}$ by a variational calculation.

February 27, 1979

Toward understanding Schwinger's transformation functions. Let's think of a quantum-mechanical system as follows. For each time t one is given a Hilbert space $\mathcal{H}_t$ whose lines are states of the system at time t. Thus one gets a Hilbert space vector bundle $\mathcal{H}$ over the time axis. Moreover in each fibre one is given <u>position</u> and <u>momentum</u> <u>operators</u> satisfying the usual commutation relations. Thus we have

$$q, p \in \Gamma(\mathrm{End}(\mathcal{H}))$$

satisfying $\quad q = q^*, \quad p = p^*, \quad \{q, p\} = i$. Finally time evolution of the system provides compatible unitary transformations (Schwinger's transf. functions)

$$W(t_2, t_1) : \mathcal{H}_{t_1} \xrightarrow{\sim} \mathcal{H}_{t_2}$$

The nice way to express these is as parallel transport with respect to a <u>connection</u> on the bundle $\mathcal{H}$ over $\mathbb{R}$. This amounts to a differentiation operator D on sections of $\mathcal{H}$ over $\mathbb{R}$. The connection ~~and~~ position and momentum operators are connected by the Heisenberg formulas:

$$Dq = [-iH, q]$$
$$Dp = [-iH, p]$$

where D is defined on $\Gamma(\mathrm{End}\,\mathcal{H})$ by $Dq = Dq - qD$.

Here $H \in \Gamma(\mathrm{End}(\mathcal{H}))$ is the Hamiltonian operator. It is more or less equivalent to D. Finally to complete the Heisenberg picture, one adds that the state vector $\psi(t)$ is a section of $\mathcal{H}$ which is horizontal:

$$D\psi = 0.$$

The actual Heisenberg description results by trivializing $\mathcal{H}$ using D, whereas the Schroedinger description trivializes $\mathcal{H}$ using the Stone-von-Neumann thm.

Actually perhaps the point is that the bundle $\mathcal{H}$ has two trivializations. Thus at each time t we can compare the two connections, which gives us the operator H_t on $\mathcal{H}_t$.

March 2, 1979

Recall we were studying the Hamiltonian

$$H = a^*a - \tfrac{1}{2}\varepsilon(t) q^2$$

where $\varepsilon(t)$ has compact support. Here we are using the Schroedinger picture where $q = x$, $p = \frac{1}{i}\frac{\partial}{\partial x}$ on $L^2(\mathbb{R})$. Let $U(t,t')$ denote the time-evolution operator between the times t' and t. It is determined by

$$\begin{cases} \dfrac{\partial}{\partial t} U(t,t') = \dfrac{1}{i} H(t) U(t,t') \\[2ex] U(t',t') = I \end{cases}$$

One verifies transitivity:

$$U(t,t') U(t',t'') = U(t,t'')$$

because both satisfy the DE: $\dfrac{\partial}{\partial t}\Phi = iH(t)\Phi$ and because they coincide at $t=t'$. Hence

$$U(t,t') = U(t',t)^{-1}$$

$$\frac{\partial}{\partial t'} U(t,t') = -U(t',t)^{-1}\left\{\frac{1}{i}H(t')\,U(t',t)\right\} U(t',t)^{-1}$$

$$= -U(t,t')\,\frac{1}{i}H(t')$$

Now let H undergo an infinitesimal variation δH, e.g. vary $\varepsilon(t)$ infinitesimally: $\delta\varepsilon(t)$. Then

$$\frac{\partial}{\partial t}\,\delta U(t,t') = \frac{1}{i}H(t)\,\delta U(t,t') + \frac{1}{i}\,\delta H(t)\,U(t,t')$$

$$\delta U(t',t') = 0.$$

The forward Green's function for $\frac{\partial}{\partial t} + iH(t)$ is

$$G(t,t') = \begin{cases} U(t,t') & t > t' \\ 0 & t < t' \end{cases}$$

so

$$\delta U(t,t') = \int^{t} U(t,t_1) \tfrac{1}{i} \delta H(t_1)\, U(t_1,t')\, dt_1 \;+\; \underset{\text{in } \operatorname{Ker} \frac{\partial}{\partial t} + iH}{\text{something}}$$

To get the boundary condition $\delta U(t',t') = 0$ take the lower limit to be t', whence

$$\boxed{\delta U(t,t') = \int_{t'}^{t} U(t,t_1) \tfrac{1}{i} \delta H(t_1)\, U(t_1,t')\, dt_1}$$

Now we are interested in the scattering matrix which results from comparing H to $H_0 = a^* a$:

$$S = e^{iH_0 t_2}\, U(t_2,t_1)\, e^{-iH_0 t_1}$$

where $t_1 < \operatorname{Supp} \varepsilon < t_2$. Since $H_0 |0\rangle = 0$, that is, the ground state for H_0 has energy 0 we have

$$\langle 0|S|0\rangle = \langle 0|\, U(t_2,t_1)\, |0\rangle$$

hence

$$\delta \langle 0|S|0\rangle = \langle 0|\, \delta U(t_2,t_1)\, |0\rangle$$

$$= \int_{t_1}^{t_2} \langle 0|\, U(t_2,t) \tfrac{1}{i} \delta H(t)\, U(t,t_1)\, |0\rangle\, dt$$

Now the idea is to express $\delta\langle 0|S|0\rangle$ as a trace using that

$$\frac{1}{i}\delta H(t) = i\,\delta\varepsilon(t)\,\frac{q^2}{2}$$

Set

$$\varphi(t) = \langle 0|\, U(t_2,t)\,\frac{q^2}{2}\,U(t,t_1)\,|0\rangle$$

Then

$$\frac{d}{dt}\varphi(t) = \langle 0|\,U(t_2,t)\underbrace{\left(iH(t)\frac{q^2}{2} - \frac{q^2}{2}iH(t)\right)}_{}U(t,t_1)\,|0\rangle$$

$$i\left[\frac{p^2}{2},\frac{q^2}{2}\right] = \frac{i}{2}\left(\left[\frac{p^2}{2},q\right]q + q\left[\frac{p^2}{2},q\right]\right)$$

$[p,q] = \frac{1}{i}$

$$= \frac{i}{4}\left\{\left([p,q]p + p[p,q]\right)q + q\left([p,q]p + p[p,q]\right)\right\}$$

$$= \frac{1}{2}\,[redacted]\,(pq+qp)$$

So

$$\frac{d}{dt}\varphi = \langle 0|\,U(t_2,t)\,\frac{1}{2}(pq+qp)\,U(t,t_1)\,|0\rangle$$

