Let try to understand $D/D'$ in the continuous case where the wave equation is
\[
\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} - V u
\]
where $V$ has support in $(-b, b)$. Start with smooth solutions of the wave equations whose support near $t = 0$ is contained in $(-b, b)$. Closely this up suitably, possibly weakening the vanishing at $-b, b$ should give us $D/D'$.

To simplify suppose $V = 0$. Then we know any solution is of the form
\[
u(x, t) = f(x-t) + g(x+t)
\]

To solve the Cauchy problem we solve
\[
\begin{align*}
\frac{\partial u}{\partial t}(x, 0) &= -f'(x) + g''(x) \\
\frac{\partial u}{\partial x}(x, 0) &= f'(x) + g'(x)
\end{align*}
\]
for $f', g'$ and then we integrate to get $f, g$. This determines $f, g$ up to arbitrary constants, but since the Cauchy data is compact we can fix the constants by requiring that $f, g$ vanish to the right.

Recall the energy norm
\[
\frac{1}{2} \int \left( \left( \frac{\partial u}{\partial t} \right)^2 + \left| \frac{\partial u}{\partial x} \right|^2 + Vu^2 \right) dx
\]

makes Cauchy data into a pre-Hilbert space (assuming
Consider a solution \( u(x,t) = f(x-t) + g(x+t) \) which near \( t = 0 \) is supported in \((-b,b)\). Then \( f, g \) are supported for \( x < b \) and for \( x \leq -b \) they are constant of opposite sign. Now let time evolve: \( t > 0 \). After a while the wave begins to leave \([-b, b]\) and we have to somehow project it back into this interval. So it is clear that we want to truncate \( f' \) at \( x = b \) and \( g' \) at \( x = -b \), in some sense.

So let us consider solutions \( u(x,t) = f(x-t) + g(x+t) \) such that \( f \) is smooth, defined on \([-b, b]\), and vanishes near \(-b\); similarly \( g \) is smooth, defined on \([-b, b]\) and vanishes near \( b \). These solutions obviously get mapped into themselves under time evolution. Moreover, one sees that \( \frac{\partial u}{\partial x} \rightarrow f', \ g' \) agrees with what we expect from the restriction of \( \frac{\partial u}{\partial x} \) to \( L^2(-b,b) \) vanishing with boundary conditions.

Somehow we are looking at the wave equation on \([-b, b]\) with the boundary conditions

\[
\frac{\partial u}{\partial x} + \frac{\partial u}{\partial t} = 0 \quad \text{at} \quad x = b
\]

\[
\frac{\partial u}{\partial x} - \frac{\partial u}{\partial t} = 0 \quad \text{at} \quad x = -b
\]

This makes sense when \( V \neq 0 \).

**Question:** Suppose we solve the wave equation with these boundary conditions, can we suppose \( f, g \) vanish at the appropriate endpoints?
It seems that $u(x,t) = \text{constant}$ is not in the right form. Here $f' = g' = 0$, so $f = g = 0$ if we require vanishing at the ends.

So we seem to get the following continuous model for $D/D'$. Take solutions of the wave equation on $[-b,b]$ with the above boundary conditions. Now it is clear that under time evolution any solution becomes constant. Observe that on $D/D'$ we have time evolution defined for $t \geq 0$. 
We are considering the wave equation
\[ \frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} - V u \quad \text{on} \quad \mathbb{R} \]
where \( \text{Supp}(V) \subset (-b, b) \). We have an analogue of \( \frac{\partial}{\partial x} \) consisting of solutions of the wave equation \( u(x,t) \) defined for \( |x| \leq b, \ t > 0 \) satisfying the boundary conditions

\[ \frac{\partial u}{\partial x} + \frac{\partial u}{\partial t} = 0 \quad x = b \]

\[ \frac{\partial u}{\partial x} - \frac{\partial u}{\partial t} = 0 \quad x = -b \]

To understand such solutions, we use Laplace transform in time:
\[ \int_0^\infty e^{i k t} u(x,t) \, dt = \mathcal{F}(x,k) \]

Here \( s = \frac{k}{i} \), so \( -s = ik \). Then \( \mathcal{F} \) satisfies
\[ \left( \frac{d^2}{dx^2} - V \right) \mathcal{F} = -k^2 \mathcal{F} - (-ik) \mathcal{F}(0) - \frac{\partial \mathcal{F}}{\partial x}(0) \]

\[ \frac{d \mathcal{F}}{dx} - i k \mathcal{F} = 0 \quad \text{at} \quad x = b \]

\[ \frac{d \mathcal{F}}{dx} + i k \mathcal{F} = 0 \quad \text{at} \quad x = -b. \]

and \( \mathcal{F} \) should be analytic in a half-plane \( \text{Im} k > a \).

