February 2, 1979: Phys Rev 93, 615

Today I began to understand Schwinger V. Consider Dirac field $\psi(x)$ being perturbed by a time dependent electromagnetic field $A_\mu(x)$ (elementary guage $A=0$ for trivial field). The field equations are

$$\gamma_\mu[-i\partial_\mu - eA_\mu(x)]\langle\psi(x)\rangle + m\langle\psi(x)\rangle = \eta(x)$$

$$[i\partial_\mu - eA_\mu(x)]\langle\bar\psi(x)\rangle\gamma_\mu + m\langle\bar\psi(x)\rangle = \bar\eta(x)$$

I think this is a non-homogeneous system of the form

$$d\psi = \eta$$
$$d^*\psi = \bar\eta$$

but the significance of $\langle\ \rangle$ is unclear. In any case, ultimately the only thing that matters is the Greens function $G^+(x,x')$ which satisfies

$$\gamma_\mu[-i\partial_\mu - eA_\mu(x)]G_+(x,x') + mG_+(x,x') = \delta(x-x')$$

$$[i\partial_\mu - eA_\mu(x')]G_+(x,x')\gamma_\mu + mG_+(x,x') = \delta(x-x')$$

with suitable outgoing boundary conditions (G_+ as a function of x contain only positive frequencies for $x_0 > x_0'$ and times from Supp A, and only neg. freq. for $x_0 < x_0'$ and times from Supp A.

The basic quantity of interest is a number e^{iw} which turns out to be an infinite determinant obtained as follows. Connect the field A to 0 by a sequence of infinitesimal steps δA, and multiply the corresponding determinants $1+i\delta w$, where $i\delta w$ is given by a trace

$$\delta w = \int_{-\infty}^{\infty} (dx)\ \mathrm{tr}\ \ ie\gamma_\mu\, \delta A_\mu(x)\, G^+(x,x)$$

New notation:

$$[\gamma(p-eA)+m]\,G^+ = G^+[\gamma(p-eA)+m] = I$$

$$\delta w = \mathrm{Tr}\,(ie\gamma\,\delta A\, G^+) = i\,\mathrm{Tr}\,(G^+ e\gamma\,\delta A)$$

On the other hand if G_0^+ is the free Green's function (for $A=0$), then we get L.S.

$$G_0^+ = [1 - G_0^+ e\gamma A]\,G^+$$

$$G^+ = (1 - G_0^+ e\gamma A)^{-1}\,G_0^+$$

so

$$\delta w = i\ \mathrm{Tr}\big((1-G_0^+ e\gamma A)^{-1}\, G_0^+ e\gamma\,\delta A\big)$$

$$= -i\ \mathrm{Tr}\,(X^{-1}\cdot\delta X) \qquad X = 1 - G_0^+ e\gamma A$$

Thus

$$i\,\delta w = \mathrm{Tr}\,(X^{-1}\cdot\delta X) = \delta(\log \det X)$$

so integrating

$$e^{iw} = \det(X) = \det(1 - G_0^+ e\gamma A)$$

$$= \det(1 - e\gamma A G_0^+)$$

By symmetry considerations (<u>charge conjugation</u>) one knows w is an even fn. of e so

$$e^{2iw} = \det(1 - e\gamma A G_0^+)\det(1 + e\gamma A G_0^+)$$

$$= \det(1 - e^2\gamma A G_0^+ \gamma A G_0^+)$$

If one puts

$$\lambda K = -e^2 \gamma A G_0^+ \gamma A G_0^+$$

then it turns out that under suitable conditions

$$tr\,(KK^\dagger) < \infty$$

so that the poison tooth Fredholm determinant

$$\det{}'(1+\lambda K) = e^{-Tr(\lambda K)} \det(1+\lambda K)$$

is defined. However $Tr(\lambda K) = -e^2\, Tr\,(\gamma A G_0^+ \gamma A G_0^+)$ diverges logarithmicly. The divergent quantity is real so that $|e^{i\omega}|^2$ is finite. In fact it turns out that $|e^{i\omega}|^2 \leq 1$ and that it is the probability of the field remaining in the vacuum state.

February 3, 1979:

Recall the Dirac field $\psi(x)$ perturbed by an external electromagnetic field $A(x)$ is understood in terms of a Green's function satisfying

$$\left[\gamma(p-eA)+m\right]G^{+} = G^{+}\left[\gamma(p-eA)+m\right] = I$$

where $p=\left(\frac{1}{i}\frac{\partial}{\partial x_\mu}\right)$ is a momentum(?) operator. The free Hamiltonian is

$$H_0 = \gamma p + m$$

the perturbation is $V = e\gamma A$, and G^{+} is an inverse for

$$H = H_0 - V = \gamma p + m - e\gamma A$$

defined via outgoing bdry conditions. According to the Schwinger paper the quantity of interest is the Lipmann-Schwinger determinant

$$e^{iw} = \det(H_0^{-1}H) = \det(1-G_0^{+}V) = \det(1-G_0^{+}e\gamma A)$$

Notice that this differs from what you've looked at in that one is working with a hyperbolic DE and not a Helmholz DE where time has been replaced by a frequency k. So the obvious thing is to go back to the one-(space)-diml case and see if the above determinant makes sense.

So consider on the line a D-system

$$\frac{d}{dx}\begin{pmatrix}u_1\\u_2\end{pmatrix} = \begin{pmatrix}ik & p\\ \bar{p} & -ik\end{pmatrix}\begin{pmatrix}u_1\\u_2\end{pmatrix}$$

or rather the associated hyperbolic DE, which means $-ik$

should be replaced by $\frac{\partial}{\partial t}$. In self-adjoint form:

$$\begin{pmatrix} \frac{1}{i}\frac{d}{dx} & -\frac{1}{i}p \\ \frac{1}{i}\bar{p} & -\frac{1}{i}\frac{d}{dx} \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = k\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$$

Skew-adjoint form:

$$-\frac{\partial}{\partial t}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} \frac{\partial}{\partial x} & -p \\ \bar{p} & -\frac{\partial}{\partial x} \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$$

or

$$\left\{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\frac{\partial}{\partial t} + \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\frac{\partial}{\partial x} + \begin{pmatrix} 0 & -p \\ \bar{p} & 0 \end{pmatrix}\right\}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = 0$$

However we can also write it

$$(*) \qquad \left\{\underbrace{\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}}_{\gamma_0}\frac{\partial}{\partial t} + \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}_{\gamma_1}\frac{\partial}{\partial x} + \underbrace{\begin{pmatrix} 0 & -p \\ -\bar{p} & 0 \end{pmatrix}}_{-V}\right\}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = 0$$

but then we get a γ_0 fact when we compute the adjoint. Anyway, to keep close to Schwinger's notation we use $(*)$, so that G_0^+ is going to satisfy

$$\begin{pmatrix} \frac{\partial}{\partial x} + \frac{\partial}{\partial t} & 0 \\ 0 & \frac{\partial}{\partial x} - \frac{\partial}{\partial t} \end{pmatrix} G_0^+ = \begin{pmatrix} \delta & 0 \\ 0 & \delta \end{pmatrix}$$

So now we want to compute this Green's fn. and make sense of the outgoing bdry. conditions. First do for $\frac{\partial}{\partial y}$:

$$\frac{\partial}{\partial y} u = f$$

$$u(x,y) = \int_{-\infty}^{y} f(x,y')\,dy'$$

$$= \iint dx'dy' \; \delta(x-x')\,\eta(y-y')\,f(x',y')$$

so

$$G(x,y;x',y') = \delta(x-x')\,\eta(y-y')$$

So the Green's function for $\frac{\partial}{\partial x} + \frac{\partial}{\partial t}$ should be something like

$$\delta(x-t-(x'-t'))\,\eta(x+t-(x'+t'))$$

Now

$$\iint dx'dt' \; \delta(x-t-x'+t')\,\eta(x+t-x'-t')\,f(x',t')$$

$$= \int dt' \; \eta(\underbrace{x+t-(x-t+t')-t'}_{2t-2t'})\,f(x-t+t',t')$$

$$= \int_{-\infty}^{t} dt'\, f(x-t+t',t')$$

and

$$\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial t}\right)\int_{-\infty}^{t} dt'\, f(x-t+t',t') = \int_{-\infty}^{t} dt'\, f_1(x-t+t',t') + f(x,t) - \int_{-\infty}^{t} f_1(x-t+t',t')\,dt'$$

$$= f(x,t)$$

so this works:

$$\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial t}\right)^{-1} = \delta(x-t-(x'-t'))\,\eta(x+t-(x'+t'))$$

$$+ \text{ any function of } x-t$$

Similarly

$$\left(\frac{\partial}{\partial x} - \frac{\partial}{\partial t}\right)^{-1} = \delta(x+t-(x'+t'))\,\eta(x-t-(x'-t'))$$

$$+ \text{ any function of } x+t.$$

Now we have to understand the outgoing boundary conditions.

Let's compute G function via F.T. in time

$$\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial t}\right)G = \delta(x)\,\delta(t)$$

$$\left(\frac{\partial}{\partial x}-ik\right)\hat{G} = \delta(x)$$ [redacted]

$$\hat{G} = e^{ikx}\,\eta(x)$$

$$\therefore\quad G = \int e^{-ikt}e^{ikx}\eta(x)\,dk/2\pi$$

$$= \delta(x-t)\,\eta(x)$$

which seems to be the same as $\delta(x-t)\,\eta(x+t)$. Any other Green's function differs from this one by a function of $x-t$.

