February 13, 1979

The following analogy has occurred to me: There is something similar in the way one constructs the path and loop spaces in DG rational homotopy theory with the construction of an interacting quantum field. Somehow 2nd quantization is like the exterior alg. on odd degrees $\otimes$ symmetric alg. on even degrees. Also $\mathcal{H}_I$ is reminiscent of the twisting cochain. Finally

$$\det(1-\lambda A) = \operatorname{tr}(\Lambda A)$$

$$\frac{1}{\det(1-\lambda A)} = \operatorname{tr}(SA).$$

(Another point might be that the t variable is a path variable?)

$\mathcal{D}/\mathcal{D}'$: Is this related to propagation in "proper time"?

It seems necessary to go through 2nd quantization. Take the KG equation

$$\left(\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} + m^2\right)\psi = 0$$

Global solutions in t, x are F.T. of distributions supported on hyperboloid

$$k^2 = \xi^2 + m^2$$

and hence one can split any solution into a "positive energy" solution supported on $k = \sqrt{\xi^2+m^2}$ and a negative energy solution. If we take F.T. wrt x alone we get

$$\left(\frac{d^2}{dt^2} + \xi^2 + m^2\right)\hat{\varphi}(t,\xi) = 0$$

so

$$\hat{\varphi}(t,\xi) = \text{lin. comb. of } e^{-i\omega(\xi)t} \text{ and } e^{i\omega t}$$

$$\text{where } \omega(\xi) = \sqrt{\xi^2+m^2}.$$

Positive energy solutions have only $e^{-i\omega t}$ time dependence and there is a unique positive energy solution $\varphi(t,x)$ with given $\varphi(0,x)$, namely

$$\varphi(t,x) = \int \frac{d\xi}{2\pi} e^{-i\xi x} e^{-i\omega t} \hat{\varphi}(0,\xi)$$

Positive energy solutions are made into a Hilbert space in the following way. The hyperboloid $k = \sqrt{\xi^2+m^2}$ is a homogeneous space under the Lorentz group preserving the form $t^2 - x^2$, time reversal not allowed. It turns out that $\frac{d\xi}{\omega(\xi)}$ is an invariant measure on the hyperboloid. Thus if we use the formula

$$\varphi(t,x) = \int \frac{d\xi}{2\pi\,\omega(\xi)} e^{-i\xi x - i\omega t}\, \Phi(\xi)$$

to set up the correspondence of pos. energy KG solns with functions Φ on the hyperboloid, then the good norm is the L^2 norm on the hyperboloid.

$$\|\varphi\|^2 = \int \frac{d\xi}{2\pi\omega} |\Phi(\xi)|^2$$

Denote this Hilbert space by $\mathcal{H}$. One forms the associated Fock space which is the symmetric algebra on $\mathcal{H}$ completed in some nice way. Let f_α be an orthonormal basis for $\mathcal{H}$ and a_α the corresponding destruction operators and a_α^* the corresp. creation operators. Then the key operator seems to be

$$\phi^+(\underline{x}) = \sum f_\alpha(\underline{x})\, a_\alpha$$

and it is independent of the choice of $\{f_\alpha\}$. Notice that $\phi^+(\underline{x})$ is a kind of destruction operator, because it is a linear combination of the a_α. Here $\underline{x} = (t, x)$ is a point of space time, so that $\phi^+(\underline{x})$ is a destruction operator depending on a point of space time.

The adjoint of $\phi^+(\underline{x})$ is

$$\left(\phi^+(\underline{x})\right)^* = \phi^-(\underline{x}) = \sum \overline{f_\alpha(\underline{x})}\, a_\alpha^*$$

and one has the commutation relation

$$\left[\phi^+(\underline{x}), \phi^-(\underline{x}')\right] = \sum_\alpha f_\alpha(\underline{x})\, \overline{f_\beta(\underline{x}')}\, \underbrace{[a_\alpha, a_\beta^*]}_{\delta_{\alpha\beta}}$$

$$= \underbrace{\sum_\alpha f_\alpha(\underline{x})\, \overline{f_\alpha(\underline{x}')}}_{}$$

kernel of the projection operator on positive energy solutions of KG equation.

February 15, 1979

Recall that one constructs a Hilbert space out of positive energy solutions of the KG equation, then if $f_\alpha(\underline{x})$ is an orthonormal basis, one has

$$\sum_\alpha f_\alpha(\underline{x})\,\overline{f_\alpha(\underline{x}')} = K(\underline{x},\underline{x}')$$

is the Green's function for the KG equation with "outgoing" boundary conditions. Why is this true and how general is this phenomenon? NOT TRUE

Take the F.T. wrt x, x':

$$\int dx\, e^{i\xi x}\int dx'\, e^{-i\xi' x'}\sum_\alpha f_\alpha(t,x)\,\overline{f_\alpha(t',x')}$$

$$= \sum_\alpha \hat{f}_\alpha(t,\xi)\,\overline{\hat{f}_\alpha(t',\xi')}$$

$$= \sum_\alpha e^{-i\omega(\xi)t}\,\hat{f}_\alpha(0,\xi)\;\overline{e^{+i\omega(\xi')t'}\hat{f}_\alpha(0,\xi')}$$

$$= e^{-i\omega(\xi)t+i\omega(\xi')t'}\sum_\alpha \hat{f}_\alpha(0,\xi)\,\overline{\hat{f}_\alpha(0,\xi')}$$

So far we have used the positive energy solution condition when we put

$$\hat{f}_\alpha(t,\xi) = e^{-i\omega(\xi)t}\hat{f}_\alpha(0,\xi)$$

since $\omega(\xi) = +\sqrt{\xi^2+m^2} > 0$. Now the Hilbert space structure on these solutions is defined to be the L^2 norm with respect to a measure $\rho(\xi)\,d\xi$ of $\hat{f}(0,\xi)$, and for any such measure time evolution, which is given by mult. by $e^{-i\omega(\xi)t}$, will be unitary. The condition that f_α be an orthonormal basis means that for any $g(\xi) \in C_0^\infty$

$$g = \sum_\alpha f_\alpha (f_\alpha | g)$$

where $(f_\alpha | g)$ is the physicists notation for (g, f_α). Thus

$$g(\xi) = \sum_\alpha \hat{f}_\alpha(0,\xi) \int \rho(\xi')\,d\xi'\, \overline{\hat{f}_\alpha(0,\xi')}\, g(\xi')$$

so

$$\sum_\alpha{}' \hat{f}_\alpha(0,\xi)\, \overline{\hat{f}_\alpha(0,\xi')}\, \rho(\xi') = \delta(\xi-\xi')$$

So

$$\int dx\, e^{ix\xi} \int dx'\, e^{-ix'\xi'} \sum_\alpha f_\alpha(t,x)\, \overline{f_\alpha(t',x')}$$

$$= e^{-i\omega(\xi)(t-t')} \frac{1}{\rho(\xi)} \delta(\xi-\xi')$$

so

$$\sum_\alpha f_\alpha(t,x)\, \overline{f_\alpha(t',x')} = \int \frac{d\xi}{2\pi} e^{-i\xi x} \int \frac{d\xi'}{2\pi} e^{+i\xi' x'} e^{-i\omega(\xi)(t-t')} \times \frac{1}{\rho(\xi)} \delta(\xi-\xi')$$

$$= \int \frac{d\xi}{(2\pi)^2 \rho(\xi)} e^{-i\xi(x-x')} e^{-i\omega(\xi)(t-t')}$$

But this is not a Green's function — instead it satisfies the KG equation.

The Feynman style Green's function comes in only with time-ordered contractions.

It seems, at first glance, that it should be possible to [redacted] discuss [redacted] for the K-G equation perturbation by a potential. The idea is to do the case of the Dirac field perturbed by an external EM field [redacted] but for the KG field. The modified Dirac equation is

$$\left\{\left(\gamma \frac{1}{i}\partial - e\gamma A\right) + m\right\}\phi = 0.$$

What [redacted] should be the modified KG equation [redacted]? The obvious candidate is

$$\left\{\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} + \overset{m^2+}{\overbrace{g}}\right\}\phi = 0$$

However in Schweber's book (p.63) the modified equation is obtained by replacing:

$$\left(\frac{1}{i}\frac{\partial}{\partial t}, \frac{1}{i}\frac{\partial}{\partial x}\right) \longmapsto \left(\frac{1}{i}\frac{\partial}{\partial t} - eA_0, \frac{1}{i}\frac{\partial}{\partial x} - eA_1\right)$$

i.e $\quad \frac{1}{i}\partial \longmapsto$ [redacted] $\frac{1}{i}\partial - eA$

but this is for <u>charged</u> particles: ϕ charged scalar field.