For example if \( V = 0 \), then we have
\[ \left( \frac{d^2}{dx^2} + k^2 \right) \mathcal{F} = -S u(x,0) - u_t(x,0) \]
and the Green's function for \( \frac{d^2}{dx^2} + k^2 \) with the above outgoing boundary conditions in \( G_k(x,x') = \frac{e^{ik|x-x'|}}{2ik} \), so that

\[
\Psi(x,k) = \int_{-b}^{b} \frac{e^{ik|x-x'|}}{2ik} (-s(u(x';0) - u_+(x';0)) \, dx'
\]

January 31, 1979:

Yesterday I decided that \( \partial/\partial x' \) in the continuous case should be identified with solutions \( u(x,t) \), defined for \( |x| \leq b \), \( t > 0 \), of

\[
\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} - \delta u \quad \text{Supp}(\delta) \subset (-b,b)
\]

with the boundary conditions

\[
\frac{\partial u}{\partial x} + \frac{\partial u}{\partial t} = 0 \quad \text{on} \quad x = b
\]

\[
\frac{\partial u}{\partial x} - \frac{\partial u}{\partial t} = 0 \quad \text{on} \quad x = -b.
\]

This is a well-posed problem, whereas the same problem with \( t \leq 0 \) isn't. The point is that given Cauchy data, that is, smooth functions \( (u_0, u_1) \) on \([-b, b]\) such that

\[
\frac{\partial u}{\partial x} + u = 0 \quad \text{on} \quad x = b
\]

\[
\frac{\partial u}{\partial x} - u = 0 \quad \text{on} \quad x = -b
\]

one can extend this to Cauchy data on \( \mathbb{R} \) having zero incoming component for \( |x| \geq b \), and solve the resulting problem on \( \mathbb{R} \), and restrict to \([-b, b]\) to get the desired solution.

From now on I think of \( \partial/\partial x' \) as pairs \((u)\) satisfying
The infinitesimal generator of the time-evolution semi-group on $S^1$ is

$$\frac{\partial}{\partial t} (u) = \left( \begin{array}{cc} \hat{u} & \frac{\partial u}{\partial x} \\
\frac{\partial u}{\partial x} & -u \end{array} \right) = \left( \begin{array}{cc} 0 & 1 \\
\frac{\partial}{\partial x} & -u \end{array} \right) (u)$$

One can see that it preserves the boundary conditions, e.g.,

$$\frac{\partial}{\partial x} \hat{u} + \left( \frac{\partial^2 u}{\partial x^2} - u \right) = \frac{\partial}{\partial x} (\hat{u} + \frac{\partial u}{\partial x}) = 0 \quad \text{near } x = b.$$ (Note that if we were to start by defining $S^1$ to consist of $(u)$ satisfying (x) with $\frac{\partial}{\partial x}$ as above, then it is not clear that $\frac{\partial}{\partial x}$ can be integrated.) This requires some sort of existence theorem for PDE's.\[\]

Now the problem becomes to interpret the “characteristic polynomial of the Lax-Phillips semi-group” as being $\det (1 - G^t e^g)$.

Recall what we did for

$$\frac{\partial \Psi}{\partial x} = i \kappa \Psi + V \Psi \quad \text{on } |x| \leq b$$

$$\Psi = 0 \quad \text{near } -b.$$ \[
\]

Here the wave equation was

$$-\frac{\partial^2 u}{\partial t^2} = \frac{\partial u}{\partial x} - Vu$$

(because $u(x,t) = \int e^{-ikt} \Psi(x,k) dk/2\pi$), Thus $S^1$ is the functions on $[-b,b]$ vanishing near $x = -b$ with
the infinitesimal generator
\[ \frac{d}{dt} = -\frac{d}{dx} + V \]

which was denoted \(-A\) before. Then
\[ 1 - G^+_k V = (A_0 - ik)^{-1} (A - ik) \]
\[ = (1 - ik A_0^{-1})^{-1} (A_0^{-1} A - ik A_0^{-1}) \]

where both factors have determinants. If also \(\text{det} (A_0^{-1} A) \neq 0\), then we get
\[ \text{det} (1 - G^+_k V) = \frac{\text{det} (1 - ik A_0^{-1})^{-1} \cdot \text{det} (A_0^{-1} A) \cdot \text{det} (1 - ik A_0^{-1})}{\text{det} (1 - ik A_0^{-1})} \]
\[ = 1 \]

which in some sense interprets \(\text{det} (1 - G^+_k V)\) as "\(\text{det} (A - ik)\)".

so the program is to do something similar for
\[ +A = \begin{pmatrix} 0 & 1 \\ \frac{d}{dx} & 0 \end{pmatrix} \quad \text{on pairs } (u, \bar{u}) \quad \text{satisfying } (\dagger) \text{ in 523.} \]