The problem is to pin down the choice of Green's function via the words "G^{+}[redacted]$(\vec{x},\vec{x}')$ as a function of $\vec{x}$ shall contain only positive frequencies for $t>t'$ and only negative frequencies for $t<t'$." Let us recall that

$$\hat{G} = e^{ikx}\,\eta(x)$$

is the extension of the L^2-Green's function for $\operatorname{Im} k>0$ and that

(+) $$\hat{G} = -e^{ikx}\,\eta(-x)$$

is the L^2-Green's function for $\operatorname{Im} k<0$. [redacted] Corresponding to (+) we get

$$G = -\int e^{ikx}\eta(-x)\,e^{-ikt}\,dk/2\pi$$

$$= -\,\delta(x-t)\,\eta(-x)$$

so the problem is which one we want.

There exists a distinction called retarded and advanced Green's functions. Retarded means it vanishes for $t<0$, hence clearly

$$\delta(x-t)\,\eta(x)$$

is the retarded Green's function while

$$-\delta(x-t)\,\eta(-x)$$

is the advanced Green's function. Also we can get the retarded G fn by Laplace transform

$$\hat{G} = \int_0^\infty e^{ikt}\, G(x,t)\, dt \qquad \text{analytic}_{\wedge}^{\text{(meaning bounded)}} \text{ in UHP}$$

Thus

$$\hat{G}(x,k) = \begin{cases} e^{ikx} & x>0 \\ 0 & x<0 \end{cases} \qquad \text{etc.}$$

So now it's clear that for the operator

$$\begin{pmatrix} \frac{\partial}{\partial x}+\frac{\partial}{\partial t} & 0 \\ 0 & \frac{\partial}{\partial x}-\frac{\partial}{\partial t} \end{pmatrix}$$

the retarded Green's matrix will satisfy

$$\hat{\mathbb{G}}(x,k) = \begin{pmatrix} e^{ikx}\,\eta(x) & 0 \\ 0 & -e^{-ikx}\,\eta(-x) \end{pmatrix}$$

hence

$$\hat{\mathbb{G}}(x,t) = \begin{pmatrix} \delta(x-t)\,\eta(x) & 0 \\ 0 & \underbrace{-\delta(x+t)\,\eta(-x)}_{-\delta(x+t)\,\eta(t)} \end{pmatrix}$$

It seems clear that there are four types of G-fns. to consider. To find the one we want let's take the F.T. wrt x

$$\left(-i\xi + \frac{\partial}{\partial t}\right)\hat{G} = \delta(t)$$

$$\hat{G} = e^{i\xi t}\eta(t) \quad \text{or} \quad -e^{i\xi t}\eta(-t)$$

$$G = \delta(t-x)\,\eta(t) \quad \text{or} \quad -\delta(t-x)\,\eta(-t)$$

$$\overset{\parallel}{\delta(x-t)\,\eta(x)}$$

February 4, 1979

Green's functions [redacted] for the Klein-Gordon equation:

$$(\Box + \mu^2)\psi = \left(\frac{\partial^2}{\partial t^2} - \sum_i \frac{\partial^2}{\partial x_i^2} + \mu^2\right)\psi = 0$$

We want to solve $(\Box + \mu^2)\,G = \delta$. The full Fourier transform satisfies

$$(-k_0^2 + \vec{k}^2 + \mu^2)\hat{G} = 1$$

or

$$\hat{G} = -\frac{1}{k_0^2 - \vec{k}^2 - \mu^2}$$

To accomplish this division of distributions one has to specify for each $\underset{\to}{k}$ what happens at the poles

$$k_0 = \pm\sqrt{\vec{k}^2 + \mu^2}$$

[redacted]

and one does this by inverse transforming wrt k_0 and pushing the integration into the complex plane. So this means one F.T. first wrt $\vec{x}$.

$$\left(\frac{\partial^2}{\partial t^2} + \vec{k}^2 + \mu^2\right)\hat{G} = \delta(t)$$

Put $\omega = \sqrt{\vec{k}^2 + \mu^2}$. Different solutions of the last equation are

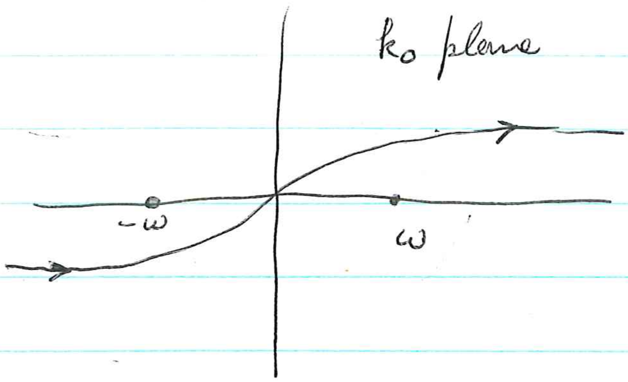
$$\frac{e^{i\omega|t|}}{2i\omega}, \qquad \frac{\sin \omega\,\text{[redacted]}\,t}{\omega}\,\eta(t)$$

denoted $\theta(t)$ by Schweber

and in general one uses contour integration:

$$\hat{G}(t, \underset{\to}{k}) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dk_0\, e^{-ik_0 t}\,\frac{(-1)}{k_0^2 - \omega^2}$$

Examples are:



Here if $t > 0$, then $e^{-ik_0 t}$ decays in the LHP, so you get the $k_0 = \omega$ pole residue

$$(-1)\frac{2\pi i}{2\pi}\,\frac{e^{-i\omega t}(-1)}{2\text{[redacted]}\omega}$$

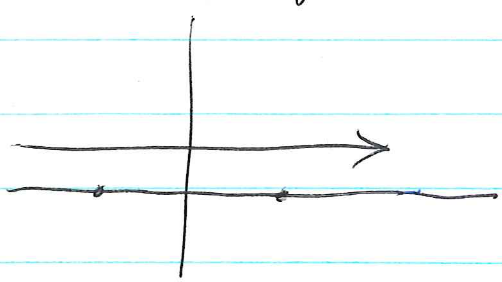
If $t < 0$, then $e^{-ik_0 t}$ decays in UHP so we get the $k_0 = -\omega$ contribution

$$\frac{2\pi i}{2\pi}\,\frac{e^{i\omega t}}{-2\omega}(-1)$$

so

$$\hat{G}(t, \underset{\rightarrow}{k}) = \text{[redacted]}\ \frac{e^{-i\omega|t|}}{-2i\omega} \qquad \omega(\underset{\rightarrow}{k}) = \sqrt{\underset{\rightarrow}{k}^2 + \mu^2}$$

since $\omega > 0$, this has positive frequencies for $t > 0$ and negative frequencies for $t < 0$. [redacted] The contour



gives the retarded Green's function (i.e. vanishes for $t < 0$)

$$\hat{G}_{ret}(t, \underset{\rightarrow}{k}) = \begin{cases} 0 & t < 0 \\ \frac{1}{2\pi}(+2\pi i)\frac{e^{+i\omega t}}{-2\omega}(+1) + \frac{1}{2\pi}(+2\pi i)\frac{e^{-i\omega t}}{2\omega}(+1) & \end{cases}$$

$$= \frac{\sin \omega t}{\omega} \qquad t > 0.$$

Because of the difficulties encountered in constructing the appropriate Green's function, I should work with something closer to the Dirac equation, i.e. where there is a mass parameter m.

The Dirac equation is of the form

$$\left\{\frac{\partial}{\partial t} + \sum_{k=1}^{3} \alpha^k \frac{\partial}{\partial x^k} + im\beta\right\}\psi = 0$$

Self-adjointness considerations (it should be of the form $\left\{\frac{1}{i}\frac{\partial}{\partial t} + H\right\}\psi$ with H hermitian) require that

$$(\alpha^k)^* = \alpha^k \qquad \beta^* = \beta$$

Also each component should satisfy $(\Box + m^2)\psi = 0$, so multiplying by $\frac{\partial}{\partial t} - \sum \alpha^k \frac{\partial}{\partial x^k} - im\beta$ gives

$$\left\{\frac{\partial^2}{\partial t^2} - \left(\sum \alpha^k \frac{\partial}{\partial x^k} + im\beta\right)^2\right\}\psi = 0$$

$$= \left\{\frac{\partial^2}{\partial t^2} - \sum \alpha^k \alpha^l \frac{\partial^2}{\partial x^k \partial x^l} - \sum im(\alpha^k\beta + \beta\alpha^k)\frac{\partial}{\partial x^k} + m^2\beta^2\right\}\psi$$

so the matrices α^k, β satisfy

$$\frac{1}{2}\left(\alpha^k\alpha^l + \alpha^l\alpha^k\right) = \delta^{kl} \qquad \beta^2 = I$$

$$\alpha^k\beta + \beta\alpha^k = 0$$

In 2 space dimensions these become

$$\alpha^2 = \beta^2 = 1, \quad \alpha\beta + \beta\alpha = 0$$

so for example

$$\alpha = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad \beta = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

It is customary to write the Dirac equation in the form

$$\Big\{ \frac{1}{i}\big(\underbrace{\beta}_{\gamma^0} \frac{\partial}{\partial t} + \sum_{k=1}^{3} \underbrace{\beta \alpha^k}_{\gamma^k} \frac{\partial}{\partial x^k} \big) + m \Big\} \psi = 0$$

or

$$\left(-i \gamma^\mu \frac{\partial}{\partial x^\mu} + m \right) \psi = 0$$

In 2-space dims we get the equation (before mult. by $-i\beta$)

$$\begin{pmatrix} \frac{\partial}{\partial t} + \frac{\partial}{\partial x} & im \\ im & \frac{\partial}{\partial t} - \frac{\partial}{\partial x} \end{pmatrix} \psi = 0$$

Now I want to see that I can define a Green's function G_0^+ with only positive frequencies for positive times, etc. The point will be to understand the singularities of this Green's function, not necessarily to get an explicit formula.

