January 23, 1979

Look at the discrete case again: Let $H$ be a $J$-matrix which is a finite perturbation $H_0-V$ of $H_0$:

$$(H_0 u)_n = \frac{1}{2} (u_{n+1} + u_{n-1})$$

Then we get rank 2 vector bundles over the $z$-plane with fibres

1) $\text{Ker } (H_0 - \lambda)$ and $\text{Ker } (H - \lambda)$

Denote by $G^+_z$ the Green's function for $H_0 - \lambda$ given by

$$G^+_z(n, n') = \frac{z^{1-n-n'}}{\frac{1}{2}(z-z') \lambda}$$

where $\frac{1}{2}(z+z') = \lambda$. Better: define $G^+_z$ by this formula and then state that for $\lambda = \frac{1}{2}(z+z')$ one has

$$(H_0 - \lambda) G^+_z(n, n') = \delta(n-n')$$

Thus if we pull back the vector bundles described in (1) to the $z$-plane with $0, \infty, \pm 1$ removed, we get a map

$$(1 - G^+_z V) : \text{Ker } (H - \lambda) \to \text{Ker } (H_0 - \lambda)$$

Let $C = C(\mathbb{Z})$ denote all sequences $c_0 \in C_0(\mathbb{Z})$ all finite support sequences. Over the $z$-plane we have a diagram of fibre bundles as follows:
\[ 0 \rightarrow \text{Ker}(H - \lambda) \rightarrow C \xrightarrow{H - \lambda} C \rightarrow 0 \]
\[ 0 \rightarrow \text{Ker}(H_0 - \lambda) \rightarrow C \xrightarrow{H_0 - \lambda} C \rightarrow 0 \]

**Question:** Are \( H - \lambda, H_0 - \lambda \) onto? **Yes**. For \( H_0 - \lambda \) it is clear as follows. It suffices to solve \((H_0 - \lambda)u = f\) when \( f_n = 0 \) for \( n < 0 \). Whence one can take \( u_n = 0 \) for \( n < 0 \) and then grind the rest out using the recursion formula:

\[ \Delta u_{n+1} - \lambda u_n + \frac{1}{2} u_{n-1} = f_n \]

This amounts to using the Green's function:

\[ G(n, n') = \begin{cases} 0 & n \leq n' \\ \frac{2^{n-n'} - 2^{n'-n}}{2 - 2^{-1}} & n > n' \end{cases} \]

The same argument works for \( H - \lambda \), since \( H \) is a \( J \)-matrix.

**Question:** Can you interpret \( \det(1 - G_z^+V) \) somehow? **Mumford idea:** Work with \( C_0 \) and the transposed situation:
It's not clear that we get a map on \( \mathfrak{C}_0 \) from \( 1 - G_2^+ V \).

It seems that \( \mathfrak{C}_0 \) should be the space of global sections of the rank 2 vector bundle over \( \mathcal{C} \) with fibre \( \mathfrak{C}_0 / (H_0 - \lambda) \mathfrak{C}_0 \) at \( \lambda \). Thus \( \mathfrak{C}_0 \) should be a free module of rank 2 over \( \mathcal{C}[H_0, 1] \). Similarly \( \mathfrak{C}_0 \) should be a free rank 2 module over \( \mathcal{C}[H, 1] \). Now the map \( \text{Ker} (H_0 - \lambda) \rightarrow \text{Ker} (H_0 - 1) \) furnished by \( 1 - G_2^+ V \) should be some sort of correspondence between these vector bundles.