This time \(A\) is the actual infinitesimal generator. Begin by computing
\[ (A_0 + ik)^{-1} = \left( \begin{pmatrix} +ik & 1 \\ \frac{d}{dx} & +ik \end{pmatrix} \right)^{-1} \]
on this space \(D/D'\).
\[
\left( \begin{array}{c}
-ik \\
\frac{d^2}{dx^2} - ik
\end{array} \right) \left( \begin{array}{c}
u \\
\dot{u}
\end{array} \right) = \left( \begin{array}{c}
f \\
\dot{f}
\end{array} \right)
\]

\[
\text{subject to}
\begin{align*}
-ik u + \dot{u} &= f \\
\frac{d^2 u}{dx^2} + ik u &= \dot{f}
\end{align*}
\]

\[
\left( \frac{d^2}{dx^2} + k^2 \right) u = \dot{f} - ik f
\]

\[
u = \sum_{n=0}^{\infty} \left( c_n e^{ikx} + c_n e^{-ikx} + \int G_k^+ (f - ik f) dx' \right)
\]

Now suppose \( f, \dot{f} \) supported inside \((-b, b)\). Then

\[
-ik u + \dot{u} = 0 \quad \text{at} \quad x = b
\]

\[
\frac{\partial u}{\partial x} + \dot{u} = 0 \quad \text{at} \quad x = b
\]

so

\[
\frac{\partial u}{\partial x} - ik u = 0 \quad \text{at} \quad x = b,
\]

so \( u = \text{const.} e^{ikx} \) near \( x = b \). Similarly

\[
\frac{\partial u}{\partial x} = \dot{u} = -ik u \quad \text{at} \quad x = b
\]

so \( u = \text{const.} e^{-ikx} \) near \( x = b \). Thus

\[
u = \int G_k^+ (f - ik f) dx'
\]

\[
\dot{u} = f - ik u = f - ik \int G_k^+ (f - ik f) dx'
\]

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\[
\begin{pmatrix}
u \\
\dot{u}
\end{pmatrix} = \begin{pmatrix}
-ik G_k^+ & G_k^+ \\
1 - k^2 G_k^+ & -ik G_k^+
\end{pmatrix} \begin{pmatrix}
f \\
\dot{f}
\end{pmatrix}
\]

\[
= \begin{pmatrix}
+ik & -1 \\
-\frac{d^2}{dx^2} + ik
\end{pmatrix} \begin{pmatrix}
f \\
\dot{f}
\end{pmatrix}
\]

(Cramer's Rule)
I ought to check that if \( \left( \frac{f_k}{p} \right) \) satisfies the boundary conditions, then so does \( \left( \frac{u}{\hat{u}} \right) \). The reason this should be so is that if we take the flow-line beginning with \( \left( \frac{f}{p} \right) \), and take Laplace transform,

\[
\int_0^\infty e^{ikt} \mathcal{L}\left( A_0(f) \right) \, dt = -\left( A_0 + ik \right)^{-1} \left( \frac{f}{p} \right)
\]

which is \( \left( \frac{u}{\hat{u}} \right) \). We know this is the case when \( \left( \frac{f}{p} \right) \) has support inside \( (-b, b) \), so it's enough to check when \( \hat{f} = -\frac{df}{dx} \) and \( f \) vanishes near \( -b \), and when \( \hat{f} = \frac{df}{dx} \), \( f \) vanishes near \( x = b \). Assume for the moment there is no problem—it's messy.

Next,

\[
\left( A + ik \right) \left( A_0 + ik \right)^{-1} = \begin{pmatrix} i k & 1 \\ -\frac{d^2}{dx^2} - ik & 0 \end{pmatrix} \begin{pmatrix} i k & -1 \\ -\frac{d^2}{dx^2} & ik \end{pmatrix} \left( -G_k^+ \right)
\]

\[
= \begin{pmatrix} -k^2 - \frac{d^2}{dx^2} & 0 \\ -ikq & -\frac{d^2}{dx^2} + g - k^2 \end{pmatrix} \left( -G_k^+ \right)
\]

\[
= \begin{pmatrix} 1 & 0 \\ ikqG_k^+ & 1 - qG_k^+ \end{pmatrix}
\]

So now we see that

\[
\det \left( \left( A + ik \right) \left( A_0 + ik \right)^{-1} \right) = \det \left( 1 - qG_k^+ \right)
\]
Consider the discrete case again. The model for $D/D'$ consists of $(u_n^\alpha)$ defined on $[-N,N]$ where the support of the perturbation is inside. Picture an element