Similarly

$$\frac{d^2}{dt^2}\varphi = \langle 0|\,U(t_2,t)\left[iH, \frac{1}{2}(pq+qp)\right]U(t,t_1)\,|0\rangle$$

$$[iH,q] = i\left[\frac{p^2}{2},q\right] = p$$

Now

$$[iH,p] = i\frac{(1-\varepsilon)}{2}[q^2,p] = -(1-\varepsilon)q = (-1+\varepsilon)q$$

So

$$\left[iH,\frac{1}{2}(pq+qp)\right] = \frac{1}{2}\left\{(-1+\varepsilon)qq + p^2 + p^2 + q(-1+\varepsilon)q\right\}$$

$$= [redacted]\quad p^2 + (-1+\varepsilon)q^2$$

$\therefore$

$$\frac{d^2}{dt^2}\varphi = \langle 0|\,U(t_2,t)\left\{p^2 + (-1+\varepsilon)q^2\right\}U(t,t_1)\,|0\rangle$$

To get rid of p^2, we use

$$\frac{d}{dt}\langle 0|u(t_2,t)\,H\,u(t,t_1)|0\rangle = \cancel{\langle 0|u(t_2,t)[iH,H]\,u(t,t_1)|0\rangle}$$
$$+\langle 0|u(t_2,t)\,\frac{\partial H}{\partial t}\,u(t,t_1)|0\rangle$$
$$= -\varepsilon'(t)\langle 0|u(t_2,t)\,\frac{g^2}{2}\,u(t,t_1)|0\rangle$$
$$= -\varepsilon'(t)\,\varphi(t) \qquad ?$$

This turns out to be too complicated. Also it seems pointless to look for the DE for $G(t,t)$. You want $G(t,t')$, so you are still missing an idea.

March 3, 1979

I would like to understand Schwinger's variational principle. He somehow starts with a Lagrangian and ends with 1) field equations 2) infinitesimal generators for changes in $\psi, \bar{\psi}$ on boundary surfaces.

Consider some classical examples:

1) $L = L(q, \dot{q}, t)$ and

$$\text{Action} = A(t \mapsto q(t)) = \int_{t_1}^{t_2} L(q, \tfrac{dq}{dt}, t)\, dt. \quad \text{Then}$$

$$\delta A = \int_{t_1}^{t_2} \frac{\partial L}{\partial q}\delta q + \frac{\partial L}{\partial \dot{q}}\left(\delta \text{[redacted]} \frac{dq}{dt}\right) dt$$

$$= \left[\frac{\partial L}{\partial \dot{q}}\delta q\right]_{t_1}^{t_2} + \int_{t_1}^{t_2}\left[\frac{\partial L}{\partial q} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right)\right] \text{[redacted]}\, \delta q\, dt$$

Then the equations of motion are

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q} = 0$$

and the infinitesimal generator for changes in q [redacted] is

$$\frac{\partial L}{\partial \dot{q}}\delta q = p\,\delta q$$

2) KG: $\mathcal{L}(\phi) = \frac{1}{2}(\phi_t^2 \text{[redacted]} - \phi_x^2 - m^2\phi^2)$

$$A = \int_{t_1}^{t_2} dt \int dx\, \mathcal{L}(\phi)$$

$$\delta A = \int_{t_1}^{t_2} dt \int dx \left[\text{[redacted]}\, \phi_t\,\delta\phi_t \underbrace{- \phi_x\,\delta\phi_x}_{+\phi_{xx}\delta\phi} - m^2\phi\,\delta\phi \right]$$

$$= \int dx \left[\phi_t \,\delta\phi\right]_{t_1}^{t_2} - \int_{t_1}^{t_2} dt \int dx \left[\phi_{tt} - \phi_{xx} + m^2\phi\right]\delta\phi$$

This leads to the equation of motion:

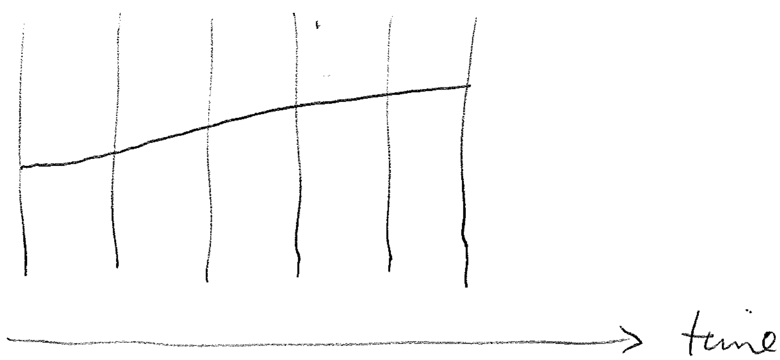
$$\phi_{tt} - \phi_{xx} + m^2\phi = 0$$

and to the infinitesimal generator for changes in ϕ:

$$G(\phi) = \int dx \ \phi_t \,\delta\phi$$

Somehow these inf. generators express the classical symplectic structure, because the commutation relations get <u>derived</u> from them.

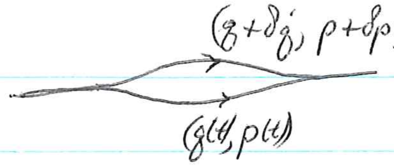
Back to classical mechanics. Suppose we are given a Hamiltonian $H(q,p,t)$. Let us view Hamilton's equations as giving us a connection on (q,p,t) space over t-space.



Hamilton's principle says that the integral curves of this connection are those along which the 1-form

$$\omega = p\,dq - H\,dt$$

when integrated over the curve is stationary. So if we have a small variation



then by Stokes' theorem $d\omega$ integrated over the spanning surface must vanish. This means that $d\omega$ vanishes ~~spanned by the tangent vector to the~~ on 2-planes containing a tangent vector to the curve $(q,p)(t)$. Thus the vector field X over $\frac{d}{dt}$ which is the time flow satisfies

$$i(X)\,d\omega = 0$$

Note that $d\omega = dp\,dq - \frac{\partial H}{\partial q}\,dq\,dt - \frac{\partial H}{\partial p}\,dp\,dt$ is of maximum rank 2 in the 3-manifold of (q,p,t)-space, hence its kernel as a skew-symmetric form is a line. In general $d\omega$ will have rank $2n$ in a $(2n+1)$-diml. space and its kernel will be a line in each tangent space, and this line is the connection.

I think this is an example of a <u>contact structure</u> on a $(2n+1)$-manifold, namely, a 1-form ω such that $d\omega$ has the maximal rank $2n$.

