Let's try to reconcile LS and Lax-Phillips in the case of a radial Dirac system, which we write in the form

$$\frac{d\psi}{dx} = ik\psi + V\psi$$

where $(V\psi)(x) = h(x)u(-x)$ and h [redacted] satisfies

$$h(-x) = -\overline{h(x)}$$

so that iV is hermitian. Then

$$(1 - G_k^+ V)\psi\ (x) = \psi(x) - \int_{-\infty}^{x} e^{ik(x-x')}(V\psi)(x')\,dx'.$$

Recall the way we computed $\det(1 - G_k^+ V)$. We note that $G_k^+ V\psi$ is supported to the [redacted] right of the obstacle, and that

$$\left(\frac{d}{dx} - ik\right)(1 - G_k^+ V) = \frac{d}{dx} - ik - V$$

[redacted] If we denote by A_0 the operator induced on $L^2(a,\infty)$ by $\frac{d}{dx}$ with domain cut down to f with $f(a) = 0$, then G_k^+ is simply [redacted] $(A_0 - ik)^{-1}$, and

$$1 - G_k^+ V = (A_0 - ik)^{-1}(A - ik)$$

Finally A_0^{-1} is a Volterra operator

$$A_0^{-1} f = \int_{a}^{x} f(x')\,dx'$$

so $$1 - G_k^+ V = (1 - ikA_0^{-1})^{-1}(A_0^{-1}A)(1 - ikA^{-1})$$

where I assume that A^{-1} exists. Now

$$\det(1 - ikA_0^{-1}) = 1$$

so

$$\det(1 - G_k^+ V) = \det(A_0^{-1}A)\det(1 - ikA^{-1}).$$

Here A denotes the operator $\frac{d}{dx}$ [redacted] $-V$ on $L^2(a,\infty)$ with domain restricted to f vanishing at a; hence $A = A_0 - V$. The basic determinant we are after is $\det(1 - ikA^{-1}) = \det(1 - k\tilde{H}^{-1})$ where $\tilde{H} = \frac{1}{i}A$, is the operator on $L^2(a,\infty)$ [redacted] generating the semi-group of contractions.

What is the Lax-Phillips approach? [redacted] Let $\mathcal{D} = L^2(a,\infty)$, $\mathcal{D}' = L^2(b,\infty)$ where V is supported between a, b. To be precise one replaces the Hilbert space by solutions of the wave equation

$$\frac{\partial\tilde{\psi}}{\partial x} = -\frac{\partial\tilde{\psi}}{\partial t} + V\tilde{\psi}$$

by associating to any $u \in L^2(\mathbb{R})$, the solution of the wave equation beginning at u at $t=0$. Then elements of $\mathcal{D}$ correspond to solutions $\tilde{\psi}(x,t)$ vanishing for [redacted] $x < t$ [redacted] $+a$, $t \le 0$, and $\mathcal{D}'$ corresponds to solutions supported in $[b+t, \infty)$ for $t \ge 0$.

Let us next shift to the stationary picture which means that we Fourier transform solutions of the wave equation with respect to time.

$$\tilde{\psi}(x,t) = \int e^{-ikt}\psi(x,k)\,dk/2\pi$$

Then we have the scattering

$$e^{ikx} \longleftrightarrow R(k)\, e^{-ikx}$$

Denote the eigenfunction with above behavior by $\phi(x,k)$, so that solutions of the wave equation are given by

$$\psi(x,k) = \alpha(k)\, \phi(x,k)$$

Then $\psi \in \mathcal{D}$ means that for $x < t + a$, $t \leq 0$

$$\int_{-\infty}^{\infty} e^{-ikt} e^{ikx} \alpha(k)\, dk = 0$$

$\forall \varepsilon > 0$ $\quad \int e^{+ika} \alpha(k) e^{-ik\varepsilon}\, dk = 0 \implies e^{+ika} \alpha \in H^-$

or $\quad \alpha \in e^{-ika} H^-$

Similarly $\psi \in \mathcal{D}'$ means that for $b < x < b + t$, $t \geq 0$ one [redacted] has

$$\int \underbrace{e^{-ikt} e^{ikx}}_{e^{ikb} e^{-ik\varepsilon}} R(k) \alpha(k)\, dk = 0$$

or $\quad R(k)\alpha(k) \in e^{-ikb} H^-$

So therefore

$$\mathcal{D} \simeq e^{-ika} H^-$$

$$\mathcal{D}' \simeq e^{-ikb} R^{-1} H^-$$

[redacted] $\quad \mathcal{D}/\mathcal{D}' \simeq H^- / e^{-ik(b-a)} R^{-1} H^-$

Now when the original system comes from a Dirac [redacted] system pieced end to end we have

$$A(k)e^{-ikx} \longleftrightarrow B(k)e^{-ikx}$$

where $B(k) = \overline{A(\bar{k})}$, and $a=-b$, and so

$$R = \frac{B(k)}{A(k)}$$

[redacted] Thus if we revert to a simpler notation

$$\overline{(\mathcal{D}/\mathcal{D}')} \simeq H^+ \Big/ e^{ik2b}\frac{B}{A}H^+ \qquad T(t) = \text{mult. by } e^{ikt}$$

Now A is non-vanishing on the UHP, so only the zeroes of B matter.