Idea: p. 267 of Schweber gives the following types of interactions between a neutral scalar field ϕ and a charged spinor field ψ:

$$\begin{cases} \bar{\psi}\psi\phi & \text{scalar coupling} \\ \bar{\psi}\gamma_5\psi\phi & \text{pseudo-scalar coupling} \\ \bar{\psi}\gamma_\mu\psi\partial^\mu\phi & \text{vector} \quad " \\ \bar{\psi}\gamma_5\gamma_\mu\phi\partial^\mu\phi & \text{pseudo-vector} \quad " \end{cases}$$

So therefore it is clear that $g\phi$ is an acceptable coupling between a neutral scalar field and an external scalar field.

There seems to be a procedure for quantizing a classical field theory derived from a Lagrangian. Let's review Lagrange's & Hamilton's equation and how they get transformed in the field case.

Take a single particle on the line

$$L(q,\dot{q},t) = \underbrace{\frac{m}{2}\dot{q}^2}_{K.E.} - \underbrace{V(q,t)}_{P.E.}$$

action $\quad A = \int L(q,\dot{q},t)\,dt \qquad$ with $\dot{q} = \frac{dq}{dt}$

$$\delta A = \int\left(\frac{\partial L}{\partial q}\delta q + \frac{\partial L}{\partial \dot{q}}\delta\dot{q}\right)dt$$

$$= \int\left\{\frac{\partial L}{\partial q} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right)\right\}\delta q\,dt = 0$$

Hamilton's Principle

So you get

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q} = \frac{d}{dt}\left(m\frac{dq}{dt}\right) + \frac{\partial V}{\partial q} = 0$$

for the equation of motion. This is Newton's law with force $F = -\frac{\partial V}{\partial q}$

The Hamiltonian is

$$H(p,q,t) = p\dot q - L(q,\dot q,t)$$

where $\dot q$ is regarded as a function of q, p, t via

$$p = \frac{\partial L}{\partial \dot q}$$

In this example $p = m\dot q$ so $\dot q = \frac{p}{m}$ and

$$H = \frac{p^2}{2m} + V(q,t).$$

In general

$$\frac{\partial H}{\partial q} = \cancel{p\frac{\partial \dot q}{\partial q}} - \frac{\partial L}{\partial q} - \cancel{\frac{\partial L}{\partial \dot q}\frac{\partial \dot q}{\partial q}} = -\frac{d}{dt}\left(\frac{\partial L}{\partial \dot q}\right) = -\frac{dp}{dt}$$

$$\frac{\partial H}{\partial p} = \dot q + p\frac{\partial \dot q}{\partial p} - \frac{\partial L}{\partial \dot q}\frac{\partial \dot q}{\partial p} = \dot q = \frac{dq}{dt}$$

which are Hamilton's equations. Maybe a simpler derivation is

$$0 = -\delta A = \delta \int \{H - p\dot q\}\,dt = \int \left\{\frac{\partial H}{\partial p}\delta p + \frac{\partial H}{\partial q}\delta q - \delta p\,\dot q - p\,\delta \dot q\right\} dt$$

$$= \int \left[\left\{\frac{\partial H}{\partial p} - \dot q\right\}\delta p + \left\{\frac{\partial H}{\partial q} + \dot p\right\}\delta q\right] dt \qquad \text{etc.}$$

Next consider typical field situation such as the vibrating string. Let $\phi(t,x)$ be the displacement; think of $\phi(t,x)$ as the value of the x-th coordinate q_x at time t. Then

$$K.E. = \int dx\, \frac{\rho}{2}(\dot\phi)^2 \qquad \rho \text{ density}$$

$$P.E. = \int dx \, \frac{T}{2}\left(\frac{\partial \phi}{\partial x}\right)^2 \qquad T \cdot \text{tension}$$

so

$$A = \iint dt\, dx \; \frac{1}{2}\left[\rho(\dot\phi)^2 - T(\phi_x)^2\right]$$

$$0 = \delta A = \iint dt\, dx \left[\rho\dot\phi(\delta\phi)^{\cdot} - T\phi_x(\delta\phi)_x\right]$$

$$= \iint dt\, dx \left[-(\rho\dot\phi)^{\cdot} + (T\phi_x)_x\right]\delta\phi$$

yielding $\quad \dfrac{\partial}{\partial t}\left(\rho\dfrac{\partial\phi}{\partial t}\right) = \dfrac{\partial}{\partial x}\left(T\dfrac{\partial\phi}{\partial x}\right).$

The Lagrangian is

$$L = \int dx \; \underbrace{\frac{1}{2}\left[\rho(\dot\phi)^2 - T(\phi_x)^2\right]}_{\mathcal{L}(\phi,\dot\phi) \;=\; \text{Lagrangian density}}$$

The momentum conjugate to the variable $\phi(x)$ is

$$\pi(x) = \frac{\partial L}{\partial \dot\phi_x} = \left(\frac{\partial \mathcal{L}}{\partial \dot\phi}\right)(x)$$

$$= (\rho\dot\phi)(x)$$

The Hamiltonian is

$$H = \int dx\, \pi(x)\,\dot\phi(x) - L$$

$$= \int dx \; \underbrace{\frac{1}{2}\left[\frac{\pi(x)^2}{\rho} - T\left(\frac{\partial\phi}{\partial x}\right)^2\right]}_{\mathcal{H}(\phi,\pi) \;=\; \text{Hamiltonian density}}$$

Let us take the field equation

$$\left\{ \frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} + (m^2 + g) \right\} \phi = 0$$

where $g = g(t, x)$, and find a suitable $\mathcal{L}$. Then if

$$\mathcal{L}(\phi, \dot\phi, t, x) = \frac{1}{2}\left[(\dot\phi)^2 - \left(\frac{\partial \phi}{\partial x}\right)^2 - (m^2 + g) \phi^2\right]$$

$$\delta A = \iint dt\,dx\,\delta\mathcal{L} = \iint dt\,dx \left[\dot\phi (\delta\phi)^{\cdot} - \phi_x (\delta\phi)_x - (m^2+g)\phi\,\delta\phi\right]$$

$$= \iint dt\,dx \left[-\ddot\phi + \phi_{xx} - (m^2+g)\phi\right]\delta\phi ,$$

so it appears the good Lagrangian density is

$$\mathcal{L}(\phi, \dot\phi, t, x) = \frac{1}{2}\left[\left(\frac{\partial\phi}{\partial t}\right)^2 - \left(\frac{\partial\phi}{\partial x}\right)^2 - (m^2+g)\phi^2\right].$$

The momentum conjugate to $\phi(x)$ is

$$\pi(x) = \left(\frac{\partial\mathcal{L}}{\partial\dot\phi}\right)(x) = \dot\phi(x)$$

and the Hamiltonian density is

$$\mathcal{H} = \pi\dot\phi - \mathcal{L} = \frac{1}{2}\left[(\dot\phi)^2 + \left(\frac{\partial\phi}{\partial x}\right)^2 + (m^2+g)\phi^2\right]$$

February 17, 1979

Consider Segal's viewpoint toward quantum field theory: The solutions of the wave equation are made into a symplectic manifold which one then quantizes.

Review the usual formalism. Start with Lagrangian $L(q,\dot q,t)$ which gives a flow on the tangent bundle of configuration space:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot q}\right) = \frac{\partial L}{\partial q}$$

Then one identifies tangent bundle with the cotangent bundle by defining momentum:

$$p = \frac{\partial L}{\partial \dot q} \quad = \text{fn. of } q, \dot q, t$$

This identification [redacted] is time dependent in general. So it seems better to work at the outset with the states of the system which are paths in configuration space specified by initial data at time 0. [redacted] Using the momentum p conjugate to q the states becomes paths in the cotangent bundle specified by their value at time zero. Thus

This isn't quite correct because if the Lagrangian is time-dep. there is no flow in T_X

States $\cong$ solutions curves of Lagrangian flow in T_X

$\downarrow$

solution curves for Hamilton's flow in T_X^*

in good cases this is an isomorphism (i.e. when one can solve for $\dot q$ as a fn. of q, p, t)

Thus the states [redacted] form a symplectic manifold with time-evolution generated by a Hamiltonian. Now to quantize this classical system one seeks a representation where functions $f(p,q)$ on T_X^* become operators $\underline{f}$ such that

$$[\underline{f}, \underline{g}] = \frac{1}{i}\underline{\{f,g\}}$$

in some sense.