Return to

$$H = a^* a - \varepsilon(t)\frac{q^2}{2}$$

Recall that if H undergoes an infinitesimal change $\delta H = -\delta\varepsilon\, q^2/2$, then $U(t,t')$ undergoes the change

$$\delta U(t,t') = \int_{t'}^{t} dt_1\, U(t,t_1)\frac{1}{i}\delta H(t_1)\, U(t_1,t')$$

We are interested in the S-operator:

$$S = e^{iH_0 t_2}\, U(t_2,t_1)\, e^{-iH_0 t_1}$$

where $H_0 = a^* a$. Since $H_0|0\rangle = 0$ we have

$$\langle 0|S|0\rangle = \langle 0|U(t_2,t_1)|0\rangle$$

$$\delta\langle 0|S|0\rangle = \langle 0|\delta U(t_2,t_1)|0\rangle$$

$$= \int_{t_1}^{t_2} \langle 0|U(t_2,t)\underbrace{\tfrac{1}{i}\delta H(t)}_{i\delta\varepsilon(t)\frac{q^2}{2}}U(t,t_1)|0\rangle\, dt$$

where $t_1 < \mathrm{Supp}\ \varepsilon, \delta\varepsilon < t_2$.

Now the argument I missed yesterday goes as follows: Put

$$g(t,t') = \langle 0|U(t_2,t)\, q\, U(t,t')\, q\, U(t',t_1)|0\rangle$$

and recall that

$$[iH(t), q] = i[p^2/2, q] = \frac{i}{2}\big([p,q]p + p[p,q]\big) = p$$

$$[iH(t), p] = i(1-\varepsilon)[q^2/2, p] = i(1-\varepsilon)q[q,p] = (-1+\varepsilon)q$$

$$\frac{d}{dt}g(t,t') = \langle 0|U(t_2,t)\underbrace{[iH(t),q]}_{p}U(t,t')\, q\, U(t',t_1)|0\rangle$$

$$\frac{d^2}{dt^2}g(t,t') = (-1+\varepsilon(t))\, g(t,t')$$

similarly
$$\frac{d^2}{dt'^2} g(t,t') = (-1+\varepsilon(t'))\, g(t,t')$$

Now introduce
$$G(t,t') = c \begin{cases} g(t,t') & \text{if } t>t' \\ g(t',t) & \text{if } t<t' \end{cases}$$

where [redacted] c is a constant to be determined later. Then

$$\frac{d}{dt} G(t,t') \Big]_{t'-}^{t'+} = c \langle 0 | U(t_2,t') \, p \, q \, U(t',t_1) | 0 \rangle$$
$$- c \langle 0 | U(t_2,t') \, q \, p \, U(t',t_1) | 0 \rangle$$
$$= \frac{c}{i} \langle 0 | U(t_2,t_1) | 0 \rangle$$

so that if
$$c = \frac{i}{\langle 0 | S | 0 \rangle}$$
then $G(t,t')$ is a Green's function for the operator
$$\frac{d^2}{dt^2} + 1 - \varepsilon(t) .$$

Now
$$g(t_2,t') = \langle 0 | q \, U(t_2,t') \, q \, U(t',t_1) | 0 \rangle$$
$$\frac{dg}{dt}(t_2,t') = \langle 0 | p \, U(t_2,t') \, q \, U(t',t_1) | 0 \rangle$$

and
$$\langle 0 | q - \langle 0 | ip = \langle 0 | a^* = 0 .$$

Thus
$$-i \frac{dg}{dt}(t,t') + g(t,t') = 0 \qquad \text{for } t \gg 0$$

which means
$$g(t,t') = \text{const } e^{-it} \qquad t \gg 0 .$$

Thus we see that the Green's function $G(t,t')$ satisfies the "outgoing" boundary conditions:

$$G(t,t') = \begin{cases} \text{const. } e^{-it} & t \gg 0 \\ \text{const. } e^{+it} & t \ll 0 \end{cases}$$

So the final step is to go back to the formula for $\delta\langle 0|S|0\rangle$:

$$\delta\langle 0|S|0\rangle = \int_{t_1}^{t_2} \frac{i\delta\varepsilon(t)}{2} \underbrace{\langle 0|U(t_2,t)\, q^2\, U(t,t_1)|0\rangle}_{g(t,t) = \frac{\langle 0|S|0\rangle}{i} G(t,t)} dt$$

so

$$\boxed{\frac{\delta\langle 0|S|0\rangle}{\langle 0|S|0\rangle} = \frac{1}{2}\int_{t_1}^{t_2} G(t,t)\,\delta\varepsilon(t)\,dt = \frac{1}{2}\,\mathrm{tr}\,(G\,\delta\varepsilon)}$$

Integrating this à la Schwinger one should get

$$\boxed{(*) \qquad \langle 0|S|0\rangle = \left[\det(1-G_0\varepsilon)\right]^{-1/2}}$$

where G_0 is the Green's function for $\varepsilon = 0$:

$$G_0(t,t') = \frac{e^{-i|t-t'|}}{-2i}$$

Justification for $(*)$:

$$\delta \det(1-G_0\varepsilon) = \det(1-G_0\varepsilon - G_0\,\delta\varepsilon) - \det(1-G_0\varepsilon)$$

$$\frac{\delta\det(1-G_0\varepsilon)}{\det(1-G_0\varepsilon)} = \det\left(1-(1-G_0\varepsilon)^{-1}G_0\,\delta\varepsilon\right) - 1$$

$$= -\,\mathrm{tr}\left((1-G_0\varepsilon)^{-1}G_0\,\delta\varepsilon\right).$$

But $\quad \left(\frac{d^2}{dt^2}+1\right)G = \varepsilon G + 1 \Rightarrow \quad G = G_0\varepsilon G + G_0 \Rightarrow G = (1-G_0\varepsilon)^{-1}G_0$

Thus $\quad \delta \log\det(1-G_0\varepsilon) = -\,\mathrm{tr}\,(G\,\delta\varepsilon) \quad$ and so

$$\delta \log \left[\det(1-G_0\varepsilon)\right]^{-1/2} = +\frac{1}{2}\operatorname{tr}(G\delta\varepsilon) = \delta \log \langle 0|S|0\rangle$$

whence (*) follows because both sides agree when $\varepsilon = 0$.