Let's take simpler case. Let \( U_0 = \text{shift in} \mathfrak{C}_0 \) and let \( U \) be a finite perturbation. We know \( \mathfrak{C}_0 \) is a free \( \mathcal{C}[U_0, U_0^{-1}] \) module of rank 1. Assuming \( U \) bijective on \( \mathfrak{C}_0 \), \( \mathfrak{C}_0 \) will be a module of rank 1 over \( \mathcal{C}[U, U^{-1}] \), so if there is no torsion (bound states) \( \mathfrak{C}_0 \) will be free of rank 1. So \( \text{Ker} (U - \frac{1}{2}) \) will be one-dimensional for each \( z \neq 0, \infty \). (Actually I am ignoring the transposes.)
Let $C_0 = C_0(\mathbb{Z}) \cong C[1, 1^{-1}]$ with $U_0 = \text{right shift} = \text{multiplication by } 1$. We identify the dual of $C_0$ with $C = C(\mathbb{Z})$. The transpose of $U_0$ is the left shift:

$$(U_0^t(a_n), (b_n)) = ((a_n), (b_{n-1})) = ((a_{n+1}), (b_n))$$

and so $\text{Ker } (U_0^t - z)$ is spanned by the sequence $(z^n)$. Recall the diagram

$$
\begin{array}{ccccccccc}
0 & \rightarrow & \text{Ker } (U_0^t - z) & \rightarrow & C & \overset{U_0^t - z}{\rightarrow} & C & \rightarrow & 0 \\
& & \uparrow & & \uparrow & & \downarrow & & 0 \\
0 & \rightarrow & \text{Ker } (U_0^t - z) & \rightarrow & C & \overset{U_0^t - z}{\rightarrow} & C & \rightarrow & 0
\end{array}
$$

where $1 - G_+^tV = (U_0^t - z)^{-1}(U_t - z)$ in this notation. Recall that the left vertical arrow is the map on eigenfunctions induced by the wave operator

$$\Omega = \lim_{n \to \infty} U_0^{-n}$$

on $C_0$.

So I ought to be able to claim that the above diagram results by applying $\text{Hom}_{C(\mathbb{Z})}(?, C)$ to

$$
\begin{array}{ccccccccc}
0 & \rightarrow & C[\mathbb{Z}] \otimes C_0 & \overset{U_0 - z}{\rightarrow} & C[\mathbb{Z}] \otimes C_0 & \overset{T_0}{\rightarrow} & C_0 & \rightarrow & 0 \\
& & \downarrow \text{id} & & \downarrow & & \downarrow \Omega & & 0 \\
0 & \rightarrow & C[\mathbb{Z}] \otimes C_0 & \overset{U_0 - z}{\rightarrow} & C[\mathbb{Z}] \otimes C_0 & \overset{T}{\rightarrow} & C_0 & \rightarrow & 0
\end{array}
$$

I have to check that $1 - V_t(G_+^t)^t$ is really well-defined.
Start with deriving the LS equation
\[
(tU - z)\psi = 0 \\
(tU - z)\psi = (tU_0 - tU)\psi = tU_0 V\psi \\
(1 - ztU_0^{-1})\psi = V\psi
\]
or
\[
\psi - \psi = G_z^+ V\psi = \sum_{n \geq 0} z^n (tU_0^{-1})^n V\psi
\]
We need the transpose of this operator \( G_z^+ V \) which is
\[
\sum_{n \geq 0} z^n (tU_0^{-1})^n V = tV (1 - zU_0^{-1})^{-1}
\]
Notice that this operator is well-defined on \( C[z] \otimes C_0 \) because if \( f \in C_0 \), then \( tV U_0^{-n} f \) will be zero for \( n \) large. As a check note that
\[
\left[ 1 - tV (1 - zU_0^{-1})^{-1} \right] (U_0 - z) = U_0 - z - tV U_0 = U - z
\]
Since \( U = (I - tV) U_0 \).