Write G fn equation in the form

$$\left\{ \frac{\partial}{\partial t} + \begin{pmatrix} \frac{\partial}{\partial x} & im \\ im & -\frac{\partial}{\partial x} \end{pmatrix} \right\} G = \delta$$

and take the F.T. wrt x:

$$\left\{ \frac{\partial}{\partial t} + i \underbrace{\begin{pmatrix} -\xi & m \\ m & \xi \end{pmatrix}}_{A} \right\} \hat{G} = \delta(t)$$

The matrix A has the eigenvalues $\pm\omega = \pm\sqrt{\xi^2 + m^2}$.
Corresponding eigenvectors are found:

$$\begin{pmatrix} -\xi & m \\ m & \xi \end{pmatrix}\begin{pmatrix} m \\ \xi+\omega \end{pmatrix} = \begin{pmatrix} -\xi m + m\xi + m\omega \\ m^2+\xi^2+\xi\omega \end{pmatrix} = \omega\begin{pmatrix} m \\ \xi+\omega \end{pmatrix}$$

$$\begin{pmatrix} -\xi & m \\ m & \xi \end{pmatrix}\begin{pmatrix} m \\ \xi-\omega \end{pmatrix} = -\omega\begin{pmatrix} m \\ \xi-\omega \end{pmatrix}$$

Put $T = \begin{pmatrix} m & m \\ \xi+\omega & \xi-\omega \end{pmatrix}$ so that $AT = T\begin{pmatrix} \omega & 0 \\ 0 & -\omega \end{pmatrix}$.

Then

$$T^{-1}\left(\frac{\partial}{\partial t} + iA\right)\hat{G}T = T^{-1}\delta(t)T = \delta(t)$$

$$\left\{\frac{\partial}{\partial t} + i\begin{pmatrix} \omega & 0 \\ 0 & -\omega \end{pmatrix}\right\} T^{-1}\hat{G}T = \delta(t)\cdot I$$

Now

$$\left(\frac{\partial}{\partial t} + i\omega\right) g = \delta(t)$$

has two different solutions:

$$e^{-i\omega t}\eta(t), \quad -e^{-i\omega t}\eta(-t).$$

The one we want has only positive frequences for $t>0$ and only negative ones for $t<0$, and since $\omega>0$ this means $e^{-i\omega t}\eta(t)$. Similarly

$$\left(\frac{\partial}{\partial t} - i\omega\right) g = \delta(t)$$

has the solutions $e^{i\omega t}\eta(t)$, $-e^{i\omega t}\eta(-t)$ and we want the latter. Thus

$$T^{-1}\hat{G}T = \begin{pmatrix} e^{-i\omega t}\eta(t) & 0 \\ 0 & -e^{i\omega t}\eta(-t) \end{pmatrix}$$

and so

$$\hat{G} = \begin{pmatrix} m & m \\ \xi+\omega & \xi-\omega \end{pmatrix} \begin{pmatrix} e^{-i\omega t}\eta(t) & 0 \\ 0 & -e^{i\omega t}\eta(-t) \end{pmatrix} \begin{pmatrix} \xi-\omega & -m \\ -\xi-\omega & m \end{pmatrix} \frac{1}{-2m\omega}$$

$$= \begin{pmatrix} me^{-i\omega t}\eta & me^{i\omega t}(-)\eta(-t) \\ (\xi+\omega)e^{-i\omega t}\eta & (\xi-\omega)e^{i\omega t}(-)\eta(-t) \end{pmatrix} \begin{pmatrix} \xi-\omega & -m \\ -\xi-\omega & m \end{pmatrix} \frac{1}{-2m\omega}$$

too complicated.

February 6, 1979:

So I've been wasting time trying to understand Green's functions, especially the Feynman Green's function. Let us take the 2 diml. space-time KG equation:

$$\left(\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} + m^2 \right) G = \delta$$

Now the idea is to understand the singularities of any solution of this, which by general thms. should lie on the lines $x = \pm t$. (One of the things you ought to understand by means of examples is [redacted] all the new ways to work with hyperbolic DE's: (Hörmander, Duistermaat-Guillemin, etc.))

Start by F.T. in x:

$$\left(\frac{d^2}{dt^2} + \xi^2 + m^2 \right) \hat{G}(t,\xi) = \delta(t)$$

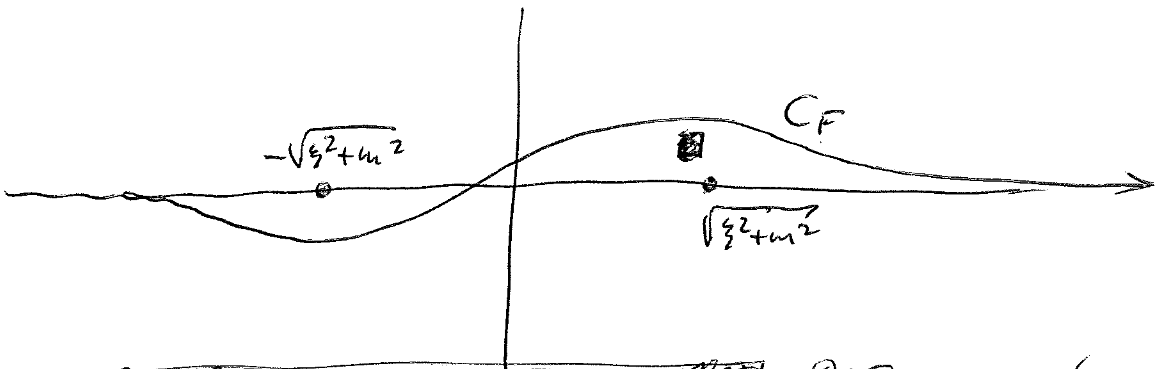
and then F.T. in t:

$$\left(-k^2 + \xi^2 + m^2 \right) \hat{G}(k,\xi) = 1$$

so

$$\hat{G}(t,\xi) = \frac{1}{2\pi}\int dk\, e^{-ikt}\frac{-1}{k^2-(\xi^2+m^2)}$$

The Feynman Green's function is obtained by taking the contour below $k=-\sqrt{\xi^2+m^2}$ and above $k=\sqrt{\xi^2+m^2}$.



We get (see 541)

$$\hat{G}^{+}(t,\xi) = \frac{e^{-i\sqrt{\xi^2+m^2}\,|t|}}{-2i\sqrt{\xi^2+m^2}}$$

and the corresponding Green's function is

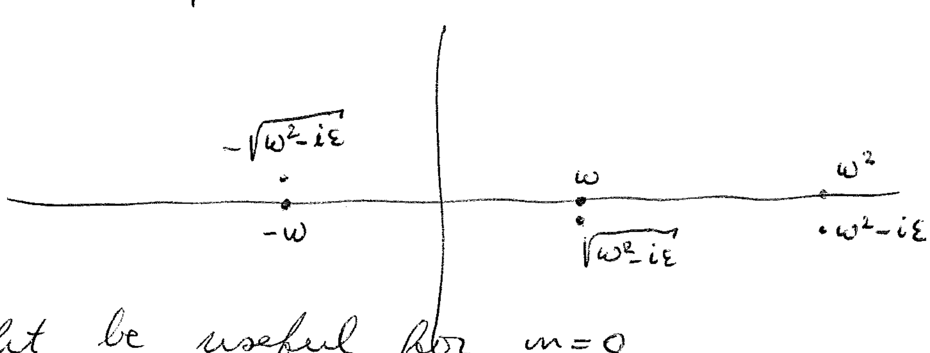
$$G^{+}(t,x) = \int\frac{d\xi}{2\pi}\, e^{-i\xi x}\,\frac{e^{-i\sqrt{\xi^2+m^2}\,|t|}}{-2i\sqrt{\xi^2+m^2}}.$$

What does this look like?

Notice that

$$\frac{1}{2\pi}\int_{C_F} dk\, e^{-ikt}\frac{-1}{k^2-\omega^2} = \lim_{\varepsilon\downarrow 0}\frac{1}{2\pi}\int_{-\infty}^{\infty} dk\, e^{-ikt}\frac{-1}{k^2+i\varepsilon-\omega^2}$$

because



This might be useful for $m=0$.

Let's try to understand the singularities of $G^+(t,x)$ for fixed $t > 0$ by means [redacted] of an asymptotic expansion as $|\xi| \to \infty$.

$$\sqrt{\xi^2 + m^2} = |\xi| \sqrt{1 + \frac{m^2}{|\xi|^2}} = |\xi| \left\{ 1 + \frac{1}{2}\frac{m^2}{|\xi|^2} + O(|\xi|^{-4}) \right\}$$

$$\frac{e^{-i\sqrt{\xi^2+m^2}\,|t|}}{-2i\sqrt{\xi^2+m^2}} = \frac{e^{-i|\xi|\left\{1+\frac{1}{2}\frac{m^2}{|\xi|^2}\right\}|t| + O(|\xi|^{-3}}}{-2i|\xi|\left\{1+\frac{1}{2}\frac{m^2}{|\xi|^2}\right\}}$$ [redacted]

$$\doteq \frac{e^{-i|\xi||t| - \frac{i}{2}\frac{m^2}{|\xi|}|t|}}{-2i|\xi|} \left\{1 - \frac{1}{2}\frac{m^2}{|\xi|^2}\right\}.$$

$$\doteq \frac{e^{-i|\xi|t}}{-2i|\xi|} \left\{1 - \frac{i}{2}\frac{m^2}{|\xi|}t + O\left(\frac{1}{|\xi|^2}\right)\right\} \qquad t>0$$

Now in 1-dimension $\displaystyle\int_{|\xi|\geq \text{const}} \frac{d\xi}{|\xi|^2}$ is convergent, [redacted]

whence [redacted] the difference between $G^+(t,x)$ and

$$\int_{|\xi|\geq\text{const}} \frac{d\xi}{2\pi} \frac{e^{-i|\xi|t}}{-2i|\xi|} e^{-i\xi x}$$

will be continuous.