March 4, 1979

Generalize the above to a coupled system of oscillators:

$$H = \frac{1}{2}\left\{\sum p_j^2 + (\gamma q)_j^2 - \varepsilon(t)\, q_j^2\right\}$$

where γ is a positive ~~def~~ definite real matrix.

$$H_0 = \frac{1}{2}\left[\sum p_j^2 + (\gamma q)_j^2\right]$$

$$\frac{\partial}{\partial x_i} e^{-\frac{1}{2}\sum x_i \gamma_{ij} x_j} = e^{-\frac{1}{2}x\cdot\gamma x}\left(-\frac{1}{2}\right)\left(\sum_j \gamma_{ij}x_j + \sum_j x_j \gamma_{ji}\right)$$

$$= -(\gamma x)_i\, e^{-\frac{1}{2}x\cdot\gamma x}$$

Hence we have destruction operators

$$a_j = \frac{1}{\sqrt{2}}\left(\frac{\partial}{\partial x_j} + (\gamma x)_j\right) = \frac{1}{\sqrt{2}}\left((\gamma q)_j + i p_j\right)$$

creation ops:

$$a_j^* = \frac{1}{\sqrt{2}}\left((\gamma q)_j - i p_j\right)$$

commutation relations:

$$[a_j, a_k] = \frac{1}{2}\left[\frac{\partial}{\partial x_j}(\gamma x)_k - \frac{\partial}{\partial x_k}(\gamma x)_j\right] = \frac{1}{2}\left(\gamma_{jk} - \gamma_{kj}\right) = 0$$

$$[a_j^*, a_k^*] = 0$$

$$[a_j, a_k^*] = \gamma_{jk}$$

and

$$H_0 = \sum a_j^* a_j + \underbrace{\frac{1}{2}\operatorname{tr}(\gamma)}_{\text{ground state energy.}}$$

Equations of motion

$$[iH, q_j] = \frac{i}{2}[p_j^2, q_j] = p_j$$

$$[iH, p_j] = \frac{i}{2}\Big[\sum_{jk} q_j (\gamma^2)_{jk} q_k - \sum_j \varepsilon q_j^2 \; , p_j\Big]$$

$$= +\varepsilon q_j + \frac{i^2}{2}\Big\{\sum_k (\gamma^2)_{jk} q_k + q_k (\gamma^2)_{kj}\Big)$$

$$= -(\gamma^2 q)_j + \varepsilon q_j$$

[redacted] We are interested in $\langle 0|S|0\rangle$ where

$$S = e^{+iH_0 t_2} U(t_2, t_1)\, e^{-iH_0 t_1}$$

[redacted] Remove the ground state energy from H_0, H so that

$$\langle 0|S|0\rangle = \langle 0|U(t_2,t_1)|0\rangle$$

and then consider the effect of an infinitesimal change $\delta\varepsilon$ in ε.

[redacted] $$\delta U(t_2,t_1) = \int_{t_1}^{t_2} U(t_2,t)\, \underbrace{\frac{1}{i}\delta H(t)}_{\frac{i\varepsilon(t)}{2}\sum q_j^2}\, U(t,t_1)\, dt$$

$$\delta\langle 0|S|0\rangle = \langle 0|\delta U(t_2,t_1)|0\rangle.$$

As before introduce

$$g(t,j,t',j') = \langle 0|U(t_2,t) q_j\, U(t,t') q_{j'}\, U(t',t_1)|0\rangle$$

and verify using the equations of motion that

$$\frac{d^2}{dt^2} g(t,j,t',j') = \sum_k -\gamma^2{}_{jk}\, g(t,k,t',j') + \varepsilon\, g(t,j,t',j')$$

i.e.
$$\left(\frac{d^2}{dt^2} + \gamma^2 - \varepsilon\right) g(t,t') = 0$$

And similarly on the other side. Then put

$$\tilde{G}(t,j,t',j') = \begin{cases} g(t,j,t',j') & t > t' \\ g(t',j',t,j) & t < t' \end{cases}$$

$$\frac{d}{dt}\tilde{G}(t,j,t',j')\Big]_{t'^-}^{t'^+} = -i\,\delta_{jj'}\,\langle 0|S|0\rangle.$$

Thus
$$G = C\tilde{G} \qquad \text{with} \quad C = \frac{i}{\langle 0|S|0\rangle}$$

is a Green's matrix for $\frac{d^2}{dt^2} + \gamma^2 - \varepsilon$. Check the boundary conditions at t_2:

$$\frac{d}{dt} g(t,j,t',j')\Big|_{t=t_2} = \langle 0|\, p_j\, U(t_2,t')\, q_{j'}\, U(t',t_1)|0\rangle$$

$$\gamma_{jk}\, g(t_2,k,t',j') = \gamma_{jk}\langle 0|\, q_k\, U(t_2,t')\, q_{j'}\, U(t',t_1)|0\rangle$$

Now
$$\langle 0|\, a^* = 0 \qquad \sqrt{2}\,a^* = (\gamma q)_j - i p_j$$

$$\therefore \left(\frac{d}{dt} + i\gamma\right) g = 0 \quad \text{at } t_2$$

so that the Green's function has the exponential dependence $e^{-i\gamma t}$ for $t > 0$.

It's clear that all the rest of the calculations are the same. Moreover ε can be replaced by any real symmetric matrix.

March 6, 1979

Schwinger VI deals with time-independent EM field. The analogue for me to look at is

$$H = \frac{1}{2}\left[p^2 + (\gamma q)^2 - q\varepsilon q \right]$$

where $\gamma > 0$ and ε is a real symmetric matrix. As long as $\gamma^2 - \varepsilon > 0$ this system is a coupled system of harmonic oscillators. The basic quantity of interest to Schwinger is the vacuum energy $E(0)$, that is, the ground state energy for H. I know this should be

$$\frac{1}{2} \operatorname{tr}\left(\sqrt{\gamma^2 - \varepsilon}\right)$$

or perhaps $\frac{1}{2}\operatorname{tr}\left(\sqrt{\gamma^2-\varepsilon} - \gamma\right)$ since in the field case one subtracts off the infinite constant $\frac{1}{2}\operatorname{tr}\gamma$ from H_0.

Schwinger constructs $\delta E(0)$ as follows:

$$\langle 0_{t_2} | 0_{t_1} \rangle = \langle e^{+it_2 H} 0 \,|\, e^{+it_1 H} 0 \rangle \qquad (?)$$

$$= e^{-iE(0)(t_2 - t_1)}$$

and by time-dep. theory (see p. 643)

$$\frac{\delta \langle 0_{t_2} | 0_{t_1} \rangle}{\langle 0_{t_2} | 0_{t_1} \rangle} = \frac{1}{2}\operatorname{tr}(\delta\varepsilon \cdot G) = \frac{1}{2}\int_{t_1}^{t_2} \operatorname{tr}\left(G(t,t)\,\delta\varepsilon(t)\right) dt$$

where $G(t,t')$ is the Green's function for $\frac{d^2}{dt^2} + \gamma^2 - \varepsilon$

$$G(t,t') = \frac{e^{-i\tilde{\gamma}|t-t'|}}{-2i\tilde{\gamma}} \qquad \tilde{\gamma} = \sqrt{\gamma^2 - \varepsilon}$$

Thus

$$-i\delta E(0)(t_2-t_1) = \frac{1}{2}(t_2-t_1)\, tr\left(\frac{1}{-2i\tilde{\gamma}}\delta\varepsilon\right)$$

$$\delta E(0) = \frac{1}{2} tr\left(-\frac{1}{2}\frac{1}{\sqrt{\gamma^2-\varepsilon}}\delta\varepsilon\right)$$

On the other hand

$$\delta\, \frac{1}{2} tr\left(\sqrt{\gamma^2-\varepsilon}\right) = \frac{1}{2} tr\left(\frac{1}{2}(\gamma^2-\varepsilon)^{-1/2}(-\delta\varepsilon)\right)$$

so it all checks.

