Today I began to understand Schwinger IV.
Consider Dirac field $\Psi(x)$ being perturbed by a
time dependent electromagnetic field $A_\mu(x)$ (elementary
gauge $A=0$ for trivial field). The field equations are

$$\gamma_\mu \left[-i\partial_\mu - e A_\mu(x)\right] \Psi(x) + m \Psi(x) = \gamma(x)$$
$$\left[i\partial_\mu - e A_\mu(x)\right] \bar{\Psi}(x) \gamma_\mu + m \bar{\Psi}(x) = \bar{\gamma}(x)$$

I think this is a non-homogeneous system of the form

$$d\Psi = \gamma$$
$$d^* \Psi = \bar{\gamma}$$

but the significance of $\langle \rangle$ is unclear. In any case, ultimately the only thing that matters is the
Greens function $G^+(x,x')$ which satisfies

$$\gamma_\mu \left[-i\partial_\mu - e A_\mu(x)\right] G^+(x,x') + m G^+(x,x) = \delta(x-x')$$
$$\left[i\partial_\mu - e A_\mu(x)\right] \bar{G}^+(x,x') \gamma_\mu + m \bar{G}^+(x,x') = \delta(x-x')$$

with suitable outgoing boundary conditions ($G^+$ as a
function of $x$ contain only positive frequencies for $x_0 > x'_0$!
and times from Supp A, and only neg. freq. for $x_0 < x'_0$!
and times from Supp A.

The basic quantity of interest is a number $e^{i\omega}$
which turns out to be an infinite determinant obtained
as follows. Connect the field $A$ to $0$ by a sequence of infinitesimal steps $iS\delta w$, where $i\delta w$ is given by a trace

\[
\delta w = \int_{-\infty}^{\infty} (dx) \text{tr} \ i e \gamma A \mathcal{G}_0^+(x, x)
\]

New notation:

\[
[V(p-eA)+m] \mathcal{G}_0^+ = \mathcal{G}_0^+ [V(p-eA)+m] = I
\]

\[
\delta w = \text{Tr} (i e \gamma A \mathcal{G}_0^+) = i \text{Tr} (\mathcal{G}_0^+ e^y A)
\]

On the other hand, if $\mathcal{G}_0^+$ is the free Green's function (for $A=0$), then we get L.S.

\[
\mathcal{G}_0^+ = [1 - \mathcal{G}_0^+ e^y A] \mathcal{G}_0^+
\]

\[
\mathcal{G}_0^+ = \mathbf{1} (1-\mathcal{G}_0^+ e^y A)^{-1} \mathcal{G}_0^+
\]

so

\[
\delta w = i \text{Tr} ((1-\mathcal{G}_0^+ e^y A)^{-1} \mathcal{G}_0^+ e^y \delta A)
\]

\[
= -i \text{Tr} (X^{-1} \delta X) \quad X = 1 - \mathcal{G}_0^+ e^y A
\]

Thus

\[
i \delta w = \text{Tr} (X^{-1} \delta X) = \delta (\log \det X)
\]

so integrating

\[
e^{i\omega} = \det (X) = \det (1 - \mathcal{G}_0^+ e^y A)
\]

\[
= \det (1 - e^y A \mathcal{G}_0^+)
\]

By symmetry considerations (charge conjugation) one knows $\omega$ is an even fn. of $e$, so

\[
e^{2i\omega} = \det (1 - e^y A \mathcal{G}_0^+) \det (1 + e^y A \mathcal{G}_0^+)
\]

\[
= \det (1 - e^2 y A \mathcal{G}_0^+ A \mathcal{G}_0^+)
\]
If one puts
\[ \lambda K = -e^2 \mathcal{F} \mathcal{A} \mathcal{G}_o \mathcal{F} \mathcal{A} \mathcal{G}_o^+ \]
then it turns out that under suitable conditions
\[ \text{tr} (KK^+) < \infty \]
so that the poison tooth Fredholm determinant
\[ \det'(1+\lambda K) = e^{-\text{tr}(\lambda K)} \det(1+\lambda K) \]
is defined. However \( \text{tr}(\lambda K) = -e^2 \text{tr} (\mathcal{F} \mathcal{A} \mathcal{G}_o \mathcal{F} \mathcal{A} \mathcal{G}_o^+) \)
diverges logarithmically. The divergent quantity is real so that \( |e^{i\omega}|^2 \) is finite. In fact it turns
out that \( |e^{i\omega}|^2 \leq 1 \) and that it is the probability of the field remaining in the vacuum state.
February 3, 1979:

Recall the Dirac field perturbed by an external electromagnetic field $A(x)$ is understood in terms of a Green's function satisfying

$$\left[ \gamma(p-eA)+m \right] G^+ = G^+ \left[ \gamma(p-eA)+m \right] = I$$

where $p = \left( \frac{i}{\hbar} \frac{\partial}{\partial x_\mu} \right)$ is a momentum operator. The free Hamiltonian is

$$H_0 = \gamma p + m$$

the perturbation is $V = e \gamma A$, and $G^+$ is an inverse for

$$H = H_0 - V = \gamma p + m - e \gamma A$$

defined via outgoing boundary conditions.

According to the Schwinger paper the quantity of interest is the Hipmann–Schwinger determinant

$$e^{i\omega} = \det \left( H_0^{-1} H \right) = \det(1 - G^+ V) = \det(1 - G^+ e \gamma A)$$

Notice that this differs from what you've looked at in that one is working with a hyperbolic DE and not a Helmholtz DE where time has been replaced by frequency $k$. So the obvious thing is to go back to the one-(space)-dimensional case and see if the above determinant makes sense.

So consider as the base a D-system

$$\frac{d}{dx} \left( \begin{array}{c} u_1 \\ u_2 \end{array} \right) = \left( \begin{array}{cc} ik & p \\ -p & -ik \end{array} \right) \left( \begin{array}{c} u_1 \\ u_2 \end{array} \right)$$

or rather the associated hyperbolic DE, which means $-ik$
should be replaced by \( \frac{2}{\partial t} \). In self-adjoint form:

\[
\begin{pmatrix}
\frac{1}{i} \frac{d}{dx} & -\frac{1}{i} p \\
\frac{1}{i} \bar{p} & -\frac{1}{i} \frac{d}{dx}
\end{pmatrix}
\begin{pmatrix}
u_1 \\
u_2
\end{pmatrix} = k
\begin{pmatrix}
u_1 \\
u_2
\end{pmatrix}
\]

For self-adjoint form:

\[-\frac{2}{\partial t} (u^1) = \begin{pmatrix}
\frac{\partial}{\partial x} & -p \\
\bar{p} & -\frac{\partial}{\partial x}
\end{pmatrix} (u^1)
\]

or

\[
\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \frac{\partial}{\partial t} + \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \frac{\partial}{\partial x} + \begin{pmatrix} 0 & -p \\ -\bar{p} & 0 \end{pmatrix} \right\} (u^1) = 0
\]

However, we can also write it

\[
\left\{ \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \frac{\partial}{\partial t} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \frac{\partial}{\partial x} + \begin{pmatrix} 0 & -p \\ -\bar{p} & 0 \end{pmatrix} \right\} (u^1) = 0
\]

\[
\Gamma_0 \quad \Gamma_1 \quad -V
\]

but then we get a fact when we compute the adjoint. Anyway, to keep close to Schrödinger's notation we use (x), so that \( G_0^+ \) is going to satisfy

\[
\begin{pmatrix}
\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial t^2} & 0 \\
0 & \frac{\partial^2}{\partial x^2}
\end{pmatrix}
\begin{pmatrix}
G_0^+
\end{pmatrix} = \begin{pmatrix}
\delta \\
0
\end{pmatrix}
\]

So now we want to compute this Green's fn. and make sense of the outgoing bdry. conditions. First do for \( \frac{\partial}{\partial y} \):

\[
\frac{\partial}{\partial y} u = f
\]
\[ u(x, y) = \int_{-\infty}^{\infty} f(x, y') \, dy' \]
\[ = \int \int dx' dy' \delta(x-x') \eta(y-y') f(x',y') \]

so
\[ G(x,y;x',y') = \delta(x-x') \eta(y-y') \]

so the Green's function for \( \frac{\partial}{\partial x} + \frac{\partial}{\partial t} \) should be something like
\[ \delta(x-t-(x'-t')) \eta(x+t-(x'+t')) \]

