January 23, 1979   comm. alg. interp. of $\det(1 - G^+V)$.

Look at the discrete case again: Let H be a J-matrix which is a finite perturbation $H_0 - V$ of H_0:

$$(H_0 u)_n = \tfrac{1}{2}(u_{n+1} + u_{n-1})$$

(∞ removed)

Then we get rank 2 vector bundles over the λ-plane with fibres

1) $\quad \mathrm{Ker}(H_0 - \lambda) \quad$ and $\quad \mathrm{Ker}(H - \lambda)$

Denote by G_z^+ the Green's function for $H_0 - \lambda$ given by

$$G_z^+(n, n') = \frac{z^{|n-n'|}}{\frac{1}{2}(z - z^{-1})}$$

where $\frac{1}{2}(z + z^{-1}) = \lambda$. Better: define G_z^+ by this formula and then state that for $\lambda = \frac{1}{2}(z + z^{-1})$ one has

$$(H_0 - \lambda)\, G_z^+(n, n') = \delta(n - n')$$

Thus if we pull-back the vector bundles described in 1) to the z-plane with $0, \infty, \pm 1$ removed, we get a map

$$(1 - G_z^+ V) : \mathrm{Ker}(H - \lambda) \longrightarrow \mathrm{Ker}(H_0 - \lambda)$$

Let $C = C(\mathbb{Z})$ denotes all sequences, $C_0 = C_0(\mathbb{Z})$ all finite support sequences. Over the z plane we have a diagram of fibre bundles as follows:

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Ker}(H-\lambda) \arrow[r] \arrow[d] & C \arrow[r, "H-\lambda"] \arrow[d, "1-G_z^+V"] & C \arrow[r] \arrow[d, "id"] & 0 \\
0 \arrow[r] & \mathrm{Ker}(H_0-\lambda) \arrow[r] & C \arrow[r, "H_0-\lambda"] & C \arrow[r] & 0
\end{tikzcd}$$

Question: Are $H-\lambda$, $H_0-\lambda$ onto? YES For $H_0-\lambda$ it is clear as follows: It suffices to solve $(H_0-\lambda)u=f$ when f_n [redacted] $=0$ for $n<0$ whence one can take $u_n=0$ for $n\leq 0$ and then grind the rest out using the recursion [redacted] formula

$$\frac{1}{2}u_{n+1}-\lambda u_n+\frac{1}{2}u_{n-1}=f_n$$

This amounts to using the Green's function

$$G(n,n')=\begin{cases} 0 & n\leq n' \\ 2\dfrac{z^{n-n'}-z^{n'-n}}{z-z^{-1}} & n>n' \end{cases}$$

The same argument works for $H-\lambda$, since H is a J-matrix.

Question: Can you interpret $\det(1-G_z^+V)$ somehow?

Mumford idea: Work with C_0 and the transposed situation: [redacted]

$$\begin{tikzcd}
0 \arrow[r] & C_0 \arrow[r, "H_0-\lambda"] \arrow[d, equal, "id"] & C_0 \arrow[r] & C_0/(H_0-\lambda)C_0 \arrow[r] & 0 \\
0 \arrow[r] & C_0 \arrow[r, "H-\lambda"] & C_0 \arrow[r] & C_0/(H-\lambda)C_0 \arrow[r] & 0
\end{tikzcd}$$

It's not clear that we get a map [redacted] on C_0 from $1-G_z^+V$ [redacted].

[redacted] It seems that C_0 should be the space of global sections of the rank 2 vector bundle over $\mathbb{C}$ with fibre $C_0/(H_0-\lambda)C_0$ at λ. Thus C_0 should be a free module of rank 2 over $\mathbb{C}[H_0]$. Similarly C_0 should be a free rank 2 module over $\mathbb{C}[H]$. Now the [redacted] map $\operatorname{Ker}(H-\lambda) \longrightarrow \operatorname{Ker}(H_0-\lambda)$ furnished by $1-G_z^+V$ should be some sort of correspondence between these vector bundles.

Let's take simpler case. Let U_0 = shift on C_0 and let U be a finite perturbation. We know C_0 is a free $\mathbb{C}[U_0, U_0^{-1}]$ module of rank 1. [redacted] Assuming U bijective on C_0, C_0 will be a module of rank 1 over $\mathbb{C}[U, U^{-1}]$, so if there is no torsion (bound states) C_0 will be [redacted] free of rank 1. So $\operatorname{Ker}(U-z)$ will be one-dimensional for each $z \neq 0, \infty$. (Actually I am ignoring the transposes.)

[redacted]

Let $C_0 = C_0(\mathbb{Z}) \simeq \mathbb{C}[\zeta,\zeta^{-1}]$ with $U_0 =$ right shift $\simeq$ multiplication by ζ. [redacted] We identify the dual of C_0 with $C = C(\mathbb{Z})$. The transpose U_0^t of U_0 is the left shift:

$$\left(U_0^t(a_n), (b_n)\right) = \left((a_n), (b_{n-1})\right) = \left((a_{n+1}), (b_n)\right)$$

and so $\operatorname{Ker}(U_0^t - z)$ is spanned by the sequence (z^n). [redacted] Recall the [redacted] diagram

$$\begin{tikzcd}
0 \arrow[r] & \operatorname{Ker}(U_0^t - z) \arrow[r] & C \arrow[r, "U_0^t - z"] & C \arrow[r] & 0 \\
0 \arrow[r] & \operatorname{Ker}(U^t - z) \arrow[r] \arrow[u] & C \arrow[r, "U^t - z"] \arrow[u, "1 - G_z^+ V"'] & C \arrow[r] \arrow[u, equal] & 0
\end{tikzcd}$$

where $1 - G_z^+ V = (U_0^t - z)^{-1}(U^t - z)$ in this notation. Recall that the left vertical arrow is the map on eigenfunctions induced by the wave operator

$$\Omega = \lim_{n\to\infty} U^n U_0^{-n} \quad \text{on } C_0 .$$

So I ought to be able to claim that the above diagram results by applying $\operatorname{Hom}_{\mathbb{C}[z]}(\,?\,, \mathbb{C})$ to