Next consider the wave equation

$$\left\{\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} + (m^2 - q)\right\}\phi = 0$$

which comes from the Lagrangian

$$L(\phi, \dot\phi, t) = \int dx \; \underbrace{\frac{1}{2}\left[\dot\phi^2 - \left(\frac{\partial\phi}{\partial x}\right)^2 - (m^2 - q)\phi^2\right]}_{\mathcal{L}(\phi, [redacted] \nabla\phi, t, x)}$$

Check: $$0 = \frac{\partial}{\partial t}\left(\frac{\partial\mathcal{L}}{\partial\phi_t}\right) + \frac{\partial}{\partial x}\left(\frac{\partial\mathcal{L}}{\partial\phi_x}\right) - \frac{\partial\mathcal{L}}{\partial\phi}$$

$$= \ddot\phi - \phi_{xx} + (m^2 - q)\phi$$

Now states [redacted] are global solutions $\phi(t,x)$ of the wave equation, and they form a vector space $\mathcal{S}$. Any solution is specified by its Cauchy data at time $t = 0$.

The momentum variable conjugate to $\phi(x)$ is

$$\pi(x) = \frac{\partial\mathcal{L}}{\partial\dot\phi}(x) = \dot\phi(x).$$

Because [redacted] everything is linear, the [redacted] vector space S should carry a symplectic form.

To understand this, [redacted] I maybe have to review the basic formulas. So let us consider the canonical 2-form Ω of T^*_X; in local coordinates $q^1, \dots, q^n$ it is

$$\Omega = \sum_i dp_i \, dq^i$$

One uses Ω to identify 1-forms ω with vector fields X:

$$\omega = i(X)\Omega$$

Because of Cartan's formula $\theta(X) = d\,i(X) + i(X)\,d$ one has

$$\theta(X)\Omega = d\,i(X)\Omega$$

so that X is Hamiltonian $\Longleftrightarrow$ the corresp. form is closed. Locally in this case $\omega = df$, [redacted] so X coincides with X_f defined by

$$i(X_f)\Omega = df.$$

In [redacted] local coords

$$\overset{\|}{\sum} \frac{\partial f}{\partial p_i} dp_i + \frac{\partial f}{\partial q^i} dq^i$$

so that

$$X_f = \sum_i \frac{\partial f}{\partial q^i}\frac{\partial}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial}{\partial q^i}$$

and so

$$X_f g = \sum \frac{\partial f}{\partial q_i}\frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q_i} = \{f, g\} \quad \text{Poisson bracket}$$

$$\| $$

$$i(X_f)dg = i(X_f)\,i(X_g)\Omega$$

In the field case we have coord. variables [redacted] $q_x^{(\phi)} = \phi(x)$ and momentum variables $p_x^{\phi} = \dot{\phi}(x) = \pi(x)$.

and the canonical form is

$$\Omega = \int dx \; dp_x \, dq_x$$

Notice that I am thinking of p_x, q_x as functions on the states ϕ, and that a state is determined by its Cauchy data $\phi(0, \;)$, $\dot\phi(0, \;)$ at time 0.

So now to get the 2-form on S take a pair of states ϕ, ψ and regard them as vector fields and form $i(\phi) i(\psi)\Omega$ which is a constant function on S.

$$i(\phi)\, dp_x = p_x(\phi) = \dot\phi(x)$$

$$i(\phi)\, dq_x = q_x(\phi) = \phi(x)$$

So

$$i(\phi) i(\psi)\Omega = \int dx \; i(\phi)\left[\dot\psi(x)\, dq_x - dp_x\, \psi(x)\right]$$

$$= \int dx \left[\phi(x)\dot\psi(x) - \dot\phi(x)\psi(x)\right]$$

should be the canonical 2-form on S. Check that it doesn't vary in time.

$$\int dx \left[\phi\ddot\psi - \ddot\phi\psi\right] = \int dx \left[\phi(\psi_{xx} - \cancel{V\psi}) - (\phi_{xx} - \cancel{V\phi})\psi\right]$$

$$= \int dx \left[-\phi_x\psi_x + \phi_x\psi_x\right] = 0$$

Summary: The solutions of

$$\left[\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} + V\right]\phi = 0$$

defined for all t and which are smooth and of compact support in x for any fixed t form a vector space S on which we have a skew-symmetric form which is invariant under time evolution. Now the program is to quantize S. What one would like is a unique Hilbert space representation of the commutation relations, such as that provided by the Stone von Neumann thm.

Let recall the algebra. Let W be a vector space and $S(W^*)$ the polynomial algebra on W^*. Then for each $\lambda \in W^*$ we have multiplication by λ, call it $e(\lambda)$, and for each $w \in W$ we have differentiation in the direction w call it $\partial(w)$. These satisfy the commutation relations

$$[e(\lambda), e(\lambda')] = [\partial(w), \partial(w')] = 0$$

$$[\partial(w), e(\lambda)] = (w, \lambda).$$

Let $V = W \oplus W^*$ act on $S(W^*)$ by associating to $v = \binom{w}{\lambda}$ the operator $\rho(v) = \partial(w) + e(\lambda)$. Then

$$\begin{aligned}[\rho(v), \rho(v')] &= [\partial(w) + e(\lambda), \partial(w') + e(\lambda')] \\ &= (w, \lambda') - (w', \lambda) \\ &= \{v, v'\}\end{aligned}$$

where the skew-form $\{\,,\,\}$ is defined on V by

$$\{w \oplus \lambda\ ,\ w' \oplus \lambda'\} = (w, \lambda') - (w', \lambda).$$

Now we can form the Weyl algebra $W(V)$ of V by dividing $T(V)$ out by the ideal generated by

$$v_1 \otimes v_2 - v_2 \otimes v_1 - \{v_1, v_2\}.$$

Then ρ defines an algebra map

$$W(V) \longrightarrow \operatorname{End}(S(W^*)).$$

One of the examples I once understood well is [redacted] the poly ring $\mathbb{C}[z_1, \ldots, z_n]$ equipped with the Gaussian measure inner product

$$\|f\|^2 = \int |f|^2 e^{-|z|^2}\, dV$$

so that $\quad (z_i f, g) = \int z_i f \bar{g}\, e^{-\sum |z_j|^2}\, dV$

$$(\text{[redacted]} f, \tfrac{\partial}{\partial z_i} g) = \int f \frac{\partial}{\partial \bar{z}_i} \bar{g}\ e^{-|z|^2}\, dV$$

$$= \int \bar{g} \left(-\frac{\partial}{\partial \bar{z}_i}\right)\left(f e^{-|z|^2}\right) dV$$

$$= \int \bar{g} f e^{-|z|^2}(+z_i)\, dV = (z_i f, g).$$

In this case V is generated by the operators $\frac{\partial}{\partial z_i} = a_i$ and their adjoints $a_i^* = z_i$. So V is closed under $*$ which is a conjugate linear involution. Thus V is the complexification of a [redacted] real symplectic vector space.

$$(z^n, z^n) = (z^{n-1}, \tfrac{d}{dz} z^n) = n(z^{n-1}, z^{n-1})$$

$$\therefore\ \|z^n\|^2 = n! \qquad \|z_1^{\alpha_1} \cdots z_n^{\alpha_n}\|^2 = \alpha_1! \cdots \alpha_n!$$

<u>February 18, 1979</u>

Solutions of the KG equation which are of compact support in x form a complex vector space with the skew-symmetric form.