Now
\[ \int \int dx'dt' \delta(x-t-(x'+t')) \eta(x+t-x-t') f(x',t') \]
\[ = \int dt' \eta(x+t-(x-t+t')-t') f(x-t+t',t') \]
\[ = \int_{-\infty}^{t} dt' f(x-t+t',t') \]

and
\[ \left( \frac{\partial}{\partial x} + \frac{\partial}{\partial t} \right) \int_{-\infty}^{t} dt' f(x-t+t',t') = \int_{-\infty}^{t} dt' f(x-t+t',t') + f(x,t) - \int_{-\infty}^{t} f(x-t+t',t') dt' \]
\[ = f(x,t) \]

so this works:
\[ \left( \frac{\partial}{\partial x} + \frac{\partial}{\partial t} \right)^{-1} = \delta(x-t-(x'-t')) \eta(x+t-(x'+t')) \]
+ any function of \( x-t \)

Similarly
\[ \left( \frac{\partial}{\partial x} - \frac{\partial}{\partial t} \right)^{-1} = \delta(x+t-(x'+t')) \eta(x-t-(x'-t')) \]
+ any function of \( x+t \).

Now we have to understand the outgoing boundary conditions.
Let's compute a function via F.T. in time

\[
\left( \frac{\partial}{\partial x} + \frac{\partial}{\partial t} \right) \hat{G} = \delta(x) \delta(t)
\]

\[
\left( \frac{\partial}{\partial x} - i k \right) \hat{G} = \delta(x)
\]

\[
\hat{G} = e^{ikx} \eta(x)
\]

\[
\therefore \quad G = \int e^{-ikt} e^{ikx} \eta(x) \, dk / 2\pi
\]

\[
= \delta(x-t) \eta(x)
\]

which seems to be the same as \( \delta(x-t) \eta(x+t) \). Any other Green's function differs from this one by a function of \( x-t \).

The problem is to pin down the choice of Green's function via the words "\( G^+(x,x') \) as a function of \( x \) shall contain only positive frequencies for \( t > t' \) and only negative frequencies for \( t < t' \)." Let us recall that

\[
\hat{G} = e^{ikx} \eta(x)
\]

is the extension of the \( L^2 \)-Green's function for Im \( k > 0 \) and that

\[
\hat{G} = -e^{ikx} \eta(-x)
\]

is the \( L^2 \)-Green's function for Im \( k < 0 \). Corresponding to \( (4) \) we get

\[
G = -\int e^{ikx} \eta(-x) e^{-ikt} \, dk / 2\pi
\]

\[
= -\delta(x-t) \eta(-x)
\]
so the problem is which one we want.

There exists a distinction called retarded and advanced Green's functions. Retarded means it vanishes for \( t < 0 \), hence clearly

\[
\delta(x-t)\eta(x)
\]

is the retarded Green's function while

\[
-\delta(x-t)\eta(-x)
\]

is the advanced Green's function. Also we can get the retarded Gfn by Laplace transform

\[
\hat{G} = \int_0^{\infty} e^{-ikt} G(x,t) \, dt \quad \text{meaning bounded}
\]

Thus

\[
\hat{G}(x,k) = \begin{cases} 
  e^{ikx} & \quad x > 0 \\
  0 & \quad x < 0
\end{cases}
\]

dc. these are analytic in \( \text{u} + \text{i} \in \text{u} + \text{i} \).

So now it's clear that for the operator

\[
\begin{pmatrix}
\frac{\partial}{\partial x} + \frac{\partial}{\partial t} & 0 \\
0 & \frac{\partial}{\partial x} - \frac{\partial}{\partial t}
\end{pmatrix}
\]

the retarded Green's matrix will satisfy

\[
\hat{G}(x,k) = \begin{pmatrix}
  e^{ikx} \eta(x) & 0 \\
  0 & -e^{-ikx} \eta(-x)
\end{pmatrix}
\]

hence

\[
\hat{G}(x,t) = \begin{pmatrix}
  \delta(x-t)\eta(x) & 0 \\
  0 & -\delta(x+t)\eta(-x)
\end{pmatrix}
\]
It seems clear that there are four types of $G$-fus. to consider. To find the one we want let's take the F.T. \[ \eta(x) \]

\[
\left( -i \frac{\partial}{\partial t} + \frac{\partial^2}{\partial x^2} \right) \hat{G} = \delta(t)
\]

\[
\hat{G} = e^{i \frac{\pi}{2} t} \eta(t) \quad \text{or} \quad -e^{i \frac{\pi}{2} t} \eta(-t)
\]

\[
G = \delta(t-x) \eta(t) \quad \text{or} \quad \delta(t-x) \eta(-t)
\]

\[
\delta(x-t) \eta(x)
\]
Green's functions $G$ for the Klein-Gordon equation:

$$(\Box + \mu^2) \psi = \left( \frac{\partial^2}{\partial t^2} - \sum_{i} \frac{\partial^2}{\partial x_i^2} + \mu^2 \right) \psi = 0$$

We want to solve $$(\Box + \mu^2) \hat{G} = \delta.$$ The Fourier transform satisfies

$$(-k_0^2 + \vec{k}^2 + \mu^2) \hat{G} = 1$$

or

$$\hat{G} = -\frac{1}{k_0^2 - \vec{k}^2 - \mu^2}$$

To accomplish this division of distributions one has to specify for each $k_0$ what happens at the poles $k_0 = \pm \sqrt{k^2 + \mu^2}$

and one does this by inverse transforming at $k_0$ and pushing the integration into the complex plane. So this means one F.T. first w.r.t $\vec{x}$:

$$\left( \frac{\partial^2}{\partial t^2} + \vec{k}^2 + \mu^2 \right) \hat{G} = \delta(t)$$

Put $w = \sqrt{k^2 + \mu^2}$. Different solutions of the last equation are

$$\frac{e^{iwt}}{2i\omega}, \quad \frac{\sin \omega \delta t}{\omega} \eta(t)$$

denoted $\Theta(t)$ by Schwinger.

and in general one uses contour integration:

$$\hat{G}(t, \vec{k}) = \frac{1}{2\pi} \intdk \frac{e^{-ik_0 t} (\pi)}{k_0^2 - \omega^2}$$
Examples are:

\[ \kappa_0 \text{ plane} \]

Here if \( t > 0 \), then \( e^{-i\kappa_0 t} \) decays in the LHP, so you get the \( \kappa_0 = \omega \) pole residue

\[ (-1) \frac{2\pi i}{2\pi} \frac{e^{-i\omega t}}{-2i\omega} \]

If \( t < 0 \), then \( e^{-i\kappa_0 t} \) decays in UHP so we get the \( \kappa_0 = -\omega \) contribution

\[ \frac{2\pi i}{2\pi} \frac{e^{i\omega t}}{-2i\omega} (-1) \]

So

\[ \hat{G}(t, k^3) = \mathcal{M} \frac{e^{-i\omega t}}{-2i\omega} \]

since \( \omega > 0 \), this has positive frequencies for \( t > 0 \) and negative frequencies for \( t < 0 \).

The contour gives the retarded Green's function (i.e. vanishes for \( t < 0 \))

\[ \hat{G}_{\text{ret}}(t, k^3) = \begin{cases} 0 & t < 0 \\ \frac{1}{2\pi} (e^{-i\omega t} - e^{-i\omega t}) + \frac{1}{2\pi} \frac{e^{-i\omega t}}{-2i\omega} & t > 0 \end{cases} \]

\[ = \frac{\sin \omega t}{\omega} \]

\( t > 0 \).
Because of the difficulties encountered in constructing the appropriate Green's function, I should work with something closer to the Dirac equation, i.e., where there is a mass parameter \( m \).