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{C}[z]\otimes C_0 \arrow[r, "U_0 - z"] \arrow[d, "id"] & \mathbb{C}[z]\otimes C_0 \arrow[r, "\pi_0"] \arrow[d, "1 - V^t (G_z^+)^t"] & C_0 \arrow[r] \arrow[d, "\Omega"] & 0 \\
0 \arrow[r] & \mathbb{C}[z]\otimes C_0 \arrow[r, "U - z"] & \mathbb{C}[z]\otimes C_0 \arrow[r, "\pi"] & C_0 \arrow[r] & 0
\end{tikzcd}$$

I have to check that $1 - V^t (G_z^+)^t$ is really well-def'd.

Start with deriving the LS equation

$$({}^tU - z)\psi = 0 \qquad\qquad {}^tU = {}^tU_0(I - V)$$

$$({}^tU_0 - z)\psi = ({}^tU_0 - {}^tU)\psi \qquad \text{Put } [redacted]$$

$$= {}^tU_0 V\psi$$

$$(1 - z\,{}^tU_0^{-1})\psi = V\psi$$

or

$$\psi - \varphi = G_z^+ V\psi = \sum_{n\geq 0} z^n ({}^tU_0^{-1})^n V\psi$$

We need the transpose of this operator $G_z^+ V$ which is

$$\sum_{n\geq 0} z^n\, {}^tV U_0^{-n} = {}^tV(1 - zU_0^{-1})^{-1}$$

Notice that this operator is well-defined on $\mathbb{C}[z]\otimes C_0$ because if $f \in C_0$, then ${}^tVU_0^{-n}f$ will be zero for n large. As a check note that

$$\{1 - {}^tV(1 - zU_0^{-1})^{-1}\}(U_0 - z) = U_0 - z - {}^tVU_0 = U - z$$

since

$$U = (I - {}^tV)U_0 .$$

Now what the diagram at the bottom of preceding page suggests is that in some sense Ω has a determinant which is an element in $\mathbb{C}[z]$ (or at least in $\mathbb{C}[z, z^{-1}]$) and

$$\det(\Omega) = \det(1 - {}^tV\,{}^t(G_z^+)) = \det(1 - G_z^+V)$$

I see I have forgotten to check Ω is compatible with $1 - {}^tV(1 - zU_0^{-1})^{-1}$. But $\pi : \mathbb{C}[z]\otimes C_0 \to C_0$ sends z to U

so for $f \in C_0$

$$\pi\left(1 - {}^tV(1 - zU_0^{-1})^{-1}\right) f = \overset{f}{\text{[redacted]}} - \sum_{n \geq 0} U^n\, {}^tV\, U_0^{-n} f$$

$$= f - \sum_{n\geq 0} U^n \underbrace{{}^tV U_0}_{-(U-U_0)} U_0^{-n-1} f$$

$$= f + \sum_{n \geq 0} U^n (U - U_0) U_0^{-n-1} f$$

$$= f + (UU_0^{-1} - 1) f + U(U - U_0) U_0^{-2} f + \cdots$$

$$= \lim_{n \to +\infty} U^n U_0^{-n} f = \Omega f.$$

The question arises as to whether it is possible to define the determinant of Ω more directly. The idea is that Ω is a map between [redacted] free rank 1 modules over [redacted] $\mathbb{C}[z, z^{-1}]$, so up to a choice of bases it has a determinant. Can the identity map provide[redacted] you with compatible bases?

January 24, 1979 (David is 15)

Question: Let M be a free rank 1 module over $\mathbb{C}[z,z^{-1}]$ and let J be an endomorphism of M as a $\mathbb{C}$-vector space. [redacted] What does one have to assume so as to be able to define $\det(J)$ as an element of the ring $\mathbb{C}[z,z^{-1}]$?

Example: Take $M = \mathbb{C}[z,z^{-1}]$ and let U_0 = mult. by z, $U = (1-V)U_0$ a finite perturbation. [redacted] Then we can take $J = \Omega = \lim_{n\to\infty} U^n U_0^{-n}$. Why should J have a determinant which is a poly in z?

Now we know that

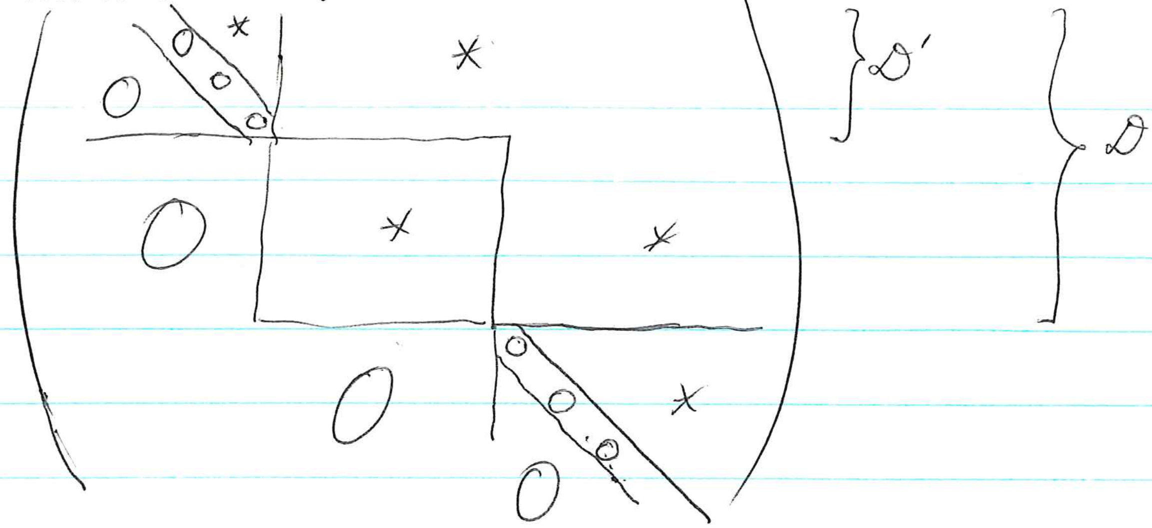
$$UJ = JU_0$$

and that [redacted] there are $\mathbb{C}[z^{-1}]$-lattices $\mathcal{D}$, $\mathcal{D}'$ such that $J = \mathrm{id}$ on $\mathcal{D}'$ and $\mathcal{D}$ is J-stable.