February 7, 1979

I was trying to understand the Green's function:

(*) $$\left(\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} + m^2\right) G = \delta$$

If we put $t = iy$, then

$$\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} + m^2 = -\frac{\partial^2}{\partial y^2} - \frac{\partial^2}{\partial x^2} + m^2$$

$$= -\left(\frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right) + m^2$$

and so solutions of the homogeneous equation are given by

$$u(r) = u\left(\sqrt{x^2 - t^2}\right)$$

where

$$\left(\frac{1}{r}\frac{d}{dr} r \frac{d}{dr} - m^2\right) u = 0$$

Recall that $\left(-\left(r\frac{d}{dr}\right)^2 + r^2\right)\psi = k^2\psi = -s^2\psi$

or $$\left(\left(r\frac{d}{dr}\right)^2 - s^2 - r^2\right)\psi = 0$$

is the modified Bessel DE. Thus

$$K_0\left(m\sqrt{x^2 - t^2}\right), \quad I_0\left(m\sqrt{x^2 - t^2}\right)$$

are solutions of $$\left(\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} + m^2\right)\psi = 0.$$

Recall the Green's function of interest was given by

$$G(t,x) = \int_{-\infty}^{\infty} \frac{d\xi}{2\pi} e^{-i\xi x} \frac{e^{-i\sqrt{\xi^2 + m^2}\,|t|}}{-2i\sqrt{\xi^2 + m^2}}$$

Substitute
$$\xi = \frac{u-u^{-1}}{2} m \qquad 0 < u < \infty$$

$$\xi^2 + m^2 = m^2\left[\frac{(u-u^{-1})^2}{4} + 1\right] = m^2\left(\frac{u+u^{-1}}{2}\right)^2$$

$$\sqrt{\xi^2+m^2} = m\,\frac{u+u^{-1}}{2}$$

$$d\xi = \frac{1+u^{-2}}{2} m\, du = \frac{u+u^{-1}}{2} m \frac{du}{u}$$

$$\frac{d\xi}{\sqrt{\xi^2+m^2}} = \frac{du}{u}$$

so
$$G(t,x) = \int_0^\infty \frac{du}{2\pi}\frac{1}{-2i} e^{-ixm\left(\frac{u-u^{-1}}{2}\right) - i|t|m\left(\frac{u+u^{-1}}{2}\right)}$$

$$= -\frac{1}{4\pi i}\int_0^\infty e^{-i(x+|t|)m\frac{u}{2} - i(|t|-x)m\frac{u^{-1}}{2}} \frac{du}{u}$$

Notice that

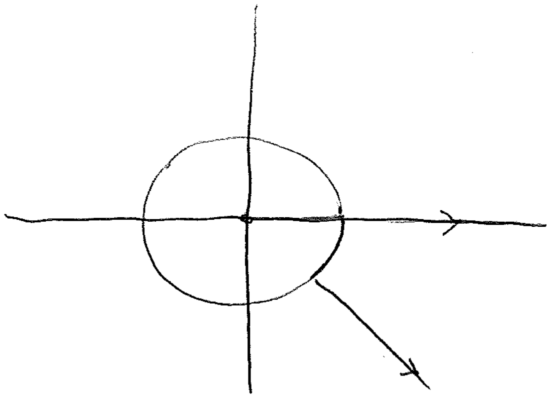
$a, b > 0$
$$\int_0^\infty e^{-\frac{a}{2}u - \frac{b}{2}u^{-1}} \frac{du}{u} = \int_0^\infty e^{-\frac{a\lambda}{2}u - \frac{b\lambda^{-1}}{2}u^{-1}} \frac{du}{u}$$

and so if λ is choosen so that $a\lambda = b\lambda^{-1}$, $\lambda^2 = \frac{b}{a}$, $a\lambda = a\sqrt{\frac{b}{a}} = \sqrt{ab}$, we get
$$\int_0^\infty e^{-\frac{\sqrt{ab}}{2}(u+u^{-1})} \frac{du}{u} = K_0\left(\sqrt{ab}\right)$$

Suppose $x > |t|$ so that $x + |t| > 0$, $|t| - x < 0$.
The integral

$a, b > 0$
$$\int_0^\infty e^{-\frac{ia}{2}u + \frac{ib}{2}u^{-1}} \frac{du}{u}$$

is conditionally convergent. As $u \to +\infty$, we can deform the path to the LHP.



The same is true for the path $u \to 0$. Thus we can evaluate the above integral over the contour $u = -it$, $0 < t < \infty$.

$$\int_0^\infty e^{-\frac{ia}{2}u + \frac{ib}{2}u^{-1}}\, du = \int_0^\infty e^{-\frac{a}{2}\cancel{t} - \frac{b}{2}t^{-1}} \frac{du}{u} = K_0(\sqrt{ab})$$

So therefore we see that

$$G(t,x) = -\frac{1}{4\pi i} K_0\left(m\sqrt{x^2 - t^2}\right) \qquad \text{for } |x| > |t|$$

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$$G(t,x) = -\frac{1}{4\pi i} \int_0^\infty e^{-i(x+|t|)\, m\frac{u}{2} - i(|t|-x)\, m\frac{u^{-1}}{2}} \frac{du}{u}$$

If $|x| < |t|$, then $a = m(|t|+x)$, $b = m(|t|-x)$ are both > 0, and one has

$$\int_0^\infty e^{-ia\frac{u}{2} - ib\frac{u^{-1}}{2}} \frac{du}{u} = \int_0^\infty e^{-i\sqrt{ab}\left(\frac{u+u^{-1}}{2}\right)} \frac{du}{u}$$

$$= K_0\left(i\sqrt{ab}\right)$$

where $K_0(r)$ which is normally defined for $r > 0$ is extended by analytic continuation to $\operatorname{Re} r \geq 0$. So we get the formula:

$$G(t,x) = \begin{cases} -\frac{1}{4\pi i} K_0\left(m\sqrt{x^2-t^2}\right) & \text{for } |x| > |t| \\ -\frac{1}{4\pi i} K_0\left(i m\sqrt{t^2-x^2}\right) & \text{for } |x| < |t| \end{cases}$$

Recall that as $r \to 0$

$$K_0(r) = \int_0^\infty e^{-r\left(\frac{t+t^{-1}}{2}\right)} \frac{dt}{t} = 2\int_1^\infty e^{-rt/2}\left(e^{-\frac{r}{2}t^{-1}}\right)\frac{dt}{t}$$

$$\cancel{= 2\int_{1/r}^\infty e^{-\frac{1}{2}-\frac{r^2}{2t}} \frac{dt}{t}} = 2\int_1^\infty e^{-rt/2}\frac{dt}{t} + O(1)$$

$$= 2\int_r^\infty e^{-t/2}\frac{dt}{t} + O(1) = 2\int_r^1 \frac{dt}{t} = -2\ln(r) + O(1)$$

Hence near $|x| = |t|$ we have

$$G(t,x) \approx \frac{1}{2\pi i}\ln\left(m\sqrt{x^2-t^2}\right) \sim +\frac{1}{4\pi i}\ln(x^2-t^2) + O(1)$$

So it seems that the singularity in the Green's function is logarithmic, hence the Green's function is integrables locally.

Now look at the case $m=0$.

$$G(t,x) = \int_{-\infty}^\infty \frac{d\xi}{2\pi} e^{-i\xi x} \frac{e^{-i|\xi||t|}}{-2i|\xi|}$$

Some process will be needed to make sense of this integral near $\xi = 0$.

$$G(t,x) = \int_0^\infty \frac{d\xi}{2\pi}\left(e^{-i\xi x} + e^{i\xi x}\right)\frac{e^{-i\xi|t|}}{-2i\xi}$$

$$\frac{\partial G}{\partial t}(t,x) = \int_0^\infty \frac{d\xi}{2\pi}\left(e^{-i\xi x} + e^{i\xi x}\right)\frac{e^{-i\xi|t|}}{2} \qquad \text{say } t>0$$

$$= \frac{1}{4\pi}\left[\frac{1}{i(x+t)} + \frac{1}{i(t-x)}\right]$$

$$= \frac{1}{4\pi i}\,\frac{2t}{t^2-x^2}$$

so

$$G(t,x) = \frac{1}{4\pi i}\log(t^2-x^2) \quad + \text{ fn. of } x$$

Also

$$\frac{\partial G}{\partial x} = \int_0^\infty \frac{d\xi}{2\pi}\left(e^{-i\xi x} - e^{i\xi x}\right)\frac{e^{-i\xi t}}{2}$$

$$= \frac{1}{4\pi i}\left[\frac{1}{x+t} - \frac{1}{t-x}\right] = \frac{1}{4\pi i}\,\frac{2x}{x^2-t^2}$$

$$\therefore \quad G = \frac{1}{4\pi i}\log(x^2-t^2) + \text{fn. of } t.$$

So what appears to be happening is that one has the function

$$\frac{1}{4\pi i}\log|x^2-t^2| = \frac{1}{4\pi i}\log|x+t| + \frac{1}{4\pi i}\log|x-t|$$

which is a solution of the homogeneous equation added to a more familiar type of Greens function.