$$\{\phi,\psi\} = \int dx\,[\phi\dot\psi - \dot\phi\psi]$$

Take the Fourier transform in x:

$$\phi(t,x) = \int \frac{d\xi}{2\pi} e^{-i\xi x}\hat\phi(t,\xi)$$

$$\Big(\frac{d^2}{dt^2} + \underbrace{\xi^2+m^2}_{\omega^2}\Big)\hat\phi(t,\xi) = 0$$

since

$$\int dx\,\phi\dot\psi = \int dx \int \frac{d\xi}{2\pi} e^{-i\xi x}\hat\phi(t,\xi)\int\frac{d\xi'}{2\pi}e^{-i\xi' x}\hat{\dot\psi}(t,\xi')$$

$$= \int\frac{d\xi}{2\pi}\int\frac{d\xi'}{2\pi}\hat\phi(t,\xi)\hat{\dot\psi}(t,\xi')\underbrace{\int dx\, e^{-i(\xi+\xi')x}}_{2\pi\delta(\xi+\xi')}$$

$$= \int\frac{d\xi}{2\pi}\hat\phi(t,\xi)\hat{\dot\psi}(t,-\xi)$$

it follows that

$$\{\phi,\psi\} = \int\frac{d\xi}{2\pi}\underbrace{\begin{vmatrix}\hat\phi(t,\xi) & \hat\psi(t,-\xi)\\ \hat{\dot\phi}(t,\xi) & \hat{\dot\psi}(t,-\xi)\end{vmatrix}}_{}$$

Wronskian of the solutions $\hat\phi(t,\xi), \hat\psi(t,-\xi)$ of $\Big(\frac{d^2}{dt^2}+\xi^2+m^2\Big)u = 0$.

Recall that ϕ is a positive energy soln. when

$$\hat{\phi}(t,\xi) = e^{-i\omega t}\hat{\phi}(0,\xi)$$

So we see that if ψ is [redacted] also a pos. energy soln.

$$\hat{\psi}(t,-\xi) = e^{-i\omega(-\xi)t}\hat{\psi}(0,-\xi)$$

$$= e^{-i\omega t}\hat{\psi}(0,-\xi)$$

since $\omega(-\xi) = \omega(\xi)$. Thus the subspace[redacted] of pos. energy solns. is isotropic for $\{\,,\,\}$, and similarly for the negative energy solutions. Moreover if $\phi \in$ P.E. then

$$\overline{\phi} = \overline{\int \frac{d\xi}{2\pi} e^{-i\xi x - i\omega t}\hat{\phi}(0,\xi)} = \int \frac{d\xi}{2\pi} e^{i\xi x + i\omega t}\overline{\hat{\phi}(0,\xi)}$$

has N.E. and

$$= \int \frac{d\xi}{2\pi} e^{-i\xi x + i\omega t}\overline{\hat{\phi}}(0,-\xi)$$

$$\{\overline{\phi},\psi\} = \int \frac{d\xi}{2\pi} \begin{vmatrix} \overline{\hat{\phi}(0,-\xi)} & \hat{\psi}(0,-\xi) \\ i\omega\,\overline{\hat{\phi}(0,-\xi)} & -i\omega\,\hat{\psi}(0,-\xi) \end{vmatrix}$$

$$= \int \frac{d\xi}{2\pi}(-2i\omega)\,\overline{\hat{\phi}}(0,-\xi)\,\hat{\psi}(0,-\xi)$$

$$= -i\int \frac{d\xi}{2\pi} 2\omega\, \overline{\hat{\phi}}\,\hat{\psi}$$

Thus $i\{\overline{\phi},\psi\}$ = inner product [redacted] $(\hat{\phi}|\hat{\psi})$ [redacted] wrt the measure $2\omega\frac{d\xi}{2\pi}$.

Let $V = \mathcal{S}$ denote the space of solutions of the [redacted] KG equation, let W^+ (resp[redacted] W^-) be the positive energy (respectively negative energy) solutions, so that

$$V = W^- \oplus W^+$$

(I assume these spaces are completed to Hilbert spaces suitably)

When we quantize we want linear functions on V to become operators. For example, the field operator $\underset{\sim}{\phi}(x)$ is the operator belonging to the linear function $q_x : \phi \longrightarrow \phi(x)$.

Recall that one wants

$$\underset{\sim}{\phi}(x) = \underset{\sim}{\phi}^+(x) + \left[\underset{\sim}{\phi}^+(x)\right]^*$$

where $\underset{\sim}{\phi}^+(x) = \sum_\alpha f_\alpha(x)\, a_\alpha$ with f_α an orth. basis for W^+ and a_α the corresponding destruction operators. [redacted] In order for this to be invariant expression the destruction operator belonging to $f \in W^+$ must be conjugate linear in f. Thus

$$\text{Fock space} = S(W^+)$$

[redacted] creation operators: $e(w)$ $\quad w \in W^+$

destruction ": $\partial(w$[redacted]$^\dagger)$ where $w^\dagger$ is the linear functional on W^+:

$w' \longmapsto (w^\dagger | w') = (w', w)$

↑ inner product.

So we need to know what inner product to put on W^+.

So on $S(W^+)$ I have the operators $e(w)$ and $\partial($[redacted]$)$ $w^\dagger$ and I want these operators to correspond to linear functions on V.

February 19, 1979

Let's try to understand the scattering business - interaction picture, S matrix, etc. on the classical level.

To fix the ideas consider 1-dimensional motion

$$\ddot{x} = -\frac{\partial V}{\partial x}$$

where V is a function of x,t. Let $\mathcal{S}$ denote the set of states, i.e. global solutions of this DE. Note that there is no time translation motion on $\mathcal{S}$. $\mathcal{S}$ is a 2-dimensional manifold; one gets coordinates by picking a time and taking the position and velocity at that time.

What structure does $\mathcal{S}$ have? Is it a symplectic manifold in a natural way? To answer this let q_t, p_t denote the functions on $\mathcal{S}$ which give the position and velocity of the state at time t. Then the question is whether the 2-form $dp_t\, dq_t$ is independent of t.

Notationally this is very difficult. Let's coordinatize $\mathcal{S}$ using $\alpha = q_0$, $\beta = p_0$ and let

$$f(t, \alpha, \beta)$$

denote the solution of the DE above with

$$f(0,\alpha,\beta) = \alpha$$
$$\frac{\partial f}{\partial t}(0,\alpha,\beta) = \beta$$

Then

$$q_t : (\alpha,\beta) \longmapsto f(t,\alpha,\beta)$$
$$p_t : (\alpha,\beta) \longmapsto \frac{\partial f}{\partial t}(t,\alpha,\beta)$$

$$dq_t = \frac{\partial f}{\partial \alpha} d\alpha + \frac{\partial f}{\partial \beta} d\beta$$

Here t is constant and we taking forms on $\mathcal{S}$.

$$dp_t = \frac{\partial^2 f}{\partial\alpha\partial t} d\alpha + \frac{\partial^2 f}{\partial\beta\partial t} d\beta$$

$$dp_t dq_t = \left[\frac{\partial f}{\partial \alpha}\frac{\partial^2 f}{\partial\beta\partial t} - \frac{\partial^2 f}{\partial\alpha\partial t}\frac{\partial f}{\partial \beta}\right] d\alpha\, d\beta$$

Now we to see if this form on the right is time-independent:

$$\frac{\partial}{\partial t}\left\{\frac{\partial f}{\partial \alpha}\frac{\partial^2 f}{\partial\beta\partial t} - \frac{\partial^2 f}{\partial\alpha\partial t}\frac{\partial f}{\partial \beta}\right\} = \frac{\partial f}{\partial \alpha}\frac{\partial^3 f}{\partial\beta\partial^2 t} - \frac{\partial^3 f}{\partial\alpha\partial t^2}\frac{\partial f}{\partial \beta}$$

$$= \frac{\partial f}{\partial \alpha}\frac{\partial}{\partial \beta}\left(-\frac{\partial V}{\partial x}(f,t)\right) - \frac{\partial}{\partial \alpha}\left(-\frac{\partial V}{\partial x}(f,t)\right)\frac{\partial f}{\partial \beta}$$

$$= \frac{\partial f}{\partial \alpha}\left(-\frac{\partial^2 V}{\partial x^2}(f,t)\frac{\partial f}{\partial \beta}\right) - \left(-\frac{\partial^2 V}{\partial x^2}(f,t)\frac{\partial f}{\partial \alpha}\right)\frac{\partial f}{\partial \beta} = 0.$$

So it works, but there should be a better way of seeing this.