The Dirac equation is of the form

\[
\left\{ \frac{\partial}{\partial t} + \sum_{k=1}^{3} \alpha^k \frac{\partial}{\partial x^k} + i m \beta \right\} \psi = 0
\]

Self-adjointness considerations (it should be of the form \( \{ -i \frac{\partial}{\partial t} + H \} \psi \) with \( H \) hermitian) require that

\[
(\alpha^k)^* = \alpha^k \quad \beta^* = \beta
\]

Also each component should satisfy \((\gamma + m^2) \psi = 0\), so multiplying by \ \frac{\partial}{\partial t} - \sum \alpha^k \frac{\partial}{\partial x^k} - i m \beta \ gives

\[
\left\{ \frac{\partial^2}{\partial t^2} - \left( \sum \alpha^k \frac{\partial}{\partial x^k} + i m \beta \right)^2 \right\} \psi = 0
\]

\[
= \left\{ \frac{\partial^2}{\partial t^2} - \sum \alpha^k \alpha^l \frac{\partial^2}{\partial x^k \partial x^l} - \sum i m (\alpha^k \beta + \beta \alpha^k) \frac{\partial}{\partial x^k} + m^2 \beta^2 \right\} \psi
\]

so the matrices \( \alpha^k, \beta \) satisfy

\[
\frac{1}{2} (\alpha^k \alpha^l + \alpha^l \alpha^k) = \delta^{kl} \quad \beta^2 = 1
\]

\[
\alpha^k \beta + \beta \alpha^k = 0
\]

In 2 space dimensions these become

\[
\alpha^2 = \beta^2 = 1 \quad \alpha \beta + \beta \alpha = 0
\]

so for example

\[
\alpha = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \beta = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}
\]
It is customary to write the Dirac equation in the form
\[
\left\{ \frac{i}{\hbar} \left( \frac{\partial}{\partial t} + \sum_{k=1}^{3} \beta \cdot \frac{x^k}{\hbar} \frac{\partial}{\partial x^k} \right) + m \right\} \psi = 0
\]
or
\[
\left( -i \gamma^\mu \frac{\partial}{\partial x^\mu} + m \right) \psi = 0
\]
In 2-space dimensions we get the equation (before mult. by \( \gamma^0 \))
\[
\left( \frac{\partial}{\partial t} + \frac{\partial}{\partial x} \begin{pmatrix} \text{im} \\ \text{im} \end{pmatrix} \right) \psi = 0
\]
Now I want to see that I can define a Green's function \( G(x, t) \) with only positive frequencies for positive times, etc. The point will be to understand the singularities of this Green's function, not necessarily to get an explicit formula.

Write the equation in the form
\[
\left\{ \frac{\partial}{\partial t} + \begin{pmatrix} \text{im} \\ -\text{im} \end{pmatrix} \right\} \psi = \delta
\]
and take the F.T. w.r.t. \( x \):
\[
\left\{ \frac{\partial}{\partial t} + i \begin{pmatrix} -\xi \\ m \end{pmatrix} \right\} \hat{\psi} = \delta(t)
\]
The matrix \( A \) has the eigenvalues \( \pm \omega = \pm \sqrt{\xi^2 + m^2} \). Corresponding eigenvectors are found.
\[
\begin{pmatrix}
-\frac{\omega}{m} & m \\
m & \frac{\omega}{m}
\end{pmatrix}
\begin{pmatrix}
m \\
\frac{\omega}{m} + m
\end{pmatrix}
= \begin{pmatrix}
-\frac{\omega}{m}m + m \frac{\omega}{m} + m \omega \\
m^2 + \frac{\omega^2}{m^2} + \frac{\omega^2}{m}
\end{pmatrix}
= \omega \begin{pmatrix}
m \\
\frac{\omega}{m} + m
\end{pmatrix}
\]

\[
\begin{pmatrix}
-\frac{\omega}{m} & m \\
m & \frac{\omega}{m}
\end{pmatrix}
\begin{pmatrix}
m \\
\frac{\omega}{m} - m
\end{pmatrix}
= \begin{pmatrix}
-\frac{\omega}{m}m + m \frac{\omega}{m} - m \omega \\
m^2 + \frac{\omega^2}{m^2} + \frac{\omega^2}{m}
\end{pmatrix}
= -\omega \begin{pmatrix}
m \\
\frac{\omega}{m} - m
\end{pmatrix}
\]

Put \[ T = \begin{pmatrix}
m & m \\
\frac{\omega}{m} + m & \frac{\omega}{m} - m
\end{pmatrix} \] so that \[ AT = T \begin{pmatrix}
\omega & 0 \\
0 & -\omega
\end{pmatrix}. \]

Then \[
T^{-1} \left( \frac{\partial}{\partial t} + iA \right) \hat{G} T = \delta(t)
\]

\[
T^{-1} \begin{pmatrix}
\omega & 0 \\
0 & -\omega
\end{pmatrix} T = \delta(t), I
\]

Now \[
\left( \frac{\partial}{\partial t} + i\omega \right) g = \delta(t)
\]

has two different solutions:
\[
e^{-i\omega t} \eta(t), -e^{-i\omega t} \eta(-t).
\]

The one we want has only positive frequencies for \( t > 0 \) and only negative ones for \( t < 0 \), and since \( \omega > 0 \) this means \( e^{-i\omega t} \eta(t) \). Similarly
\[
\left( \frac{\partial}{\partial t} - i\omega \right) g = \delta(t)
\]

has the solutions \( e^{i\omega t} \eta(t), -e^{i\omega t} \eta(-t) \) and we want the latter. Thus
\[
T^{-1} \hat{G} T = \begin{pmatrix}
e^{-i\omega t} \eta(t) & 0 \\
0 & -e^{i\omega t} \eta(-t)
\end{pmatrix}
\]

and so
\[
\begin{align*}
G &= \begin{pmatrix} m & m \\ \frac{\xi + \omega}{2} & \frac{\xi - \omega}{2} \end{pmatrix} \begin{pmatrix} e^{-i\omega t} \eta(t) & 0 \\ 0 & -e^{i\omega t} \eta(-t) \end{pmatrix} \begin{pmatrix} \frac{\xi - \omega}{2} & -m \\ -\frac{\xi - \omega}{2} & m \end{pmatrix} \frac{1}{-2m} \\
&= \begin{pmatrix} me^{-i\omega t} \eta & me^{i\omega t} (-\eta(-t)) \\ (\frac{\xi + \omega}{2})e^{-i\omega t} \eta & (\frac{\xi - \omega}{2})e^{i\omega t} (-\eta(-t)) \end{pmatrix} \begin{pmatrix} \frac{\xi - \omega}{2} & -m \\ -\frac{\xi - \omega}{2} & m \end{pmatrix} \frac{1}{-2m} 
\end{align*}
\]
too complicated.

February 6, 1979:

So I've been wasting time trying to understand Green's functions, especially the Feynman Green's function. Let us take the 2d Minkowski space-time KG equation:

\[
\left( \frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} + m^2 \right) G = \delta
\]

Now the idea is to understand the singularities of any solution of this, which by general things should lie on the lines \( x = \pm t \). (One of the things you ought to understand by means of examples is all the new ways to work with hyperbolic DE's: (Hörmander, Daisters, Guillemin, etc.).)

Start by F.T. in \( x \):

\[
\left( \frac{\partial^2}{\partial t^2} + \frac{\xi^2}{2} + m^2 \right) \hat{G}(\xi, t) = \delta(t)
\]

and then F.T. in \( t \):

\[
\left( -k^2 + \frac{\xi^2}{2} + m^2 \right) \hat{G}(k, \xi) = 1
\]
\[ \hat{G}(t, \xi) = \frac{1}{2\pi} \int dk \frac{e^{-ikt} - 1}{k^2 - (\xi^2 + m^2)} \]

The Feynman Green's function is obtained by taking the contour below \( k = -\sqrt{\xi^2 + m^2} \) and above \( k = \sqrt{\xi^2 + m^2} \).

We get (see 541)

\[ \hat{G}^+(t, \xi) = \frac{e^{-i\sqrt{\xi^2 + m^2} |t|}}{-2i\sqrt{\xi^2 + m^2}} \]

and the corresponding Green's function is

\[ G^+(t, x) = \int \frac{d\xi}{2\pi} e^{-i\xi x} \frac{e^{-i\sqrt{\xi^2 + m^2} |t|}}{-2i\sqrt{\xi^2 + m^2}} \]

What does this look like?