Now the problem is to compute the characteristic polynomial of the infinitesimal generator of the semi-group $T(t)$.

Example: suppose we consider

$$\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}(x) = \begin{pmatrix} e^{ikx} & 0 \\ 0 & e^{-ikx} \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} \qquad \text{for } 0 \le x < \varepsilon$$

then

$$= \begin{pmatrix} e^{ik(x-\varepsilon)} & 0 \\ 0 & e^{-ik(x-\varepsilon)} \end{pmatrix}$$ [redacted] $$\frac{1}{\text{"}k\text{"}}\begin{pmatrix} 1 & -h \\ -\bar{h} & 1 \end{pmatrix}\begin{pmatrix} e^{ik\varepsilon} & 0 \\ 0 & e^{-ik\varepsilon} \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} \qquad \text{for } x > \varepsilon$$

[redacted]

$$= \begin{pmatrix} e^{ikx} & 0 \\ 0 & e^{-ikx} \end{pmatrix}\frac{1}{\text{"}k\text{"}}\begin{pmatrix} 1 & -he^{-2ik\varepsilon} \\ -\bar{h}e^{2ik\varepsilon} & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} B(k)e^{ikx} \\ A(k)e^{-ikx} \end{pmatrix} \qquad B(k) = \frac{1}{\sqrt{1-|h|^2}}\left(1 - he^{-2ik\varepsilon}\right)$$

$$A(k) = \frac{1}{\sqrt{1-|h|^2}}\left(-\bar{h}e^{2ik\varepsilon} + 1\right)$$

so that
$$R(k) = \frac{B(k)}{A(k)} = e^{-2ik\varepsilon}\left(\frac{e^{2ik\varepsilon}-h}{-\bar{h}e^{2ik\varepsilon}+1}\right)$$

Now a convenient choice for b is ε. Then
$$e^{2ikb}R(k) = \frac{e^{2ik\varepsilon}-h}{-\bar{h}e^{2ik\varepsilon}+1}$$

and so $\quad \mathcal{D}/\mathcal{D}' \simeq H^+/(e^{2ik\varepsilon}-h)H^+$

and I have to compute a determinant, if the singularity in V doesn't interfere.

Let us try to understand $\quad H^+/(e^{ik}-h)H^+$
where $|h|<1$. Recall
$$H^+ \simeq L^2(0,\infty)$$
$$\int_0^\infty e^{ikx}g(x)\,dx \longleftarrow\!\!| \; g(x)$$
$$f(k) \longmapsto \int e^{-ikx}f(k)\,dk/2\pi = \hat{f}(x)$$

So that
$$e^{ikt}f(k) \longmapsto \int e^{-ikx}\,e^{ikt}f(k)\,dk/2\pi = \hat{f}(x-t)$$

Thus multiplying by e^{ikt} corresponds to the forward shift $\hat{f} \mapsto \hat{f}(x-t)$.

Let $f \in L^2(0,\infty) \simeq H^+$ be $\perp$ $(e^{ik}-h)H^+$, i.e. $(T^*-\bar{h})f=0$ where T^* is the adjoint of the shift $g(x) \longmapsto g(x-1)$, which is $g(x) \mapsto g(x+1)$ restricted to $[0,\infty)$. Thus $f \in H^+ \ominus (e^{ik}-h)H^+ \iff$
$$f(x+1) = \bar{h}f(x)$$

Thus $H^+/(e^{ik}-h)H^+$ can be identified with the subspace $\mathcal{W}$ of $f \in L^2(0,\infty)$ satisfying $f(x+1) = \bar{h} f(x)$. This in turn is isomorphic to $L^2(0,1)$ by restriction, but with norm

$$\|f\|^2_{\mathcal{W}} = \|f\|^2_{(0,\infty)} = \|f\|^2_{(0,1)}\left(1 + |h|^2 + |h|^4 + \cdots\right)$$

$$= \frac{\|f\|^2_{(0,1)}}{1-|h|^2}$$

Let $f \in \mathcal{W}$. Choose k_0 with $e^{ik_0} = \bar{h}$, whence the solutions of $e^{ik} = \bar{h}$ are $k_n = k_0 + 2\pi n$ as $n \in \mathbb{Z}$. Also $e^{-ik_0 x} f(x)$ satisfies

$$e^{-ik_0(x+1)} f(x+1) = e^{-ik_0 x} f(x)$$

so it is periodic and hence admits a Fourier series expansion

$$e^{-ik_0 x} f(x) = \sum a_n e^{2\pi i n x}$$

$$a_n = \int_0^1 f(x) e^{-ik_0 x} e^{-2\pi i n x}\, dx$$

So

$$f(x) = \sum_{n\in\mathbb{Z}} a_n e^{ik_n x} \qquad a_n = \int_0^1 f(x) e^{-ik_n x}\, dx.$$