Let X be configuration space. One has an isom.

$$\mathcal{S} \times \mathbb{R} \xrightarrow{\ \sim\ } T_X \times \mathbb{R}$$

$$(t \mapsto x(t)\ \text{[redacted]}, t) \longmapsto (x(t), \dot{x}(t), t).$$

Somehow what is going on is that one has a flow on $T_X \times \mathbb{R}$ compatible with the translation on $\mathbb{R}$; it is given by Lagrange's equation

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q} = 0 \quad \text{and} \quad \frac{d}{dt} q = \dot{q}$$

$\mathcal{S}$ appears as the set of horizontal sections of $T_X \times \mathbb{R}/\mathbb{R}$.

On the other hand one has

$$T_X \times \mathbb{R} \longrightarrow T_X^* \times \mathbb{R}$$
$$(q, \dot q, t) \longmapsto (q, p, t)$$

given by $p = \dfrac{\partial L}{\partial \dot q}$. In good cases this map will be an isom., so we can define the Hamiltonian

$$H(q,p,t) = p\dot q - L(q,\dot q, t)$$

as a function on $T_X^* \times \mathbb{R}$. Then

$$\frac{\partial H}{\partial q} = \underbrace{\frac{\partial p}{\partial q}}_{0}\dot q + p\frac{\partial \dot q}{\partial q} - \frac{\partial L}{\partial q} - \frac{\partial L}{\partial \dot q}\frac{\partial \dot q}{\partial q} = -\frac{\partial L}{\partial q} \overset{\text{Lag. flow on } T_X\times\mathbb{R}}{=} -\frac{d}{dt}(p)$$

$$\frac{\partial H}{\partial p} = \dot q + p\frac{\partial \dot q}{\partial p} - \frac{\partial L}{\partial \dot q}\frac{\partial \dot q}{\partial p} = \dot q \overset{\text{Lag. flow on } T_X\times\mathbb{R}}{=} \frac{d}{dt}(q)$$

so the Lagrange flow on $T_X \times \mathbb{R}$ becomes Hamilton's flow on $T_X^* \times \mathbb{R}$. But the form $dp \wedge dq$ is invariant (?) under Hamilton's flow because if we are on a solution curve

$$\frac{d}{dt}(dp\,dq) = d\left(-\frac{\partial H}{\partial q}\right)dq + dp\, d\left(\frac{\partial H}{\partial p}\right)$$
$$= -\frac{\partial^2 H}{\partial t\,\partial q}\,dt\,dq + \frac{\partial^2 H}{\partial t\,\partial p}\,dp\,dt$$
$$= d\left(\frac{\partial H}{\partial t}\right)\cdot dt$$

This will die when one restricts to a fixed t ?

The above is highly confusing and so it is necessary to understand things a bit better. The first thing to note is that the Euler-Lagrange DE

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot q}\right) = \frac{\partial L}{\partial q} \qquad \text{or}$$

$$\frac{\partial^2 L}{\partial \dot q^2}\frac{d}{dt}(\dot q) + \frac{\partial^2 L}{\partial q \partial \dot q}\frac{dq}{dt} + \frac{\partial^2 L}{\partial t \partial \dot q} = \frac{\partial L}{\partial q}$$

will not give a flow on $T_X \times \mathbb{R}$ unless $\frac{\partial^2 L}{\partial \dot q^2} \neq 0$. This is the same condition required [redacted] to [redacted] solve for $\dot q$ as a function of p, q, t, at least locally. Therefore if we assume $\frac{\partial^2 L}{\partial \dot q^2} \neq 0$ we get that

$$T_X \times \mathbb{R} \longrightarrow T_X^* \times \mathbb{R}$$

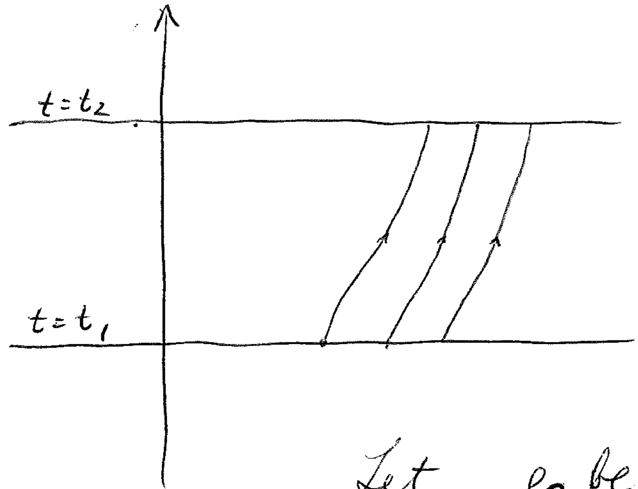
is an [redacted] immersion, [redacted] and the flow is described by Hamilton's equations, where p, q are good local coords on $T_X \times \mathbb{R}$. To simplify assume the above map is an isomorphism. Then we have the following situation. We have on $T_X^* \times \mathbb{R}$ a function $H(q,p,t)$ and the flow given by the [redacted] vector field

$$\xi = \frac{\partial H}{\partial p}\frac{\partial}{\partial q} - \frac{\partial H}{\partial q}\frac{\partial}{\partial p} + \frac{\partial}{\partial t}.$$

Thus for any "dynamical variable" F we have

$$\frac{d}{dt}F = \{F, H\} + \frac{\partial F}{\partial t}$$

Now the basic problem or question appears as follows. By solving Hamilton's equations between t_1 and t_2 we get



a mapping from T_X^* to itself. We want to know whether this map preserves the symplectic structure on T_X^*, even when H depends on t.

Let ρ_ε be the automorphism of $T_X^* \times \mathbb{R}$ given by integrating the vector field ξ thru time ε. Thus

$$(\xi f)(x) = \frac{f(\rho_\varepsilon(x)) - f(x)}{\varepsilon}$$

$$\rho_\varepsilon^* f = \exp(\varepsilon \xi) f$$

Then we are interested in showing

$$\rho_{t_2-t_1}^* (dp\,dq) \text{ pulled back to } T_X^* \text{ via } j_{t_1} : (q,p) \mapsto (q,p,t_1)$$

doesn't depend on t_2. We can simplify by supposing $t_2 = t$, $t_1 = 0$. Then

$$\frac{d}{dt} \rho_t^* (dp\,dq) = \exp(t\xi)\, \theta(\xi)\, dp\,dq$$

$$= \exp(t\xi)\left(d\left(\frac{\partial H}{\partial t}\right) dt\right) \qquad \text{(see bottom p. 594)}$$

$$= \rho_t^*\left(d\left(\frac{\partial H}{\partial t}\right) dt\right).$$

Now when this is pulled back to T_X^* via j_0 one gets the form $d\left(\frac{\partial H}{\partial t}\right) dt$ pulled back via $\rho_t j_0$ which has its image contained in $T_X^* \times \{t\}$. Since dt vanishes on this submanifold, it's clear we win.

Different proof. Think of having functions p_t, q_t given on [redacted] the manifold S which depend on t. Precisely, [redacted] describe S via initial values α, β at $t=0$, let $q(t,\alpha,\beta)$, $p(t,\alpha,\beta)$ be the solns. of Hamilton's equations with initial values α, β, and then [redacted] define q_t by $q_t(\alpha,\beta) = q(t,\alpha,\beta)$. So the problem is to show $dp_t\, dq_t$ is independent of t. [redacted] So differentiate wrt t:

$$\frac{d}{dt}(dp_t\, dq_t) = d\Big(-\frac{\partial H}{\partial q}\,\text{[redacted]}\,(q_t,p_t,t)\Big)\, dq_t + dp_t\, d\Big(\frac{\partial H}{\partial p}(q_t,p_t,t)\Big)$$

$$= -H_{qp}(q_t,p_t,t)\, dp_t\, dq_t + dp_t\big(H_{\text{[redacted]}pq}\, dq_t\big) = 0$$

This proof seems simpler. One is using that H is a function of variables q, p, t and that [redacted] S is a manifold with functions q_t, p_t satisfying Hamilton's equations for [redacted] the Hamiltonian function H.