Now what the diagram at the bottom of preceding page suggests is that in some sense \( \Omega \) has a determinant which is an element in \( C[z] \) (or at least in \( C[z, z^{-1}] \)) and
\[
det(\Omega) = \det \left( 1 - tV \otimes (G_z^+) \right) = \det \left( 1 - G_z^+ V \right)
\]
I see I have forgotten to check \( \Omega \) is compatible with \( 1 - tV (1 - zU_0^{-1})^{-1} \). But \( \Pi : C[z] \otimes C_0 \rightarrow C_0 \) sends \( z \) to \( U \).
So for \( f \in C_0 \)
\[
\pi \left( 1 - t \mathcal{V}(1-\varepsilon u_0^{-1})^{-1} \right) f = \sum_{n \geq 0} \frac{u^n \cdot t \mathcal{V} u_0^{-n} f}{(u-u_0)} \\
= f - \sum_{n \geq 0} u^n (u-u_0) u_0^{-n-1} f \\
= f + \sum_{n \geq 0} u^n (u-u_0) u_0^{-n-1} f \\
= f + (u u_0^{-1} - 1) f + u(u-u_0) u_0^{-2} f + \ldots \]
\[
= \lim_{n \to \infty} u^n u_0^{-n} f = 0 f.
\]

The question arises as to whether it is possible to define the determinant of \( \Omega \) more directly. The idea is that \( \Omega \) is a map between free rank 1 modules over \( \mathbb{Z}[z, z^{-1}] \), so up to a choice of bases it has a determinant. Can the identity map provide you with compatible bases?
January 24, 1979 (David is 15)

Question: Let $M$ be a free rank 1 module over $\mathbb{C}[z, z^{-1}]$ and let $J$ be an endomorphism of $M$ as a $\mathbb{C}$-vector space. What does one have to assume so as to be able to define $\det(J)$ as an element of the ring $\mathbb{C}[z, z^{-1}]$?

Example: Take $M = \mathbb{C}[z, z^{-1}]$ and let $U_0 = \text{mult. by } z$, $U = (1 - V)U_0$ a finite perturbation. Then we can take $J = \lim_{n \to \infty} U^nU_0^{-n}$. Why should $J$ have a determinant which is a poly in $z$?

Now we know that $UJ = JU_0$ and that there are $\mathbb{C}[z^{-1}]$-lattices $D$, $D'$ such that $J = \text{id}$ on $D'$ and $D$ is $J$-stable.

Let's try to generalize the construction. Let's see what one needs to define the determinant of $1 - V(1 - zU_0^{-1})^{-1}$. Let $z \in \mathbb{C}$. $\left((1 - zU_0^{-1})^{-1} = 1 + zU_0^{-1} + z^2U_0^{-2} + \cdots \right)$ has the matrix

\[
\begin{pmatrix}
1 & z & z^2 & \cdots \\
1 & z & z^2 & \\
1 & z & z^2 & \\
1 & 0 & 0 & \\
1 & 0 & 0 & \\
\end{pmatrix}
\]
so it would seem that we want $\mathsf{V}$ to have
the form

\[
\begin{pmatrix}
0 & \ast & \ast \\
\ast & \ast & \ast \\
\ast & \ast & \ast \\
\end{pmatrix}
\]

In other words $D, D'$ should be stable under $V$
and $V$ should be nilpotent with respect to the
filtration on $D'$ and on $C_0/D'$. Simplest way of
meeting these requirements is

\[
\begin{pmatrix}
0 & \ast \\
\ast & \ast \\
\ast & \ast \\
\end{pmatrix}
\]

i.e. $\text{Im} \ V \subset D', \ D' \subset \ker V$, or in
words, $V$ leaves the filtration $0 \subset D' \subset D \subset C_0$
fixed and induces $0$ on $C_0/D'$ and $mD'$.

So therefore we should be able to work
with a polynomial ring.

Let $D = \mathbb{C}[S]$ with $U_0 = \text{mult. by } 1$ and
let $U : D \to D$ be an endomorphism such that
\[ U = U_0 \text{ on } D' = \mathcal{F}N\mathcal{D}. \] For any \( z \in \mathbb{C} \) we consider the operator on \( D \) given by

\[
(1 - zU)(1 - zU_0)^{-1} = 1 - z(U - U_0)(1 - zU_0)^{-1} = 1 - z(U - U_0) \sum_{n=0}^{\infty} z^n U_0^n
\]

There is no problem with convergence of the series because \((U - U_0)U_0^n = 0\). This endomorphism is the identity on \( D' \), and hence its determinant can be evaluated on \( D/D' \).

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January 25, 1979

Let \( k \) be a field, let \( A \) be a \( k \)-algebra, and \( m \) an ideal in \( A \) such that \( A/m, m/m^2 \) are finite dimensional over \( k \). Let \( \theta \) be a \( k \)-linear endomorphism of \( A \) such that

1) \( \theta(m^n) \subseteq m^{n+1} \) for \( n \gg 0 \).

Then I can define \( \det(1 - z\theta) \) as follows. For \( n \gg 0 \), \( \theta \) induces an endom. \( \theta_n \) of \( A/m^n \). Since

\[
0 \to m/m^{n+1} \to A/m^{n+1} \to A/m^n \to 0
\]

it follows \( \det(1 - z\theta_n) \) doesn't depend on \( n \), so we
can define $\det(1-z\Theta)$ to be this poly.

An example where (1) holds occurs when there is an element $x \in M$ such that $\Theta - x = 0$ on $\mathfrak{m}^N$. In this case we can write formally

$$
(1-z\Theta)(1-zx)^{-1} = [1-zx + z(x-\Theta)](1-zx)^{-1}
$$

$$
= 1 + z(x-\Theta) \sum_{n \geq 0} z^n x^n
$$

$$
= 1 + z(x-\Theta) \sum_{0 \leq n < N} z^n x^n
$$

since $(x-\Theta)z^N = 0$ for $n > N$. Notice that the second term is an endomorphism of $A$ of finite rank, hence it vanishes on $\mathfrak{m}^N$. Thus if we define $(1-z\Theta)(1-zx)^{-1}$ by the above formula it has a determinant, and since this determinant can be evaluated by restriction to $A/\mathfrak{m}^N$ on which $x$ is nilpotent we see

$$
\det_A((1-z\Theta)(1-zx)^{-1}) = \det(1-z\Theta) \text{ as defined above.}
$$

Next suppose that we have a $k$-linear endomorphism $\Theta$ of the quotient field $F$ of a d.v.r. $(A, \mathfrak{m})$ such that

$$
\Theta(\mathfrak{m}^n) \subset \mathfrak{m}^{n+1} \quad \text{for } |n| > 0
$$

Then we can define $\det(1-z\Theta)$ by looking at the effect of $\Theta$ on $\mathfrak{m}^{-N}/\mathfrak{m}^N$ for $N$ large. On the other hand, suppose there is an element $x \in M$ such that
$$\ker(\theta - \alpha) \supseteq m^{-N}$$
$$\text{Im}(\theta - \alpha) \subseteq m^{-N}$$

for \( N \) large. Then the transformation of \( F \)

$$1 + z(\theta - \alpha) \sum_{n \geq 0} z^n x^n$$

is well-defined, since for any \( x \in F \) one has \( z^n x \in m^{-N} \) for \( n \gg 0 \). On the other hand, this transformation is the identity on \( m^{-N} \), and is the identity modulo \( m^{-N} \), hence it has a determinant which one can evaluate in \( m^{-N} / m^{-N} \).
January 26, 1979

Let $U_0 = \text{multiplication by } i$ on $E[1, j^{-1}] = e_0$. and let $H_0 = \frac{1}{2}(U_0 + U_0^{-1})$. Let $V$ be a finite perturbation of $H_0$. We wish to compute

\[(H_0 - \lambda)(H_0 - \lambda)^{-1} = 1 - V(H_0 - \lambda)^{-1}\]

formally. Recall that the Green's function for $H_0 - \lambda$ is

\[G^+_\lambda (u, u') = \frac{1}{2} \begin{pmatrix} 1 \end{pmatrix} \begin{pmatrix} 1 \end{pmatrix} \sum \frac{z^{1-n-n'}}{2(z-z^{-1})} f(u') \]

where $\frac{1}{2}z^{-1} = \lambda$ and for $\lambda \notin [-1, 1]$, $z$ is chosen with $|z| < 1$. Thus

\[(G^+_\lambda f)(u) = \frac{1}{2(z-z^{-1})} \sum_{n'} z^{1-n-n'} f(n') \]

\[= \frac{1}{2(z-z^{-1})} \left( \sum_m z^{1-m} U_0^{-m} f \right)(u') \]

\[G^+_\lambda = \frac{1}{2(z-z^{-1})} \left( (1-zU_0)^{-1} + zU_0^{-1} (1-zU_0)^{-1} \right) \]

\[= \frac{1}{2(z-z^{-1})} \left( (1-zU_0)^{-1} \left[ (1-zU_0^{-1} + (1-zU_0)zU_0^{-1}) \right] ^{-1} \right) \]

\[= -2z (1-zU_0)^{-1} (1-zU_0^{-1})^{-1} \]

Check:

\[H_0 - \lambda = \frac{U_0 + U_0^{-1}}{2} - \frac{z + z^{-1}}{2} = \frac{1}{2z} \left( zU_0 + zU_0^{-1} - z^2 - 1 \right) \]

\[= -\frac{1}{2z} \left( \sqrt{zU_0} - zU_0^{-1} + z \right) = -\frac{1}{2z} (1-zU_0)(1+zU_0^{-1}) \]
So we can set up the diagram:

\[ 0 \to R \otimes C_0 \xrightarrow{\lambda - H_0} R \otimes C_0 \xrightarrow{1-VG_e^+} R \otimes C_0 \xrightarrow{c[\lambda]_z} 0 \]

where \( R \) is the minimal ring over which \( VG_e^+ \) is defined. Since \( \lambda = \frac{1}{2} (z + z^{-1}) \) and \( z \in R \) and

\[ VG_e^+ = \frac{1}{2(z-z^{-1})} \left\{ V \cdot \sum_m z^m U_o^{-m} \right\} \]

it is clear that we want

\[ R = C[z, z^{-1}, \frac{1}{z-z^{-1}}] \]

What about the cokernels of the horizontal arrows in the above diagram? One knows for any \( C[\lambda] \)-module \( M \) that

\[ 0 \to C[\lambda] \otimes M \xrightarrow{\rho \otimes \text{id}} C[\lambda] \otimes M \to M \to 0 \]

is exact. Also \( R \) is flat over \( C[\lambda] \). Hence the above can be filled in as indicated in ink.

The cokernels should have something to do with the wave equation and so perhaps they have appropriate lattices \( D, D' \).

Note that \((*)\) is a perfectly good finite rank index of \( C_0 \) for any \( z \in \mathbb{C} \setminus \{0,1,-1\} \), so our problem is to interpret the determinant.
There is a chance that
\[ C[u, u^{-1}] \otimes_{C[H]} C_0 \]
\[ H = \frac{1}{2} (u + u^{-1}) \]
can be interpreted via a wave equation. Note that its dual
\[ \text{Hom}_{C[z, z^{-1}]} \left( C[u, u^{-1}] \otimes_{C[H]} C_0, X \right) = \text{Hom}_{C[H]} \left( C_0, X \right) \]
can be identified with the set of sequences \[ (x_n) \text{ in } X \]
such that
\[ H x = \left( \frac{z + z^{-1}}{2} \right) x \]
This is what one gets when one takes the F.T. of a solution of the wave equation:
\[ H u(t) = \frac{u(t+1) + u(t-1)}{2} \]

This namely
\[ u(t) = \int t^{\frac{\pi}{2}} x_n(z) \frac{dz}{2\pi} \]