~~But~~

$$\cancel{\int_0^\infty f(x) e^{-ik_n x}\, dx = \sum_{j=0}^{\infty} \int_j^{j+1} f(x) e^{-ik_n x}\, dx}$$

$$\cancel{= \sum_{j=0}^{\infty} \int_0^1 f(x+j) e^{-ik_n x} e^{-ik_n j}\, dx}$$

January 14, 1979

$\mathcal{W} = H^+ \ominus (e^{ik} - h)H^+$ contains the point-evaluators at the zeroes of $e^{ik} - h$. Since

$$\hat{f}(k) = \int_0^\infty f(x) e^{ikx}\, dx$$

$\mathcal{W}$ contains $e^{-i\bar{k}_a x}$ when $e^{+ik_a} = h$, i.e. $e^{-i\bar{k}_a} = \bar{h}$. So again we see that $\mathcal{W}$ contains e^{ikx} when $e^{ik} = \bar{h}$.

Let's change notation slightly and denote by [redacted] $\Lambda = \{k \mid e^{ik} = h\}$, so that $\mathcal{W}$ is spanned by $e^{-i\bar{\lambda}x}$, $\lambda \in \Lambda$. The evaluation map

$$f \longmapsto \hat{f}(\lambda) = (f, e^{-i\bar{\lambda}x})$$

satisfies $\quad f(x-t) \longmapsto (e^{ikt}\hat{f})(\lambda) = e^{i\lambda t}\hat{f}(\lambda) \quad$ for $t \geq 0$

so it [redacted] picks up the eigenspace for $T(t)$ corresponding to λ.

Now

$$\left(T(t)f, e^{-i\bar{\lambda}x}\right) = e^{i\lambda t}\left(f, e^{-i\bar{\lambda}x}\right)$$

$$\parallel$$

$$\left(f, T(t)^* e^{-i\bar{\lambda}x}\right)$$

for all $f \in \mathcal{W}$, so that

$$T(t)^*\left(e^{-i\bar{\lambda}x}\right) = e^{-i\bar{\lambda}t} e^{-i\bar{\lambda}x}$$

hence $\quad T(t)^*(f) = f(x+t) \quad$ for all $f \in \mathcal{W}$.

Hence $T(t)^*$ is diagonal with the eigenvalues $e^{-i\bar{\lambda}t}$ $\lambda \in \Lambda$. I expect $T(t)$ to have the eigenvalues $e^{i\lambda t}$.

The infinitesimal generator iH of $\{T(t)\}$ has the eigenvalues $\{i\lambda\,,\ \lambda \in \Lambda\}$, and thus we form $1-kH^{-1}$, which [redacted] has the eigenvalues $1-k/\lambda$ $\quad \lambda \in \Lambda$ and so

$$\det(1-kH^{-1}) = \prod_{\lambda\in\Lambda}(1-k/\lambda)$$

The problem here is that this infinite product is not convergent without making an Eisenstein summation of some sort.

Question: Is there a well-defined LS determinant in this case?

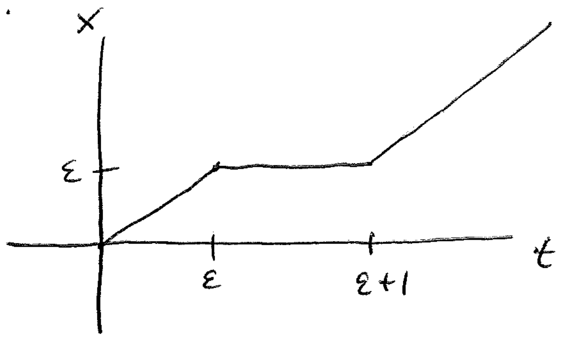
January 15, 1979

Consider a Dirac system on $0 \le x < \infty$

$$\frac{d}{dx}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} ik & p \\ \bar{p} & -ik \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \qquad u_1(0) = u_2(0)$$

where $p(x) = c\,\delta(x-\varepsilon)$.