Notice that

\[ \frac{1}{2\pi} \int dk \frac{e^{-ikt} - 1}{k^2 - \omega^2} = \lim_{\varepsilon \to 0} \frac{1}{2\pi} \int dk e^{-ikt} - \frac{1}{k^2 + i\varepsilon - \omega^2} \]

because

\[ \begin{array}{c}
-\sqrt{\omega^2 + \varepsilon} \\
-\omega \\
\sqrt{\omega^2 + \varepsilon} \\
\omega \\
\omega^2 - i\varepsilon \\
\omega^2 + i\varepsilon \\
\end{array} \]

This might be useful for \( m=0 \).
Let's try to understand the singularities of $G^+(t, x)$ for fixed $t > 0$ by means of an asymptotic expansion as $|\xi| \to \infty$.

$$\sqrt{\frac{x^2}{\xi^2} + m^2} = |\xi| \sqrt{1 + \frac{m^2}{|\xi|^2}} = |\xi| \left\{ 1 + \frac{1}{2} \frac{m^2}{|\xi|^2} + O\left(\frac{1}{|\xi|^4}\right) \right\}$$

$$e^{-i \sqrt{\frac{x^2}{\xi^2} + m^2} |t|} = \frac{e^{-i |\xi||t|} \left\{ 1 + \frac{1}{2} \frac{m^2}{|\xi|^2} |t| \right\}}{-2i \sqrt{\frac{x^2}{\xi^2} + m^2}} = \frac{e^{-i |\xi||t|} \left\{ 1 \right\}}{-2i |\xi|} \left\{ 1 + \frac{1}{2} \frac{m^2}{|\xi|^2} \right\}.$$  

Now in 1-dimension, \( \int \frac{d\xi}{|\xi|^2} \) is convergent, $|\xi| > \text{const}$.

Whence the difference between $G^+(t, x)$ and

$$\int \frac{d\xi}{2\pi} \frac{e^{-i |\xi||t|} e^{-i \xi x}}{-2i |\xi|}$$

$|\xi| > \text{const}$

will be continuous.
February 7, 1979

I was trying to understand the Green's function:

\[(\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} + m^2) G = \delta\]

If we put \( t = iy \), then

\[\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} + m^2 = -\frac{\partial^2}{\partial y^2} - \frac{\partial^2}{\partial x^2} + m^2\]

\[= -\left(\frac{1}{r} \frac{\partial}{\partial r} \frac{r}{\partial r} + \frac{1}{\hbar^2} \frac{\partial^2}{\partial \theta^2}\right) + m^2\]

and so solutions of the homogeneous equation are given by

\[u(r) = u(\sqrt{x^2 - t^2})\]

where

\[\left(\frac{1}{r} \frac{d}{dr} \frac{d}{dr} - m^2\right) u = 0\]

Recall that \( -\left(\frac{r}{\partial r}\right)^2 + r^2 \) \( \psi = k^2 \varphi = -s^2 \varphi \)

or \( \left(\frac{r}{\partial r}\right)^2 - s^2 - k^2 \) \( \psi = 0 \)

is the modified Bessel DE. Thus

\[K_0\left(m\sqrt{x^2 - t^2}\right), \quad I_0\left(m\sqrt{x^2 - t^2}\right)\]

are solutions of

\[\left(\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} + m^2\right) \varphi = 0.\]

Recall the Green's function of interest was given by

\[G(t, x) = \int_0^\infty \frac{d\xi}{2\pi} e^{-i \xi x} \frac{e^{-i \sqrt{\xi^2 + m^2} t}}{-2i \sqrt{\xi^2 + m^2}}.\]
\[ \varphi = \frac{u - u^{-1}}{2}, \quad 0 < u < \infty \]

\[ \frac{x^2 + m^2}{\sqrt{x^2 + m^2}} = m \frac{u + u^{-1}}{2} \]

\[ \frac{d}{dx} = \frac{1 + u^{-2}}{2} m \frac{du}{u} = \frac{u + u^{-1}}{2} m \frac{du}{u} \]

\[ \frac{d}{\sqrt{x^2 + m^2}} = \frac{du}{u} \]

so

\[ G(t, x) = \int_0^\infty \frac{e^{-ixm(u-u^{-1})}}{2\pi - 2i} \frac{-i\lambda m (u-u^{-1})}{2} - i|t|/m \frac{(u+u^{-1})}{2} \]

\[ = \frac{-1}{4\pi i} \int_0^\infty e^{-i(x+|t|)m (u-u^{-1})/2} - i(|t|-x)m (u+u^{-1})/2 \frac{du}{u} \]

Notice that

\[ a,b > 0 \]

\[ \int_0^\infty e^{-\frac{a}{2} u - \frac{b}{2} u^{-1}} \frac{du}{u} = \int_0^\infty e^{-\frac{a}{2} u - \frac{b}{2} u^{-1}} \frac{du}{u} \]

and so if \( \lambda \) is chosen so that \( a\lambda = b\lambda^{-1}, \quad \lambda^2 = \frac{b}{a} \), \( a\lambda = a\sqrt{\frac{b}{a}} = \sqrt{ab} \), we get

\[ \int_0^\infty e^{-\frac{1}{2} (u+u^{-1})} \frac{du}{u} = K_0(\sqrt{ab}) \]

Suppose \( x > |t| \) so that \( x+|t| > 0 \), \( |t| - x < 0 \).

The integral

\[ a,b > 0 \]

\[ \int_0^\infty e^{-\frac{ia}{2} u + \frac{ib}{2} u^{-1}} \frac{du}{u} \]

is conditionally convergent. As \( u \to \infty \), we can deform the
The same is true for the path \( u \to 0 \). Thus we can evaluate the above integral over the contour \( u = -it, \infty \to 0 \).

\[
\int_{-\infty}^{\infty} e^{-i\frac{a}{2} u + \frac{ib}{2} u^{-1}} \, du = \int_{0}^{\infty} e^{-\frac{a}{2} t - \frac{b}{2} t^{-1}} \, \frac{dt}{t} = K_0(\sqrt{ab})
\]

So therefore we see that

\[
G(t, x) = -\frac{1}{4\pi i} K_0\left(\frac{m\sqrt{x^2 - t^2}}{t}\right) \quad \text{for} \quad |x| > |t|
\]

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\[
G(t, x) = -\frac{1}{4\pi i} \int_{-\infty}^{\infty} e^{i(x+it) \frac{m}{2} u - i(1-t)} \frac{m \frac{u}{2} - i(1-t)x \frac{u^{-1}}{2}}{u} \, du
\]

If \( |x| < |t| \), then \( a = m(|t|+x), \quad b = m(|t|-x) \) are both \( >0 \), and one has

\[
\int_{0}^{\infty} e^{-ia \frac{u}{2} - ib \frac{u^{-1}}{2}} \, \frac{du}{u} = \int_{0}^{\infty} e^{-i\sqrt{ab}(u + u^{-1})} \, \frac{du}{u} = K_0(i\sqrt{ab})
\]

where \( K_0(r) \) which is normally defined for \( r > 0 \) is extended by analytic continuation to \( \text{Re } r > 0 \). So we get the formula:
\[ G(t, x) = \begin{cases} \frac{-1}{4\pi i} \text{K}_0 \left( m \sqrt{x^2 - t^2} \right) & \text{for } |x| > |t| \\ \frac{-1}{4\pi i} \text{K}_0 \left( i m \sqrt{x^2 - t^2} \right) & \text{for } |x| \leq |t| \end{cases} \]

Recall that as \( r \to 0 \)

\[ \text{K}_0(r) = \int_0^\infty e^{-r(t + t^{-1})} \frac{dt}{t} = 2 \int_0^\infty e^{-rt^2} \left( e^{-\frac{r}{2} t^{-1}} \right) \frac{dt}{t} \]

\[ = 2 \int_0^\infty e^{-rt^2} \frac{dt}{t} + O(1) = 2 \int_0^\infty e^{-rt^2} \frac{dt}{t} = -2 \ln(r) + O(1) \]

Hence near \( |x| = |t| \) we have

\[ G(t, x) \approx \frac{1}{2\pi i} \ln \left( m \sqrt{x^2 - t^2} \right) \sim \frac{1}{4\pi i} \ln(x^2 - t^2) + O(1) \]

So it seems that the singularity in the Green's function is logarithmic, hence the Green's function is integrable locally.