Compute forward Green's fn.

$$\int_{-\infty}^{\infty}\frac{d\xi}{2\pi}\, e^{-i\xi x}\,\frac{\sin \xi t}{\xi}\,\eta(t) = \eta(t)\begin{cases}\frac{1}{2} & \text{for } |x|<t \\ 0 & \text{for } |x|>t.\end{cases}$$

Since $\displaystyle\int_{-t}^{t} \frac{1}{2} e^{i\xi x}\, dx = \frac{1}{2}\,\frac{e^{i\xi t} - e^{-i\xi t}}{i\xi} = \frac{\sin \xi t}{\xi}$

The backwards Green's function is the reflection of this under $t \mapsto -t$. The average of the forward and backward Green's functions is

$$\frac{1}{2}G_f + \frac{1}{2}G_b = \begin{cases} \frac{1}{4} & \text{if } |x| < |t| \\ 0 & \text{otherwise} \end{cases}$$

Note that

$$G^+ = \begin{cases} \frac{2}{4\pi i} \log\left(\sqrt{x^2 - t^2}\right) & |x| > |t| \\ \frac{2}{4\pi i} \log\left(i\sqrt{t^2 - x^2}\right) & |x| < |t| \end{cases}$$

is just

$$\frac{1}{4\pi i} \log|x^2 - t^2| + \frac{1}{2}(G_f + G_b)$$

So we seem to get

$$\boxed{G^+(t,x) = \frac{1}{4\pi i} \log|x^2 - t^2| + \frac{1}{4}\,\eta(t^2 - x^2)}$$

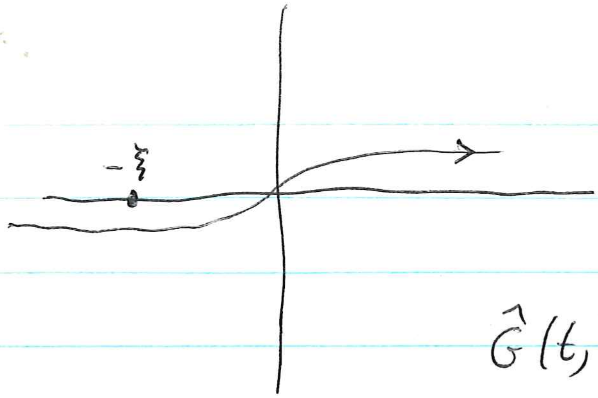
So now let us return to

$$\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial x}\right) G = \delta$$

Taking F.T. in x gives

$$\left(\frac{d}{dt} - i\xi\right)\hat{G}(t,\xi) = \delta(t).$$

Using the Feynman prescription



$$\hat{G}(t,\xi) = \int \frac{dk}{2\pi}\, e^{-ikt}\, \frac{1}{-i(k+\xi)}$$

If $t>0$, then push into LHP

$$\hat{G}(t,\xi) = \begin{cases} 0 & \xi > 0 \\ \dfrac{-2\pi i}{-2\pi i}\, e^{i\xi t} = e^{i\xi t} & \xi < 0. \end{cases}$$

But if $t<0$, then push into UHP to get

$$\hat{G}(t,\xi) = \begin{cases} \dfrac{2\pi i}{-2\pi i}\, e^{i\xi t} = -e^{i\xi t} & \xi > 0 \\ 0 & \xi < 0. \end{cases}$$

Thus

$$\hat{G}(t,\xi) = e^{i\xi t}\left[\eta(t)\eta(-\xi) - \eta(-t)\eta(\xi)\right].$$

Notice this is a tempered distribution, so it has a Fourier transform wrt ξ. If $t>0$

$$G(t,x) = \int_{-\infty}^{0} \frac{d\xi}{2\pi}\, e^{-i\xi x}\, e^{i\xi t} = \frac{1}{2\pi}\,\frac{1}{-ix+it} = \frac{-1}{2\pi i}\,\frac{1}{x-t}$$

and if $t<0$, then

$$G(t,x) = \int_{0}^{\infty} \frac{d\xi}{2\pi}\, e^{-i\xi x}\, e^{i\xi t}(-1) = (-1)\frac{1}{2\pi}\,\frac{1}{ix-it} = \frac{-1}{2\pi i}\,\frac{1}{x-t}$$

This formal calculation makes one suspect that $G(t,x)$ is $\frac{-1}{2\pi i} P\left(\frac{1}{x-t}\right)$ + conventional Green's function made up of δ functions.

Let us compute the F.T. of $-\frac{1}{2\pi i} P\left(\frac{1}{x-t}\right)$:

$$\left(-\frac{1}{2\pi i}\right)\int_{-\infty}^{\infty} P\left(\frac{1}{x-t}\right) e^{i\xi x}\, dx = \left(-\frac{1}{2\pi i}\right) e^{i\xi t}\int_{-\infty}^{\infty} P\left(\frac{1}{x}\right) e^{i\xi x}\, dx$$

the P means you average

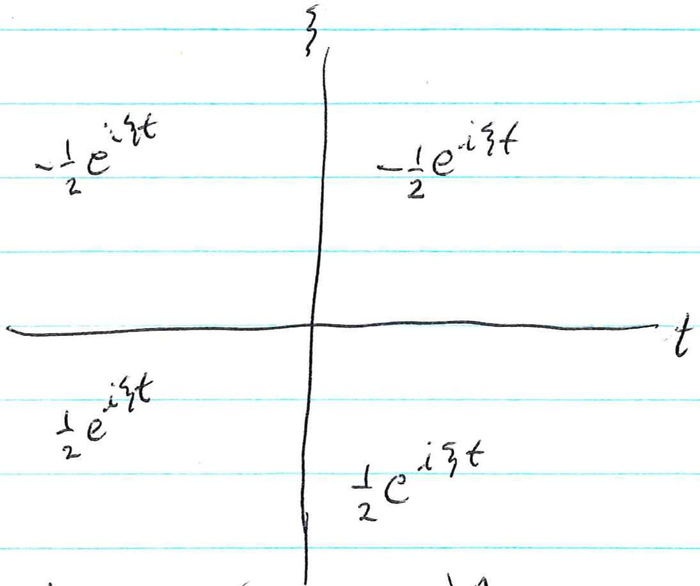
If $\xi > 0$ use UHP to get

$$\frac{1}{2}\left(-\frac{1}{2\pi i}\right) e^{-i\xi t} \Big(\underset{\substack{\uparrow \\ \mathrm{res}_0 (e^{-i\xi x})}}{2\pi i} + \underset{\substack{\uparrow \\ \text{upper contour}}}{0} \Big) = -\frac{1}{2} e^{i\xi t}$$

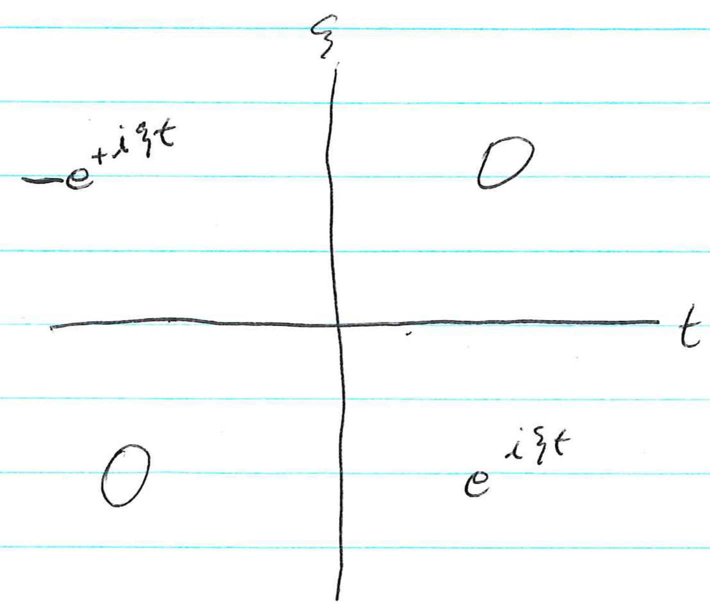
If $\xi < 0$ use LHP to get

$$\frac{1}{2}\left(-\frac{1}{2\pi i}\right) e^{-i\xi t} \left(-2\pi i\right) = \frac{1}{2} e^{-i\xi t}$$

So to get $\hat{G}(t, \xi)$ we want to add $\frac{1}{2} e^{i\xi t} \eta(t)$ $-\frac{1}{2} e^{i\xi t} \eta(-t)$



↑ here is $\left(-\frac{1}{2\pi i} P\left(\frac{1}{x-t}\right)\right)^\wedge$



$\hat{G}(t, \xi)$

Thus

$$\boxed{G(t, x) = \frac{-1}{2\pi i} P\left(\frac{1}{x-t}\right) + \frac{1}{2}\delta(x-t)\eta(t) - \frac{1}{2}\delta(x-t)\eta(-t)}$$

So at this point we understand somewhat how to make sense of the Green's function for

$$\begin{pmatrix} \frac{\partial}{\partial t} + \frac{\partial}{\partial x} & 0 \\ 0 & \frac{\partial}{\partial t} - \frac{\partial}{\partial x} \end{pmatrix}$$

having only positive frequencies for $t>0$ and only negative frequencies for $t<0$. Now the next point is to take the equation with potential

$$\left\{ \begin{pmatrix} \frac{\partial}{\partial t}+\frac{\partial}{\partial x} & 0 \\ 0 & \frac{\partial}{\partial t}-\frac{\partial}{\partial x} \end{pmatrix} - \underbrace{\begin{pmatrix} 0 & p \\ -\bar{p} & 0 \end{pmatrix}}_{V} \right\} u = 0$$

and to see if $\det(1-GV)$ is defined. This leads to

$$\det\left(I - \begin{pmatrix} 0 & \left(\frac{\partial}{\partial t}+\frac{\partial}{\partial x}\right)^{-1} p \\ -\left(\frac{\partial}{\partial t}-\frac{\partial}{\partial x}\right)^{-1}\bar{p} & 0 \end{pmatrix} \right) = \det\left(1+\left(\frac{\partial}{\partial t}-\frac{\partial}{\partial x}\right)^{-1}\bar{p}\left(\frac{\partial}{\partial t}+\frac{\partial}{\partial x}\right)^{-1}p\right)$$

which brings up the problem of whether the operator

$$\left(\frac{\partial}{\partial t}-\frac{\partial}{\partial x}\right)^{-1}\bar{p}\left(\frac{\partial}{\partial t}+\frac{\partial}{\partial x}\right)^{-1}p$$

is of trace class.

A simpler problem would be the equation

$$\left(\frac{\partial^2}{\partial t^2}-\frac{\partial^2}{\partial x^2}+q\right)u = 0$$

Here one deals with the [redacted] operator

[redacted]

$$\left(G_q^+ f\right)(t,x) = \iint dt'dx' \; G(t-t', x-x')\, q(t',x')\, f(t',x')$$

whose trace should be obtained by integrating over the diagonal:

$$tr(G^+ g) = \iint dt\, dx\;\; G(0,0)\, g(t', x')$$

obviously undefined.