In order to study this we introduce a new independent variable t:



$$x(t) = \begin{cases} t & 0 \le t \le \varepsilon \\ \varepsilon & \varepsilon \le t \le \varepsilon+1 \\ t-1 & \varepsilon+1 \le t < \infty \end{cases}$$

and we write the above DE as

$$d\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} ik\,dx & p\,dx \\ \bar{p}\,dx & -ik\,dx \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$$

and replace $p\,dx$ by $c\chi(t)\,dt$ where $\chi(t) = 1$ on $[\varepsilon, \varepsilon+1]$ and zero elsewhere. This makes sense since

$$\int_a^b p\,dx = \begin{cases} c & \varepsilon \in [a,b] \\ 0 & \varepsilon \notin [a,b] \end{cases}$$

Notice

$$\frac{dx}{dt} = 1 - \chi(t) \qquad \text{a.e.}$$

Thus our DE becomes

$$\frac{d}{dt}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} ik\frac{dx}{dt} & c\,\chi(t) \\ \bar{c}\,\chi(t) & -ik\frac{dx}{dt} \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}.$$

[redacted] We can [redacted] put this in the form

$$(*) \qquad \frac{du}{dt}(t) = ik\frac{dx}{dt}\cdot u(t) + g(t)\,u(-t) \qquad t \in \mathbb{R}$$

where

$$g(t) = \begin{cases} c\,\chi(t) & \text{for } t \geqslant 0 \\ \text{[redacted]}\ -\bar{c}\,\chi(-t) & \text{for } t \leqslant 0. \end{cases}$$

and $x(-t) = -x(t)$.

Integrate (*) using outgoing boundary conditions; which means $u = 0$ for $t \ll 0$.

$$e^{-ik\,\text{[redacted]}\,x(t)}\frac{du(t)}{dt} - ik\frac{dx}{dt}u = e^{-ikx(t)}g(t)\,u(-t)$$

$$\|$$

$$\frac{d}{dt}\left\{e^{-ikx}u\right\}$$

$$e^{-ikx(t)}u(t) = \int_{-\infty}^{t} e^{-ikx(t')}g(t')\,u(-t')\,dt'$$

$$\therefore\ u(t) = e^{ik\,\text{[redacted]}\,x(t)}\int_{-\infty}^{t} e^{-ikx(t')}g(t')\,u(-t')\,dt'$$

$$\boxed{u(t) = e^{ikx(t)}\int_{-t}^{\infty} e^{ikx(t')}g(-t')\,u(t')\,dt'}$$

Now the kernel

$$K(t,t') = \begin{cases} e^{-ik\left(x(t)+x(t')\right)}\, q(-t') & t+t' \geq 0 \\ 0 & < 0 \end{cases}$$

should have a Fredholm determinant.
For example

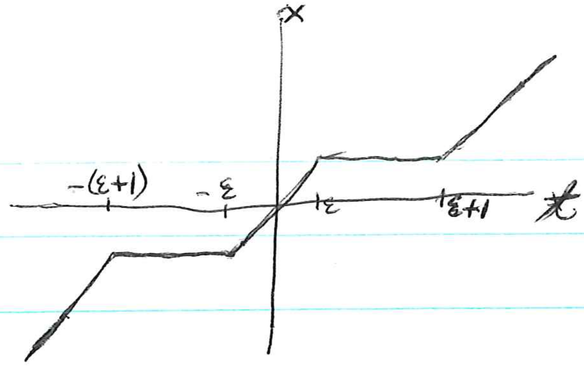
$$\text{trace} = \int_0^\infty e^{2ik\,x(t)}\, q(-t)\, dt$$

$$= \int_\varepsilon^{\varepsilon+1} e^{2ik\varepsilon}\,(-\bar{c})\,dt = -\bar{c}\,e^{2ik\varepsilon}$$

So it is clear more or less that a determinant exists for the integral equation at the bottom of the preceding page

January 19, 1979:

$$x(t) = \begin{cases} t & 0 \le t \le \varepsilon \\ \varepsilon & \varepsilon \le t \le \varepsilon+1 \\ t-1 & \varepsilon+1 \le t < \infty \end{cases}$$



$$\frac{dx}{dt} = 1 - \pi(t) \qquad \pi(t) = \begin{cases} 1 & t \overset{\text{or } -t}{\in} [\varepsilon, \varepsilon+1] \\ 0 & \text{otherwise} \end{cases}$$

$$\frac{d}{dt}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} ik\frac{dx}{dt} & c\pi(t) \\ \bar{c}\pi(t) & -ik\frac{dx}{dt} \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$$

or

$$\left(\frac{d}{dt} - ik\frac{dx}{dt}\right)(\psi) = \text{[redacted]}\ V\psi$$

where $$(V\psi)(t) = q(t)\psi(-t)$$

$$q(t) = \begin{cases} c\pi(t) & t \ge 0 \\ -\bar{c}\pi(t) & t \le 0 \end{cases}$$

So this leads to the integral equation

$$\psi(t) = \text{const.}\, e^{ikx} + \int_{-\infty}^{t} e^{ik(x-x')} q(t')\psi(-t')\,dt'$$

which still has the form

$$\psi = \varphi + G_k^+ V\psi$$

except that G_k^+ is the Green's function for

$$\left(\frac{d}{dt} - ik\frac{dx}{dt}\right).$$

Can we interpret $\det(1 - G_k^+ V)$ as a char. poly. This time we work with functions of t. Restrict to functions on $[a, \infty)$ where $a < -\varepsilon - 1$. Then $\frac{d}{dt}$ with boundary condition $u(a) = 0$ gives an operator A_0 with inverse

$$A_0^{-1} = \int_a^x f(x')\,dx'$$

If $\alpha(t) = \frac{dx}{dt}$, then

$$A_0^{-1}(A_0 - ik\alpha) = 1 - ik\, A_0^{-1}\alpha$$

where $$\left(\left(A_0^{-1}\alpha\right) f\right)(t) = \int_a^t \alpha(t') f(t')\,dx'$$ is a Volterra operator. Thus $A_0^{-1}(A_0 - ik\alpha)$ has determinant 1. Thus it seems that

$$1 - G_k^+ V = (A_0 - ik\alpha)^{-1}(A - ik\alpha) \qquad A = A_0 - V$$

$$= \left(1 - ik\, A_0^{-1}\alpha\right)^{-1} A_0^{-1}(A - ik\alpha)$$

so that

$$\det(1 - G_k^+ V) = \det\left(A_0^{-1}(A - ik\alpha)\right)$$

$$= \det\left(A_0^{-1} A\right) \det\left(1 - ik\, A^{-1}\alpha\right)$$

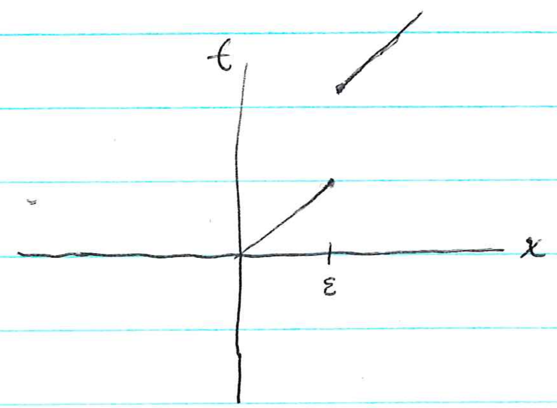
assuming $A_0^{-1}A$ is invertible. If this formal argument is correct, it ought to follow that $e^{ik\,\cdot}h$ is of the form of a characteristic polynomial.

Check more carefully:

Start with
$$\frac{d}{dx}\begin{pmatrix}u_1\\u_2\end{pmatrix} = \begin{pmatrix}ik & p\\ \bar{p} & -ik\end{pmatrix}\begin{pmatrix}u_1\\u_2\end{pmatrix}$$
on $0 \le x < \infty$ with $u_1(0) = u_2(0)$

and [redacted] where $p = c\,\delta(x-\varepsilon)$, ε fixed [redacted] > 0. We write this
$$d\begin{pmatrix}u_1\\u_2\end{pmatrix} = \begin{pmatrix}ik\,dx & p\,dx\\ \bar{p}\,dx & -ik\,dx\end{pmatrix}\begin{pmatrix}u_1\\u_2\end{pmatrix}$$

where $p\,dx = c\,\delta(x-\varepsilon)\,dx$. We interpret this by choosing a measure dt such that dx and $\delta(x-\varepsilon)dx$ are abs. continuous wrt it, e.g.
$$dt = dx + \delta(x-\varepsilon)dx$$
$$t = \begin{cases} x & 0 \le x < \varepsilon \\ x+1 & \varepsilon < x \end{cases}$$



Then we put
$$\frac{dx}{dt} = \alpha(t)\,, \quad \frac{\delta(x-\varepsilon)\,dx}{dt} = [redacted]\ \beta(t)$$

and we get
$$\frac{d}{dt}\begin{pmatrix}u_1\\u_2\end{pmatrix} = \begin{pmatrix}ik\alpha & c\beta\\ \bar{c}\beta & -ik\alpha\end{pmatrix}\begin{pmatrix}u_1\\u_2\end{pmatrix}$$

Now we put this onto $-\infty < t < \infty$ in the usual way
$$\frac{du}{dt} = ik\alpha\,u + [redacted]\ Vu$$

where
$$(Vu)(t) = q(t)\,u(-t)$$

and
$$q(t) = \begin{cases} c\,\beta(t) & t \ge 0 \\ -\bar{c}\,\beta(-t) & t \le 0 \end{cases}$$

Solve the DE: using $\frac{dx}{dt} = \alpha$:

$$e^{-ikx}\left(\frac{du}{dt} - ik\frac{dx}{dt}u\right) = e^{-ikx}Vu$$

$$u = \text{const.}\, e^{ikx} + \int_{-\infty}^{t} e^{ik(x-x')} \underbrace{(Vu)(t')}_{q(t')u(-t')} dt'$$

[redacted] where

$$G_k^+(t,t') = \begin{cases} e^{ik(x-x')} & t > t' \\ 0 & t < t' \end{cases}$$

is the Green's function for $\frac{d}{dt} - ik\alpha$.

Now my goal is to compute $\det(1 - G_k^+ V)$. Since $q(t)$ is supported to the right of $a = -\varepsilon - 1$, this determinant can be computed on the interval $[a, \infty)$. Let $A_0 = \frac{d}{dt}$ on $L^2(a,\infty)$ with boundary condition $f(a) = 0$. Then A_0^{-1} is a bounded operator

$$(A_0^{-1}f)(t) = \int_a^t f(t')\,dt'$$

and $G_k^+(t,t')$ is the kernel for the operator $(A_0 - ik\alpha)^{-1}$.

$$\left((A_0 - ik\alpha)^{-1}f\right)(t) = [redacted] \left[\left(e^{ikx}A_0 e^{-ikx}\right)^{-1} f\right](t)$$

$$= \left[\left(e^{ikx}A_0^{-1}e^{-ikx}\right)f\right](t)$$

$$= e^{ikx}\int_a^t e^{-ikx(t')} f(t')\,dt'$$

Thus on $L^2((a,\infty))$, $1 - G_k^+ V = 1 - (A_0 - ik\alpha)^{-1} V$ [redacted].

January 20, 1979

Review: I started with a Dirac system

$$\frac{d}{dx}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} ik & p \\ \bar{p} & -ik \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \qquad \begin{matrix} 0 \le x < \infty \\ u_1(0) = u_2(0) \end{matrix}$$

where $p = c\delta(x-\varepsilon)$, $\varepsilon > 0$. Introduce new independent variable t in order to make sense of this DE:

$$dt = dx + \delta(x-\varepsilon)dx$$

and then the system becomes $\frac{d}{dt}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} ik\alpha & c\beta \\ \bar{c}\beta & -ik\alpha \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$ on $0 \le t < \infty$, same bdy condition at $t=0$, where $\alpha = \frac{dx}{dt}$, $\beta = 1-\alpha$. Then I piece $u_1(t), u_2(-t)$ together to get $u(t)$ defined on R satisfying

$$(*) \qquad \frac{du}{dt} = ik\alpha u + Vu$$

where $(Vu)(t) = g(t)\,u(-t)$ $\qquad g(t) = \begin{cases} c\beta(t) & t \ge 0 \\ -\overline{g(-t)} & t \le 0 \end{cases}$

The Green's function for $\frac{d}{dt} - ik\alpha$ on $L^2(-\infty < t < \infty)$ is

$$G_k^+(t,t') = \begin{cases} e^{ik(x(t)-x(t'))} & t > t' \\ 0 & t < t' \end{cases}$$

for $\operatorname{Im} k > 0$. Consequently $(*)$ is equivalent to

$$u = \varphi + G_k^+ V u$$

where $\varphi(t) = \text{const} \cdot e^{ikx(t)}$.

The problem is now to compute and understand $\det(1 - G_k^+ V)$. Since

$$\left(G_k^+ V f\right)(t) = \int_{-\infty}^{t} e^{ik(x(t)-x(t'))}\, g(t') f(-t')\, dt'$$

$$= \int_t^\infty \underbrace{e^{ik(x(t)+x(t'))} g(-t')}_{\text{supported in } |t'| \leq \text{const.}} f(t')\,dt'$$

it would seem that the determinant is well-defined by the Fredholm formulas.

To compute it, or rather, to interpret this det as a char. poly., let's restrict to functions defined on $[a,\infty)$ where $a \ll 0$, and let A_0 be the operator on $L^2(a,\infty)$ given by $\frac{d}{dt}$ with the bdry condition $f(a)=0$. Then

$$(A_0^{-1}f)(t) = \int_a^t f(t')\,dt'$$

and

$$((A_0 - ik\alpha)^{-1}f)(t) = \int_a^t e^{ik(x(t)-x(t'))} f(t')\,dt'$$

$$\therefore \quad (A_0 - ik\alpha)^{-1} = e^{ikx} A_0^{-1} e^{-ikx}$$

Then on $L^2(a,\infty)$

$$1 - G_k^+ V = 1 - (A_0 - ik\alpha)^{-1} V$$

and

$$(1 - A_0^{-1} ik\alpha)(1 - G_k^+ V) = 1 - A_0^{-1} ik\alpha - \underbrace{(1 - A_0^{-1} ik\alpha)(A_0 - ik\alpha)^{-1}}_{A_0^{-1}} V$$

The proof of the assertion is

$$(1 - A_0^{-1} ik\alpha)(A_0 - ik\alpha)^{-1} = A_0^{-1}(A_0 - ik\alpha)(A_0 - ik\alpha)^{-1} = A_0^{-1}$$

where one uses that ~~the domain~~ $A_0^{-1}(A_0 - ik\alpha) = 1 - A_0^{-1} ik\alpha$ on $\mathcal{D}_{A_0}$.

Next note that

$$(A_0^{-1}\alpha f)(t) = \int_a^t \alpha(t')f(t')dt'$$

is a Volterra kernel, hence $\det(1 - A_0^{-1}ik\alpha) = 1$.
Thus

$$\det(1 - G_k^+ V) = \det\left(1 - A_0^{-1}ik\alpha - A_0^{-1}V\right)$$

which has the gen. characteristic polynomial form, since k appears linearly inside.

There might be some problem ultimately with $\det(1 - A_0^{-1}ik\alpha) = 1$, since the operator $A_0^{-1}\alpha$ and even A_0^{-1} might not be of trace class. It is not obvious how to define

$$\det(1 - A_0^{-1}ik\alpha)$$

So maybe it is necessary to work on a finite interval, in which case things should be OK.

Let's compute to the first order

$$(G_k^+ Vf)(t) = \int_a^t e^{ik(x(t) - x(t'))} q(t')f(-t')\,dt'$$

$$= \int_{-t}^{-a} e^{ik(x(t) + x(t'))} q(-t')f(t')\,dt' \qquad t' \geq -t$$

$-a = \varepsilon + 1$. $\qquad K(t,t') = \theta(t+t')e^{ik(x(t)+x(t'))} q(-t')$

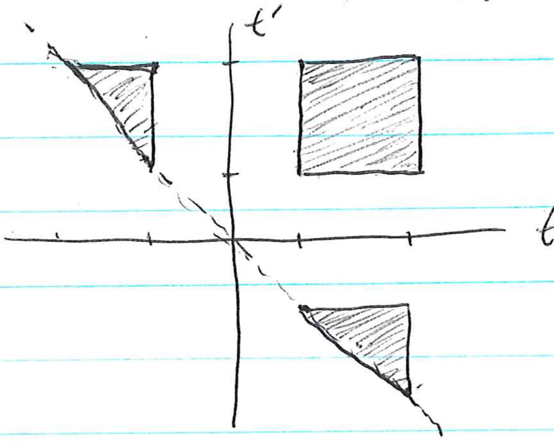
$$\operatorname{tr} K = \int_a^{-a} K(t,t)\,dt = \int_a^{-a} \theta(2t)e^{ik2x(t)}(-q(t))\,dt$$

$$= \int_{\text{[redacted]}\,\varepsilon}^{\varepsilon+1} e^{2ik\varepsilon}(-\bar{c})\,dt = -\bar{c}e^{2ik\varepsilon}$$

2nd order term: $\operatorname{tr}(\Lambda^2 K) = \text{[redacted]} \iint_{t \geq t'} \begin{vmatrix} K(t,t) & K(t,t') \\ K(t',t) & K(t',t') \end{vmatrix} dt\,dt'$

$$= \frac{1}{2}\left((\operatorname{tr}K)^2 - \operatorname{tr}(K^2)\right)$$

Now
$$\operatorname{tr}(K^2) = \iint \theta(t+t')e^{ik(x+x')}g(-t')\,\theta(t'+t)e^{ik(x'+x)}g(-t)\,dt\,dt'$$



$$= \frac{1}{2}(-\bar{c})(c) + \frac{1}{2}(c)(-\bar{c}) + e^{4ik\varepsilon}(-\bar{c})^2$$

$$= -|c|^2 + \bar{c}^2 e^{4ik\varepsilon}$$

Thus
$$\det(1 - G_k^+ V) = 1 + \bar{c}e^{2ik\varepsilon} + \frac{1}{2}\left(\bar{c}e^{2ik\varepsilon}\right)^2 - \frac{1}{2}\left(-|c|^2 + \bar{c}^2 e^{4ik\varepsilon}\right) + $$

$$= 1 + \bar{c}e^{2ik\varepsilon} + \frac{1}{2}|c|^2 + O(c^3)$$

The question to answer: Do you have an analogue of the Wronskian formula for $\det(1-G_k^+V)$ in the Dirac setup?

Solve the Dirac system with $\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}(0) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and take the Wronskian with the good solution $\begin{pmatrix} e^{ikx} \\ 0 \end{pmatrix}$ behavior for $x \gg 0$. Do this both for the free & perturbed systems and compare.

We have already solved the Dirac system and found (p. 478)

$$\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} e^{ikx} \\ e^{-ikx} \end{pmatrix} \qquad \text{for} \quad 0 \le x < \varepsilon$$

$$= \begin{pmatrix} B(k)e^{ikx} \\ A(k)e^{-ikx} \end{pmatrix} \qquad \text{for} \quad \varepsilon < x$$

where

$$B(k) = \frac{1}{\sqrt{1-|h|^2}}\left(1 - h e^{-2ik\varepsilon}\right)$$

$$A(k) = \frac{1}{\sqrt{1-|h|^2}}\left(1 - \bar{h} e^{2ik\varepsilon}\right)$$

and

$$\frac{1}{\sqrt{1-|h|^2}}\begin{pmatrix} 1 & -h \\ -\bar{h} & 1 \end{pmatrix} = \exp\begin{pmatrix} 0 & c \\ \bar{c} & 0 \end{pmatrix}$$

so the two Wronskians are:

$$\begin{vmatrix} B(k)e^{ikx} & e^{ikx} \\ A(k)e^{-ikx} & 0 \end{vmatrix} = -A(k), \quad \begin{vmatrix} e^{ikx} & e^{-ikx} \\ e^{-ikx} & 0 \end{vmatrix} = -1$$

and so therefore we conjecture

$$\det(1 - G_k^+ V) = A(k) = \frac{1}{\sqrt{1-|h|^2}}\left(1 - \bar{h}e^{2ik\varepsilon}\right).$$

Now

$c = |c|e^{i\theta}$

$$\exp\begin{pmatrix} 0 & c \\ \bar{c} & 0 \end{pmatrix} = \exp\left\{\begin{pmatrix} e^{i\theta} & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & |c| \\ |c| & 0 \end{pmatrix}\begin{pmatrix} e^{-i\theta} & 0 \\ 0 & 1 \end{pmatrix}\right\}$$

$$= \begin{pmatrix} e^{i\theta} & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \cosh|c| & \sinh|c| \\ \sinh|c| & \cosh|c| \end{pmatrix}\begin{pmatrix} e^{-i\theta} & 0 \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} \cosh|c| & e^{i\theta}\sinh|c| \\ e^{-i\theta}\sinh|c| & \cosh|c| \end{pmatrix}$$

$$= \begin{pmatrix} 1 + \frac{1}{2}|c|^2 & c \\ \bar{c} & 1 + \frac{1}{2}|c|^2 \end{pmatrix} + O(c^3)$$

$$\therefore \quad -h = e^{i\theta} \tanh |c| = c + O(c^3)$$

$$\therefore \quad A(k) = \cosh |c| + e^{-i\theta} \sinh |c| \, e^{2ik\varepsilon}$$

$$= 1 + \frac{|c|^2}{2} + \bar{c}\, e^{2ik\varepsilon} + O(c^3)$$

which checks.

January 21, 1979

Yesterday we found for

$$\left(\frac{d}{dx} - ik\right) u = Vu \qquad \text{on } \mathbb{R}$$

where $(Vu)(x) = p(x)\,u(-x)$ $\qquad p(-x) = -\overline{p(x)}$
$\qquad p = 0 \quad \text{for } |x| >> 0$

the conjectural formula

$$\det\left(1 - G_k^+ V\right) = A(k)$$

where $u(x) = A(k)e^{ikx}$ $\quad x << 0$ and u is the solution with $u(0) = 1$. The motivation for the formula comes by applying our prescription for the Schroed. DE to the Dirac DE

$$\frac{d}{dx}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} ik & p \\ \bar{p} & -ik \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \qquad \text{on } 0 \leq x < \infty$$

$$u_1(0) = u_2(0)$$

Thus we compute the Wronskian of $\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$ normalized with $u_1(0) = u_2(0) = 1$ with the solution good for $x >> 0$, and divide it by the corresponding Wronskian with $p = 0$:

$$\frac{\begin{vmatrix} B(k)e^{ikx} & e^{ikx} \\ A(k)e^{-ikx} & 0 \end{vmatrix}}{\begin{vmatrix} e^{ikx} & e^{ikx} \\ e^{-ikx} & 0 \end{vmatrix}} = A(k)$$

Now the problem is to understand what is going on here.