Thus we want to understand how to compute $tr\left(\sqrt{\gamma^2-\varepsilon} - \gamma\right)$. An obvious question is whether this is finite for

$$\gamma^2-\varepsilon = (-\Delta+m^2) - \varepsilon(x) \qquad \varepsilon \in C_0^\infty$$

in dimension 1. More generally it seems like we want to understand ζ functions:

$$tr(A^{-s})$$

and why A^{-s} is a ψDO.

Digress to learn some basics about ψDO's.

How does one calculate

$$\left(-\frac{d^2}{dx^2}+q\right)^{-s}$$

where q is a C^∞ function of x. Actually one only calculates the symbol which is a formal expression of the form $\sum_{n\geq 0} a_n(x,s)\left(\frac{1}{i}\frac{d}{dx}\right)^{-2s-n}$.

To see what we want let us use interpolation from the negative integers. Put $D = \frac{1}{i}\frac{d}{dx}$, $m = -s$

$$(D^2+q)^m = (D^2+q)\cdots\cdots(D^2+q) \qquad m \text{ times}$$

$$= D^{2m} + (qD^{2m-2} + D^2 q D^{2m-4} + \cdots + D^{2m-2} q) + q^2 D^{2m-4} +$$

Look for the coeff. in D^{2m-2}:

$$qD^{2m-2} \equiv 0$$

$$D^2 q D^{2m-4} = qD^{2m-2} + 2(Dq)D^{2m-3} + (D^2 q)D^{2m-4}$$

$$D^4 q D^{2m-6} = qD^{2m-2} + 4(Dq)D^{2m-3} + 6(D^2 q)D^{2m-4} + 4(D^3 q)D^{2m-5} + (D^4 q)D^{2m-6}$$

$$D^6 q D^{2m-8} = qD^{2m-2} + 6(Dq)D^{2m-3} + \frac{6\cdot 5}{2} D^2 q \, D^{2m-4}$$

So therefore we see that

$$\text{coeff of } D^{2m-1} = 0$$

$$\text{coeff of } D^{2m-2} = mq$$

$$\text{coeff of } D^{2m-3} = [0 + 2 + 4 + 6 + \cdots + (2m-2)]\, Dq$$

$$= m(m-1)(Dq)$$

$$\text{coeff. of } D^{2m-4} = \left[0 + \frac{2\cdot 1}{2} + \frac{4\cdot 3}{2} + \frac{6\cdot 5}{2} + \cdots + \frac{(2m-2)(2m-4)}{2}\right] D^2 q + \frac{m(m-1)}{2} q^2$$

So formally we seem to get

$$(D^2+q)^{-s} = D^{-2s} + (-s)qD^{-2s-2} + (-s)(-s-1)(Dq)D^{-2s-3} + \cdots$$

Another procedure: Put
$$(D^2+q)^{-s} = \sum_{n\geq 0} a_{ns} D^{-2s-n}$$
Then
$$(D^2+q)(D^2+q)^{-s} = \sum_{n\geq 0} (D^2+q) a_{ns} D^{-2s-n}$$
$$= \sum_{n\geq 0} a_{ns} D^{2-2s-n} + 2(D\cdot a_{ns}) D^{1-2s-n} + \left[(D^2\cdot a_{ns}) + q a_{ns}\right] D^{-2s-n}$$
$$= \sum_{n\geq 0} \left\{a_{ns} + 2(Da_{n-1,s}) + (D^2+q) a_{n-2,s}\right\} D^{2-2s-n}$$
So we get the recursion relation:
$$\boxed{a_{n,s-1} = a_{n,s} + 2\,Da_{n-1,s} + (D^2+q)a_{n-2,s}}$$
where $a_{n,s} = 0$ for $n < 0$. Thus
$$a_{0,s-1} = a_{0,s}$$
so a_{0s} is constant. Since $(D^2+q)^0 = 1$ we have
$$a_{00} = 1, \quad a_{n0} = 0 \quad n \neq 0.$$
Thus
$$a_{0s} = 1 \quad \text{for all } s.$$
Next $a_{1,s-1} = a_{1,s} + \cancel{2Da_{0s}}$, so a_{1s} is constant and so
$$a_{1s} = 0 \quad \text{for all } s.$$
Next
$$a_{2,s-1} = a_{2,s} + q \quad \text{and so}$$
$$a_{2s} = -sq$$
Next
$$a_{3,s-1} = a_{3,s} + 2D(-sq) = a_{3s} - 2s\,Dq$$

and $\qquad a_{3s} = a_{3,s-1} + 2sDq$

so $\qquad a_{3s} = s(s+1)Dq$

so we get again

$$(D^2+q)^{-s} = D^{-2s} - sqD^{-2s-2} + s(s+1)(Dq)D^{-2s-3} + \cdots\cdots$$

From the recursion relation one sees that a_{ns} is a polynomial of degree $n-1$ in s. In fact I already used the fact that a_{ns} is a polynomial in s in order to solve the recursion relation. We also know that $(D^2+q)^m = \sum a_{n,-m} D^{+2m-n}$ has no negative powers of D, so

$$a_{n,-m} = 0 \quad \text{for} \quad n > 2m$$
$$\Updownarrow$$
$$m < \frac{n}{2}$$

and hence $a_{n,s}$ vanishes for $s = 0, -1, \dots, -(\text{int} < \frac{n}{2})$,

so
$$\begin{cases} a_{2k,s} & \text{divisible by } s(s+1)\cdots(s+k-1) \\ a_{2k-1,s} & \quad " \qquad " \quad s(s+1)\cdots(s+k-1). \end{cases}$$

Now what does one do with this formal expansion for $(D^2+q)^{-s}$? This idea is this: Let's suppose q is real + periodic of period 2π, so that D^2+q is an operator over S^1. Assume the spectrum of D^2+q is > 0 and let the eigenvalues be $\lambda_1, \lambda_2, \cdots\cdots$ and the corresponding normalized eigenfunctions be ϕ_n. Then there is a definite operator A^{-s}, $A = D^2+q$, defined

by
$$A^{-s} = \overbrace{\sum_n \lambda_n^{-s} \phi_n(x)\phi_n(x')}^{\text{operator with kernel}} .$$

This sum is convergent for $Re(s) \gg 0$ and then one can analytically continue. The point is that this operator A^{-s} is a pseudo-differential operator with [redacted] symbol given by the above asymptotic expansion.