To calculate $u(t)$, we adjoin $u(N+1,1)$, $u(-N-1,1)$ in the way indicated and use the wave equation

$$\frac{u(2) + u(0)}{2} = H u(1)$$

Assuming that $(Hy)_N = \frac{1}{2} y_{N+1} + \frac{1}{2} y_N + a_{N-1} y_{N-1}$ and analogously at the other end, so that $H-H_0 = -V$ has its image inside $[-N,N]$, one sees that one gets the same result for $u(t)$ if one changes $u(N,0) = u(N+1,1)$ to zero, and similarly at the other end. Let $T$ be the operator on sequences supported in $[-N,N]$ which kills the $-N,N$ components. Then we have

$$u(t) = 2Hy_1 - T u_0$$

where $H$ is restricted to $[-N,N]$ by forcing other components to be zero, e.g.

$$H_0 = \frac{1}{2} (u_0 + u_0^*)$$

Thus time-evolution on $D/D'$ is given by the operator
\[ \overrightarrow{U}(u_0) = (u_1, u_2) = \begin{pmatrix} 0 & 1 \\ -z & 2H \end{pmatrix}(u_1) \]

We want to compute \( \det(I - z \overrightarrow{U}) \). Now

\[ I - z \overrightarrow{U} = \begin{pmatrix} 1 & -z \\ z & 1 - 2zH \end{pmatrix} \]

\[ (I - z \overrightarrow{U})(I - z \overrightarrow{U}_0)^{-1} = I + 2z \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}(I - z \overrightarrow{U}_0)^{-1} \]

Because of \( T \) it is not immediately clear how to calculate \( (I - z \overrightarrow{U}_0)^{-1} \).

However suppose one assumes \(|z| < 1\) and computes the inverse in \( \mathbb{Z} \), not the finite-dimensional \( D/B' \). In this case

\[ I - z \overrightarrow{U}_0 = \begin{pmatrix} 1 & -z \\ z & 1 - zU_0 - zU_0^{-1} \end{pmatrix} \]

and we can compute the inverse by Cramer's

\[ \det = 1 - zU_0 - zU_0^{-1} + z^2 = (1 - zU_0)(1 - zU_0^{-1}) \]

Thus

\[ (I - z \overrightarrow{U}_0)^{-1} = \begin{pmatrix} 1 - zU_0 - zU_0^{-1} & z \\ -z & 1 \end{pmatrix} \frac{1}{(1 - zU_0)(1 - zU_0^{-1})} \]

So it follows that
\[(I-\tilde{u})(I-\tilde{u}_0)^{-1} = I - \begin{pmatrix} 0 & 0 \\ -2z & V \end{pmatrix} \begin{pmatrix} 1-\tilde{u}_0 & 1-\tilde{u}_0 \\ -2z(1-\tilde{u}_0)^{-1} & 1-\tilde{u}_0^{-1} \end{pmatrix} \]

But we see that
\[H_0 - \lambda = \frac{1}{2z} (1-\tilde{u}_0)(1-\tilde{u}_0^{-1}) \]

so that
\[(I-\tilde{u})(I-\tilde{u}_0)^{-1} = \begin{pmatrix} 1 & 0 \\ 2zG_k^+ & 1 - G_k^+ \end{pmatrix} \]

It is interesting to do this calculation on the line in the continuous case.

Thus
\[\frac{\partial}{\partial t} \begin{pmatrix} u \\ \dot{u} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ \frac{d^2}{dx^2} & 0 \end{pmatrix} \begin{pmatrix} u \\ \dot{u} \end{pmatrix} \]

\[A = \begin{pmatrix} 0 & 1 \\ \frac{d^2}{dx^2} & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = A_0 - V \]

Moreover
\[\left( A_0 + ik \right)^{-1} = \begin{pmatrix} ik & 1 \\ \frac{d^2}{dx^2} & ik \end{pmatrix}^{-1} = \begin{pmatrix} ik & -1 \\ -\frac{d^2}{dx^2} & ik \end{pmatrix} \frac{1}{k^2 - \frac{d^2}{dx^2}} \]

where the inverse is computed with outgoing boundary conditions \(= \text{inverse in } L^2 \text{ when } \text{Im} k > 0 \). Thus
\[\left( A + ik \right)\left( A_0 + ik \right)^{-1} = I - \begin{pmatrix} 0 & 0 \\ \frac{d^2}{dx^2} & ik \end{pmatrix} \begin{pmatrix} -1 \\ -G_k^+ \end{pmatrix} \]

\[= \begin{pmatrix} 1 & 0 \\ ikG_k^+ & 1 - G_k^+ \end{pmatrix} \]
which shows that
\[
\det \left( (A+ik)(A_0+ik)^{-1} \right) = \det \left( 1 - g \delta_k^+ \right) = \det \left( (H+ik)(H_0+k^2)^{-1} \right)
\]

Notice that this calculation doesn't yield anything about assigning a determinant to \((A_0+ik)^{-1}\). Nor do we have an interpretation of \(\det \left( (A+ik)(A_0+ik)^{-1} \right)\) as the characteristic polynomial of some endomorphism of cohomology.