Summarizing: On the space of ~~solutions~~ states S of a classical mechanical system one has a canonical symp. [redacted] form. So S is a symplectic manifold and it has [redacted] Poisson bracket defined on functions. For each time t the functions p_t, q_t on S give a complete description, i.e. coordinate system on S. The transformation relating p_{t_1}, q_{t_1} and p_{t_2}, q_{t_2} is a symplectic transf.

February 20, 1979

Review yesterday. We consider a classical mechanical system described by Hamilton's equations

$$\dot{q} = \frac{\partial H}{\partial p} \qquad \dot{p} = -\frac{\partial H}{\partial q}$$

where H is function of q, p, t. States of the system are solutions of these DE's; they form a manifold S on which one has functions q_t, p_t for each time t, which give a complete description in the sense that

$$S \xrightarrow{(q_t, p_t)} \mathbb{R}^2$$

is an isomorphism. The form $dp_t\, dq_t$ on S is independent of t, so S is a symplectic manifold in a canonical way, hence equipped with a Poisson bracket $\{\ ,\ \}$. We get a function H_t on S by $H_t = H(q_t, p_t, t)$.

$$\{q_t, H_t\} = (q_t, p_t)^*\{q, H\} = (q_t, p_t)^*\left(\frac{\partial H}{\partial p}\right)$$
$$= \frac{\partial H}{\partial p}(q_t, p_t, t) = \frac{dq_t}{dt}.$$

similarly $\quad \{p_t, H_t\} = \dfrac{dp_t}{dt}$. So therefore

a state of the system is a point of S and the functions giving position and momentum vary in time according to the above DE's where $\{\ \}$ is the natural Poisson bracket on S and H_t is the Hamiltonian function. This description of the system is the classical Heisenberg picture.

The quantum Heisenberg picture looks as follows. Quantum states are lines in a Hilbert space $\mathcal{H}$, the position and momentum at time t are operators q_t, p_t such that

$$[p_t, q_t] = \hbar/i$$

The dynamics are given by

$$\dot{p} = -\frac{i}{\hbar}[p, H] = [+\frac{i}{\hbar}H, p]$$

$$\dot{q} = -\frac{i}{\hbar}[q, H] = [+\frac{i}{\hbar}H, q]$$

where $H = H_t$ is a Hermitian operator possibly depending on times (In practice $\mathcal{H}$ is a quantization of $\mathcal{S}$ which means that there is a correspondence of sorts between functions f on $\mathcal{S}$ and operators $\underline{f}$ on $\mathcal{H}$ such that

$$-\frac{i}{\hbar}[\underline{f}, \underline{g}] \text{ corresponds to } \{f, g\}$$

$$[\underline{f}, \underline{g}] \quad \text{"} \quad \text{"} \quad i\hbar\{f, g\} \text{ .)}$$

If we fix $\mathcal{H}$, then by the Stone-von Neumann theorem we get a complete description of the quantum states using the eigenvectors of q_t in the sense that we get an isomorphism (unique up to a scalar)

$$\theta_t : \mathcal{H} \xrightarrow[\sim]{} L^2(\mathbb{R})$$

$$q_t \longmapsto \text{mult. by } x$$

$$p_t \longmapsto \frac{\hbar}{i}\frac{d}{dx}$$

Assume the Θ_t can be picked nicely. Also since

$$q_t = \Theta_t^{-1} x \Theta_t$$ [redacted]

$$\dot{q}_t = -\Theta_t^{-1}\dot{\Theta}_t\Theta_t^{-1} x \Theta_t + \Theta_t^{-1} x \dot{\Theta}_t$$

$$= \left[-\Theta_t^{-1}\dot{\Theta}_t , q_t\right]$$

and similarly for p_t, or any poly in p_t, q_t, we will suppose we can arrange

$$-\Theta_t^{-1}\dot{\Theta}_t =$$ [redacted] $$\frac{i}{\hbar} H_t$$

(This is really [redacted] no restriction since H_t is determined only by its effect as a commutator, so could start by picking the Θ_t and then defining H_t by this formula.) But now if $\psi \in \mathcal{H}$ is the state vector of the system, then $\psi_t = \Theta_t \psi$ will be the state in the position representation, [redacted] and

$$\frac{d}{dt}\psi_t = \dot{\Theta}_t \psi = -\frac{i}{\hbar}\Theta_t H_t \psi = -\frac{i}{\hbar}\Theta_t H_t \Theta_t^{-1}\psi_t$$

which is the Schroedinger equation. Notice that if H_t the Heisenberg Hamiltonian doesn't depend on t, then neither does the Schroedinger Hamiltonian:

$$\frac{d}{dt}\left(\Theta_t H_t \Theta_t^{-1}\right) = \dot{\Theta}_t H_t \Theta_t^{-1} -$$ [redacted] $$\Theta_t H_t \Theta_t^{-1}\dot{\Theta}_t\Theta_t^{-1}$$

$$= \left[\dot{\Theta}_t\Theta_t^{-1}, \Theta_t H_t \Theta_t^{-1}\right] = \left[-\frac{i}{\hbar}\Theta_t H_t \Theta_t^{-1}, \Theta_t H_t \Theta_t^{-1}\right] = 0.$$

February 21, 1979

Consider a classical mechanical system described by Hamilton's equations

$$\dot{q} = \frac{\partial H}{\partial p} \qquad \dot{p} = -\frac{\partial H}{\partial q}$$

where $H = H(p,q,t)$. The states are solutions of these equations; they form a manifold S. The values of the variables q, p at any time t give a complete description of the states in the sense that

$$\Theta_t : S \xrightarrow[\sim]{(q_t, p_t)} \mathbb{R}^2$$

is an isomorphism. For two times t_1, t_2 there is a transformation function W:

$$\begin{tikzcd} & S \arrow[dl, "\Theta_{t_1}"'] \arrow[dr, "\Theta_{t_2}"] & \\ \mathbb{R}^2 \arrow[rr, "W^{t_1}_{t_2}"] & & \mathbb{R}^2 \end{tikzcd}$$

relating these description which is a symplectic automorphism of $\mathbb{R}^2$.

Quantum viewpoint: Here the states are the lines in a Hilbert space $\mathcal{H}$ and a complete description at time t consists of [redacted] a Hilbert space isomorphism

$$\Theta_t : \mathcal{H} \xrightarrow{\sim} L^2(\mathbb{R})$$

such that operators $\begin{cases} q_t \\ p_t \end{cases}$ correspond to $\begin{matrix} x \\ \frac{\hbar}{i}\frac{\partial}{\partial x} \end{matrix}$

Time evolution is described by Schroedinger's equation

$$\frac{\partial \psi}{\partial t} = -\frac{i}{\hbar} H\psi$$

Here the transformation operator

$$\begin{tikzcd}
& \mathcal{H} \arrow[dl, "\theta_{t_1}"'] \arrow[dr, "\theta_{t_2}"] & \\
L^2(\mathbb{R}) \arrow[rr, "W_{t_2}^{t_1}"'] & & L^2(\mathbb{R})
\end{tikzcd}$$

is a unitary operator which we can represent as a kernel

$$(W_{t_2 t_1} f)(x) = \int K(x, t_2; x', t_1)\, f(x')\, dx'$$

The kernel $K(x,t; x', t')$ is not a Green's function for the Schroedinger equation because it satisfies the (homog.) equation in x, t. It is the solution with Cauchy data $\delta(x-x')$ at $t=t'$. However if we redefine it to be zero for $t<t'$, then it jumps by $\delta(x-x')$ as t crosses t', so we get the forward Green's function. Thus

$$K = \frac{1}{2}\left(G_{forw} - G_{back}\right).$$

<u>Example:</u> [redacted] The harmonic oscillator:

$$H_0 = \frac{1}{2}(p^2 + q^2)$$

With $\hbar = 1$, the Schroedinger DE is

$$i\frac{\partial \psi}{\partial t} = \frac{1}{2}\left(-\frac{\partial^2}{\partial x^2} + x^2\right)\psi.$$

The spectrum is discrete:

$$H_0 u_n = E_n u_n$$

where u_n is an orthonormal basis for $L^2(\mathbb{R})$, essentially a Gaussian factor times Hermite polys. General soln of Schroedinger equation is

$$\psi(x,t) = \sum_n e^{-iE_n t} u_n(x)\, a_n \qquad a_n \text{ const.}$$

so it is clear that the propagator or transformation function is

$$K_0(x,t,x',t') = \sum_n e^{-iE_n(t-t')} u_n(x)\overline{u_n(x')}$$

February 22, 1979:

The idea now is to understand the field equation
$$\ddot{\phi} = -(-\Delta + m^2)\phi$$

as a continuous system of coupled harmonic oscillators. The latter would be described by
$$\ddot{\vec{q}} = -P\vec{q}$$

where $\vec{q} = (q_1, \ldots, q_n)$ and P is a positive definite matrix.