An obvious generalization of the above "spectral" formula is
$$A^{-s} = \int_0^\infty \lambda^{-s}\, dE_\lambda$$
where E_λ is the spectral resolution of the self-adjoint operator A. The operator A^{-s} is obviously bounded in L^2 for $Re(S) \geq 0$

March 9, 1979

Review: I was looking at a perturbed system of harmonic oscillators:

$$H = \frac{1}{2}\left[p^2 + (\gamma q)^2 - q \cdot \varepsilon q\right]$$

[redacted] with ε time-independent, and found the ground state energy to be

$$\frac{1}{2} tr\left(\sqrt{\gamma^2 - \varepsilon}\right)$$

In the applications to field theory γ^2 is a differential operator like $A_0 = -\Delta + m^2$ and ε is a potential smooth with compact support, so that

$$A = \gamma^2 - \varepsilon = -\Delta + m^2 + g \qquad \text{One assumes } A > 0.$$

The above trace is infinite, [redacted] because $tr(\gamma)$ is infinite, so one removes $\frac{1}{2} tr(\gamma)$ from H, and hence the ground state energy is

$$\frac{1}{2} tr\left(\sqrt{\gamma^2 - \varepsilon} - \gamma\right)$$

This is a special case of the ζ-function

$$tr\left(A^{-s} - A_0^{-s}\right),$$

which I would like to understand at least on the line.

So let's assume that A is a self-adjoint operator with spectrum contained in $[\varepsilon, \infty)$ with $\varepsilon > 0$. Then the resolvant $(z-A)^{-1}$ is analytic off the spectrum and we can obtain A^{-s} from it formally by contour integration

$$A^{-s} = \frac{1}{2\pi i} \int z^{-s} (z-A)^{-1} dz$$

Here z^{-s} is defined using the principal branch with cut along the negative real axis, and the contour is as follows:

(Check on the above formula:

$$A \text{ [redacted]} = \int_\sigma \lambda \, dE_\lambda$$

$$\frac{1}{2\pi i} \int z^{-s} \int_\sigma \text{[redacted]} \frac{1}{z-\lambda} dE_\lambda \, dz = \int_\sigma \left\{ \frac{1}{2\pi i} \int_{\circlearrowleft} z^{-s} \frac{1}{z-\lambda} dz \right\} dE_\lambda$$

$$= \int \lambda^{-s} dE_\lambda \quad)$$

So

$$\operatorname{tr}(A^{-s} - A_0^{-s}) = \frac{1}{2\pi i} \int z^{-s} \operatorname{tr} \left[(z-A)^{-1} - (z-A_0)^{-1} \right] dz$$

and so we are reduced to understanding the trace of the resolvant.

But now note that formally one has

$$\operatorname{tr}(z-A)^{-1} = \operatorname{tr} \frac{d}{dz} \log(z-A)$$

$$= \frac{d}{dz} \log \det(z-A)$$

hence

$$\mathrm{tr}\left[(z-A)^{-1}-(z-A_0)^{-1}\right] = \frac{d}{dz}\log\left(\frac{\det(z-A)}{\det(z-A_0)}\right)$$

$$\cancel{= \frac{d}{dz}\log(1-G_0 q)} = \frac{d}{dz}\log\det(1-G_0 q)$$

where I use the old notation

$$G_0 = (z-A_0)^{-1} \qquad \text{and } A = A_0 + q$$

There doesn't seem to be any real reason why I can't take $A_0 = -\frac{d^2}{dx^2}$. In this case let's recall what we know about $\det(1-G_0 q)$. Let $z=k^2$ where $\mathrm{Im}\, k > 0$ corresponds to $z \notin \mathbb{R}_{\geq 0}$.

$$z - A_0 = \frac{d^2}{dx^2} + k^2 \qquad G_0(x,x') = \frac{e^{ik|x-x'|}}{2ik}$$

$$z - A = \frac{d^2}{dx^2} + k^2 - q$$

$$(z-A_0)^{-1}(z-A) = 1 - G_0 q.$$

Recall that

$$\det(1-G_0 q) = \frac{W(\varphi,\psi)}{W(\varphi^0,\psi^0)} = \frac{W(Ae^{-ikx}, e^{ikx})}{W(e^{-ikx}, e^{ikx})}$$

$$= A(k) = \frac{1}{T(k)}$$

where

$$e^{-ikx} \longleftrightarrow A(k)e^{-ikx} + B(k)e^{ikx}.$$

Next let's understand the jump in the resolvant.

(Note that for k real: conjugation same as $k \to -k$, so $\overline{A(k)} = A(-k)$

$$(x - i\varepsilon - A)^{-1} - (x + i\varepsilon - A)^{-1} = \int \left[\frac{1}{x - i\varepsilon - \lambda} - \frac{1}{x + i\varepsilon - \lambda}\right] dE_\lambda$$

$$= 2\pi i \int \frac{1}{\pi} \cdot \frac{\varepsilon}{(x-\lambda)^2 + \varepsilon^2}\, dE_\lambda \xrightarrow[\varepsilon \downarrow 0]{} 2\pi i \frac{dE_x}{dx}$$

Thus
$$\operatorname{tr}\left(A^{-s} - A_0^{-s}\right) = \int_0^\infty x^{-s} \operatorname{tr}(dE_x - dE_x^0)$$

$$= \frac{1}{2\pi i} \int_0^\infty x^{-s} \operatorname{tr}\Big[(x - i0^+ - A)^{-1} - (x + i0^+ - A)^{-1} - (x - i0^+ - A_0)^{-1} + (x + i0^+ - A_0)^{-1}\Big] dx$$

$$= \frac{1}{2\pi i} \int_0^\infty x^{-s} \frac{d}{dx} \log \det\left((1 - G_0^- q)(1 - G_0^+ q)\right) dx$$

where G_0^+ means that z approaches the real axis from above.

Since $\det(1 - G_0^+ q) = A(k)$, $\quad k = +\sqrt{x}$

$\det(1 - G_0^- q) = A(-k)$. Thus

$$\operatorname{tr}\left(A^{-s} - A_0^{-s}\right) = \int_0^\infty x^{-s} \frac{d \log}{2\pi i} \left(\frac{T(k)}{\bar{T}(k)}\right)$$

$$\boxed{\operatorname{tr}\left(A^{-s} - A_0^{-s}\right) = \int_0^\infty k^{-2s}\, d \arg\left(\frac{T(k)}{\bar{T}(k)}\right)}$$

Recall that $\dfrac{T(k)}{\bar{T}(k)}$ is the determinant of the scattering matrix.

$$\det \begin{pmatrix} R & T \\ T & -\frac{T}{\bar{T}} \bar{R} \end{pmatrix} = |R|^2\left(-\frac{T}{\bar{T}}\right) - T^2 = \frac{T}{\bar{T}}\left(-|R|^2 - |T|^2\right) = -\frac{T}{\bar{T}}$$

You want $-$ this, so as to get I when $q = 0$.