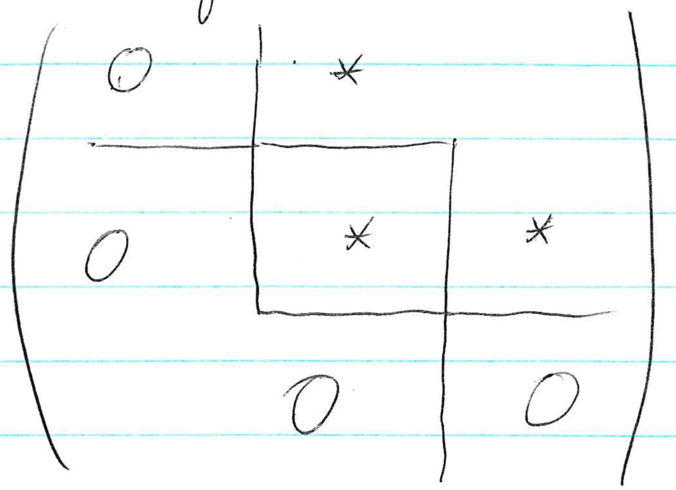
Let's try to generalize the construction. Let's see what one needs to define the determinant of $1 - V(1-zU_0^{-1})^{-1}$. Let $z \in \mathbb{C}$. $(1-zU_0^{-1})^{-1} = 1 + zU_0^{-1} + z^2U_0^{-2} + \cdots\cdots$ has the matrix

$$\begin{pmatrix} 1 & z & z^2 & \cdot & \cdot \\ & 1 & z & z^2 & \cdot \\ & & 1 & z & z^2 \\ & & & 1 & z \\ & & & & 1 \end{pmatrix}$$

so it would seem that we want V to have the form



In other words $\mathcal{D}, \mathcal{D}'$ should be stable under V and V should be nilpotent with respect to the filtration on $\mathcal{D}'$ and on $C_0/\mathcal{D}$. Simplest way of meeting these requirements is



i.e. $\quad \operatorname{Im} V \subset \mathcal{D}, \quad \mathcal{D}' \subset \operatorname{Ker} V.$ or in words, V leaves the filtratun $0 \subset \mathcal{D}' \subset \mathcal{D} \subset C_0$ [redacted] fixed and induces O on $C_0/\mathcal{D}$ and on $\mathcal{D}'$.

So therefore we should be able to [redacted] work with a polynomial ring.

Let $\mathcal{D} = \mathbb{C}[\zeta]$ with $U_0 = $ mult. by ζ and let $U : \mathcal{D} \to \mathcal{D}$ be an [redacted] endomorphism such that

$U = U_0$ on $\mathcal{D}' = \xi^N \mathcal{D}$. For any $z \in \mathbb{C}$ we consider the operator on $\mathcal{D}$ given by

$$
\begin{aligned}
(1 - zU)(1 - zU_0)^{-1} &= 1 - z(U - U_0)(1 - zU_0)^{-1} \\
&= 1 - z(U - U_0) \sum_{n \geq 0}' z^n U_0^n
\end{aligned}
$$

There is no problem with convergence of the series because $(U - U_0)U_0^N = 0$. This endomorphism is the identity on $\mathcal{D}'$, and hence its determinant can be evaluated on $\mathcal{D}/\mathcal{D}'$.

<u>January 25, 1979</u>

Let k be a field, let A be a k-algebra, and $\mathfrak{m}$ an ideal in A such that $A/\mathfrak{m}$, $\mathfrak{m}/\mathfrak{m}^2$ are finite dimensional over k. Let Θ be a k-linear endomorphism of A such that

1) $\quad \Theta(\mathfrak{m}^n) \subset \mathfrak{m}^{n+1} \quad$ for $\quad n \gg 0$.

Then I can define $\det(1 - z\Theta)$ as follows. For $n \gg 0$, Θ induces an endom. Θ_n of $A/\mathfrak{m}^n$. Since

$$
\begin{tikzcd}
0 \arrow[r] & \mathfrak{m}^n/\mathfrak{m}^{n+1} \arrow[r] \arrow[loop below, "\mathrm{id}"] & A/\mathfrak{m}^{n+1} \arrow[r] \arrow[loop below, "1 - z\Theta_{n+1}"] & A/\mathfrak{m}^n \arrow[r] \arrow[loop below, "1 - z\Theta_n"] & 0
\end{tikzcd}
$$

it follows $\det(1 - z\Theta_n)$ doesn't depend on n, so we can define $\det(1-z\theta)$ to be this poly.