Now look at the case \( m = 0 \).

\[ G(t, x) = \int_{-\infty}^{\infty} \frac{d\xi}{2\pi} e^{-i\xi x} e^{-i\xi |t|} \]

Some process will be needed to make sense of this integral near \( \xi = 0 \).
\[ G(t, x) = \int_0^\infty \frac{d\xi}{2\pi} \left( e^{-i\xi x} + e^{i\xi x} \right) \frac{e^{-i\xi|t|}}{-2i\xi} \]

\[ \frac{\partial G}{\partial t} = \int_0^\infty \frac{d\xi}{2\pi} \left( e^{-i\xi x} + e^{i\xi x} \right) \frac{e^{-i\xi|t|}}{2} \]

\[ = \frac{1}{4\pi^2} \left[ \frac{1}{i(x+t)} + \frac{1}{i(t-x)} \right] \]

\[ = \frac{1}{4\pi i} \frac{2t}{x^2-t^2} \]

so

\[ G(t, x) = \frac{1}{4\pi i} \log \left( t^2 - x^2 \right) + \text{fn. of } x \]

Also

\[ \frac{\partial G}{\partial x} = \int_0^\infty \frac{d\xi}{2\pi} \left( e^{-i\xi x} - e^{i\xi x} \right) \frac{e^{-i\xi t}}{2} \]

\[ = \frac{1}{4\pi i} \left[ \frac{1}{x+t} - \frac{1}{t-x} \right] = \frac{1}{4\pi i} \frac{2x}{x^2-t^2} \]

\[ \therefore G = \frac{1}{4\pi i} \log \left( x^2 - t^2 \right) + \text{fn. of } t. \]

So what appears to be happening is that one has the function

\[ \frac{1}{4\pi i} \log |x^2 - t^2| = \frac{1}{4\pi i} \log |x+t| + \frac{1}{4\pi i} \log |x-t| \]

which is a solution of the homogeneous equation added to a more familiar type of Greens function.

Compute forward Greens fn.

\[ \int_{-\infty}^{\infty} \frac{d\xi}{2\pi} e^{-i\xi x} \sin \frac{\xi t}{\xi} \eta(t) = \eta(t) \left\{ \begin{array}{ll} \frac{1}{2} & \text{for } |x| < t \\ 0 & \text{for } |x| > t. \end{array} \right. \]
Since \[ \frac{1}{2} \int_{-t}^{t} e^{\frac{i}{\hbar} x^2} dx = \frac{1}{2} \frac{e^{i \xi t} - e^{-i \xi t}}{i \xi} = \frac{\sin \xi t}{\xi} \]
The backwards Green's function is the reflection of this under \( t \rightarrow -t \). The average of the forward and backward Green's functions is

\[ \frac{1}{2} G_f + \frac{1}{2} G_b = \begin{cases} \frac{1}{4} & \text{if } |x| < |t| \\ 0 & \text{otherwise} \end{cases} \]

Note that \[ G^+ = \begin{cases} \frac{2}{4 \pi i} \log |x^2 - t^2| & |x| \geq |t| \\ \frac{2}{4 \pi i} \log |\sqrt{t^2 - x^2}| & |x| < |t| \end{cases} \]

is just

\[ \frac{1}{4 \pi i} \log |x^2 - t^2| + \frac{1}{2}(G_f + G_b) \]

So we seem to get

[boxed]
G^+(x, \xi) = \frac{1}{4 \pi i} \log |x^2 - t^2| + \frac{1}{4} \eta(x^2 - \xi^2)
[/boxed]

So now let us return to

\[ \left( \frac{\partial}{\partial t} + \frac{\partial}{\partial x} \right) G = \delta \]

Taking F.T. in \( x \) gives

\[ \left( \frac{\partial}{\partial t} - i \xi \right) \hat{G}(t, \xi) = \delta(t) \]

Using the Feynman prescription
\[ \hat{G}(t, \xi) = \int \frac{dk}{2\pi} e^{-ikt} \frac{1}{-i(k+\xi)} \]

If \( t > 0 \), then push into L.H.P.

\[ \hat{G}(t, \xi) = \begin{cases} 
0 & \xi > 0 \\
\frac{2\pi i}{-2\pi i} e^{i\xi t} = e^{i\xi t} & \xi < 0.
\end{cases} \]

But if \( t < 0 \), then push into U.H.P. to get

\[ \hat{G}(t, \xi) = \begin{cases} 
\frac{2\pi i}{-2\pi i} e^{i\xi t} = -e^{i\xi t} & \xi > 0 \\
0 & \xi < 0.
\end{cases} \]

Thus

\[ \hat{G}(t, \xi) = e^{i\xi t} \left[ \eta(t) \eta(-\xi) - \eta(-t) \eta(\xi) \right]. \]

Notice this is a tempered distribution, so it has a Fourier transform w.r.t \( \xi \).

If \( t > 0 \)

\[ G(t, x) = \int_{-\infty}^{\infty} \frac{d^3}{2\pi} e^{-ix\xi} e^{i\xi t} = \frac{1}{2\pi} \frac{1}{-ix+it} = \frac{-1}{2\pi i} \frac{1}{x-t} \]

and if \( t < 0 \), then

\[ G(t, x) = \int_{-\infty}^{\infty} \frac{d^3}{2\pi} e^{-ix\xi} e^{i\xi t} (-1) = (-1) \frac{1}{2\pi} \frac{1}{ix-it} = \frac{-1}{2\pi i} \frac{1}{x-t} \]

This formal calculation makes one suspect that \( G(t, x) \) is

\[ \frac{-1}{2\pi i} P\left(\frac{1}{x-t}\right) \]

plus conventional Green's function made up of \( \delta \) functions.

Let us compute the F.T. of \( -\frac{1}{2\pi i} P\left(\frac{1}{x-t}\right) \):

\[ (-\frac{1}{2\pi i}) \int_{-\infty}^{\infty} P\left(\frac{1}{x-t}\right) e^{i\xi x} dx = (-\frac{1}{2\pi i}) e^{i\xi t} \int_{-\infty}^{\infty} P\left(\frac{1}{x}\right) e^{i\xi x} dx \]

the \( P \) means you average
If \( \xi > 0 \) use UHP to get
\[
\frac{1}{2} \left( -\frac{1}{2\pi i} \right) e^{i\xi t} \left( 2\pi i + 0 \right) = -\frac{1}{2} e^{i\xi t}
\]
\[\text{Res}_0(e^{i\xi x}) \quad \text{upper contour}\]

If \( \xi < 0 \) use LHP to get
\[
\frac{1}{2} \left( -\frac{1}{2\pi i} \right) e^{i\xi t} (-2\pi i) = \frac{1}{2} e^{i\xi t}
\]

So to get \( \hat{G}(t, \xi) \) we want to add \( \frac{1}{2} e^{i\xi t} \gamma(t) \)
\[-\frac{1}{2} e^{i\xi t} \gamma(-t)\]

\[
\hat{G}(t, \xi)
\]

Thus
\[
G(t, x) = \frac{-1}{2\pi i} \frac{1}{\xi - t} + \frac{1}{2} \delta(x - t) \gamma(t) - \frac{1}{2} \delta(x - t) \gamma(-t)
\]

So at this point we understand somewhat how to make sense of the Green's function for
\[
\begin{pmatrix}
\frac{\partial}{\partial t} + \frac{\partial}{\partial x} \\
0 & \frac{\partial}{\partial t} - \frac{\partial}{\partial x}
\end{pmatrix}
\]
having only positive frequencies for \( t > 0 \) and only negative frequencies for \( t < 0 \). Now the next point is to take the equation with potential

\[
\begin{pmatrix}
\frac{\partial}{\partial t} + \frac{\partial}{\partial x} & 0 \\
0 & \frac{\partial}{\partial t} - \frac{\partial}{\partial x}
\end{pmatrix} - \begin{pmatrix}
0 & P \\
-P & 0
\end{pmatrix} \mathbf{u} = 0
\]

and to see if \( \det(I - GV) \) is defined. This leads to

\[
\det \left( I - \begin{pmatrix} \mathbf{0} & (\frac{\partial}{\partial t} + \frac{\partial}{\partial x})^{-1} \mathbf{P} \\ \frac{\partial}{\partial x} \mathbf{P}^{-1} & \mathbf{0} \end{pmatrix} \right) = \det \left( I + \left( \frac{\partial}{\partial t} - \frac{\partial}{\partial x} \right)^{-1} \mathbb{P} \left( \frac{\partial}{\partial t} + \frac{\partial}{\partial x} \right)^{-1} \right)
\]

which brings up the problem of whether the operator

\[
\left( \frac{\partial}{\partial t} - \frac{\partial}{\partial x} \right)^{-1} \mathbb{P} \left( \frac{\partial}{\partial t} + \frac{\partial}{\partial x} \right)^{-1}
\]

is of trace class.