March 10, 1979

Up to now, I have been looking at the field theory obtained from the KG equation

$$\left[\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} + m^2\right]\phi = 0.$$

The approach has been to view this [redacted] DE as the classical equation of motion for a coupled harmonic oscillator system

1) $$\ddot{q} = -\gamma^2 q \qquad\qquad \gamma^2 = -\frac{\partial^2}{\partial x^2} + m^2$$

and then to quantize.

Now I would like to understand how to quantize the Dirac DE

$$\left[\frac{1}{i}\frac{\partial}{\partial t} + \alpha \frac{1}{i}\frac{\partial}{\partial x} + m\beta\right]\psi = 0$$

Using the Fourier transform in x, we are led [redacted] to ask how to consider

2) $$i\frac{\partial}{\partial t}\psi = \gamma\psi$$

as a classical mechanical system. Here γ is a self-adjoint matrix of trace 0, because I have seen before that

$$\alpha = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad \beta = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

The first problems is how to deal with the fact ψ has to be complex. The solutions of 2) form a 2-dimensional complex vector space. I need a conjugation and a symplectic form on this vector space. From these I obtain a real symplectic vector space, and can then quantize.

$$i\frac{\partial}{\partial t}\begin{pmatrix}u_1\\u_2\end{pmatrix} = \begin{pmatrix}a & b\\ \bar b & -a\end{pmatrix}\begin{pmatrix}u_1\\u_2\end{pmatrix} \qquad a=\bar a$$

Conjugate and flip

$$+i\frac{\partial}{\partial t}\begin{pmatrix}\bar u_2\\ \bar u_1\end{pmatrix} = \begin{pmatrix}+a & -b\\ -\bar b & -a\end{pmatrix}\begin{pmatrix}\bar u_2\\ \bar u_1\end{pmatrix}$$

$$\therefore i\frac{\partial}{\partial t}\begin{pmatrix}\bar u_2\\ -\bar u_1\end{pmatrix} = \begin{pmatrix}a & b\\ \bar b & -a\end{pmatrix}\begin{pmatrix}\bar u_2\\ -\bar u_1\end{pmatrix}$$

Thus $J: \begin{pmatrix}u_1\\u_2\end{pmatrix} \longmapsto \begin{pmatrix}\bar u_2\\ -\bar u_1\end{pmatrix}$ is a symmetry of the space of solns. It is conjugate linear with square $-I$, so that the solution space is a 1-diml. vector space over $\mathbb{H}$.

Because the matrix $\begin{pmatrix}a & b\\ \bar b & -a\end{pmatrix}$ is self-adjoint, the form $|u_1|^2+|u_2|^2$ is constant in time for any solution, hence the solution space has a natural inner product. Call V the solution space. Then one finds a bilinear form on V

$$(u, Jv) = u_1v_2 - u_2v_1 = \begin{vmatrix}u_1 & v_1\\ u_2 & v_2\end{vmatrix}$$

(Check

$$\frac{d}{dt}\begin{vmatrix}u_1 & v_1\\ u_2 & v_2\end{vmatrix} = \begin{vmatrix}\check a u_1 + \check b u_2 & v_1\\ \check{\bar b} u_1 - \check a u_2 & v_2\end{vmatrix} + \begin{vmatrix}u_1 & \check a v_1 + \check b v_2\\ u_2 & \check{\bar b} v_1 - \check a v_2\end{vmatrix} = 0$$

In fact this would work for $\frac{\partial}{\partial t}\psi = A\psi$ with $\operatorname{tr} A = 0$.)

March 11, 1979:

The problem now is to understand the Dirac situation studied by Schwinger. The actual Dirac equation is

$$\left\{ \frac{1}{i}\frac{\partial}{\partial t} + \sum_{k=1}^{3} \alpha_k \frac{1}{i}\frac{\partial}{\partial x_k} + m\beta \right\} \psi = 0$$

where ψ has four components and where the matrices $\alpha_1, \alpha_2, \alpha_3, \beta$ belong to a Clifford algebra:

$$\alpha_k \alpha_\ell + \alpha_\ell \alpha_k = 2\delta_{k\ell} \qquad \alpha_4 = \beta.$$

Let's assume that there is no difficulty in doing Schwinger's theory with a 2 component ψ in 1-space dimension. In this case the equation becomes

$$\left\{ \frac{1}{i}\frac{\partial}{\partial t} + \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \frac{1}{i}\frac{\partial}{\partial x} + m \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\} \psi = 0$$

If one has a time-dependent EM field, then the Dirac equation is changed by replacing

$$\frac{1}{i}\frac{\partial}{\partial x} \longmapsto \frac{1}{i}\frac{\partial}{\partial x} - eA_1(t,x)$$

$$\frac{1}{i}\frac{\partial}{\partial t} \longmapsto \frac{1}{i}\frac{\partial}{\partial t} + eA_0(t,x).$$

[redacted] View the equation in the form

$$i\frac{\partial \psi}{\partial t} = \left[eA_0 + \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \left(\frac{1}{i}\frac{\partial}{\partial x} - eA_1 \right) + m \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right] \psi$$

$$= H\psi$$

where H is a 2×2 matrix of operators on functions of x which is hermitian and which depends on t.

It would be nice if I could obtain a simple model of the phenomena in 0-space dimension, i.e. using the DE

$$(*) \qquad i\frac{\partial \psi}{\partial t} = H(t)\psi$$

where $H(t)$ is a 2×2 hermitian matrix depending on t. I want to think of the above as a classical equation and then to quantize it. One problem is how to get anti-commutation relations [redacted].

So we look at the space V of solutions of $(*)$; it a 2-dimensional complex vector space, with a canonical inner product $\psi^*\psi$, since

$$\frac{d}{dt}\psi^*\psi = (-iH\psi)^*\psi + \psi^*(-iH\psi)$$
$$= i\psi^*H^*\psi - i\psi^*H\psi = 0$$

as H is hermitian. There doesn't seem to be any additional structure at this level, because clearly the transformation function $U(t,t')$ can be an arbitrary path in $U(2)$.