An example where 1) holds occurs when there is an element $\alpha \in \mathfrak{m}$ such that $\theta - \alpha = 0$ on $\mathfrak{m}^N$. In this case we can write formally

$$(1-z\theta)(1-z\alpha)^{-1} = [1-z\alpha + z(\alpha-\theta)](1-z\alpha)^{-1}$$

$$= 1 + z(\alpha-\theta)\sum_{n\geq 0} z^n \alpha^n$$

$$= 1 + z(\alpha-\theta)\sum_{0\leq n<N} z^n\alpha^n$$

since $(\alpha-\theta)\alpha^n = 0$ for $n \geq N$. Notice that the second term is an [redacted] endomorphism of A of finite rank since it vanishes on $\mathfrak{m}^N$. Thus if we define $(1-z\theta)(1-z\alpha)^{-1}$ by the above formula it has a determinant, and since this determinant can be evaluated by restriction to $A/\mathfrak{m}^N$ on which α is nilpotent we [redacted] see

$$\det_A\left((1-z\theta)(1-z\alpha)^{-1}\right) = \det(1-z\theta) \quad \text{as defined above.}$$

Next suppose that we have [redacted] a k-linear endomorphism θ of the quotient field F of a d.v.r. $(A,\mathfrak{m})$ such that

$$\theta(\mathfrak{m}^n) \subset \mathfrak{m}^{n+1} \qquad \text{for } |n| \gg 0$$

Then we can define $\det(1-z\theta)$ by looking at the effect of θ on $\mathfrak{m}^{-N}/\mathfrak{m}^N$ for N large. On the other hand [redacted] suppose there is an element $\alpha \in \mathfrak{m}$ such that

$$\operatorname{Ker}(\theta - \alpha) \supset \mathfrak{m}^N$$
$$\operatorname{Im}(\theta - \alpha) \subset \mathfrak{m}^{-N}$$

for N large. Then the transformation of F

$$1 + z(\theta - \alpha) \sum_{n \geq 0} z^n \alpha^n$$

is well-defined, since for any $x \in F$ one has $\alpha^n x \in \mathfrak{m}^N$ for $n \gg 0$. On the other hand, this transformation is the identity on $\mathfrak{m}^{-N}$, and is the identity modulo $\mathfrak{m}^{-N}$, hence it has a determinant which one can evaluate on $\mathfrak{m}^{-N}/\mathfrak{m}^N$.

January 26, 1979

Let U_0 = multiplication by ζ on $\mathbb{C}[\zeta, \zeta^{-1}] = C_0$ and let $H_0 = \frac{1}{2}(U_0 + U_0^{-1})$. Let H be a finite perturbation of H_0. We wish to compute

$$(\overset{H_0-V}{H} - \lambda)(H_0 - \lambda)^{-1} = 1 - V(H_0-\lambda)^{-1}$$

formally. Recall that the Green's function for $H_0 - \lambda$ is

$$G_z^+(n, n') = \frac{z^{|n-n'|}}{\frac{1}{2}(z - z^{-1})}$$

where $\frac{z+z^{-1}}{2} = \lambda$ and for $\lambda \notin [-1,1]$, z is chosen with $|z| < 1$. Thus

$$(G_z^+ f)(n) = \frac{1}{\frac{1}{2}(z-z^{-1})} \sum_{n'} z^{|n-n'|} f(n')$$

$$= \frac{1}{\frac{1}{2}(z-z^{-1})} \left(\sum_m z^{|m|} U_0^{-m} f \right)(n')$$

$$G_z^+ = \frac{1}{\frac{1}{2}(z-z^{-1})} \left((1 - zU_0)^{-1} + zU_0^{-1}(1 - zU_0^{-1})^{-1} \right)$$

$$= \frac{1}{\frac{1}{2}(z-z^{-1})} (1 - zU_0)^{-1} \Big[\underbrace{1 - zU_0^{-1} + (1 - zU_0) z U_0^{-1}}_{1 - z^2} \Big] (1 - zU_0^{-1})^{-1}$$

$$= -2z \, (1 - zU_0)^{-1} (1 - zU_0^{-1})^{-1}$$

Check:

$$H_0 - \lambda = \frac{U_0 + U_0^{-1}}{2} - \frac{z + z^{-1}}{2} = +\frac{1}{2z}\left\{ zU_0 + zU_0^{-1} - z^2 - 1 \right\}$$

$$= -\frac{1}{2z}\left(1 - zU_0 - zU_0^{-1} + z^2\right) = -\frac{1}{2z}(1 - zU_0)(1 - zU_0^{-1})$$

So we can [redacted] set up the diagram

$$\begin{tikzcd}
0 \arrow[r, dotted] & R\otimes C_0 \arrow[r, "\lambda - H_0"] \arrow[d, equal] & R\otimes C_0 \arrow[r, dashed] \arrow[d, "1-VG_z^+"] & R \underset{\mathbb{C}[\lambda]}{\otimes} (C_0)_{\lambda=H_0} \arrow[r, dotted] \arrow[d, dashed, "\Omega"] & 0 \\
0 \arrow[r, dotted] & R\otimes C_0 \arrow[r, "\lambda - H"] & R\otimes C_0 \arrow[r, dotted] & R \underset{\mathbb{C}[\lambda]}{\otimes} (C_0)_{\lambda=H} \arrow[r, dotted] & 0
\end{tikzcd}$$

where R is the minimal ring over which VG_z^+ is defined. Since $\lambda = \frac{1}{2}(z+z^{-1})$ and $z \in R$ and

$$(*) \qquad VG_z^+ = \frac{1}{\frac{1}{2}(z-z^{-1})}\left\{V \sum_m z^{|m|} U_0^{-m}\right\}$$

it is clear that we want

$$R = \mathbb{C}[z, z^{-1}, \tfrac{1}{z-z^{-1}}]$$

What about the cokernels of the horizontal arrows in the above diagram? One knows for any $\mathbb{C}[\lambda]$-module M that

$$0 \longrightarrow \mathbb{C}[\lambda]\otimes M \xrightarrow[\text{[redacted]}]{\lambda\otimes 1 - 1\otimes\lambda} \mathbb{C}[\lambda]\otimes M \longrightarrow M \longrightarrow 0$$

is exact. Also R is flat over $\mathbb{C}[\lambda]$. Hence the above can be filled in as indicated in pen.