February 9, 1979

State vector $\psi(t)$ in Schröd picture satisfies

$$\frac{d}{dt}\psi_S(t) = -iH\psi_S(t) = -i(H_0 - V)\psi_S(t)$$

or

$$\left(\frac{d}{dt} + iH_0\right)\psi_S = iV\psi_S$$

$$\frac{d}{dt}\left(e^{-iH_0t}\psi_S(t)\right) = e^{iH_0t}\, iV\, e^{-iH_0t}\;\; e^{iH_0t}\psi_S(t)$$

The state vector in the Dirac or interaction picture is

$$\psi_D(t) = e^{-iH_0t}\psi_S(t).$$

It satisfies $\quad \frac{d}{dt}\psi_D(t) = H_I(t)\,\psi_D(t)$

where $\quad H_I(t) = e^{iH_0t}(iV)e^{-iH_0t}$

Thus integration gives the integral equation

$$\psi_D(t) = \psi_{in} + \int_{-\infty}^{t} dt' H_I(t')\psi_D(t')$$

hence iterating

$$\psi_D(t) = \psi_{in} + \int_{-\infty}^{t} dt_1 H_I(t_1)\psi_{in} + \int_{-\infty}^{t} dt_1 \int_{-\infty}^{t_1} dt_2\;\; H_I(t_1)H_I(t_2)\psi_{in} + \cdots$$

which leads to the following formula for S

$$S = 1 + \int_{-\infty}^{\infty} dt_1 H_I(t_1) + \int_{-\infty}^{\infty} dt_1 \int_{\infty}^{t_1} dt_2 \; H_I(t_1) H_I(t_2) + \cdots$$

Then one introduces the time ordering operator

$$P\left(H_I(t_1) H_I(t_2)\right) = \begin{cases} H_I(t_1)\, H_I(t_2) & \text{if } t_1 > t_2 \\ H_I(t_2)\, H_I(t_1) & \text{if } t_1 < t_2 \end{cases}$$

and one formally gets

$$S = P\left(e^{\int dt\, H_I(t)}\right).$$

In the case of the Dirac field perturbed by an external electromagnetic field the operators will be working on some kind of Fock space, and the quantity of interest will be

$$e^{i\omega} = \langle 0 | S | 0 \rangle$$

where $|0\rangle$ denotes the vacuum state[redacted].

Let's go over Schwinger's formulas relating a vacuum expectation value to a determinant. Let V be a vector space with a basis [redacted] $e_1, e_2, \ldots\ldots$, [redacted] and dual basis $\check{e}_i$. On ΛV we have the creation operators

$$X_i^- = e(e_i) \qquad\qquad e(v) = \text{left mult. by } v$$

and destruction op:

$$X_i^+ = i(\check{e}_i)$$

satisfying the canonical commutation relations

$$\{x_i^-, x_j^-\} = \{x_i^+, x_j^+\} = 0 \qquad \{A,B\} = AB + BA$$

$$\{x_i^+, x_j^-\} = i(\check{e}_i)e(e_j) + e(e_j)i(\check{e}_i) = \delta_{ij}$$

Let K be a linear transformation on V given by matrix $\{K_{rs}\}$, and λ a parameter. Schwinger considers the transformation on ΛV which is an ordered exponential:

$$S = P\left(\exp\left(\lambda \sum_{r,s} x_r^+ K_{rs}\, x_s^-\right)\right) = \sum_{n \geq 0} \frac{\lambda^n}{n!}\, P\left(\left(\sum x_r^+ K_{rs}\, x_s^-\right)^n\right)$$

$$= \sum_{n\geq 0} \frac{\lambda^n}{n!} \sum_{\substack{r_1,\dots,r_n \\ s_1,\dots,s_n}} x_{r_1}^+ \cdots x_{r_n}^+ K_{r_1 s_1} \cdots K_{r_n s_n}\, x_{s_n}^- \cdots x_{s_1}^-$$

For this operator S on ΛV it is clear that

$$\langle 0 | S | 0 \rangle = \det(1 + \lambda K)$$

where $|0\rangle$ denotes the element $1 \in \Lambda^0 V$.

Perhaps the above can be interpreted via the Clifford algebra.

February 10, 1979

Propagator approach to the ordinary Schröd equation.

$$\left(-\frac{d^2}{dx^2}+V\right)u = Eu \qquad \text{time-ind.}$$

$$\left(-\frac{\partial^2}{\partial x^2}+V\right)\psi = i\frac{\partial\psi}{\partial t} \qquad \text{time-dep.}$$

Think of $\psi(t)=\psi(t,x)$ [redacted] as the probability amplitude of finding a particle at time t and position x. The Green's function or propagator is the kernel expressing $\psi(t,x)$ in terms of $\psi(t',x')$ for $t'<t$:

$$\psi(t,x) = \int K(t,x;t',x')\,\psi(t',x')\,dx'$$

Thus K represents $e^{-iH_{\text{[redacted]}}(t-t')}$. Better notation $K(t-t';x,x')$

Let H_0 be the Hamiltonian with $q=0$, and let $\psi_0(t)$ denote [redacted] a free wave function. Suppose the interacting potential q is switched on for a small time interval dt_1 at t_1. Let $\psi(t)$ be the new wave function. Then

$$i\,\frac{\psi(t_1+dt_1)-\psi(t_1)}{dt} = (H_0-V)\psi(t_1)$$

$$\psi(t_1+dt_1) = \underbrace{\psi(t_1)}_{\psi_0(t_1)} + dt_1\frac{1}{i}(H_0-V)\underbrace{\psi(t_1)}_{\psi_0(t_1)}$$

$$\psi(t_1+dt_1) = \psi_0(t_1)\text{[redacted]} - i\,dt_1 H_0\psi(t_1) + dt_1\, iV\psi_0(t_1)$$

$$= \psi_0(t_1+dt_1) + dt_1\, iV\psi_0(t_1)$$

so
$$\psi(t) = \psi_0(t) + dt_1\, e^{-iH_0(t-t_1)}\, iV\, e^{-iH_0(t_1-t')}\psi(t')$$

and so we conclude that

[redacted] .

$$K(t-t';x,x') = K_0(t-t',x,x') + dt_1 \int K_0(t-t_1,x,x_1)\, iV(t_1,x_1)\, K_0(t_1-t',x_1,x')\, dx_1$$

Suppose next the interaction is switched on for t_1 to t_1+dt_1 and t_2 to t_2+dt_2, where $t_1<t_2$
Then

$$\psi(t) = \psi_0(t) \qquad t'<t<t_1$$
$$= \psi_0(t) + dt_1\, e^{-iH_0(t-t_1)}\, iV \text{ [redacted] } \psi_0(t_1) \qquad t_1<t<t_2$$

It would seem to be simpler to use propagator notation:

$$K(t-t') = K_0(t-t') \qquad t'<t<t_1$$
$$= K_0(t-t_1)\left[1+dt_1\, iV(t_1)\right]K_0(t_1-t') \qquad t_1<t<t_2$$
$$= K_0(t-t_2)\left[1+dt_2\, iV(t_2)\right]K_0(t_2-t_1)\left[1+dt_1 iV(t_1)\right]K_0(t_1-t')$$

for $t_2 < t$ [redacted]

Expand the last expression:

$$K(t-t') = K_0(t-t') + \left\{K_0(t-t_1)\, dt_1 iV(t_1)\, K_0(t_1-t') + K_0(t-t_2)\, dt_2 iV(t_2) K_0(t_2-t')\right\}$$

$$+ K_0(t-t_2)\, dt_2\, iV(t_2) K_0(t_2-t_1)\, dt_1 iV(t_1)\, K_0(t_1-t')$$

So now let us pass to the limit so as to get

$$K(t-t') = K_0(t-t') + \int_{t'}^{t} dt_1 \; K_0(t-t_1)\, iV(t_1)\, K_0(t_1-t')$$

$$(*) \qquad + \int_{t'}^{t} dt_1 \int_{t'}^{t_1} dt_2 \;\; K_0(t-t_1)\, iV(t_1)\, K_0(t_1-t_2)\, iV(t_2)\, K_0(t_2-t')$$

$$+ \cdots$$

which is $e^{-iH_0 t}\, U(t,t')\, e^{+iH_0 t'}$, $U(t,t')$ denoting the propagator in the Dirac picture. Another version of the formula is the integral equation

$$K(t-t') = K_0(t-t') + \int_{t'}^{t} dt_1 \, K_0(t-t_1)\, iV(t_1)\, K(t_1-t')$$

[redacted] If we introduce the space variables into the picture and put $\underline{x} = (t, x)$, then $(*)$ becomes

$$K(\underline{x}, \underline{x}') = K_0(x-x') + \int_{t'}^{t} d\underline{x}_1 \, K_0(\underline{x}-\underline{x}_1)\, iV(\underline{x}_1)\, K_0(\underline{x}_1-\underline{x}')$$

$$+ \int_{t'}^{t} d\underline{x}_1 \int_{t'}^{t} d\underline{x}_2 \; K_0(\underline{x}-x_1)\, iV(\underline{x}_1)\, K_0(\underline{x}_1-\underline{x}_2)\, iV(\underline{x}_2)\, K_0(\underline{x}_2-x')$$

$$+ \cdots\cdots$$

where it has to be [redacted] that $K_0(\underline{x}_1-\underline{x}_2) = 0$ if $t_1 < t_2$.