If we assume $tr(H(t)) = 0$, then V has a canonical [redacted] symplectic form

$$\psi, \psi' \longmapsto \begin{vmatrix} \psi_1 & \psi_1' \\ \psi_2 & \psi_2' \end{vmatrix}$$

and $U(t,t')$ is a path in $SU(2) \simeq$ unit quaternions.

[redacted]

There are two problems with quantizing the Dirac field. The first is the fact that ψ is complex-valued and the second is that we must end up with anti-

Commutation relations. It makes sense to [redacted] separate the difficulties and to first discuss the charged scalar field. This means means that I am interested in the [redacted] system whose classical equation-of-motion-solutions are solutions of

$$\left(\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} + m^2\right)\phi = 0$$

with ϕ complex. This is just two copies of the real KG field.

Perhaps the way to understand this is to look at the DE

$$\ddot{\phi} = -\gamma^2 \phi$$

where now γ^2 is a hermitian positive-def. matrix ~~operator~~ with pos. square root γ. Before we handled the case where ~~where~~ γ was real.

We handle this by treating the real and imaginary parts of ϕ as indep. dynamical variables:

$$\phi_j = \frac{1}{\sqrt{2}}\left(q^{(1)}_j + i q^{(2)}_j\right) \qquad \bar{\phi}_j = \frac{1}{\sqrt{2}}\left(q^{(1)}_j - i q^{(2)}_j\right)$$

Actually, where possible, we work with $\phi, \bar{\phi}$. The equations of motion are

$$\ddot{\phi}_j = -\sum_k (\gamma^2)_{jk}\phi_k = (\gamma^2\phi)_j$$

$$\ddot{\bar{\phi}}_j = -\sum_k \overline{(\gamma^2_{jk})}\,\bar{\phi}_k = -\sum_k \bar{\phi}_k (\gamma^2)_{kj} = -(\bar{\phi}\gamma^2)_j$$

We take our kinetic energy to be

$$\dot\phi\cdot\dot\phi = \sum_j \dot{\bar\phi}_j\dot\phi_j = \sum_j \tfrac{1}{2}\left[(\dot q_j^{(1)})^2 + (\dot q_j^{(2)})^2\right]$$

and the potential energy to be

$$\phi\cdot\gamma^2\phi = \sum_{jk} \bar\phi_j(\gamma^2)_{jk}\phi_k . \quad \text{Then}$$

$$L = \sum_j \dot{\bar\phi}_j\dot\phi_j - \sum_{jk}\bar\phi_j(\gamma^2)_{jk}\phi_k = \dot{\bar\phi}\cdot\dot\phi - \bar\phi\cdot\gamma^2\phi$$

The Lagrange equations are

$$\frac{d}{dt}\left(\frac{\partial L}{\partial\dot\phi_j}\right) = \frac{\partial L}{\partial\phi_j} \qquad \frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\bar\phi}_j}\right) = \frac{\partial L}{\partial\bar\phi_j}$$

$$\frac{d}{dt}\dot{\bar\phi}_j = -\sum_k\bar\phi_k(\gamma^2)_{kj} \qquad \frac{d}{dt}\dot\phi_j = -\sum_k\gamma^2_{jk}\phi_k$$

as they should be. Also

$\dot{\bar\phi}_j$ = momentum conjugate to ϕ_j

$\dot\phi_j$ = " " " $\bar\phi_j$.

$$H = \dot{\bar\phi}\cdot\dot\phi + \bar\phi\cdot\gamma^2\phi = |\dot\phi|^2 + |\gamma\phi|^2$$

When we quantize $\phi_j, \bar\phi_j$ become operators, and

$$\phi_j^* = \frac{1}{\sqrt2}\left(q_j^{(1)} + iq_j^{(2)}\right)^* = \bar\phi_j = \frac{1}{\sqrt2}\left(q_j^{(1)} - iq_j^{(2)}\right)^*$$

and we have the commutation relations

$$[\phi_j, \phi_j^*] = 0 = [\dot\phi_j, \dot\phi_j^*], \quad [\phi_j, \dot\phi_j] = \tfrac12[q^1+iq^2, p^1+ip^2] =$$

$$\tfrac12(i + i^2 i) = 0$$

$$[\phi_j, \dot{\phi}_j^*] = \frac{1}{2}[q^1 + iq^2, p^1 - ip^2] = i$$

The ground state energy is $tr(\gamma)$ since the trace over $\mathbb{R}$ is twice the trace over $\mathbb{C}$.

We can realize the commutation relations by operators on $L^2(\mathbb{C}^n)$ as follows: $q_j^{(1)}$ = mult by x_i; $q_j^{(2)}$ = mult. by y_j, so

$$\phi_j = \text{mult by } \frac{1}{\sqrt{2}} z_j$$

$$\dot{\phi}_j = \frac{1}{\sqrt{2}}(p_j^{(1)} + i p_j^{(2)}) = \frac{1}{\sqrt{2}}\left(\frac{1}{i}\frac{\partial}{\partial x_j} + i\frac{1}{i}\frac{\partial}{\partial y_j}\right)$$

$$= \frac{1}{i}\frac{1}{\sqrt{2}}\left(\frac{\partial}{\partial x_j} + i\frac{\partial}{\partial y_j}\right)$$

Recall

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} + \frac{1}{i}\frac{\partial}{\partial y}\right) \qquad \frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} - \frac{1}{i}\frac{\partial}{\partial y}\right)$$

So

$$\phi_j = \frac{1}{\sqrt{2}} z_j \qquad \dot{\phi}_j = \frac{\sqrt{2}}{i}\frac{\partial}{\partial \bar{z}_j}$$

These commute as they should. Also

$$[\phi_j, \dot{\phi}_j^*] = \left[z_j, i\frac{\partial}{\partial z_j}\right] = +i$$

since $\left(\frac{\partial}{\partial \bar{z}_j}\right)^* = -\frac{\partial}{\partial z_j}$

Maybe a simpler representation is

$$\phi_j = z_j \qquad \dot{\phi}_j = \frac{1}{i}\frac{\partial}{\partial \bar{z}_j}$$

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Idea: A complex vector space V of dimension n equipped with a hermitian inner product can be viewed as a real $2n$ vector space with either a quadratic form $Q(v) = |v|^2$ or with a symplectic form $\mathrm{Im}(u,v)$. A real vector space with quadratic form leads to anti-commutation relations

$$uv + vu = B(u,v) \qquad u = u^*$$

whereas a real vector space with symplectic form leads to commutation relations

$$[u,v] = iB(u,v) \qquad u = u^*.$$

Therefore if I take the solutions of

$$i\frac{\partial\psi}{\partial t} = H\psi$$

these form a complex vector space with inner product, hence I have available two ways of quantizing it.

Question: Do we get interesting dynamics?