The cokernels should have something to do with the wave equation and so perhaps they have appropriate lattices $\mathcal{D}, \mathcal{D}'$.

Note that $(*)$ is a perfectly good finite rank endo of C_0 for any $z \in \mathbb{C} - \{0, 1, -1\}$, so our problem is to interpret the determinant.

There is a chance that

$$\mathbb{C}[u,u^{-1}] \otimes_{\mathbb{C}[H]} C_0 \qquad\qquad H = \tfrac{1}{2}(u+u^{-1})$$

can be interpreted via a wave equation. Note that its dual

$$\mathrm{Hom}_{\mathbb{C}[z,z^{-1}]}\left(\mathbb{C}[u,u^{-1}] \otimes_{\mathbb{C}[H]} C_0, X\right) = \mathrm{Hom}_{\mathbb{C}[H]}(C_0, X)$$

can be identified with the set of sequences (x_n) in X such that

$${}^tHx = \left(\frac{z+z^{-1}}{2}\right)x$$

This is what one gets when one takes the F.T. of a solution of the wave equation:

$${}^tHu(t) = \frac{u(t+1)+u(t-1)}{2}$$

namely

$$u(t) = \int z^t \, x_n(z) \, \frac{d\theta}{2\pi} .$$

On the other hand we can get a direct interpretation as follows. Note that $\mathbb{C}[u,u^{-1}]$ is a free module over $\mathbb{C}[H]$ with basis $1, u^{-1}$, hence any element of $\mathbb{C}[u,u^{-1}] \otimes_{\mathbb{C}[H]} C_0$ can be uniquely written

$$u^{-1} \otimes a - 1 \otimes b$$

with $a, b \in C_0$. Then

$$u(u^{-1} \otimes a - 1 \otimes b) = 1 \otimes a - u \otimes b$$

and

$$u \otimes b + u^{-1} \otimes b = 1 \otimes 2Hb \qquad \text{so}$$

$$u(u^{-1} \otimes a - 1 \otimes b) = 1 \otimes a - 1 \otimes 2Hb + u^{-1} \otimes b$$

$$= \mathcal{U}^{-1} \otimes b - 1 \otimes (2Hb - a)$$

Thus I can think of $\mathbb{C}[\mathcal{U},\mathcal{U}^{-1}] \otimes_{\mathbb{C}[H]} C_0$ as pairs $\binom{a}{b}$ in C_0 with

$$\mathcal{U}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} b \\ 2Hb - a \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 2H \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix}$$

On the other hand a solution of the wave equation

$$\frac{u(t+1) + u(t-1)}{2} = Hu(t)$$

where $u : \mathbb{Z} \longrightarrow C_0$ is determined by $u(0), u(1)$ and

$$\begin{pmatrix} u(1) \\ u(2) \end{pmatrix} = \begin{pmatrix} u(1) \\ 2Hu(1) - u(0) \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 2H \end{pmatrix}\begin{pmatrix} u(0) \\ u(1) \end{pmatrix}$$

Thus elements of $\mathbb{C}[\mathcal{U},\mathcal{U}^{-1}] \otimes_{\mathbb{C}[H]} C_0$ can be interpreted in an ad hoc fashion as solutions of a wave equation.

What I am trying to do is to identify $\det(1 - VG_z^+)$ with a Lax-Phillips determinant $\det(1 - z\mathcal{U})$ where $\mathcal{U}$ is time evolution on a space $\mathcal{D}/\mathcal{D}'$. I still seem to be missing the key point.

January 27, 1979:

Let us consider a J-matrix H which is a finite perturbation of $H_0 = \frac{1}{2}(U_0 + U_0^{-1})$ where U_0 is the shift on $C_0(\mathbb{Z})$. Think of $C_0(\mathbb{Z})$ as contained in $C(\mathbb{Z})$ = all sequences. [redacted]

We consider solutions of the wave equation:

$$(*) \qquad \frac{u(t+1) + u(t-1)}{2} = Hu(t)$$

where $u(t)$ is a sequence $u(n,t)$ for each $t \in \mathbb{Z}$. Obviously [redacted] any solution is determined by two consecutive t values.

[redacted] The above equation can be written

$$\frac{1}{2}u(n,t+1) + \frac{1}{2}u(n,t-1) = a_n u(n+1,t) + b_n u(n,t) + a_{n-1} u(n-1,t)$$

[redacted] For $|n|$ large $a_n = \frac{1}{2}$, $b_n = 0$ so if we plot a solution in the n,t plane at each point (n,t), with $|n| > N$ say, [redacted] we have the sum of the up + down neighbors is the sum of the right + left ones

$\bullet(n,t+1)$

$(n-1,t)\bullet \quad \bullet(n,t) \quad \bullet(n+1,t)$

$\bullet(n,t-1)$

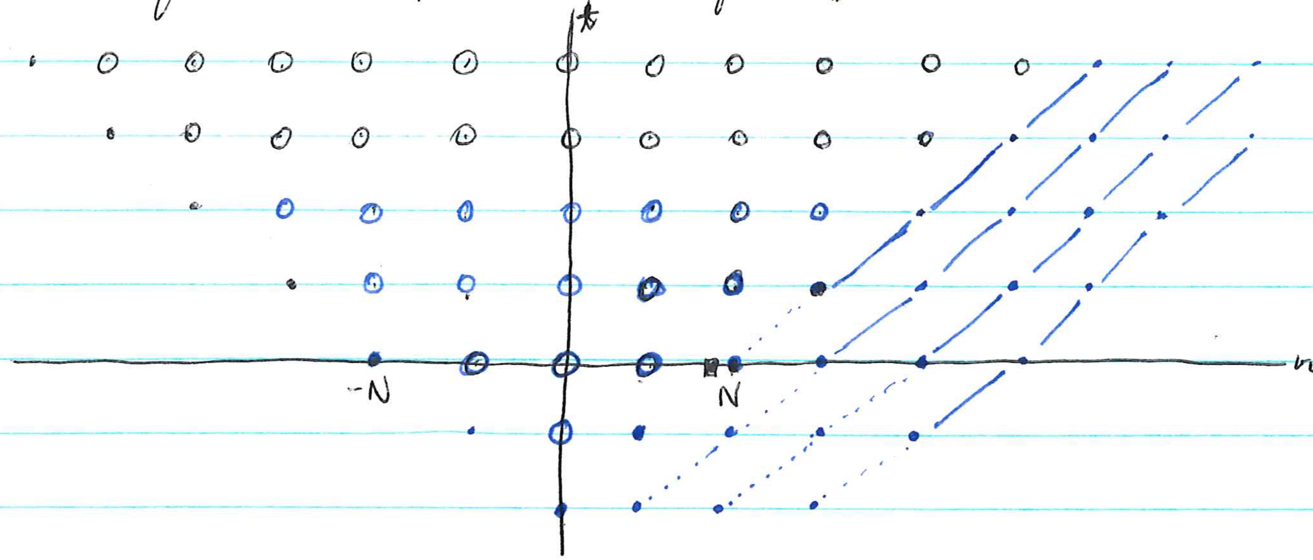
~~[redacted] It's [redacted] clear what one means about domain of dependence for solutions of the wave equations. For example [redacted] if one has~~

~~$u(n,0) = 0 \qquad n \le 0$~~

~~$u(n,-1) = 0 \qquad n \le -1$~~

~~Then $u(n,t) = 0$ for $n \le t$~~

My problem is to get at the Lax-Phillips space D/D'. So the first thing to understand is the outgoing space D'. This consists of solutions $u(n,t)$ which leave the obstacle, so that they vanish for $|n| < N+t$ for all $t \gg 0$



Such a solution is a sum

$$u(n,t) = f(n-t) + g(n+t) \qquad \begin{matrix} t \gg 0 \text{ all } n \\ \text{or } |n| > N \text{ all } t \geq 0 \end{matrix}$$

with $\quad \mathrm{Supp}\, f \subset [N, \infty) \quad$ and $\quad \mathrm{Supp}\, g \subset [-\infty, -N]$.

Notice also that if g is zero, then $u(n,t) = 0$ for all points above $t = n - N$, i.e.

$$t > n - N \quad \text{or} \quad n - t < N.$$

Maybe if I change D' so that $u(n,t) = 0$ for $|n| \leq N+t$ and $t \gg 0$, I can describe D by conditions on $u(n,0)$ and $u(n,1)$:

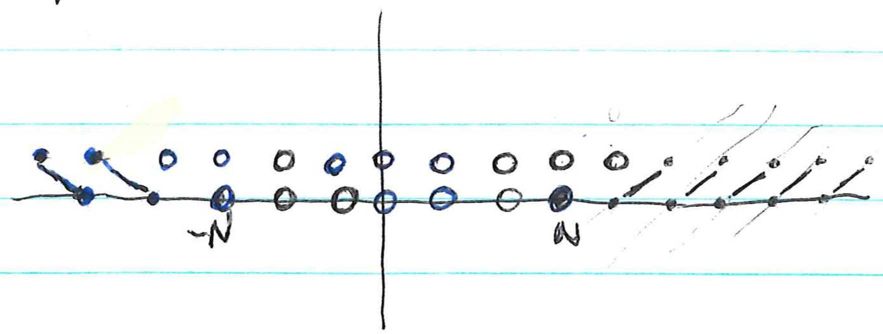
$$u(n,1) = u(n-1,0) \qquad n \geq N+1$$
$$u(n,1) = u(n+1,0) \qquad n \leq -N-1$$

other values zero.

Better description: D' consists of $u(n,t)$ such that

$$\begin{aligned} u(n,0) &= 0 & |n| &\leq N \\ u(n,1) &= 0 & |n| &\leq N+1 \\ u(n,0) &= u(n+1,1) & \text{if } n &> N \\ u(n,0) &= u(n-1,1) & \text{if } n &< -N \end{aligned}$$

and the picture is:



Next we want to determine D. The idea to keep in mind is that we want to think of a solution of the free wave equation as being a sum of a left-moving and a right-moving waves.

Let's define D tentatively to consist of all u such that for $t \ll 0$, $u(n,t)$ is supported in $[-N+t, N-t]$. Another possibility is to kill the waves running the wrong way. For $n \gg 0$ we want as $t \to -\infty$ to kill the waves running to the right, i.e. those of the form

$$f(n-t)$$

hence killed by $u \mapsto u(n+1,t+1) - u(n,t)$. Thus we might be interested in $v(n,t) = u(n+1,t+1) - u(n,t)$ which satisfies

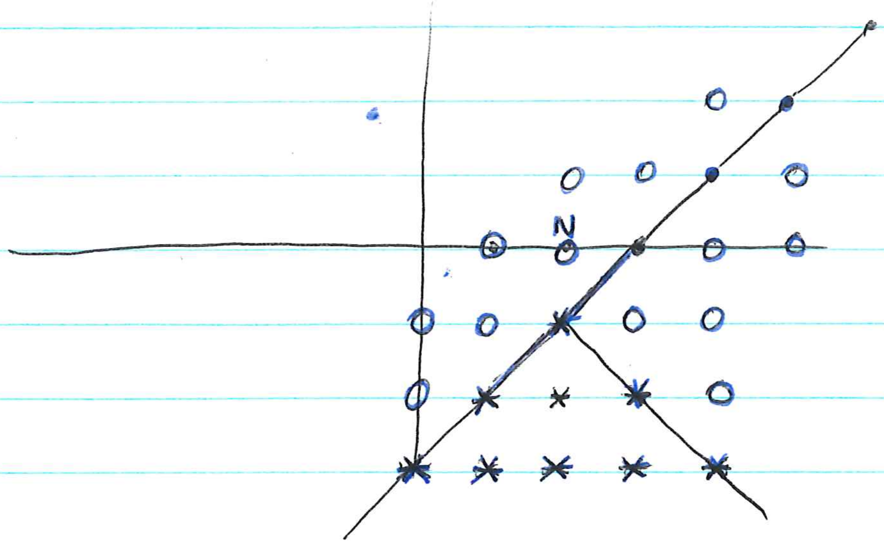
$$\begin{aligned} v(n,t) &= -u(n-1,t+1) + u(n,t+2) \\ &= +v(n-1,t+1) \qquad \text{away from the obstacle.} \end{aligned}$$

Then we could define $\mathcal{D}$ by requiring that $v(n,t) = 0$ for $n > N - t$ and a similar condition on the other side.

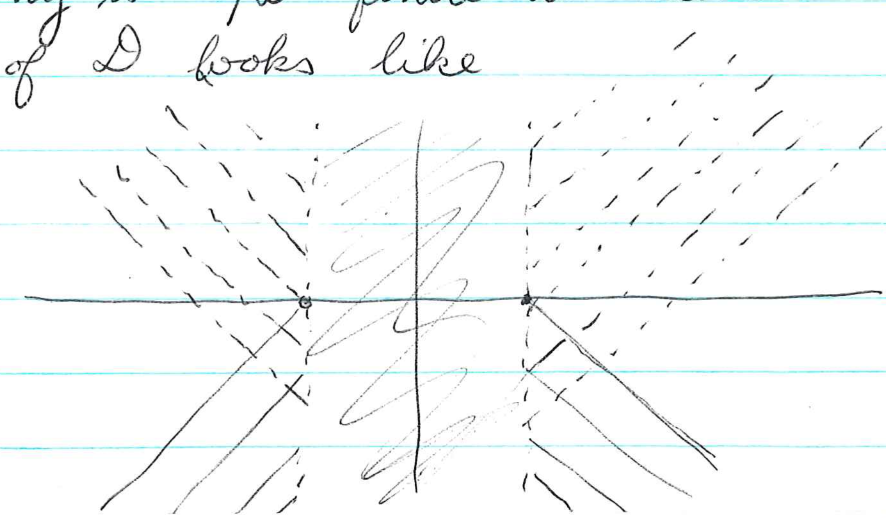
So what else can we do? We should show $\mathcal{D}' \subset \mathcal{D}$ and that $\mathcal{D}/\mathcal{D}'$ is finite-dimensional.

The two possibilities for $\mathcal{D}$ are probably the same if we restrict to compact support solutions, i.e. $u(n,t) = 0$ for $|n| \gg 0$ and t fixed.

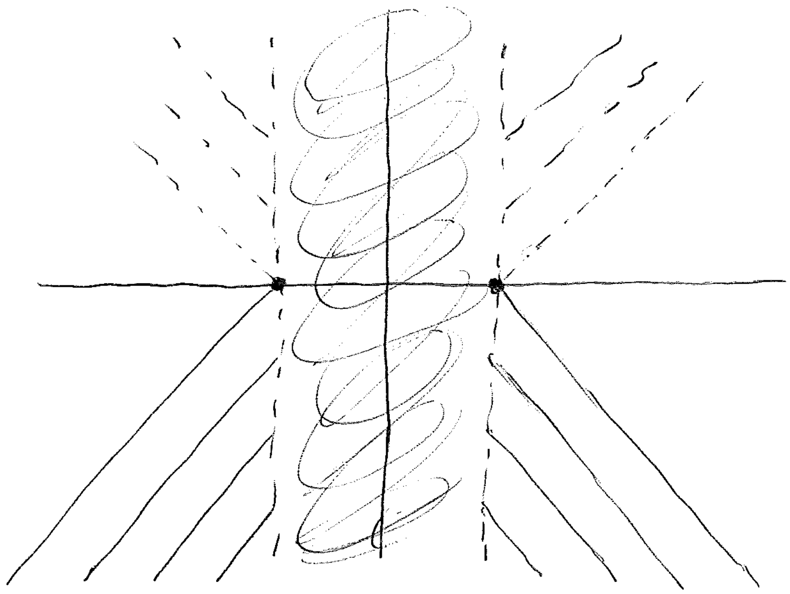
Why is $\mathcal{D}' \subset \mathcal{D}$? It suffices to take $u(n,t) = f(n-t)$ where $\mathrm{Supp}(f) \subset [N+1, N']$. Take $f = \delta$-function on $N+1$. Then the support of u looks as follows



Why is $\mathcal{D}/\mathcal{D}'$ finite-dimensional? A typical element of $\mathcal{D}$ looks like