Begin with the simple harmonic oscillator
$$\ddot{q} = -\gamma^2 q$$

The Hamiltonian is $\quad H(p,q) = \frac{1}{2}(p^2 + \gamma^2 q^2)$ and the solutions are
$$\begin{cases} q = c_1 e^{-i\gamma t} + c_2 e^{i\gamma t} \\ p = c_1(-i)\gamma e^{-i\gamma t} + c_2\, i\gamma e^{i\gamma t} \end{cases}$$

or more simply
$$q + \frac{i}{\gamma}p = \text{const } e^{-i\gamma t}$$
$$q - \frac{i}{\gamma}p = \text{const } e^{i\gamma t}$$

Next consider the quantum situation
$$H = \frac{1}{2}\left(-\frac{\partial^2}{\partial x^2} + \gamma^2 x^2\right)$$
$$\frac{1}{2}\left(-\frac{\partial}{\partial x} + \gamma x\right)\left(\frac{\partial}{\partial x} + \gamma x\right) = \frac{1}{2}\left\{-\frac{\partial^2}{\partial x^2} + \gamma^2 x^2 + \gamma\right\}$$

Thus if we put

$$a = \frac{1}{\sqrt{2\gamma}}\left(\frac{\partial}{\partial x} + \gamma x\right)$$ destruction [redacted] operator

$$a^* = \frac{1}{\sqrt{2\gamma}}\left(-\frac{\partial}{\partial x} + \gamma x\right)$$ creation operator

then we have

$$[a, a^*] = \frac{1}{2\gamma}\left[\frac{\partial}{\partial x} + \gamma x, -\frac{\partial}{\partial x} + \gamma x\right] = 1$$

and

$$a^* a = \frac{1}{2\gamma}\left(-\frac{\partial^2}{\partial x^2} + \gamma^2 x^2 \text{[redacted]} \gamma\right) = \frac{1}{\gamma} H \text{[redacted]} \frac{1}{2}$$

The eigenfunctions are obtained by starting with the lowest energy state, which given by

$$a\psi_0 = \frac{1}{\sqrt{2\gamma}}\left(\frac{\partial}{\partial x} + \gamma x\right)\psi_0 = 0$$

$$\psi_0 = \text{const} \cdot e^{-\frac{1}{2}\gamma x^2}$$

$$\int \text{[redacted]} \left|e^{-\frac{1}{2}\gamma x^2}\right|^2 dx = \int e^{-\gamma x^2} dx = \int e^{-x^2} \frac{dx}{\sqrt{\gamma}} = \sqrt{\frac{\text{[redacted]}\pi}{\gamma}}$$

$$\therefore \quad \psi_0 = \sqrt{\frac{\pi}{\gamma}}\, e^{-\frac{1}{2}\gamma x^2}$$

Then

$$(a^* a)(a^*)^n \psi_0 = a^*[a, a^{*n}]\psi_0$$

$$\left(\frac{1}{\gamma} H \text{[redacted]} \frac{1}{2}\right)(a^*)^n \psi_0 = n(a^*)^n \psi_0$$

$$H(a^*)^n \psi_0 = \gamma\left(n + \tfrac{1}{2}\right)(a^*)^n \psi_0$$

so the eigenvalues of H are

$$\gamma\left(n + \tfrac{1}{2}\right) \qquad n = 0, 1, 2, \ldots$$

Since $\|a^{*n}\psi_0\|^2 = (a a^{*n}\psi_0, (a^*)^{n-1}\psi_0) = n\|a^{*n-1}\psi_0\|^2 = n!$

orthonormalized eigenfns. are

$$u_n = \frac{1}{\sqrt{n!}} \text{[redacted]} (a^*)^n u_0 \quad , \qquad u_0 = \phi_0 \text{ above.}$$

[redacted] Next suppose we have a system of independent harmonic oscillators:

$$H = \sum_j \frac{1}{2}\left(p_j^2 + \gamma_j^2 q_j^2\right)$$

and we define annihilation + creation ops. by

$$a_j = \frac{1}{\sqrt{2\gamma_i}}\left(\frac{\partial}{\partial x_j} + \gamma_j x_j\right)$$

$$a_j^* = \frac{1}{\sqrt{2\gamma_i}}\left(-\frac{\partial}{\partial x_j} + \gamma_j x_j\right)$$

Then

$$H = \sum \gamma_j a_j^* a_j + \frac{1}{2}\sum \gamma_j .$$

We get an orthonormal basis for our Hilbert space consisting of

$$\frac{1}{\sqrt{n_1! \, n_2! \cdots\cdots}} (a_1^*)^{n_1} (a_2^*)^{n_2} \cdots\cdots u_0 .$$

Next let us consider quantizing the field theory given by

$$\ddot\phi = -(-\Delta + m^2)\phi$$

Using the F.T. in x, we can regard this as a continuum of independent harmonic oscillators

$$\ddot{\hat\phi}(t,\xi) = -(\xi^2 + m^2)\hat\phi(t,\xi)$$

Let's consider things classically. Let our classical states be certain global solutions of the field equation

$$\phi^{\cdot\cdot} = -(-\Delta + m^2)\phi$$

say the ones which belong to C_0^∞ as functions of x. Call $\mathcal{S}$ the set of classical states. Introduce on $\mathcal{S}$ the functions

$$(*) \qquad \begin{aligned} q_{\xi,t}(\phi) &= \int \frac{dx}{\sqrt{2\pi}}\, e^{i\xi x}\, \phi(t,x) \\ p_{\xi,t}(\phi) &= \int \frac{dx}{\sqrt{2\pi}}\, e^{i\xi x}\, \phi^{\cdot}(t,x) \end{aligned}$$

THESE AREN'T REAL SO THERE ARE PROBLEMS.

giving the ξ-th position and momentum at time t. Then

$$\dot q_{\xi,t} = p_{\xi,t}$$

$$\dot p_{\xi,t}(\phi) = \int \frac{dx}{\sqrt{2\pi}}\, e^{i\xi x}\, \phi^{\cdot\cdot}(t,x) = \int \frac{dx}{\sqrt{2\pi}}\, e^{i\xi x}(-\Delta + m^2)\phi(t,x)$$

$$= -(\xi^2 + m^2) \int \frac{dx}{\sqrt{2\pi}}\, e^{i\xi x}\, \phi(t,x) \qquad \text{so}$$

$$\dot p_{\xi,t} = -(\xi^2 + m^2)\, q_{\xi,t}$$

These are Hamilton's equations with Hamiltonian function

$$H = \int d\xi \tfrac{1}{2}\left(p_\xi^2 + (\xi^2 + m^2) q_\xi^2\right)$$

[redacted] where p_ξ, q_ξ are to be thought of as variables. The pull-back of H to $\mathcal{S}$ by the function $\phi \mapsto q_{\xi,t}(\phi), p_{\xi,t}(\phi)$ is

$$H_t(\phi) = \cancel{2\pi} \int dx\, \frac{1}{2}\left\{ \phi^{\cdot}(t,x)^2 + \frac{\partial\phi}{\partial x}(t,x)^2 + m^2 \phi(t,x)^2 \right\}$$

where the 2π cancels by ~~comes from~~ Parseval's formula. ($\sqrt{2\pi}$ in $(*)$ comes from the fact that one wants q_ξ, p_ξ to define the same symplectic structure as

$$q_{x,t}(\phi) = \phi(t,x) \qquad p_{x,t}(\phi) = \dot{\phi}(t,x)$$

in the sense that $\displaystyle\int dx\, dp_x\, dq_x = \int d\xi\, dp_\xi\, dq_\xi$.