A simpler problem would be the equation

\[
\left( \frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} + q \right) \mathbf{u} = 0
\]

Here one deals with the operator

\[
(G^q f)(t, x) = \int \int dt' dx' \ G(t-t', x-x') q(t', x') f(t', x')
\]

whose trace should be obtained by integrating over
the diagonal:
\[ \text{tr}(G^+G) = \iint dt \, dx \, G(0,0) \, g(t', x') \]

obviously undefined.

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State vector \( \psi(t) \) in Schröd picture satisfies

\[ \frac{d}{dt} \psi_S(t) = -iH_S \psi_S(t) = -i(H_0 - V) \psi_S(t) \]

or

\[ \left( \frac{d}{dt} + iH_0 \right) \psi_S = iV \psi_S \]

\[ \frac{d}{dt} \left( e^{iH_0 t} \psi_S(t) \right) = e^{iH_0 t} iV e^{-iH_0 t} e^{iH_0 t} \psi_S(t) \]

The state vector in the Dirac or interaction picture is

\[ \psi_D(t) = e^{iH_0 t} \psi_S(t) \]

It satisfies

\[ \frac{d}{dt} \psi_D(t) = H_I(t) \psi_D(t) \]

where \( H_I(t) = e^{iH_0 t} (iV) e^{-iH_0 t} \)

Thus integration gives the integral equation

\[ \psi_D(t) = \psi_{in} + \int_{-\infty}^{t} dt' H_I(t') \psi_D(t') \]

hence iterating

\[ \psi_D(t) = \psi_{in} + \int_{-\infty}^{t} dt' H_I(t') \psi_{in} + \int_{-\infty}^{t} dt_1 \int_{-\infty}^{t} dt_2 H_I(t_1) H_I(t_2) \psi_{in} + \ldots \]

which leads to the following formula for \( S \)
\[ S = 1 + \int_{-\infty}^{\infty} dt_1 I H_I(t_1) + \int_{-\infty}^{\infty} dt_1 \int_{-\infty}^{t_1} I H_I(t_1) H_I(t_2) + \ldots \]

Then one introduces the time ordering operator

\[ \mathcal{P} \left( H_I(t_1) H_I(t_2) \right) = \begin{cases} 
H_I(t_1) H_I(t_2) & \text{if } t_1 > t_2 \\
H_I(t_2) H_I(t_1) & \text{if } t_1 < t_2 
\end{cases} \]

and one formally gets

\[ S = \mathcal{P} \left( e^{\int dt H_I(t)} \right) . \]

In the case of the Dirac field, perturbed by an external electromagnetic field the operators will be working on some kind of Fock space, and the quantity of interest will be

\[ e^{i\omega} = \langle 0 | S | 0 \rangle \]

where \( | 0 \rangle \) denotes the vacuum state.

Let's go over Schwinger's formulas relating a vacuum expectation value to a determinant. Let \( V \) be a vector space with a basis \( e_1, e_2, \ldots \), and dual basis \( \tilde{e}_1 \). On \( \Lambda V \) we have the creation operators

\[ X_i^- = e(\tilde{e}_i) \quad e(V) = \text{left mult by } v \]

and destruction of:

\[ X_i^+ = i(e_i) \]

satisfying the canonical commutation relations...
\[ \{ x_i^-, x_j^- \} = \{ x_i^+, x_j^+ \} = 0 \quad \{ A, B \} = AB + BA \]

\[ \{ x_i^+, x_j^- \} = i(\epsilon_i^c) e(e_j^c) + e(e_i^c) i(\epsilon_j^c) = \delta_{i,j} \]

Let \( K \) be a linear transformation on \( V \) given by \( \{ K_{as} \} \) and \( \lambda \) a parameter. Schwinger considers the transformation on \( \Lambda V \) which is an ordered exponential:

\[
S = P \left( \exp \left( \sum_{n,s} x_n^+ K_{ns} x_s^- \right) \right) = \sum_{n \geq 0} \frac{\lambda^n}{n!} P \left( \left( \sum x_n^+ K_{ns} x_s^- \right)^n \right)
\]

\[
= \sum_{n \geq 0} \frac{\lambda^n}{n!} \sum_{i_1, i_2, \ldots, i_n \atop s_1, s_2, \ldots, s_n} x_{i_1}^+ x_{i_2}^+ \ldots x_{i_n}^+ K_{i_1 s_1} K_{i_2 s_2} \ldots K_{i_n s_n} x_{s_1}^- x_{s_2}^- \ldots x_{s_n}^-
\]

For this operator \( S \) on \( \Lambda V \) it is clear that

\[
\langle 0 | S | 0 \rangle = \det(1 + \lambda K)
\]

where \( | 0 \rangle \) denotes the element \( 1 \in \Lambda^0 V \).

Perhaps the above can be interpreted via the Clifford algebra.
Propagator approach to the ordinary Schröd equation.

\[
\left(-\frac{d^2}{dx^2} + \mathcal{Q}\right) u = E u \quad \text{time-ind.}
\]

\[
\left(-\frac{\partial^2}{\partial x^2} + \mathcal{Q}\right) \psi = i \frac{\partial \psi}{\partial t} \quad \text{time-dep.}
\]

Think of \( \psi(t) = \psi(t, x) \) as the probability amplitude of finding a particle at time \( t \) and position \( x \). The Green's function or propagator is the kernel expressing \( \psi(t, x) \) in terms of \( \psi(t', x') \) for \( t' < t \):

\[
\psi(t, x) = \int K(t, x; t', x') \psi(t', x') \, dx'
\]

Thus \( K \) represents \( e^{-iH_0(t-t')}. \) Better notation \( K(t-t'; x, x') \)

Let \( H_0 \) be the Hamiltonian with \( g = 0 \), and let \( \psi_0(t) \) denote a free wave function. Suppose the interacting potential \( g \) is switched on for a small time interval \( dt_1 \) at \( t_1 \). Let \( \psi(t) \) be the new wave function. Then

\[
i \frac{\psi(t_1 + dt_1) - \psi(t_1)}{dt} = (H_0 - V) \psi(t_1)
\]

\[
\psi(t_1 + dt_1) = \psi(t_1) + dt_1 \frac{i}{\psi_0(t_1)} (H_0 - V) \psi(t_1)
\]

\[
\psi(t_1 + dt_1) = \psi_0(t_1) e^{-i dt_1 H_0} \psi(t_1) + dt_1 i V \psi_0(t_1)
\]

\[
= \psi_0(t_1 + dt_1) + dt_1 i V \psi_0(t_1)
\]
\[ \psi(t) = \psi_0(t) + dt_1 e^{-iH_0(t-t_1)} iV e^{-iH_0(t_1-t')} \psi(t') \]

and so we conclude that

\[ K(t,t';x,x') = K_o(t-t',x,x') + dt_1 \int K_o(t-t_1,x,x') iV(t_1,x) K(t_1-t',x',x') dt_1 \]

Suppose next the interaction is switched on for \( t_1 \) to \( t_1 + dt_1 \) and \( t_2 \) to \( t_2 + dt_2 \), where \( t_1 < t_2 \).