So [redacted] far we have looked at the ordinary Schroedinger equation as a first order "hyperbolic" system and found the forward Green's function

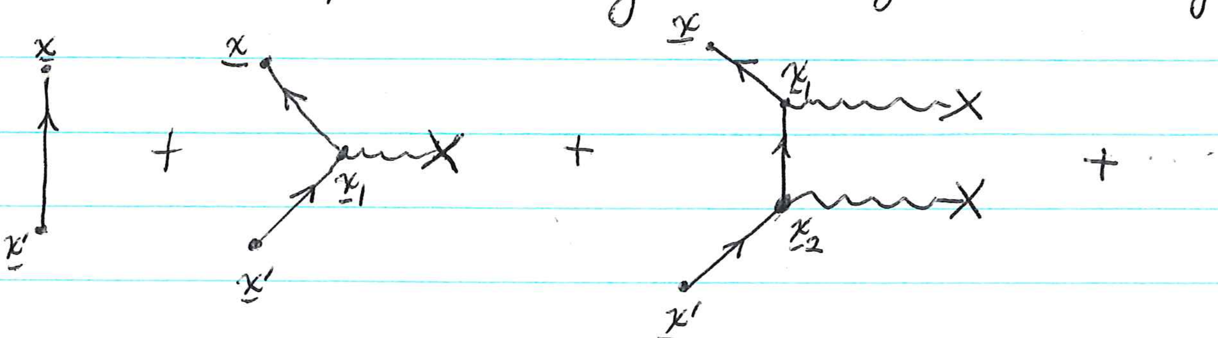
$$K(t-t') = \begin{cases} e^{-iH(t-t')} & t>t' \\ 0 & t<t' \end{cases}$$

satisfying

$$\left(\frac{\partial}{\partial t} + iH\right) K(t-t') = \delta(t-t')$$

Somehow Feynman uses the same Green's function methods but with different boundary conditions.

Thus $K(\underline{x},\underline{x}')$ represents the relative probability amplitude for finding an electron at $\underline{x}$ initially [redacted] in the state $\psi(x')|\Phi_0\rangle$, and the terms in the series are represented by the Feynman diagrams.



February 11, 1979

Review S-matrix formalism: We have Schrödinger equation

$$i\frac{\partial}{\partial t}\psi(t) = H\psi(t) = (H_0 - V)\psi(t)$$

which we convert to an integral equation:

$$\left(\frac{\partial}{\partial t} + iH_0\right)\psi = iV\psi$$

$$\frac{\partial}{\partial t}\Big(\underbrace{e^{iH_0 t}\psi}_{\substack{\text{Dirac state}\\ \text{vector } \psi_D(t)}}\Big) = \underbrace{e^{iH_0 t}\, iV\, e^{-iH_0 t}}_{\mathcal{H}_I(t)}\; e^{iH_0 t}\psi$$

$$e^{iH_0 t}\psi(t) = e^{iH_0 t'}\psi(t') + \int_{t'}^{t} dt_1\, \mathcal{H}_I(t_1)\, e^{iH_0 t_1}\psi(t_1)$$

Thus if $U(t,t')$ denotes the propagator for the Dirac state vector from time t' to time t we have

$$U(t,t') = I + \int_{t'}^{t} dt_1\, \mathcal{H}_I(t_1)\, U(t_1,t')$$

$$= I + \int_{t'}^{t} dt_1\, \mathcal{H}_I(t_1) + \int_{t'}^{t} dt_1 \int_{t'}^{t_1} dt_2\, \mathcal{H}_I(t_1)\,\mathcal{H}_I(t_2) + \cdots$$

$$= I + \int_{t'}^{t} dt_1\, \mathcal{H}_I(t_1) + \int_{t'}^{t} dt_1 \int_{t'}^{t} dt_2\, P\{\mathcal{H}_I(t_1)\mathcal{H}_I(t_2)\} + \cdots$$

$$= P\left\{ e^{\int_{t}^{t'} dt_1 \mathcal{H}_I(t_1)} \right\}.$$

So the problem for me is to understand the interaction Hamiltonian in the case of the Dirac field ψ perturbed by an external EM field A. According to Schweber

$$\mathcal{H}_I(x) = N\left(\tilde\psi(x)\,\gamma A(x)\,\psi(x)\right) \qquad A\!\!\!/ = \gamma A$$

with the following explanation. Here $x = (t, \vec{x})$ is a point of space time. One has

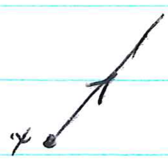
$$\psi(x) = \psi^+(x) + \psi^-(x)$$

destroys elec. / creates positron at x:


$$\tilde\psi(x) = \widetilde{\psi^+} + \widetilde{\psi^-}$$

$$= \tilde\psi^-(x) + \tilde\psi^+(x)$$

creates elec. / destroys pos.




Read as follows + means destroy, − means create

$\psi(x)$ means arrow heads toward x

$\tilde\psi(x)$ " " " away from x

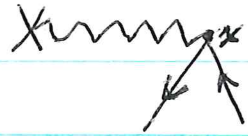
Now the N arranges things so that the creation operators appear to the left of the destruction operators. Thus

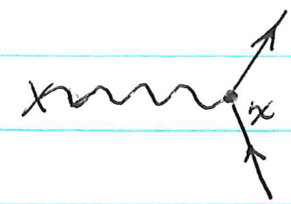
$$N(\tilde\psi A\!\!\!/ \psi) = N((\tilde\psi^+ + \tilde\psi^-)A\!\!\!/(\psi^+ + \psi^-))$$

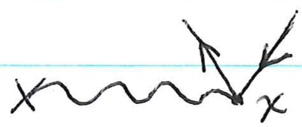
$= \tilde\psi^+ A\!\!\!/$ [redacted]$^+ + \tilde\psi^- A\!\!\!/ \psi^+ + \tilde\psi^- A\!\!\!/ \psi^- \pm \psi^- A\!\!\!/ \tilde\psi^+$

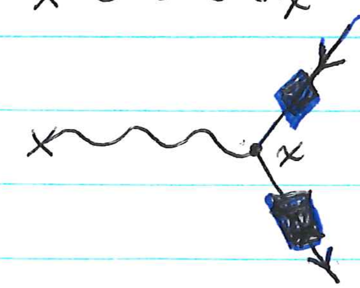
The last term is $N(\tilde{\psi}^+ A\!\!\!/ \psi^-)$ which has to be rearranged by N and I don't know whether a sign is introduced or not.

The four terms in $N(\tilde{\psi} A\!\!\!/ \psi)$ are represented by the following Feynman diagrams:

$\tilde{\psi}^+ A\!\!\!/ \psi^+(x)$  pair destruction

$\tilde{\psi}^- A\!\!\!/ \psi^+(x)$  electron scattering

$\tilde{\psi}^- A\!\!\!/ \psi^-(x)$  pair creation

$\psi^- A\!\!\!/ \tilde{\psi}^+(x$  positron scattering

These diagrams represents the first order processes, that is, the term

$$\int d^4x_1 \; \mathcal{H}_I(x_1)$$

in the S-matrix.

Let's next consider the 2nd order term

$$\frac{1}{2!} \int d^4x_1 \int d^4x_2 \; T\{ N(\tilde{\psi} A\!\!\!/ \psi)(x_1) \cdot N(\tilde{\psi} A\!\!\!/ \psi)(x_2) \}$$

where T is the time ordering operator. According to Wick's theorem one has

$$T\{N(\tilde\psi(x_1)\not{A}(x_1)\psi(x_1))\cdot N(\tilde\psi(x_2)\not{A}(x_2)\psi(x_2))\}$$

$$= N(\tilde\psi(x_1)\not{A}(x_1)\psi(x_1)\tilde\psi(x_2)\not{A}(x_2)\psi(x_2))$$

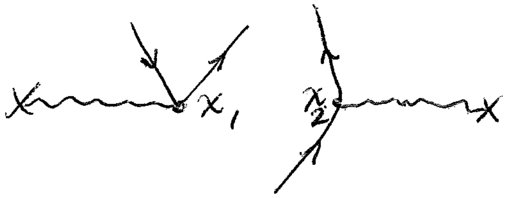
$$+ N(\tilde\psi(x_1)\not{A}(x_1)\underbrace{\psi(x_1)^{\circ}\tilde\psi(x_2)^{\circ}}_{\text{contract these two factors to get } K_+(x_1-x_2)}\not{A}(x_2)\psi(x_2))$$

$$+ N(\underbrace{\tilde\psi(x_1)^{\circ}\not{A}(x_1)\psi(x_1)\tilde\psi(x_2)\not{A}(x_2)\psi(x_2)^{\circ}}_{\text{contract}})$$

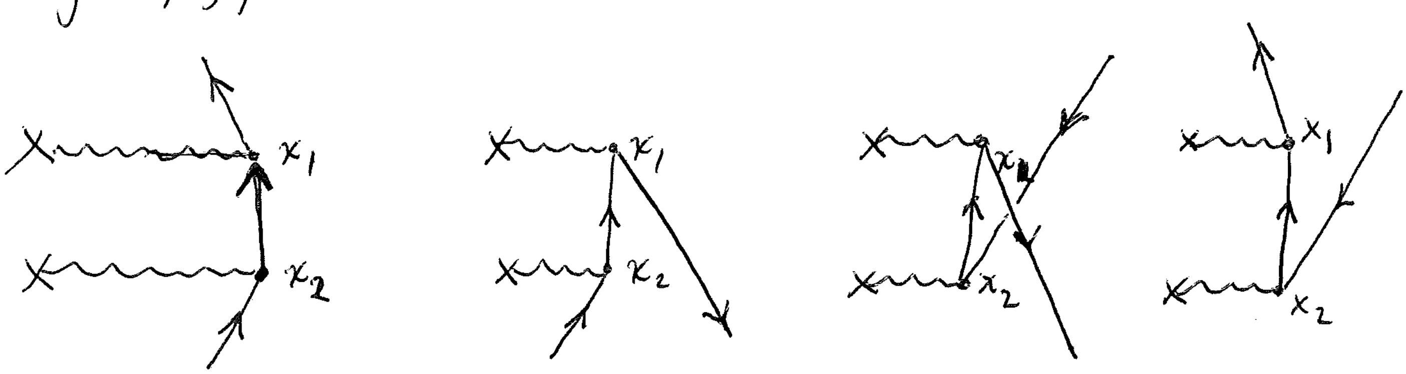
$$+ N(\tilde\psi(x_1)^{\circ}\underbrace{\not{A}(x_1)\psi(x_1)^{\bullet}\underbrace{\tilde\psi(x_2)^{\bullet}}\not{A}(x_2)\psi(x_2)^{\circ}})$$

The contracted terms result from the commutation relations between field operators.