So at this point our classical field theory has been decomposed into a continuous family of harmonic oscillators indexed by ξ. So now we pass to the quantum situation which means that we have Herm. operators q_ξ, p_ξ satisfying the commutation relations

$$[p_\xi, q_{\xi'}] = \frac{1}{i}\delta(\xi,\xi')$$

In terms of these we can introduce annihilation and creation operators by

$$a_\xi = \frac{1}{\sqrt{2\gamma_\xi}}\left(\gamma_\xi q_\xi + i p_\xi\right) \qquad q_\xi = \frac{1}{\sqrt{2\gamma_\xi}}(a_\xi + a_\xi^*)$$

$$a_\xi^* = \frac{1}{\sqrt{2\gamma_\xi}}\left(\gamma_\xi q_\xi - i p_\xi\right) \qquad p_\xi = \frac{\sqrt{2\gamma_\xi}}{2i}(a_\xi - a_\xi^*)$$

satisfying $\quad [a_\xi, a_{\xi'}^*] = \delta(\xi-\xi')$.

But recall the formula

$$q_\xi(\phi) = \int \frac{dx}{\sqrt{2\pi}}\, e^{ix\xi}\phi(x)$$

on the classical level. Use it to define $\phi(x)$ as an operator in the quantum situation. Classically q_x $\phi \mapsto \phi(x)$ is the function giving the x-th position coordinate, i.e. the value of the field on x. In the quantum situation, this number becomes an operator related to the q_ξ by

$$\phi(x) = \int \frac{d\xi}{\sqrt{2\pi}} e^{-ix\xi} q_\xi$$

$$= \int \frac{d\xi}{\sqrt{2\pi}\sqrt{2\gamma_\xi}} e^{-ix\xi}(a_\xi + a_\xi^*)$$

Similarly the momentum operator conjugate to $\phi(x)$ is

$$\pi(x) = \dot{\phi}(x) = \int \frac{d\xi}{\sqrt{2\pi}} e^{-ix\xi} \frac{\sqrt{2\gamma_\xi}}{2i}(a_\xi - a_\xi^*)$$

At this point I perceive a problem in that the operator $\phi(x)$ is not Hermitian. The problem stems from the fact that

$$q_\xi(\phi) = \int \frac{dx}{\sqrt{2\pi}} e^{i\xi x} \phi(x)$$

is not a real function, and similarly for p_ξ. So what one must do is to diagonalize the operator $-\Delta + m^2$ over the reals.

So instead let us introduce the following functions on the space S of solutions of $(\Box + m^2)\phi = 0$

$$q_\xi^{re}(\phi) = \int \frac{dx}{\sqrt{2\pi}} \cos \xi x \, \phi(x) \qquad p_\xi^{re}(\phi) = \int \frac{dx}{\sqrt{2\pi}} \cos \xi x \, \dot{\phi}(x)$$

$$q_\xi^{im}(\phi) = \int \frac{dx}{\sqrt{2\pi}} \sin \xi x \, \phi(x) \qquad p_\xi^{im}(\phi) = \int \frac{dx}{\sqrt{2\pi}} \sin \xi x \, \dot{\phi}(x)$$

Then $\displaystyle\int d\xi \left\{ (p_\xi^{re})^2 + (p_\xi^{im})^2 \right\} = \int d\xi \left| \int \frac{dx}{\sqrt{2\pi}} e^{i\xi x} \dot{\phi}(x) \right|^2$ $\qquad \dot{\phi}$ real

$$= \int dx \, \dot{\phi}(x)^2 \qquad \text{(Parseval)}$$

So therefore

$$\int d\xi \ \frac{1}{2}\left\{ (p_\xi^{re})^2 + (p_\xi^{im})^2 + (\xi^2+m^2)\left[(q_\xi^{re})^2 + (q_\xi^{im})\right]^2 \right\}$$

$$= \int dx \ \frac{1}{2}\left\{ (\dot\phi(x))^2 + \left(\frac{\partial\phi}{\partial x}\right)^2 + m^2\phi^2 \right\} = H(\phi)$$

Similarly $\int d\xi \left\{ dq_\xi^{re}\, dp_\xi^{re} + dq_\xi^{im}\, dp_\xi^{im} \right\} = \int dx \ d\phi(x)\, d\dot\phi(x)$.

So now when we pass to the quantum situation the functions q_ξ^{re}, [redacted] p_ξ^{re}, q_ξ^{im}, p_ξ^{im} become operators in terms of which we can define a_ξ^{re}, a_ξ^{im}, $(a_\xi^{re})^*$, $(a_\xi^{im})^*$. Now ask what is the operator $\phi(x)$?

$$\phi(x) = \int \frac{d\xi}{\sqrt{2\pi}} e^{-i\xi x}\left[q_\xi^{re} + i\, q_\xi^{im} \right]$$

$$= \int \frac{d\xi}{\sqrt{2\pi}} e^{-i\xi x}\left[\frac{1}{\sqrt{2\gamma}}\left(a_\xi^{re} + (a_\xi^{re})^* + i\, a_\xi^{im} + i(a_\xi^{im})^* \right)\right]$$

Now a_ξ^{re} is even in ξ and a_ξ^{im} is odd in ξ. Thus we can write this

$$\phi(x) = \int \frac{d\xi}{\sqrt{2\pi}} \frac{e^{-i\xi x}}{\sqrt{2\gamma}}\left[(a_\xi^{re} + i\, a_\xi^{im}) + (a_{-\xi}^{re} + i\, a_{-\xi}^{im})^* \right]$$

$$= \int \frac{d\xi}{\sqrt{2\pi}\sqrt{2\gamma}} e^{-i\xi x}\left(a_\xi + a_{-\xi}^* \right)$$

$$= \phi^+(x) + \phi^+(x)^*$$

where

$$\phi^+(x) = \int \frac{d\xi}{\sqrt{2\pi}\sqrt{2\gamma}} e^{-i\xi x} a_\xi \qquad a_\xi = a_\xi^{re} + i\, a_\xi^{im}$$

Put $\phi^-(x) = \phi^+(x)^*$

$$[\phi^+(x), \phi^-(x')] = \left[\int \frac{d\xi}{\sqrt{2\pi}\sqrt{2\gamma_\xi}} e^{-i\xi x} a_\xi , \int \frac{d\xi'}{\sqrt{2\pi}\sqrt{2\gamma_{\xi'}}} e^{i\xi' x'} a^*_{\xi'} \right]$$

$$= \iint \frac{d\xi\, d\xi'}{2\pi\, 2\sqrt{\gamma_\xi}\sqrt{\gamma_{\xi'}}} e^{-i\xi x + i\xi' x'} \underbrace{[a_\xi, a^*_{\xi'}]}_{\delta(\xi-\xi')}$$

$$= \int \frac{d\xi}{2\pi\, 2\gamma_\xi} e^{-i\xi(x-x')}$$

The conjugate momentum operator to $\phi(x)$ is

$$\pi(x) = \int \frac{d\xi}{\sqrt{2\pi}} e^{-i\xi x} [p_\xi^{re} + i p_\xi^{im}]$$

$$= \int \frac{d\xi}{\sqrt{2\pi}} e^{-i\xi x} \frac{\sqrt{2\gamma_\xi}}{2i} \left[a_\xi^{re} - (a_\xi^{re})^* + i a_\xi^{im} - i(a_\xi^{im})^* \right]$$

$$\boxed{\begin{aligned} \pi(x) &= \int \frac{d\xi}{\sqrt{2\pi}\sqrt{2}\,i} \sqrt{\gamma_\xi}\, e^{-i\xi x} [a_\xi - a^*_{-\xi}] \\ \phi(x) &= \int \frac{d\xi}{\sqrt{2\pi}\sqrt{2\gamma_\xi}} e^{-i\xi x} [a_\xi + a^*_{-\xi}] \end{aligned}}$$

Schweber has similar formulas (p. 183–184), but his a_ξ is my a_ξ times $\sqrt{\gamma_\xi}$, which is somehow related to his use of the measure $\frac{d\xi}{\gamma_\xi}$. ?

Now if we compute the Hamiltonian operator

$$H = \int dx\, \frac{1}{2} \left\{ \pi(x)^2 + \left(\frac{\partial \phi(x)}{\partial x}\right)^2 + m^2 \phi^2(x) \right\}$$

using the above formula we should get

$$\int d\xi\, \gamma(\xi)\, a^*_\xi a_\xi \quad + \frac{1}{2}\int d\xi\, \gamma(\xi)$$ infinite self-energy for vacuum.