On the other hand, we can get a direct interpretation as follows. Note that \( C[u, u^{-1}] \) is a free module over \( C[H] \) with basis \( 1, u^{-1} \), hence any element of \( C[u, u^{-1}] \otimes_{C[H]} C_0 \) can be uniquely written
\[ u^{-1} \otimes a - 1 \otimes b \]
with \( a, b \in C_0 \). Then
\[ u(u^{-1} \otimes a - 1 \otimes b) = 1 \otimes a - u \otimes b \]
and
\[ u \otimes b + u^{-1} \otimes b = 1 \otimes 2Hb \]
\[ u(u^{-1} \otimes a - 1 \otimes b) = 1 \otimes a - 1 \otimes 2Hb + u^{-1} \otimes b \]
\[ U^{-1} \otimes b - I \otimes (2Hb - a) \]

Thus I can think of \([u, u^{-1}] \otimes [H] \circ C_0 \) as pairs \((b)\) in \(C_0\) with

\[
U(a) = \begin{pmatrix} b \\ 2Hb - a \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 2H \end{pmatrix} \begin{pmatrix} a \end{pmatrix}
\]

On the other hand, a solution of the wave equation

\[
\frac{u(t+1) + u(t-1)}{2} = Hu(t)
\]

where \( u : \mathbb{Z} \rightarrow C_0 \) is determined by \( u(0), u(1) \) and

\[
\begin{pmatrix} u(1) \\ u(2) \end{pmatrix} = \begin{pmatrix} u(1) \\ 2Hu(1) - u(0) \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 2H \end{pmatrix} \begin{pmatrix} u(0) \end{pmatrix}
\]

Thus elements of \([u, u^{-1}] \otimes [H] \circ C_0 \) can be interpreted in an ad hoc fashion as solutions of a wave equation.

What I am trying to do is to identify \( \det (I - VG^+) \) with a Lax-Phillips determinant \( \det (I - zU) \) where \( U \) is time evolution on a space \( D/\mathcal{D}' \). I still seem to be missing the key points.
January 27, 1979:

Let us consider a T-matrix $H$ which is a finite perturbation of $H_0 = \frac{1}{2}(U_0 + U_0^{-1})$ where $U_0$ is the shift on $C_0(\mathbb{Z})$. Think of $C_0(\mathbb{Z})$ as contained in $C(\mathbb{Z})$ = all sequences.

We consider solutions of the wave equation:

\[
\frac{u(t+1) + u(t-1)}{2} = Hu(t)
\]

where $u(t)$ is a sequence $u(n, t)$ for each $t \in \mathbb{Z}$. Obviously any solution is determined by two consecutive $t$ values.

The above equation can be written

\[
\frac{1}{2} u(n, t+1) + \frac{1}{2} u(n, t-1) = a_n u(n+1, t) + b_n u(n, t) + a_{n-1} u(n-1, t)
\]

For $|n|$ large $a_n = \frac{1}{2}$, $b_n = 0$ so if we plot a solution in the $n, t$ plane at each point $(n, t)$, with $|n| > N$ say, we have the sum of the up + down neighbors is the sum of the right + left ones.

To clear what one means about domain of dependence for solutions of the wave equation. For example one has

\[
u(n, 0) = 0 \quad n \leq 0
\]

\[
u(n, -1) = 0 \quad n \leq -1
\]

then $u(n, t) = 0$ for $n \leq t$.
My problem is to get at the Lax-Phillips space $D/D'$. So the first thing to understand is the outgoing space $D'$. This consists of solutions $u(n,t)$ which leave the obstacle, so that they vanish for $|n| < N + t$ for all $t \gg 0$.

Such a solution is a sum

$$u(n,t) = f(n-t) + g(n+t)$$

for all $n$ and $t \gg 0$.

with $\text{Supp } f \subset [N, \infty)$ and $\text{Supp } g \subset (-\infty, -N]$.

Notice also that if $g$ is zero, then $u(n,t) = 0$ for all points above $t = n - N$, i.e.

$$t > n - N \text{ or } n - t < N.$$  

Maybe if I change $D'$ so that $u(n,t) = 0$ for $|n| \leq N + t$ and $t \gg 0$, I can describe $D$ by conditions on $u(n,0)$ and $u(n,1)$:

- $u(n,1) = u(n-1,0)$ for $n > N + 1$,
- $u(n,1) = u(n+1,0)$ for $n \leq N - 1$,
- other values zero.
Better description: $\mathcal{D}$ consists of $u(n, t)$ such that:

- $u(n, 0) = 0$ for $|n| \leq N$
- $u(n, 1) = 0$ for $|n| \leq N + 1$
- $u(n, 0) = u(n+1, 1)$ if $n > N$
- $u(n, 0) = u(n-1, 1)$ if $n < -N$

and the picture is:

Next we want to determine $\mathcal{D}$. The idea to keep in mind is that we want to think of a solution of the wave equation as being a sum of a left-moving and a right-moving wave.

Let's define $\mathcal{D}$ tentatively to consist of all $u$ such that for $t << 0$, $u(n, t)$ is supported in $[-N+N, N-N]$. Another possibility is to kill the waves running the wrong way. For $n >> 0$ we want to kill the waves running to the right, i.e. those of the form $f(n+1, t)$.

Hence killed by $u \mapsto u(n+1, t+1) - u(n, t)$. Thus we might be interested in $V(n, t) = u(n+1, t+1) - u(n, t)$ which satisfies

\[
V(n, t) = -u(n-1, t+1) + u(n, t+2) = +V(n-1, t+1)
\]

away from the obstacle.
Then we could define \( D \) by requiring that \( v(n,t) = 0 \) for \( n > N-t \) and a similar condition on the other sides. So what else can we do? We should show \( D' \subset D \) and that \( D/D' \) is finite-dimensional.

The two possibilities for \( D \) are probably the same if we restrict to compact support solutions, i.e. \( u(n,t) = 0 \) for \( |n| \gg 0 \) and \( t \) fixed.

Why is \( D' \subset D \)? It suffices to take \( u(n,t) = f(n-t) \) where \( \text{Supp}(f) \subset [N+1, N'] \). Take \( f = \delta\)-function on \( N+1 \). Then the support of \( u \) looks as follows:

Why is \( D/D' \) finite-dimensional? A typical element of \( D \) looks like

[Diagram showing points and lines indicating support of \( u \) and \( D/D' \) being finite-dimensional.]
Modulo elements of $O$, it becomes

and the space of solutions of the wave equation with support $u(t)$ in $[-N, N]$ is evidently finite-dimensional.
January 28, 1979:

Yesterday we looked at solutions of
\[
\frac{u(t+1) + u(t-1)}{2} = H(u(t))
\]
such that \( u(n,t) \) has finite support for each \( t \).
\( D' \) is the space consisting of \( u(n,t) \) supported in \( |n| \leq N' - t \) for \( t < 0 \), \( D' \) is the space of solutions vanishing for \( |n| \leq N' + t \) for all \( t > 0 \).

So it is clear that modulo \( D' \) any element of \( D \) is equivalent to one with
\[
\text{Supp } u \subset [-N+1, N+1]
\]
such a solution is determined by \( u(0) \) which has support \( C[-N, N] \), and \( u(1) \) which has support in \( [-N-1, N+1] \):

These are arbitrary, so dim \( D/D' \) = \( (2N+1) + (2N+3) \)
It seems we get slightly better results if we allow $D'$ to consist of solutions vanishing for $m < N + t$ whence our model for $D/D'$ consists of solutions looking like

\[ \begin{array}{cccccccc}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{array} \]

This time $\text{dim}(D/D')$ is $2(2N+1)$ dimensional. Note that time evolution consists of computing $u(2)$ from $u(0), u(1)$ and throwing away the ends.

The problem is now to compute \( \det(1 - z U) \) on $D/D'$, and \( \text{to connect it up with the LS determinant } \det(1 - VG_2^+ V) \).

Note that if we take $D/D'$ to consist of

\[ \begin{array}{cccccccc}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{array} \]

then time shifting lands you into the subspace consisting of data of the form

\[ \begin{array}{cccccccc}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{array} \]

which is the more efficient model with $\text{dim} = 2(2N+1)$. 