Then

\[ \psi(t) = \psi_0(t) \quad t' < t < t_1 \]
\[ = \psi_0(t) + dt_1 e^{-iH_0(t-t_1)} iV \psi_0(t) \quad t_1 < t < t_2 \]

It would seem to be simpler to use propagator notation:

\[ K(t-t') = K_o(t-t') \quad t' < t < t_1 \]
\[ = K_o(t-t_1)[1 + dt_1 iV(t_1)] K_o(t_1-t') \quad t_1 < t < t_2 \]
\[ = K_o(t-t_2)[1 + dt_2 iV(t_2)] K_o(t_2-t')[1 + dt_1 iV(t_1)] K_o(t_1-t') \]

for \( t_2 < t \)

Expand the last expression:

\[ K(t-t') = K_o(t-t') + \left\{ K_o(t-t_1) dt_1 iV(t_1) K_o(t_1-t') + K_o(t-t_2) dt_2 iV(t_2) K_o(t_2-t_1) \right\} dt_1 iV(t_1) K_o(t_1-t') \]

\[ + K_o(t-t_2) dt_2 iV(t_2) K(t_2-t_1) dt_1 iV(t_1) K_o(t_1-t') \]

So now let us pass to the limit so as to get
\[ K(t-t') = K_0(t-t') + \int_{t'}^{t} dt_1 K_0(t-t_1) iV(t_1) K_0(t_1-t') \]

\[ + \int dt_1 \int dt_2 K_0(t-t_1) iV(t_1) K_0(t_1-t_2) iV(t_2) K_0(t_2-t') \]

which is \( e^{-iH_0 t} U(t, t') e^{iH_0 t'} \), \( U(t, t') \) denoting the propagator in the Dirac picture. Another version of the formula is the integral equation

\[ K(t-t') = K_0(t-t') + \int_{t'}^{t} dt_1 K_0(t-t_1) iV(t_1) K(t_1-t') \]

If we introduce the space variables into the picture and put \( x = (t, \mathbf{x}) \), then (x) becomes

\[ K(x, x') = K_0(x-x') + \int_{x'}^{t} dx_1 K_0(x-x_1) iV(x_1) K_0(x_1-x') \]

\[ + \int dx_1 \int dx_2 K_0(x-x_1) iV(x_1) K_0(x_1-x_2) iV(x_2) K_0(x_2-x') \]

\[ + \ldots \]

where it has to be \( K_0(x_1-x_2) = 0 \) if \( t_1 < t_2 \).

So far we have looked at the ordinary Schrödinger equation as a first order "hyperbolic" system and found the Green's function.
\[ K(t-t') = \begin{cases} e^{-iH(t-t')} & t > t' \\ 0 & t < t' \end{cases} \]

Satisfying
\[ \left( \frac{\partial}{\partial t} + iH \right) K(t-t') = \delta(t-t') \]

Somehow, Feynman uses the same Green's function methods but with different boundary conditions.

Thus, \( K(x, x') \) represents the relative probability amplitude for finding an electron at \( x \) initially in the state \( \psi(x') |E_0\rangle \), and the terms in the series are represented by the Feynman diagrams.
Review S-matrix formalism: We have Schrödinger equation

\[ i \frac{\partial}{\partial t} \psi(t) = H \psi(t) = (H_0 - \nu) \psi(t) \]

which we convert to an integral equation:

\[ \left( \frac{\partial}{\partial t} + i H_0 \right) \psi = i \nu \psi \]

\[ \frac{\partial}{\partial t} \left( e^{i H_0 t} \psi \right) = \frac{e^{i H_0 t} i \nu e^{-i H_0 t}}{H_0(t)} e^{i H_0 t} \psi \]

\[ e^{i H_0 t} \psi(t) = e^{i H_0 t'} \psi(t') + \int_{t'}^{t} dt_1 H_0(t_1) e^{i H_0 t} \psi(t_1) \]

Thus if \( U(t, t') \) denotes the propagator for the Dirac state vector from time \( t' \) to time \( t \) we have

\[ U(t, t') = I + \int_{t'}^{t} dt_1 H_0(t_1) U(t', t') U(t, t') \]

\[ = I + \int_{t'}^{t} dt_1 H_0(t_1) + \int_{t'}^{t} dt_1 \int_{t'}^{t} dt_2 H_0(t_1) H_0(t_2) + \ldots \]

\[ = I + \int_{t'}^{t} dt_1 H_0(t_1) + \int_{t'}^{t} dt_1 \int_{t'}^{t} dt_2 P \{ H_0(t_1) H_0(t_2) \} + \ldots \]

\[ = P \{ e^{\int_{t'}^{t} dt H_0(t)} \} \]

So the problem for me is to understand the interaction Hamiltonian in the case of the Dirac field \( \psi \) perturbed
by an external EM field $\mathbf{A}$. According to Schwinger

$$\mathcal{H}(\mathbf{A}) = N(\bar{\psi}(x) \gamma A(x) \psi(x)) \quad A = \gamma A$$

with the following explanation. Here $x = (t, \mathbf{r})$ is a point of space-time. One has

$$\psi(x) = \psi^+(x) + \psi^-(x)$$

destroys electron creates positron
decays at $x$:

$$\bar{\psi}(x) = \bar{\psi}^+ + \bar{\psi}^-$$

creates electron destroys positron

Read as follows: $+$ means destroy, $-$ means create
$\psi(x)$ means arrow heads toward $x$$\bar{\psi}(x)$ " " away from $x$

Now the $N$ arranges things so that the creation operators appear to the left of the destruction operators. Thus

$$N(\mathcal{C} \mathbf{A} \psi) = N((\bar{\psi}^+ \bar{\psi}^-) A (\psi^+ \psi^-))$$

$$= \psi^+ A \bar{\psi}^+ + \psi^- A \psi^+ + \bar{\psi}^- A \psi^- + \bar{\psi}^- A \bar{\psi}^+$$
The last term is \( N(\bar{\Psi} A \Psi^-) \) which has to be rearranged by \( N \) and I don't know whether a sign is introduced or not.

The four terms in \( N(\bar{\Psi} A \Psi) \) are represented by the following Feynman diagrams:

\[\Psi^+ A \Psi^+(x)\]

pair destruction

\[\bar{\Psi}^- A \Psi^+(x)\]

electron scattering

\[\bar{\Psi}^- A \Psi^-(x)\]

pair creation

\[\bar{\Psi}^- A \Psi^+(x)\]

position scattering

These diagrams represent the first order processes, that is, the term

\[\int d^4 x_1 \; H_I(x_1)\]

in the \( S \)-matrix.

Let's next consider the 2nd order term

\[\frac{1}{2!} \int d^4 x_1 \int d^4 x_2 \; T\{ N(\bar{\Psi} A \Psi)(x_1) \cdot N(\bar{\Psi} A \Psi)(x_2) \}\]

where \( T \) is the time ordering operator. According to Wick's theorem one has
\[
T \{ N(\tilde{\phi}(x_1) A(x_1) \psi(x_1)) \cdot N(\tilde{\phi}(x_2) A(x_2) \psi(x_2)) \}
\]
\[
= N(\tilde{\phi}(x_1) A(x_1) \psi(x_1) \tilde{\phi}(x_2) A(x_2) \psi(x_2))
\]
\[
+ N(\tilde{\phi}(x_1) A(x_1) \psi(x_1) \tilde{\phi}(x_2)^* A(x_2) \psi(x_2))
\]
contract these two factors to get \( K_+ (x_1-x_2) \)
\[
+ N(\tilde{\phi}(x_1)^* A(x_1) \psi(x_1) \tilde{\phi}(x_2) A(x_2) \psi(x_2)^*)
\]
contract
\[
+ N(\tilde{\phi}(x_1)^* A(x_1) \psi(x_1) \tilde{\phi}(x_2)^* A(x_2) \psi(x_2)^*)
\]

The contracted terms result from the commutation relations between field operators.