The first term represents pairs of elementary processes of the first order [redacted] occurring together, e.g.



The second term is $N(\tilde\psi(x_1)\not{A}(x_1)K_+(x_1-x_2)\not{A}(x_2)\psi(x_2))$ and is the sum of 4 processes [redacted] obtained by splitting $\psi, \tilde\psi$:



The third term in the T_product decomposition is for some reason equal to

$$N\left(\tilde{\psi}(x_2)\,A\!\!\!/(x_2)\,K(x_2-x_1)\,A\!\!\!/(x_1)\psi(x_1)\right)$$

and it gives the same diagrams as immediately above but with x_1, x_2 interchanged.

The last term in the T-product decomposition is

$$\circledast \qquad -\,Tr\left(A\!\!\!/(x_1)\,K_+(x_1-x_2)\,A\!\!\!/(x_2)\,K_+(x_2-x_1)\right).$$

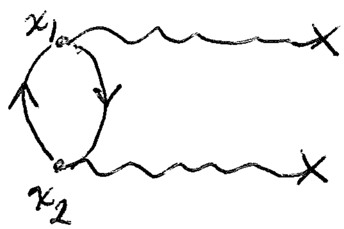
(The minus sign seems to appear because of the following steps:

$$N\big(\tilde{\psi}(x_1)^{\circ}A\!\!\!/(x_1)\,\underbrace{\psi(x_1)^{\bullet}\,\tilde{\psi}(x_2)^{\bullet}}_{K_+(x_1-x_2)}\,A\!\!\!/(x_2)\,\psi(x_2)^{\circ}\big)$$

$$= -\,N\big(A\!\!\!/(x_1)\,K_+(x_1-x_2)\,A\!\!\!/(x_2)\,\underbrace{\psi(x_2)^{\circ}\,\tilde{\psi}(x_1)^{\circ}}_{K_+(x_2-x_1)}\big)$$

because the 2 spinor factors have been interch.

$\circledast$ represents the vacuum process:



The possible 3rd order vacuum processes from the different possible ways of contracting totally in

$$\tilde{\psi}(x_1)A\!\!\!/(x_1)\psi(x_1)\;\tilde{\psi}(x_2)A\!\!\!/(x_2)\psi(x_2)\;\tilde{\psi}(x_3)A\!\!\!/(x_3)\psi(x_3)$$

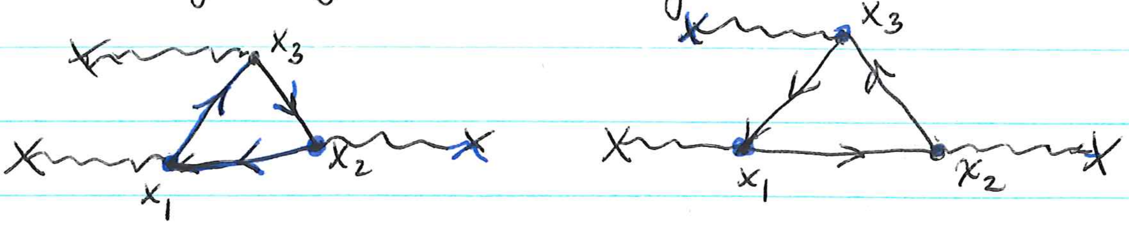
first:

yields $\quad -\mathrm{Tr}\left(\mathbb{A}(x_1)K_+(x_1-x_2)\mathbb{A}(x_2)K(x_2-x_3)\mathbb{A}(x_3)K_+(x_3-x_1)\right)$

or contract $\tilde{\psi}(x_1), \psi(x_3)$ and $\tilde{\psi}(x_3)$ with $\psi(x_2)$ etc to get

$$-\mathrm{Tr}\left(\mathbb{A}(x_1)K_+(x_1-x_3)\mathbb{A}(x_3)K(x_3-x_2)\mathbb{A}(x_2)K_+(x_2-x_1)\right)$$

[redacted] So [redacted] you get the diagrams



Can you see why all the vacuum diagrams ~~constitute~~ the Fredholm expansion of $\det(1+\mathbb{A}K_+)$?
Can you understand Schwinger's formula for $\det(1-\lambda K)$ as a vacuum expectation value?

Let's review the physicists' view of the formula

$$\det(1-\lambda K) = e^{\mathrm{tr}\log(1-\lambda K)} = e^{-\sum_{m=1}^{\infty}\frac{\lambda^m}{m}\mathrm{tr}(K^m)}$$

One has Fredholm formula:

$$\det(1-\lambda K) = \sum_{n\geq 0}\frac{(-\lambda)^n}{n!}\int dx_1\cdots\int dx_n\, \det_{(n)}(K(x_i,x_j))$$

Write out $\det_n$ as a sum over the symmetric group Σ_n

$\det_n K(x_i,x_j) = \sum_{\sigma\in\Sigma_n}$ [redacted] $(-1)^\sigma K_\sigma(x_1,x_{\sigma 1})\cdots K(x_n,x_{\sigma n})$

and analyze σ in terms of its cycles. One gets

$$\int dx_1 \int dx_2 \cdots \int dx_n \; K(x_1, x_{\sigma_1}) \cdots K(x_n, x_{\sigma_n}) = \operatorname{tr}(K^{m_1}) \cdots \operatorname{tr}(K^{m_\ell})$$

if σ is a disjoint union of cycles of lengths $m_1, m_2, \ldots, m_\ell$. So one gets

$$\int dx_1 \cdots \int dx_n \; \det_n\{K(x_i, x_j)\} = \sum_{\substack{\underline{\underline{m}} \text{ partition} \\ \text{of } n}} \operatorname{tr}(K^{m_1}) \cdots \operatorname{tr}(K^{m_\ell}) \cdot \begin{array}{l}\text{number of } \sigma \text{ in } \Sigma_n \\ \text{belonging to } \underline{\underline{m}} \\ \cdot \text{ sign of such } \sigma.\end{array}$$

Given a partition $m_1 \geq m_2 \geq \cdots \geq m_\ell$ of n, to count the number of permutations σ belonging to it, use the fact that these form a conjugacy class, so their number is $n!/\text{card of centralizer}$. We should write the partition

$$n = \underbrace{1 + \cdots + 1}_{k_1 \text{ times}} + \underbrace{2 + \cdots + 2}_{k_2 \text{ times}} + \cdots$$

so that $n = k_1 + 2k_2 + \cdots$. The centralizer of a permutation belonging to this partition is

$$\Sigma_{k_1} \times \Sigma_{k_2} \int \mathbb{Z}/2 \times \Sigma_{k_3} \int \mathbb{Z}/3 \times \cdots$$

so that the number of such permutations is

$$n! \,/\, k_1!\; k_2!\; 2^{k_2}\; k_3!\; 3^{k_3} \cdots\cdots$$

The sign of such a permutation is

$$(-1)^{(1-1)k_1 + (2-1)k_2 + (3-1)k_3} = (-1)^{n + k_1 + k_2 + k_3 + \cdots}$$

Thus
$$\det(1 - \lambda K) = \sum_{k_1, k_2, \ldots \geq 0} \frac{(+\lambda)^{k_1 + 2k_2 + \cdots}}{(k_1 + 2k_2 + \cdots)!} (\operatorname{tr} K)^{k_1} (\operatorname{tr} K^2)^{k_2} \cdots\cdots$$
$$\times \frac{(k_1 + 2k_2 + \cdots)!}{k_1!\; k_2!\; 2^{k_2}\; k_3!\; 3^{k_3}} (-1)^{k_1 + k_2 +}$$

$$= \sum_{k_1, k_2, \dots \geq 0} \frac{1}{k_1!}\left(\frac{-\lambda \,\mathrm{tr}\, K}{1}\right)^{k_1} \frac{1}{k_2!}\left(\frac{-\lambda^2 \,\mathrm{tr}\, K^2}{2}\right)^{k_2} \frac{1}{k_3!}\left(\frac{-\lambda^3 \,\mathrm{tr}\, K^3}{3}\right)^{k_3} \cdots\cdots$$

$$= e^{-\lambda \,\mathrm{tr} K - \frac{\lambda^2}{2} \mathrm{tr}\, K^2 - \frac{\lambda^3}{3} \mathrm{tr}\, K^3 - \cdots\cdots}$$

But the real point is to interpret the terms of the Fredholm expansion via diagrams.

$$\det(1-\lambda K) = 1 - \lambda\, \mathrm{tr}(K) + \frac{\lambda^2}{2}\{(\mathrm{tr}\, K)^2 - \mathrm{tr}(K^2)\}$$
$$- \frac{\lambda^3}{3!}\{(\mathrm{tr}\, K)^3 + 2\, \mathrm{tr}(K^3) - 3\, \mathrm{tr}\, K\, \mathrm{tr}(K^2)\} + \cdots$$

The last term results from

$$\det_3 = \begin{vmatrix} K(1\,1) & K(1\,2) & K(1\,3) \\ K(2\,1) & K(2\,2) & K(2\,3) \\ K(3\,1) & K(3\,2) & K(3\,3) \end{vmatrix}$$

$\mathrm{tr}(K^3)$

$(\mathrm{tr}\, K)^3$

$\mathrm{tr}(K^3)$

Picture:

$\mathrm{tr}\, K$:

$\mathrm{tr}\, K^2$:

$\mathrm{tr}\, K^3$: