December 7, 1979

Weiss theory for Ising model 482
Inverting a power series by diagrams 488

We continue with the program of the analogy between the Ising model partition function and field theory generating functions. The Ising partition function is

$$Z_\Omega = \sum_{s:\Omega\to\{\pm 1\}} e^{-\beta E_\Omega(s)}$$

where
$$\beta E_\Omega(s) = -\frac{1}{2}\sum' J_{xy}\, s_x s_y - \sum' H_x s_x$$

and the sums take place over Ω. [redacted] Here Ω is a finite number of [redacted] sites and we are interested in the $\Omega \to \infty$ limit.

Look at the average [redacted] magnetization of the x-th site

$$\langle s_x \rangle = \lim_\Omega \frac{1}{Z}\frac{\partial Z}{\partial H_x} = \lim_\Omega \frac{\partial}{\partial H_x}\log Z$$

This limit should exist, as well as the higher order averages

$$(*) \qquad \langle s_{x_1}\cdots s_{x_k}\rangle = \lim_\Omega \frac{1}{Z}\frac{\partial^k Z}{\partial H_{x_1}\cdots\partial H_{x_k}}.$$

These become [redacted] Green's functions in field theory.

Question: Let X be the set of sites for the Ising model, so that $\{\pm 1\}^X$ is the set of configurations[redacted]. Does there exist [redacted] a probability measure on $\{\pm 1\}^X$, which is in some sense the limit of the Gibbs measures [redacted] for each finite Ω, or better, which has the moments $(*)$?

One has
$$\{\pm 1\}^X = \mathrm{Hom}_{ab\,gp}\big(\underbrace{\mathbb{Z}/2\mathbb{Z}[X]}_{\simeq \text{ finite subsets of } X}, \mathbb{C}^*\big)$$

hence the characters of the compact gp $\{\pm 1\}^X$ are of the form

$$s \longmapsto s_I = \prod_{x \in I} s_x$$

for I a finite subset of X. According to Bochner's thm a measure $d\mu$ on $\{\pm 1\}^X$ can be identfied, via Fourier transform, with a function on the dual $\mathbb{Z}/2\mathbb{Z}[X]$:

$$f(I) = \int s_I \, d\mu(s) = \left\langle \prod_{x \in I} s_x \right\rangle$$

which is positive-definite, i.e. the matrix

$$f(I_i - I_j) = f(I_i \cup I_j - I_i \cap I_j)$$

is positive-definte for any finite choices $I_1, \ldots, I_n \in \mathbb{Z}/2\mathbb{Z}[X]$.

The important thing about the above example is that the moments $\left\langle \prod_{x \in I} s_x \right\rangle$, ~~determine~~ provided they are well-defined by the limiting process $\Omega \to \infty$, do determine a definite probability measure on $\{\pm 1\}^X$.

December 8, 1979

There are two kinds of limits involved with the partition function which we have to get straight. Consider the Ising partition function

$$Z_\Omega = \sum_{s:\Omega\to\{\pm 1\}}' e^{-\beta E_s + H\cdot s}$$

Then when we compute the Green's functions

$$\left\langle \prod_{x\in I} s_x \right\rangle = \lim \frac{1}{Z_\Omega}\left(\prod_{x\in I}\frac{\partial}{\partial H_x}\right) Z_\Omega$$

or the connected versions ~~[illegible]~~

$$G^{(c)}(x_1,\ldots,x_k) = \lim_\Omega \left(\prod \frac{\partial}{\partial H_x}\right)\log Z_\Omega$$

we expect this limit to exist without $\log Z_\Omega$ having a limit. Moreover, provided we deal with a constant external field we expect

$$\frac{1}{|\Omega|}\log Z_\Omega$$

to have a limit.

December 9, 1979

Consider quantum mechanical motion on the line governed by $H = \frac{p^2}{2} + V(q)$. For any [redacted] T we get a kind of partition function

$$Z_T(J) = tr\left(U_J(T/2, -T/2)\right) = \int\limits_{q \text{ periodic of period } T} \mathcal{D}q \; e^{-\int_{-T/2}^{T/2}[\frac{1}{2}\dot{q}^2 + V(q)]dt \; + \; \int_{-T/2}^{T/2} Jq\, dt}$$

[redacted]

where U_J is the propagator for $H_J = H - J(t)q$. [redacted]
It is necessary to suppose V grows as $|q| \to \infty$, so that the trace is well-defined.

For example if $V = \frac{1}{2}\mu^2 q^2$, then we know

[redacted] $$Z_T(J) = Z_T(0)\; e^{\frac{1}{2}\int J G_T J}$$

where
$$G_T(t,t') = \sum_{k \in \frac{2\pi}{T}\mathbb{Z}} \frac{e^{ik(t-t')}}{k^2+\mu^2}\frac{1}{T}$$

and
$$Z_T(0) = \frac{1}{2\sinh\left(\frac{\mu T}{2}\right)}.$$

It seems that the theory of Wiener measure really gives us an honest measure [redacted] representing

$$\mathcal{D}q\; e^{-\int[\frac{1}{2}\dot{q}^2 + V(q)]dt}.$$

The measure is on the space of distributions $q(t)$ of period T, but continuous paths have full measure in this space of distributions, so one sees the measure given on continuous paths.

Provided we normalize this measure so as to have total mass 1, we can then take the limit as $T \to \infty$, so as to get a well-defined probability measure on distributions $q(t)$ on the line. Therefore we get a stochastic process with random variables $q \longmapsto q(t)$ for each $t \in \mathbb{R}$. Concretely we have expectation values

$$\langle q(t_1) \cdots q(t_n) \rangle = \lim_{T} \frac{1}{Z_T} \frac{\partial^n}{\partial J(t_1) \cdots \partial J(t_n)} Z_T(J) \Big|_{J=0}$$

which give us the moments of the joint probability distribution for a finite set of the random variables $q(t)$.

Now from the theory of the operator $H = \frac{p^2}{2} + V(q)$ we know that the above expectation values for the stochastic process coincide with the Green's functions

$$G(t_1, \ldots, t_n) = \langle T[q(t_1) \cdots q(t_n)] \rangle .$$

I should be more careful because I am working with time-evolution e^{-tH}. Specifically for $t_1 \geq t_2 \geq \cdots \geq t_n$

$$G(t_1, \ldots, t_n) = \int \mathcal{D}q \, e^{-\int (\frac{\dot q^2}{2} + V(q)) dt} \prod_{i=1}^{n} q(t_i)$$

$$= \frac{\langle 0 | U(T/2, t_1) \, q \, U(t_1, t_2) \, q \cdots U(t_n, -T/2) | 0 \rangle}{\langle 0 | U(T/2, -T/2) | 0 \rangle} \qquad T \text{ large}$$

where $U(t, t') = e^{-(t-t')H}$ is only defined for $t \geq t'$.

~~So the main problem is to~~

<u>Example:</u> Take the 2 point function [redacted]:

$$G(t_1,t_2) = \text{[redacted]}\,\frac{\langle 0|\,U(T/2,t_1)\,q\,U(t_1,t_2)\,q\,U(t_2,-T/2)\,|0\rangle}{\langle 0\,|U(T/2,-T/2)\,|\,0\rangle}$$

$$= \text{[redacted]}\quad \frac{e^{-(T/2-t_1)E_0}\sum_n \langle 0|q|n\rangle\, e^{-(t_1-t_2)E_n}\,\langle n|q|0\rangle\, e^{-(\frac{t}{2}+\frac{T}{2})E_0}}{e^{-(T/2+T/2)E_0}}$$

$$= \sum_n |\langle n|q|0\rangle|^2\, e^{-(t_1-t_2)(E_n-E_0)} \qquad t_1 > t_2$$

For the harmonic oscillator $\langle n|q|0\rangle = 0$ for $n \neq 1$ and

$$\langle 1|\,q\,|0\rangle = \langle 0|a\cdot\left(\frac{a+a^*}{\sqrt{2\mu}}\right)|0\rangle = \frac{1}{\sqrt{2\mu}}$$

so that

$$G(t,t') = \frac{1}{2\mu}\, e^{-|t_{\text{[redacted]}}-t'_{\text{[redacted]}}|\mu}$$

<u>Question:</u> Is there any relation between the stochastic process associated to the Green's functions and the real time unitary group e^{-itH}? I seem to recall Wiener using the stochastic process as a hidden variables approach to quantum mechanics.

Case to look at: Take the harmonic oscillator. ~~[illegible]~~ This gives rise to a stationary Gaussian stochastic process such that $\langle q(t)\rangle = 0$ for all t and such that

$$\langle q(t_1)\,q(t_2)\rangle = \frac{e^{-\mu|t_1-t_2|}}{2\mu} = \int \frac{e^{i\omega(t_1-t_2)}}{\omega^2+\mu^2}\,\frac{d\omega}{2\pi}.$$

I am used to thinking of a stationary Gaussian process as a 1-parameter group $U(t)$ acting on a Hilbert space together with a cyclic vector v. Then I get a finite measure on $\mathbb{R}$ such that

$$\left(U(t_1)v, U(t_2)v\right) = \int e^{-i\omega(t_1-t_2)} d\Delta(\omega)$$

and I can realize the Hilbert space as [redacted] $L^2(\mathbb{R}, d\Delta)$.
Is there any connection between the harmonic oscillator Hilbert space and [redacted] $L^2\left(\mathbb{R}, \frac{1}{2\pi}\frac{1}{\omega^2+\mu^2}d\omega\right)$?

[redacted] Recall that for Ising models the magnetization and susceptibility are defined resp. by

$$M = \lim \frac{1}{N}\frac{\partial}{\partial H}\log Z_\Omega = \lim \frac{1}{N}\sum_{x\in\Omega}\langle s_x\rangle \qquad N = |\Omega|$$

$$\chi = \frac{\partial M}{\partial H} = [redacted] \lim \frac{1}{N}\frac{\partial^2}{\partial H^2}\log Z_\Omega$$

$$= \lim \frac{1}{N}\left(\sum_{x,y}\langle s_x s_y\rangle - \sum_x\left(\sum\langle s_x\rangle\right)^2\right)$$

$$= \lim \frac{1}{N}\sum_{x,y}\langle (s_x - \langle s_x\rangle)(s_y - \langle s_y\rangle)\rangle$$

and that if one has translation invariance $\langle s_x\rangle$ is independent of x so that

$$M = \langle s_x\rangle \qquad \text{any } x$$

$$\chi = \sum_x \underbrace{\langle (s_x - \langle s_x\rangle)(s_0 - \langle s_0\rangle)\rangle}_{\text{this is } \langle s_x s_0\rangle - \langle s_x\rangle\langle s_0\rangle}$$

which is a connected Green's fn.

In the quantum case what we get is as follows

$$-\frac{1}{T}\log Z_T(J) \longrightarrow E(J) \qquad \text{ground energy for } H - Jq$$

M is replaced by $\langle q(t)\rangle$, χ by $\int G^c(t,0)\,dt$.

We can check this as follows. From perturbation theory we know that

$$\frac{dE}{dJ} = -\langle 0|q|0\rangle = \langle q(t)\rangle \qquad \text{any } t$$

$$\frac{d^2E}{dJ^2} = 2\sum_n \frac{|\langle n|q|0\rangle|^2}{E_n - E_0}$$

But we found that

$$G(t,0) = \sum_n \langle n|q|0\rangle^2 e^{-|t|(E_n-E_0)}$$

$$G^{(c)}(t,0) = G(t,0) - \langle q(t)\rangle\langle q(0)\rangle = \sum_{n\neq 0} \langle n|q|0\rangle^2 e^{-|t|(E_n-E_0)}$$

so

$$\int G^{(c)}(t,0)\,dt = \sum_{n\neq 0} \langle n|q|0\rangle^2 \frac{2}{E_n - E_0}$$

so it all checks.

Specifically for the oscillator

$$M = \langle 0|q|0\rangle = 0 \qquad \text{at } J=0$$

$$\chi = \int \frac{e^{-\mu|t|}}{2\mu}\,dt = \frac{1}{\mu^2}$$

which agrees with

$$E(J) = \frac{1}{2}\mu - \frac{J^2}{2\mu^2}$$

December 9, 1979

Let's begin the computation for the ϕ^4-theory using Feynman diagrams. Start off in 0 space-time dimensions. The partition function is

$$Z(J) = \int e^{-\frac{1}{2}\mu^2 x^2 - \frac{g}{4!}x^4 + Jx} \frac{\mu\, dx}{\sqrt{2\pi}}$$

and it has been normalized so as to give 1 when $J = g = 0$. We compute this by expanding in a power series in g

$$Z(J) = \int e^{-\frac{1}{2}\mu^2 x^2}\left(1 - \frac{g}{4!}x^4 + \frac{1}{2}\left(\frac{g}{4!}x^4\right)^2 - \cdots\right) e^{Jx} \frac{\mu\, dx}{\sqrt{2\pi}}$$

and then using Wick's thm. to evaluate the moments.

0 th order gives $e^{\frac{1}{2}\frac{1}{\mu^2}J^2}$

This disappears if we want $Z(0)$ normalized to be 1

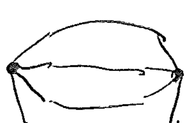
first order gives one diagram

with symmetry factor 8

so we get

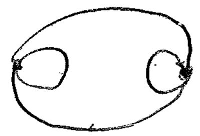
$$(-g)\, e^{\frac{1}{2}\frac{1}{\mu^2}J^2}\left(\frac{1}{8}\frac{1}{\mu^4}\right)$$

2nd order gives diagrams



$2 \cdot 4!$

$$g^2 e^{\frac{1}{2}\frac{1}{\mu^2}J^2}\, \frac{1}{\mu^8}\left(\frac{1}{2\cdot 4!} + \frac{1}{16} + \frac{1}{2\cdot 8^2}\right)$$



16

$2 \cdot 8^2$

The above can't be correct for $J \neq 0$.

Begin again. We want the second moment

$$\int e^{-\frac{1}{2}\mu^2 x^2 - \frac{g}{4!}x^4} \, x^2 \frac{dx\,\mu}{\sqrt{2\pi}}$$

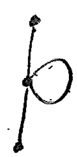
because this is the analogue of the 2 point function, except that it has to be normalized

Expand in a power series in g

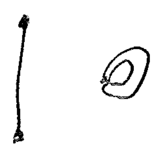
$$\int e^{-\frac{1}{2}\mu^2 x^2 \cancel{\frac{g}{4!}}} \left(1 - \frac{g}{4!}x^4 + \frac{1}{2}\left(\frac{g}{4!}\right)^2 x^8\right) x^2 \frac{dx \cdot \mu}{\sqrt{2\pi}}$$

and then use Wick's thm.

0th order:  $\frac{1}{\mu^2}$

1st order  symmetry factor 2

contribution $\frac{1}{2}\frac{1}{\mu^6}(-g)$

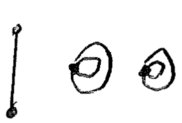
 $\frac{1}{8}\frac{1}{\mu^6}(-g)$

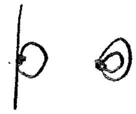
2nd order  symm factor 6 $\quad g^2 \frac{1}{6}\frac{1}{\mu^{10}}$

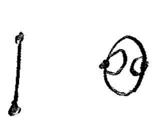
 4 $\quad g^2 \frac{1}{4}\frac{1}{\mu^{10}}$

 4 $\quad g^2 \frac{1}{4}\frac{1}{\mu^{10}}$

disconnected ones

 $g^2 \frac{1}{2 \cdot 8^2}\frac{1}{\mu^{10}}$

 $g^2 \frac{1}{2 \cdot 8}\frac{1}{\mu^{10}}$

 $g^2 \frac{1}{16}\frac{1}{\mu^{10}}$

 $g^2 \frac{1}{2 \cdot 4!}\frac{1}{\mu^{10}}$

[redacted] So we get

$$\frac{1}{\mu^2} - \frac{g}{\mu^6}\left(\frac{1}{2}+\frac{1}{8}\right) + \frac{g^2}{\mu^{10}}\left(\frac{1}{6} + \overbrace{\frac{1}{4}}^{+\frac{1}{4}} + \frac{1}{128} + \frac{1}{16} + \frac{1}{16} \text{[redacted]} + \frac{1}{48}\right)$$

On the other hand direct evaluation gives

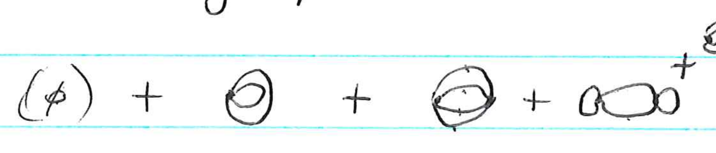
$$\frac{1}{\mu^2} + \frac{-g}{\mu^6}\left(\frac{1}{4!}\,1\cdot 3\cdot 5\right) + \frac{g^2}{\mu^{10}}\left(\frac{1}{2(4!)^2}\,1\cdot 3\cdot 5\cdot 7\cdot 9\right)$$

and [redacted] calculation shows the two numbers agree.

The two point function is

$$\langle x^2\rangle = \frac{\int e^{-\frac{1}{2}\mu^2x^2 - g\frac{x^4}{4!}}\, x^2\, dx \frac{\mu}{\sqrt{2\pi}}}{\int e^{-\frac{1}{2}\mu x^2 - g\frac{x^4}{4!}}\, dx \frac{\mu}{\sqrt{2\pi}}}$$

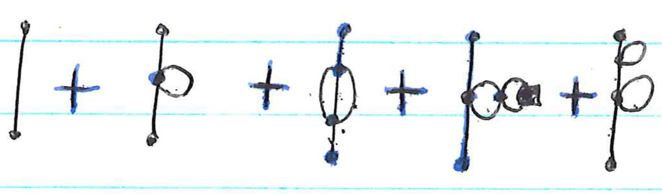
The denominator represents the contribution from the vacuum graphs



$$= 1 - \frac{g}{8}\frac{1}{\mu^4} + \frac{g^2}{\mu^8}\left(\frac{1}{48} + \frac{1}{16}\text{[redacted]} + \frac{1}{2^7}\right)$$

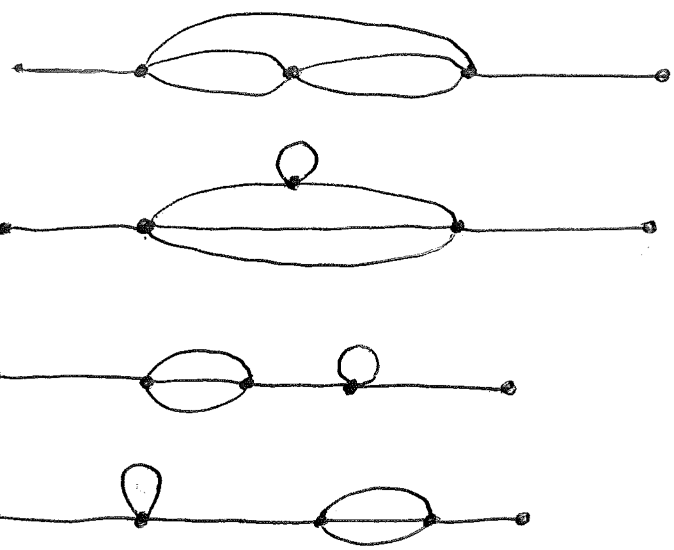
$$= 1 - \frac{g}{8}\frac{1}{\mu^4} + \frac{g^2}{\mu^8}\left(\text{[redacted]}\frac{5\cdot 7}{2^7 3}\right) - \cdots$$

see page 406 (Nov. 9, 1979), and so the 2-pt function is the sum over connected graphs



$$= \frac{1}{\mu^2} - \frac{g}{2}\frac{1}{\mu^6} + \frac{g^2}{\mu^{10}}\left(\frac{1}{6}+\frac{1}{4}+\frac{1}{4}\right)$$

There seem to be 4 kinds of 3 order connected graphs



symmetry factor

4

4

12

12

The bottom 2 are "1-particle-reducible".

December 10, 1979

I am looking at ϕ^4-theory in 0 space-dimensions which is the same quantum mechanics of the anharmonic oscillator with Hamiltonian

$$H = \frac{p^2}{2} + \frac{\mu^2}{2}q^2 + \frac{g}{4!}q^4$$

We form the generating function

$$Z(J) = \int_{q(-T/2)=q(T/2)} \mathcal{D}q \; e^{-\int\left(\frac{1}{2}\dot{q}^2 + \frac{1}{2}\mu^2 q^2 + \frac{1}{4!}g q^4\right)dt \; + \int J q\, dt}$$

where the path integral is normalized à la Feynman so as to give the trace of the propagator $\underset{J}{U}(T/2, -T/2)$ for the Schroedinger equation (imaginary time)

$$-\frac{\partial\psi}{\partial t} = \left(H - J(t)\,q\right)\psi$$

Thus

$$Z(0) = \operatorname{tr}\left(e^{-TH}\right) \sim e^{-TE_0}$$

where E_0 is the ground state energy. We can expand this path integral in powers of g, which leads to vacuum diagrams; E_0 can be computed as a series of connected vacuum diagrams:

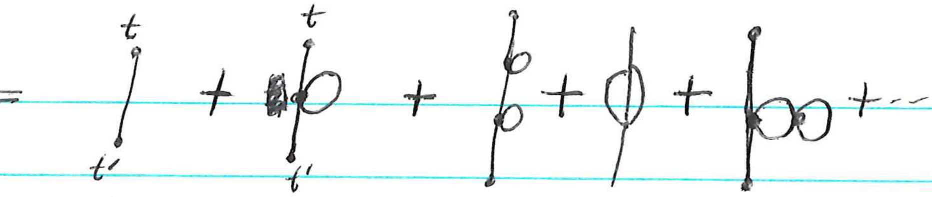
$E_0 = \frac{1}{2}\mu +$  $+$  $+$  $+ \cdots$

$$\frac{1}{8}\, g \frac{1}{T}\int_{-T/2}^{T/2} \frac{1}{(2\mu)^2}\, dt \longrightarrow \frac{g}{8}\frac{1}{(2\mu)^2}$$

t_1  t_2 $\quad -\frac{1}{2\cdot 4!}g^2 \frac{1}{T}\int\left(\frac{e^{-\mu|t_1-t_2|}}{2\mu}\right)^4 dt_1 dt_2 \rightarrow -\frac{1}{2\cdot 4!}g^2\frac{1}{(2\mu)^4}\frac{1}{2\mu}$

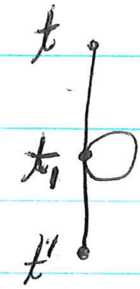
We are going to be interested in the connected 2 point function

$$\frac{\partial^2}{\partial J(t_1)\,\partial J(t_2)} \log Z(J)\Big|_{J=0} =$$



$$= G^{(c)}(t_1, t_2)$$

$$\frac{e^{-\mu|t-t'|}}{2\mu} = K(t,t') = \int \frac{e^{ik(t-t')}}{k^2+\mu^2}\,\frac{dk}{2\pi}$$



$$= -g\frac{1}{2}\int dt_1\; K(t-t_1)\,\underbrace{K(t_1-t_1)}_{1/2\mu}\,K(t_1-t')$$

$$= -\frac{g}{2}\,\frac{1}{2\mu}\int dt_1\;\frac{e^{ik_1(t-t_1)}}{k_1^2+\mu^2}\,\frac{dk_1}{2\pi}\;\frac{e^{ik_2(t_1-t')}}{k_2^2+\mu^2}\,\frac{dk_2}{2\pi}$$

$$= -\frac{g}{2}\,\frac{1}{2\mu}\int \frac{dk_1}{2\pi}\;\frac{e^{ik_1(t-t')}}{(k_1^2+\mu^2)^2}$$

The other method to evaluate these diagram terms is to work in the momentum representation. We change variables in the path integral for $Z(J)$ using the Fourier transform

$$q(t) = \frac{1}{T}\sum_{k\in\frac{2\pi}{T}\mathbb{Z}} q_k\, e^{ikt} \qquad\qquad q_k = \int_{-T/2}^{T/2} q(t)\,e^{-ikt}\,dt$$

Then $Z(J)$ is the integral of exp of

$$-\frac{1}{2}\,\frac{1}{T}\sum_k (k^2+\mu^2)|q_k|^2 - \frac{g}{4!}\,\frac{1}{T^3}\sum_k q_{k_1}\cdots q_{k_4}\,\delta(k_1+\dots+k_4)$$

$$+\frac{1}{T}\sum_k J_{-k}\,q_k$$

This way of writing things is convenient if you want to let $T \to \infty$. One gets

$$Z(J) = \int \mathcal{D}\, e^{-\int \frac{1}{2}(k^2+\mu^2)|q_k|^2 \frac{dk}{2\pi} - \frac{g}{4!}\int q_{k_1}\cdots q_{k_4}\,\delta(k_1+\cdots+k_4)\frac{dk_1\cdots dk_4}{(2\pi)^4} + \int J_{-k} q_k \frac{dk}{2\pi}}$$

December 12, 1979

Given

$$Z(J) = \int e^{\lambda(-f(x)+Jx)}\,dx$$

where λ is large, we can make the steepest descent calculation. The critical point is x_c where

$$f'(x_c) = J$$

$$f(x) = f(x_c) + J(x-x_c) + \tfrac{1}{2} f''(x_c)(x-x_c)^2 + \cdots$$

[redacted]

$$Jx - f(x) = Jx_c - f(x_c) - \tfrac{1}{2} f''(x_c)(x-x_c)^2 + \cdots$$

so

$$Z(J) \approx e^{\lambda(Jx_c - f(x_c))}\frac{1}{\sqrt{2\pi\lambda f''(x_c)}}$$

The function $g(x) = f'(x)x - f(x)$ is the Legendre transform of f, and has the property that

$$\frac{dg}{dJ} = \frac{d}{dJ}(Jx - f) = x + \cancel{J\frac{dx}{dJ} - f'(x)\frac{dx}{dJ}} = x$$

Thus

$$\frac{1}{\lambda}\log Z(J) \longrightarrow g \qquad \text{as } \lambda \to \infty$$

December 14, 1979

$$Z(J) = \int e^{\lambda(-f(x) + Jx)}\, dx$$

We evaluate this by steepest descent. The critical points for $-f(x) + Jx$ are solutions of

$$f'(x) = J$$

Assume there is only 1 critical point $\alpha = \alpha(J)$ and then use Taylor series

$$-f(x) + Jx = -f(\alpha) + J\alpha \quad \tfrac{1}{2} f''(\alpha)(x-\alpha)^2 - \tfrac{1}{3!} f'''(\alpha)(x-\alpha)^3 - \cdots$$

Then

$$Z(J) \approx e^{\lambda(-f(\alpha) + J\alpha)} \int e^{\lambda\left(-\frac{1}{2} f''(\alpha)(x-\alpha)^2 - \frac{1}{3!} f'''(\alpha)(x-\alpha)^3 - \cdots\right)}\, dx$$

The last integral ~~has~~ an asymptotic expansion in λ with terms given by Feynman diagrams. The various coefficients depend upon α. It's clear one wants to regard α as the independent variable with J determined by $f'(\alpha) = J$.

Let's work out the low terms of the asymptotic expansion of the integral

$$\int e^{-\frac{1}{2}\lambda f''(\alpha) x^2 \quad - \frac{\lambda}{3!} f'''(\alpha) x^3 - \cdots}\, dx$$

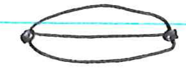
$$= \frac{1}{\sqrt{\lambda}} \int e^{-\frac{1}{2} f''(\alpha) x^2 - \frac{1}{3!} \lambda^{1-\frac{3}{2}} f'''(\alpha) x^3 - \frac{1}{4!} \lambda^{1-\frac{4}{2}} f^{(4)}(\alpha) x^4 - \cdots}\, dx$$

Put $\mu^2 = f''(\alpha)$, $g_3 = f'''(\alpha)$ etc.

It's clear that we get a series expansion in decreasing powers $\lambda^{-k/2}$ $k = 1, 2, \cdots$. Odd k disappear under integration

$$\int e^{-\frac{1}{2}\lambda f''(\alpha)x^2 - \frac{\lambda g_3}{3!}x^3 - \cdots\cdots} dx$$

$$= \frac{\sqrt{2\pi}}{\sqrt{\lambda f''(\alpha)}} \int \underbrace{\frac{dx\sqrt{\lambda f''(\alpha)}}{\sqrt{2\pi}} e^{-\frac{1}{2}\lambda f''(\alpha)x^2}}_{\substack{\text{Gaussian prob. measure} \\ \text{with covariance } \lambda f''(\alpha)}} e^{-\frac{\lambda g_3}{3!}x^3 - \frac{\lambda g_4}{4!}x^4 - \cdots}$$

$$= \frac{\sqrt{2\pi}}{\sqrt{\lambda f''(\alpha)}} \Big(1 +$$  $+$  $+ \cdots$

Recall each graph has the following power of λ associated to it. Edge vertex contributes λ and each edge contributes $\frac{1}{\lambda}$, so the total power is

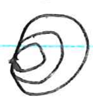
$$\lambda^v / \lambda^e = \lambda^{h^0 - h^1}$$

hence for a connected graph the power is 1 − number of loops. So list the connected diagrams according to number of loops:

$v = 1$ g_3, g_5, g_7 are not allowed

and

g_4:  has 2 loops

g_6:  has 3 loops

$v = 2$  2 loops  3 loops

 3 loops

December 15, 1979

$$Z(J) = \int e^{\lambda(-f(x) + Jx)}\, dx$$

$f'(\alpha) = J$ unique max for $-f(x) + Jx$

$$-f(x) + Jx = -f(\alpha) + f'(\alpha)\alpha - \tfrac{1}{2} f''(\alpha)(x-\alpha)^2 - \tfrac{1}{3!} f'''(\alpha)(x-\alpha)^3$$

$$\approx e^{\lambda(-f(\alpha) + f'(\alpha)\alpha)} \frac{\sqrt{2\pi}}{\sqrt{\lambda f''(\alpha)}} \left(1 + \frac{c_1}{\lambda} + \frac{c_2}{\lambda^2} + \cdots\right)$$

where the coefficients c_i can be expressed as sums [redacted] over diagrams. For

$$\int e^{-\frac{1}{2}\lambda f''(\alpha) x^2 - \frac{1}{3!}\lambda f'''(\alpha) x^3 - \cdots}\, dx$$

we have vertices of mult 3 with coefficient $\lambda f'''(\alpha)/3!$
" " " 4 " " $\lambda f^{(4)}(\alpha)/4!$
etc. and edges with the factor $\frac{1}{\lambda f''(\alpha)}$. Thus each verti[redacted] contributes λ and each edge $\frac{1}{\lambda}$ so the total λ power for a graph

$$\lambda^{v-e} = \lambda^{h^0 - h^1}$$

If k_p is the number of [redacted] vertices of mult. p, then one has

$$v = k_3 + k_4 + \cdots$$
$$e = \tfrac{1}{2}(3k_3 + 4k_4 + \cdots)$$

so

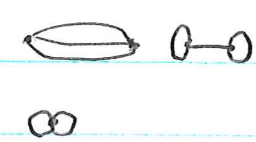
$$e - v = \tfrac{1}{2}k_3 + [redacted]\, k_4 + \tfrac{3}{2}k_5 + \cdots$$

Thus one has $e - v \geq 0$ and

$e - v = 0 \implies$ only empty graph

$e - v = 1 \implies k_3 = 2$ rest 0 or

$k_4 = 1$ " "



$e - v = 2 \implies k_3 = k_5 = 1$ rest 0

or $k_4 = 2$ rest 0

or $k_4 = 1, k_3 = 2$ "

or $k_3 = 4$ "

If connected these are all the graphs with 3 loops, and there are many possibilities.

The above case handles the situation where the integral is evaluated exactly at the maximum α for the function $-f(x) + Jx$. There are no tree graphs at all. Maybe the next thing to look at is when one expands about a point α which is not exactly the maximum. ??

Weiss theory for the Ising model.

$$Z(\beta, B) = \sum_s e^{-\beta E_s}$$

$$E_s = -\frac{1}{2}\sum_{x \neq y} J_{xy} s_x s_y - \sum_x{}' B_x s_x$$

$$-\beta E_s = \frac{1}{2}\sum{}' \beta J_{xy} s_x s_y + \sum{}' H_x s_x \qquad \text{where } H_x = \beta B_x$$

Fix a site x_0 and try to compute $\langle s_{x_0} \rangle$. ~~where $H_x = H$ is independent of x. Assuming translation invariance $\langle s_{x_0} \rangle$ = the magnetization M.~~

Let y run over sites $\neq x_0$. Then

$$\langle s_{x_0} \rangle = \frac{1}{Z}\sum_s s_{x_0} e^{-\beta E_s} = \frac{1}{Z}\sum_{s_{x_0}} s_{x_0} \sum_{s'} e^{s_{x_0}\left(\sum_y{}' \beta J_{x_0 y} s'_y + H_{x_0}\right)} \times e^{-\beta E_{s'}}$$

Notice that
$$\sum_y{}' \beta J_{x_0 y} s'_y + H_{x_0}$$
is essentially the effective field at x_0 due to the external field and the other sites. Call this $-\mathcal{H}_{x_0}(s')$. Then

$$\langle s_{x_0}\rangle = \frac{\sum_{s_{x_0}} s_{x_0} \sum_{s'} e^{s_{x_0}\mathcal{H}_{x_0}(s')} e^{-\beta E_{s'}}}{\sum_{s_{x_0}} \sum_{s'} e^{s_{x_0}\mathcal{H}_{x_0}(s')} e^{-\beta E_{s'}}}$$

$$= \frac{\langle e^{\mathcal{H}_{x_0}}\rangle - \langle e^{-\mathcal{H}_{x_0}}\rangle}{\langle e^{\mathcal{H}_{x_0}}\rangle + \langle e^{-\mathcal{H}_{x_0}}\rangle}$$

where $\langle\ \rangle$ is taken over s' ranging over spins not at x_0.

Now the basic assumption of the Weiss theory is that fluctuations are negligible, and so one may approximate:
$$\langle e^{\mathcal{H}_{x_0}}\rangle \approx e^{\langle \mathcal{H}_{x_0}\rangle} \qquad \langle e^{-\mathcal{H}_{x_0}}\rangle = e^{-\langle\mathcal{H}_{x_0}\rangle}$$

$$\langle \mathcal{H}_{x_0}\rangle = \sum{}' \beta J_{x_0 y} \langle s'_y\rangle + H_{x_0}$$

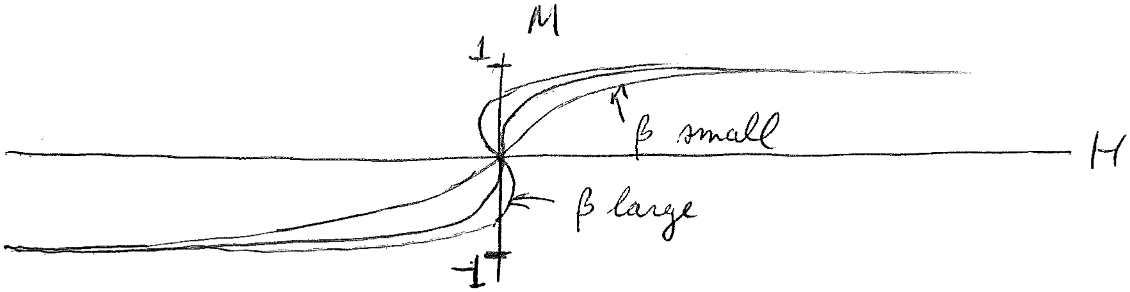
So now assume translation invariance so that $\langle s_x\rangle = M$ for all x, and $H_x = H$, and assume $\langle s'_y\rangle \approx M$. Then

$$\langle \mathcal{H}_{x_0}\rangle = \beta\Big(\sum_y{}' J_{x_0 y}\Big) M + H$$

and so one gets the implicit equation

$$M = \tanh(\beta v M + H) \qquad v \text{ constant}$$

or
$$H = \operatorname{arctanh}(M) - \beta v M$$
$$= \frac{1}{2}\log\left(\frac{1+M}{1-M}\right) - \beta v M$$



The critical temperature occurs when M becomes a multiple-valued function of H, i.e. when $\frac{dH}{dM} = 0$ or when

$$\beta v = 1$$

Compare with the linear Ising model where one finds

$$(Z_N)^{1/N} \longrightarrow e^{\beta J}\left(\cosh H + \sqrt{(\sinh H)^2 + e^{-4\beta J}}\right)$$

Now

$$M = \lim \frac{1}{N}\frac{\partial}{\partial H}\log Z_N = \lim \frac{\partial}{\partial H}\log Z_N^{1/N}$$

$$= \frac{\partial}{\partial H}\log\left(e^{\beta J}\left(\cosh H + \sqrt{}\right)\right)$$

$$= \frac{\sinh H + \frac{2\sinh H\cosh H}{2\sqrt{}}}{\cosh H + \sqrt{}}$$

$$M = \frac{\sinh H}{\sqrt{(\sinh H)^2 + e^{-4\beta J}}} \qquad \text{here } v = 2J$$

$$\geq \tanh H$$

December 16, 1979

tree graphs and Legendre transformation:

Consider

$$Z(J) = \int e^{\lambda(-f(x)+Jx)} \rho\, dx$$

$\in C_0^\infty$, $\rho \equiv 1$ near 0

say $f \in C^\infty(\mathbb{R})$ and $f(0)$ is an abs. max, and $g_2 > 0$.

where

$$f(x) = \frac{1}{2} g_2 x^2 + \frac{1}{3!} g_3 x^3 + \cdots$$

Then we know $Z(J)$ has an asymptotic expansion in λ of the form

$$Z(J) \approx e^{\lambda(-f(x_c)+Jx_c)} \sqrt{\frac{2\pi}{\lambda f''(x_c)}} \left(1 + O\left(\frac{1}{\lambda}\right)\right)$$

J, x_c near 0.

where x_c is chosen so that $f'(x_c) = J$.

On the other hand we can expand $Z(J)$ as a series in J & the g_p using diagrams, and get $\log Z(J)$ as the sum over connected diagrams. The power of λ belonging to a connected diagram with v vertices, e edges is

$$\lambda^{v-e} = \lambda^{1-\ell} \qquad \ell = \text{number of loops.}$$

If k_p = no. of vertices of mult p, then

$$v = k_1 + k_3 + k_4 + \cdots$$
$$e = \frac{1}{2}(k_1 + 3k_3 + 4k_4 + \cdots)$$

$$\therefore \quad -1 \le \ell - 1 = -\frac{1}{2}k_1 + \frac{1}{2}k_3 + k_4 + \frac{3}{2}k_5 + \cdots$$

Therefore if we take the part of $\log Z(J)$ which is the sum over all connected diagrams with no loops, i.e. trees, then we should get

$$\lambda\left(-f(x_c) + Jx_c\right)$$

Consider the example $f(x) = \frac{1}{2} g_2 x^2 + \frac{1}{4!} g_4 x^4$
and to simplify let's just write

$$f(x) = \frac{1}{2} a x^2 + \frac{g}{4!} x^4$$

First compute x_c:

$$f'(x) = ax + \frac{g}{6} x^3 = J$$

$$a \cdot \quad x = a \cdot J \left[\frac{1}{a} + c_3 J^2 + c_5 J^4 \right]$$

$$\frac{g}{6} \cdot x^3 = \frac{g}{6} \cdot J^3 \left[\frac{1}{a^3} + 3 \frac{1}{a^2} c_3 J^2 \right]$$

$$J = J \left[1 + a c_3 J^2 + a c_5 J^4 \right] + \left[\frac{g}{6} \frac{1}{a^3} J^3 + \frac{g}{2a^2} c_3 J^5 \right]$$

$$a c_3 + \frac{g}{6a^3} = 0 \qquad c_3 = -\frac{g}{6a^4}$$

$$a c_5 + \frac{g}{2a^2} c_3 = 0 \qquad c_5 = -\frac{1}{a} \frac{g}{2a^2} \left(-\frac{g}{6a^4} \right) = \frac{g^2}{12 a^7}$$

$$\boxed{x_c(J) = \frac{1}{a} J - \frac{g}{6a^4} J^3 + \frac{g^2}{12a^7} J^5 + O(J^7)}$$

Now compute the Legendre transform

$$F = x f'(x) - f(x)$$

$$= x \left(ax + \frac{g}{6} x^3 \right) - \frac{1}{2} a x^2 - \frac{g}{24} x^4$$

$$= \frac{a}{2} x^2 + \frac{g}{8} x^4$$

$$= J^2 \left\{ \frac{a}{2} \left(\frac{1}{a} - \frac{g}{6a^4} J^2 + \frac{g^2}{12 a^7} J^4 + O(J^6) \right)^2 \right\}$$

$$+ J^4 \left\{ \frac{g}{8} \left(\frac{1}{a} - \frac{g}{6a^4} J^2 + O(J^4) \right)^4 \right\}$$

$$= \frac{1}{2a}J^2 - \frac{[redacted]g}{6a^4}J^4 + \frac{g^2}{72a^7}J^6 + \frac{g^2}{12a^7}J^6 + O(J^8)$$

$$+ \frac{g}{8a^4}J^4 - \frac{g^2}{12a^7}J^6$$

$$= \frac{1}{2a}J^2 - \frac{[redacted]g}{24a^4}J^4 + \frac{g^2}{72a^7}J^6 + O(J^8)$$

$$\boxed{\hat{F}(x_c) = x_c f'(x_c) - f(x_c) = \frac{J^2}{2a} - \frac{gJ^4}{24a^4} + \frac{g^2J^6}{72a^7} + O(J^8)}$$

As a check one sees $\dfrac{dF}{dJ} = x$

Now let us compute the tree diagrams for the integral

$$\int e^{-\frac{1}{2}ax^2 - \frac{g}{24}x^4 + Jx}\,dx$$

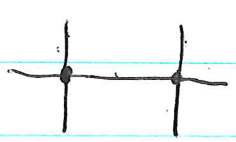
Thus we want

$$-1[redacted] = l-1 = -\tfrac{1}{2}k_1 + [redacted]k_4$$

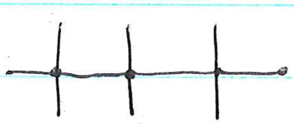
which gives the following trees

$$\frac{1}{2}\frac{J^2}{a}$$

$$\frac{1}{4!}\frac{J^4(-g)}{a^4} = -\frac{g}{4!}\frac{J^4}{a^4}$$



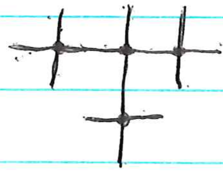
$$\frac{1}{2\cdot 3!\cdot 3!}(-g)^2\frac{J^6}{a^7} = \frac{g^2}{72}\frac{J^6}{a^7}$$



$$\frac{1}{2\cdot(3!)^2\cdot 2}(-g)^3\frac{J^8}{a^{10}}$$

Notice in fourth order for g we get the two kinds of "butane"

$$\frac{1}{2(3!)^2 2^2}(-g)^4 \frac{J^{10}}{a^{13}}$$



$$\frac{1}{3!\,(3!)^3}(-g)^4 \frac{J^{10}}{a^{13}}$$

One moral from the above calculation is that one can express the inverse under composition of a formal power series as a diagram sum. Thus to invert

$$J = x + \frac{b}{2!}x^2 + \frac{c}{3!}x^3 + \cdots$$

we integrate to get

$$f = \frac{1}{2}x^2 + \frac{b}{3!}x^3 + \frac{c}{4!}x^4 + \cdots$$

so that $f'(x) = J$. Let $F = Jx - f(x)$ be the Legendre transform of f. We know that

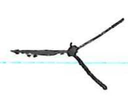
$$F(J) = \text{tree diagram sum belonging to } \int e^{-f(x)+Jx}\,dx$$

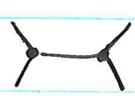
and $\dfrac{dF}{dJ} = x$, which gives x as a power series in J. The tree diagrams are obtained from

$$-1 = -\tfrac{1}{2}k_1 + \tfrac{1}{2}k_3 + k_4 + \cdots \qquad k_1 = 2 + k_3 + 2k_4 + 3k_4$$

so the first few ones are

$k_1 = 2$ $\frac{1}{2}J^2$

$k_1 = 3$  $-\frac{1}{3!}bJ^3$

$k_1 = 4$  $-\frac{1}{4!}cJ^4$  $\frac{1}{8}b^2J^4$

hence $$F(J) = \frac{1}{2}J^2 - \frac{b}{3!}J^3 + \left(-\frac{c}{4!} + \frac{b^2}{8}\right)J^4 + O(J^5)$$

and $$x(J) = J - \frac{b}{2}J^2 + \left(-\frac{c}{3!} + \frac{b^2}{2}\right)J^3 + O(J^4)$$

If $k_1 = 5$ we have three tree graphs

$k_5 = 1$

$k_3 = k_4 = 1$

$k_3 = 3$

Next let's try to understand graphs with 1-loop

$$Z(J) = \int e^{\lambda\left(-\frac{1}{2}ax^2 - \frac{1}{3!}bx^3 - \frac{1}{4!}cx^4 + Jx\right)}\,dx$$

$$= \frac{\sqrt{2\pi}}{\sqrt{\lambda a}}\, e^{\text{all connected graphs}} \simeq e^{\lambda F(J)} \frac{\sqrt{2\pi}}{\sqrt{\lambda f''(x_c)}}\left(1 + O\left(\tfrac{1}{\lambda}\right)\right)$$

Therefore it must be the case that

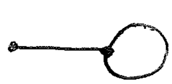
$$e^{\sum \text{all connected 1 loop diagrams}} = \sqrt{\frac{a}{f''(x_c)}}$$

or

$$\sum \text{all connected 1-loop diagrams} = -\frac{1}{2}\log\left(\frac{f''(x_c)}{a}\right)$$

For one loop we have $k_1 = k_3 + 2k_4 + 3k_4$

$k_1 = 1$

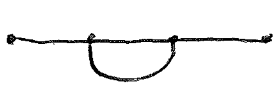


$k_3 = 1$ $\qquad \frac{1}{2}(-b)\frac{J}{a^2}$

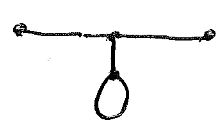
$k_1 = 2$

$k_4 = 1$ $\qquad \frac{1}{4}(-c)\frac{J^2}{a^3}$

$k_1 = k_3 = 2$



$\frac{1}{4}b^2\frac{J^2}{a^4}$



$\frac{1}{4}b^2\frac{J^2}{a^4}$

Check:

$$F = \frac{1}{2}\frac{J^2}{a} - \frac{b}{6}\frac{J^3}{a^3} \qquad f(x) = \frac{1}{2}ax^2 + \frac{1}{3!}bx^3$$

$$x = \frac{J}{a} - \frac{bJ^2}{3a^2} \qquad f''(x) = a + bx$$

$$\log \frac{f''(x)}{a} = \log\left(1 + \frac{b}{a}\frac{J}{a}\right) = \frac{bJ}{a^2}$$

so it works to first order.