The first term represents pairs of elementary processes of the first order occurring together, e.g.

\[
\begin{array}{c}
\text{x} \quad \text{x} \\
\text{x} \quad \text{x}
\end{array}
\]

The second term is \( N(\tilde{\phi}(x_1) A(x_1) K_+ (x_1-x_2) A(x_2) \psi(x_2)) \) and is the sum of 4 processes obtained by splitting \( \phi, \phi \):

\[
\begin{array}{c}
\text{x} \quad \text{x}_1 \\
\text{x} \quad \text{x}_2
\end{array}
\]

\[
\begin{array}{c}
\text{x} \quad \text{x}_1 \\
\text{x} \quad \text{x}_2
\end{array}
\]

\[
\begin{array}{c}
\text{x} \quad \text{x}_1 \\
\text{x} \quad \text{x}_2
\end{array}
\]

\[
\begin{array}{c}
\text{x} \quad \text{x}_1 \\
\text{x} \quad \text{x}_2
\end{array}
\]

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\begin{array}{c}
\text{x} \quad \text{x}_1 \\
\text{x} \quad \text{x}_2
\end{array}
\]

\[
\begin{array}{c}
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\text{x} \quad \text{x}_2
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\text{x} \quad \text{x}_2
\end{array}
\]

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\begin{array}{c}
\text{x} \quad \text{x}_1 \\
\text{x} \quad \text{x}_2
\end{array}
\]

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\text{x} \quad \text{x}_2
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\]

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\text{x} \quad \text{x}_2
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\text{x} \quad \text{x}_2
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\text{x} \quad \text{x}_2
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\text{x} \quad \text{x}_2
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\text{x} \quad \text{x}_2
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\text{x} \quad \text{x}_2
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\text{x} \quad \text{x}_2
\end{array}
\]

\[
\begin{array}{c}
\text{x} \quad \text{x}_1 \\
\text{x} \quad \text{x}_2
\end{array}
\]

\[
\begin{array}{c}
\text{x} \quad \text{x}_1 \\
\text{x} \quad \text{x}_2
\end{array}
\]
The third term in the T-product decomposition is for some reason equal to

\[ N(\widetilde{\psi}(x_2) A(x_2) K(x_2-x_1) A(x_1) \psi(x_1)) \]

and it gives the same diagrams as immediately above but with \( x_1, x_2 \) interchanged.

The last term in the T-product decomposition is

\[ \mathcal{O} = - \text{Tr} \left( A(x_1) K_+ (x_1-x_2) A(x_2) K_+^* (x_2-x_1) \right). \]

(The minus sign seems to appear because of the following steps:

\[ N(\widetilde{\psi}(x_1)^* A(x_1) \psi(x_1)^* \widetilde{\psi}(x_2)^* A(x_2) \psi(x_2)^* \frac{1}{K_+ (x_1-x_2)}) \]

\[ = - N \left( A(x_1) K_+ (x_1-x_2) A(x_2) \psi(x_2)^* \widetilde{\psi}(x_1)^* \frac{1}{K_+ (x_2-x_1)} \right) \]

\[ \mathcal{O} \] represents the vacuum process:

\[ \begin{array}{c}
\xrightarrow{x_1} \quad \hspace{1cm} \xleftarrow{x_2}
\end{array} \]

The possible 3rd order vacuum processes from the different possible ways of contracting totally in

\[ \widetilde{\psi}(x_1) A(x_1) \psi(x_1) \widetilde{\psi}(x_2) A(x_2) \psi(x_2) \widetilde{\psi}(x_3) A(x_3) \psi(x_3) \]

first:
yields

\[- \text{Tr} \left( A(x_1) K(x_1-x_2) A(x_2) K(x_2-x_3) A(x_3) K(x_3-x_1) \right) \]

or contract \( \psi(x_1), \psi(x_3) \) and \( \psi(x_3) \) with \( \psi(x_2) \) etc. to get

\[- \text{Tr} \left( A(x_1) K(x_1-x_3) A(x_3) K(x_3-x_2) A(x_2) K(x_2-x_1) \right) \]

So you get the diagrams

![Diagrams](image)

Can you see why all the vacuum diagrams constitute the Fredholm expansion of \( \det (1 + AK) \)?

Can you understand Schwinger's formula for \( \det (1 - K) \) as a vacuum expectation value?

Let's review the physicists' view of the formula

\[
\det (1 - AK) = e^{- \text{Tr} \log (1 - AK)} = e^{- \sum_{m=1}^{\infty} \frac{\lambda^m}{m} \text{tr}(K^m)}
\]

One has Fredholm formula:

\[
\det (1 - AK) = \sum_{n \geq 0} \frac{(-\lambda)^n}{n!} \int dx_1 \cdots \int dx_n \det_n (K(x_i, x_j))
\]

Write \( \det_n \) as a sum over the symmetric group \( \Sigma_n \)

\[
\det_n K(x_i, x_j) = \sum_{\sigma \in \Sigma_n} (-1)^\sigma K_{\sigma}(x_i, x_j)
\]

and analyze \( \sigma \) in terms of its cycles. One gets
\[
\int dx_1 \int dx_2 \cdots \int dx_n \quad K(x_1, x_{\sigma(1)}) \cdots K(x_n, x_{\sigma(n)}) = \text{tr}(K^{m_1}) \cdots \text{tr}(K^{m_e})
\]

if \( \sigma \) is a disjoint union of cycles of lengths \( m_1, m_2, \ldots, m_e \).

So one gets

\[
\int dx_1 \cdots \int dx_n \quad \text{det}_n \{K(x_i, x_j)\} = \sum_{\text{partition } \Pi} \text{tr}(K^{m_1}) \cdots \text{tr}(K^{m_e}) \cdot \text{number of } \sigma \in S_n \text{ belonging to } \Pi \text{ sign of such } \sigma.
\]

Given a partition \( m_1 \geq m_2 \geq \cdots \geq m_e \) of \( n \), to count the number of permutations \( \sigma \) belonging to it, use the fact that these form a conjugacy class, so their number is \( n! / \text{card } C \).

We should write the partition

\[
\begin{array}{c}
\vdots \\
i_1 & i_2 & i_3 & i_4 & \cdots \\
\text{k}_1 \\ \text{k}_2 \\ \text{k}_3 \\ \text{k}_4 \\
\vdots
\end{array}
\]

so that \( n = k_1 + 2k_2 + \cdots \). The centralizer of a permutation belonging to this partition is

\[
\sum_{k_1} \sum_{k_2} \sum_{k_3} \sum_{k_4} \cdots \text{identifiers}
\]

so that the number of such permutations is \( n! / k_1! k_2! 2^{k_2} k_3! 2^{k_3} \cdots \).

The sign of such a permutation is

\[
(-1)^{(1-1)k_1 + (2-1)k_2 + (3-1)k_3} = (-1)^{k_1 + k_2 + k_3 + \cdots}
\]

Thus

\[
\text{det} (1 - \lambda K) = \sum_{k_1, k_2, \ldots} \frac{(-1)^{k_1 + 2k_2 + \cdots}}{(k_1 + 2k_2 + \cdots)!} \left( \text{tr} K \right)^{k_1} \left( \text{tr} K^2 \right)^{k_2} \cdots \\
\times \frac{(k_1 + 2k_2 + \cdots)!}{k_1! k_2! 2^{k_2} k_3! 3^{k_3}} \left( -1 \right)^{k_1 + k_2 + \cdots}
\]
\[
\sum_{k_1, k_2, \ldots} \frac{1}{k_1!} (\frac{\lambda}{k_1})^{k_1} \frac{1}{2} (\frac{-\lambda^2}{2})^{k_2} \frac{1}{3} (\frac{-\lambda^3}{3})^{k_3} \\
= e^{-\lambda \text{tr} K - \frac{\lambda^2}{2} \text{tr} K^2 - \frac{\lambda^3}{3} \text{tr} K^3 - \ldots}
\]

But the real point is to interpret the terms of the Fredholm expansion via diagrams.

\[
\det(1-\lambda K) = 1 - \lambda \text{tr}(K) + \frac{\lambda^2}{2} \{(\text{tr} K)^2 - \text{tr}(K^2)\} \\
- \frac{\lambda^3}{3!} \{(\text{tr} K)^3 + 2 \text{tr}(K^3) - 3 \text{tr} K \text{tr}(K^2)\} + \ldots
\]

The last term results from

\[
\det_3 = \begin{vmatrix}
K(1,1) & K(1,2) & K(1,3) \\
K(2,1) & K(2,2) & K(2,3) \\
K(3,1) & K(3,2) & K(3,3)
\end{vmatrix}^{\text{tr}(K^3)}
\]

Picture:

\[
\text{tr} K : \\
\text{tr} K^2 : \\
\text{tr} K^3 :
\]