Modulo elements of $\mathcal{D}'$ it becomes



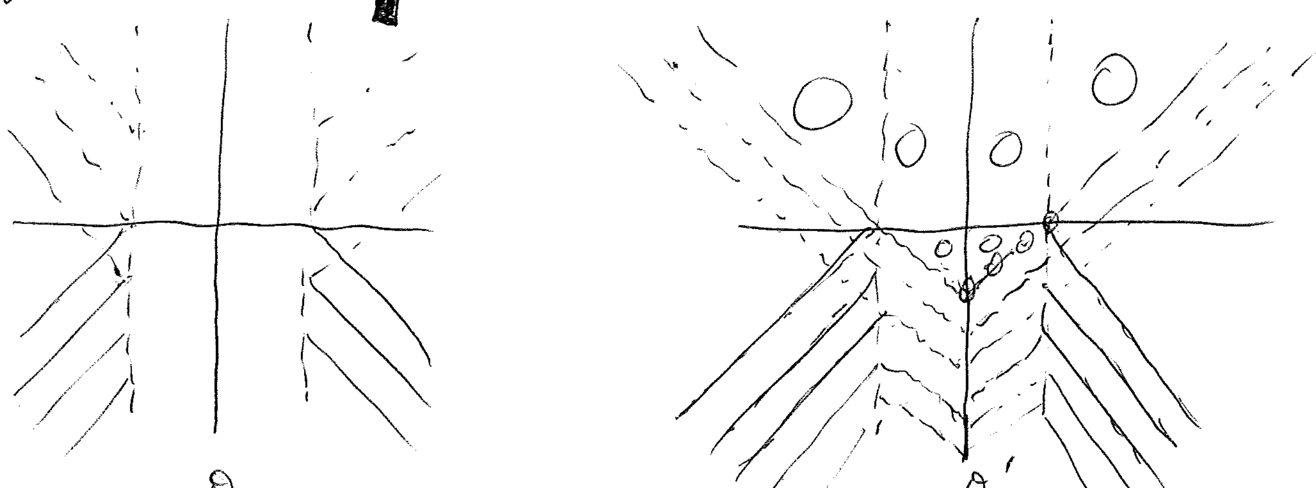
and the space of solutions of the wave equation with support $u(t)$ in $[-N-|t|, N+|t|]$ is evidently finite-dimensional.

January 28, 1979:

Yesterday we looked at solutions of

$$\frac{u(t+1)+u(t-1)}{2} = Hu(t)$$

such that [redacted] $u(n,t)$ has finite support for each t. $\mathcal{D}$ is the space consisting of $u(n,t)$ supported in $|n| \leq N$[redacted]$-t$ for $t \ll 0$, $\mathcal{D}'$ is the space of solutions vanishing for $|n| \leq N+t$ for all $t \geq 0$. [redacted] Pictures:

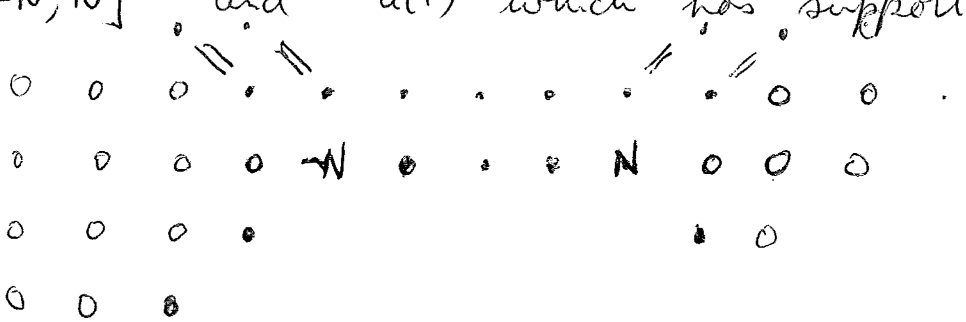


$\mathcal{D}$ $\qquad\qquad$ $\mathcal{D}'$

So it is clear that modulo $\mathcal{D}'$ [redacted] any element of $\mathcal{D}$ is equivalent to one with

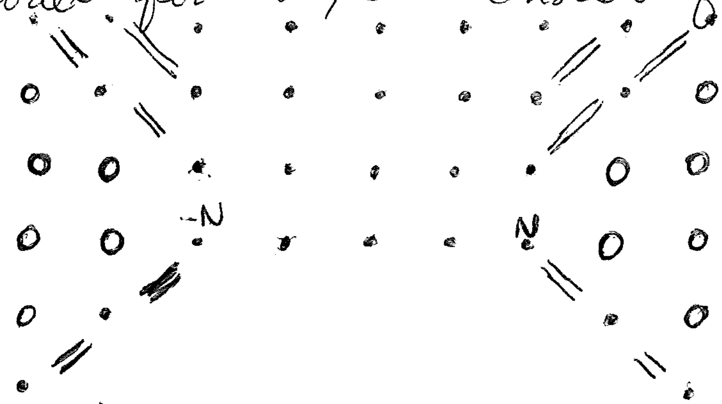
$$\operatorname{Supp} u \subset [-N-|t|, N+|t|]$$

Such a solution is determined by $u(0)$ which has support $\subset [-N,N]$ and $u(1)$ which has support in $[-N-1, N+1]$:



These are arbitrary, so $\dim \mathcal{D}/\mathcal{D}' = (2N+1)+(2N+3)$

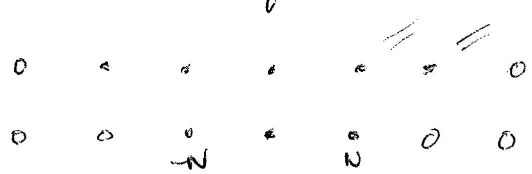
It seems we get slightly better results if we allow $\mathcal{D}'$ to consist of solutions vanishing for $|n| < N+t$ whence our model for $\mathcal{D}/\mathcal{D}'$ consists of solutions looking like



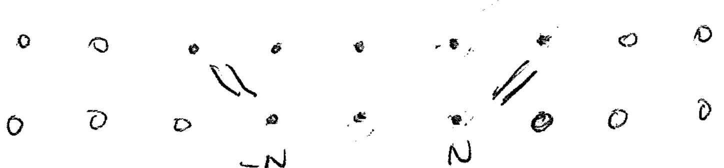
This time $\dim(\mathcal{D}/\mathcal{D}')$ is $2(2N+1)$ dimensional. Note that time evolution consists of computing $u(2)$ from $u(0), u(1)$ and throwing away the ends.

The problem is now to compute $\det(1-zU)$ on $\mathcal{D}/\mathcal{D}'$, and to connect it up with the LS determinant $\det(1-VG_z^+)$.

Note that if we take $\mathcal{D}/\mathcal{D}'$ to consist of



then time shifting lands you into the subspace consisting of data of the form



which is the more efficient model with $\dim = 2(2N+1)$.