Program: To understand the quantum mechanics of
\[ H = \frac{p^2}{2} + V(x) \]
where \( V \) looks like \( V = \frac{1}{2} \omega^2 x^2 + g x^4 \). The first problem is to compute the ground state energy. This problem is accessible by several methods I have encountered - better I have seen several ways of treating this problem.

1) The most elementary is to write
\[ H = \frac{p^2}{2} + \frac{1}{2} \omega^2 q^2 + g \delta \theta \]
and to use power series
\[ \psi = \psi_0 + g \psi_1 + g^2 \psi_2 + \ldots \]
\[ E = \frac{1}{2} \omega + g E_{11} + \ldots \]
and grind out the coefficients requiring that \( \psi_1, \psi_2, \ldots \) be orthogonal to \( \psi_0 \). For example one gets
\[ E^{(i)} = \langle \psi_0 | H' | \psi_0 \rangle \quad \text{if} \quad \langle \psi_0 | \psi_0 \rangle = 1 \]

2) Katz's method: The projection operator on the ground state is given by
\[ P_0 = \frac{1}{2\pi i} \oint \frac{d\lambda}{\lambda - H_0} \]
where the contour circles just the ground eigenvalue. We're assuming the ground state evolves nicely under the perturbation, so that at least for small \( g \) the analogous formula for \( P \) holds, but with the same contour. Then if we use the series for the resolvent:
\[ \frac{1}{\lambda - H} = \frac{1}{\lambda - H_0} + \frac{1}{\lambda - H_0} H' \frac{1}{\lambda - H_0} + \cdots \]

we get
\[ P = P_0 + \frac{1}{2\pi i} \oint \frac{1}{\lambda - H_0} H' \frac{1}{\lambda - H_0} d\lambda + \cdots \]

A similar sort of series can be found for the ground
energy
\[ E = tr(HP). \]

3) Hell-Mann-Low thm. gives the new ground
state by adiabatic perturbation: Let \( \psi_\varepsilon(t) = \text{soln of} \]
\[ i \frac{d}{dt} \psi_\varepsilon(t) = \left( H_0 + e^{i\varepsilon t} V \right) \psi_\varepsilon(t) \]
which is asymptotic to \( e^{-iH_0 t} \psi_0 \) as \( t \to \pm \infty \). Then
\[ \psi = \lim_{\varepsilon \to 0} \frac{\psi_\varepsilon(0)}{\langle \psi_0 | \psi_\varepsilon(0) \rangle} \]
is the ground state for \( H \). There's another formula
for \( E_\varepsilon \) and using it one can derive Goldstone's thm.
asserting that \( E \) is given by a sum of connected
Feynman diagrams. (Fetter-Walecka book).

4)  
If we put \( H_\lambda = H_0 + \lambda \mathcal{H} \)
and denote by \( \psi_\lambda \) the ground state of \( H_\lambda \) normalized so
that \( \| \psi_\lambda \|^2 = 1 \), then one has for \( E_\lambda = \langle \psi_1 | H_\lambda | \psi_\lambda \rangle \), the formula
\[ \frac{dE_\lambda}{d\lambda} = \langle \psi_\lambda | H' | \psi_\lambda \rangle \]
by first order perturbation theory. Hence integrating
\[ E = \int \langle \psi_\lambda | H' | \psi_\lambda \rangle d\lambda \]
5) Functional integral approach. Although not essential for what follows, let's start off with the thermal partition function:

\[ Z(\beta) = \text{tr}(e^{-\beta H}) \]

which can be expressed as a path integral

\[ Z(\beta) = \int dx(t) e^{-\frac{1}{2} \beta \dot{x}^2 + \frac{1}{2} \omega^2 x^2 + V(x)} \] \( x(0) = x(\beta) \)

Now split \( V(x) \) into \( \frac{1}{2} \omega^2 x^2 + W(x) \) and we get

\[ Z(\beta) = \int dx(t) e^{-\frac{1}{2} \beta \dot{x}^2 + \frac{1}{2} \omega^2 x^2} e^{-\int W(x) dt} \]

Introduce

\[ Z_0(\beta, J) = \int dx(t) e^{-\frac{1}{2} \beta \dot{x}^2 + \frac{1}{2} \omega^2 x^2 - \int J(t) x(t) dt} \]

Using the formula

\[ f(\frac{1}{i} \frac{d}{dJ}) \int e^{i J x} d\mu(x) = \int f(x) e^{-i J x} d\mu(x) \]

we see that

\[ Z(\beta, J) = e^{-\int W(\frac{1}{i} \frac{d}{dJ}) dt} Z_0(\beta, J) \]

Now \( Z_0(\beta, J) \) is the Fourier transform of a Gaussian measure so we have

\[ Z_0(\beta, J) = e^{-\frac{1}{2} \int J(t) G(t, t') J(t') dt dt'} Z_0(\beta) \]

where \( G \) is the inverse of \( -\frac{d^2}{dt^2} + \omega^2 \) on periodic functions on \([0, \beta]\). Also we know \( Z_0(\beta) = \text{tr}(e^{-\beta H_0}) \)

\[ Z_0(\beta) = \sum e^{-\beta (n + \frac{1}{2}) \omega} = \frac{1}{2 \sinh(\frac{\beta \omega}{2})} \]

but maybe this isn't very important.
The following relevant:

\[ Z_0(\beta) = \int Dx(t) e^{-\int \left( \frac{1}{2} \dot{x}^2 + \frac{1}{2} \omega^2 x^2 \right) dt} \frac{\det (-\partial_t^2 + \omega^2)}{\det (-\partial_t^2)} \]

where the ratio of determinants is to be evaluated on periodic functions of period \( \beta \). One has the basis \( e^{\frac{in\pi x}{\beta}} \) of eigenfunctions so

\[
\frac{\det (-\partial_t^2 + \omega^2)}{\det (-\partial_t^2)} = \prod_{n \in \mathbb{Z}} \left( \frac{(\frac{n^2 \pi^2}{\beta})^2 + \omega^2}{(\frac{n^2 \pi^2}{\beta})^2} \right) = \frac{(\frac{2\pi}{\beta})^2 + \omega^2}{(\frac{2\pi}{\beta})^2} \left( \frac{\sinh \left( \frac{\beta \omega}{2} \right)}{\frac{\beta \omega}{2}} \right)^2
\]

but there is some trouble at \( n = 0 \). No missing from (4) is \( \int Dx(t) e^{-\int \frac{1}{2} \dot{x}^2} = \text{tr} (e^{-\beta \frac{\partial^2}{2}}) \) which is an infinity cancelling the \( n = 0 \).

Returning to the derivation at hand we have

\[
\frac{Z(\beta)}{Z_0(\beta)} = e^{-\int W(t) dt} e^{-\frac{1}{\beta} \int\mathbb{J} G \mathbb{J}}
\]

where in this formula \( W(x) = g x^4 \) is the interaction and \( G \) is the periodic Green's func for \(-\partial_t^2 + \omega^2\) of period \( \beta \). Now the right side has a diagram interpretation, and the logarithm of the right side has a connected diagram interpretation. Now recall that

\[
Z(\beta) = e^{-\beta F}
\]

where \( F \) is the "free energy", hence the shift in free energy has a connected diagram interpretation. Finally letting \( \beta \to \infty \)

\[
\frac{Z(\beta)}{Z_0(\beta)} \approx \frac{e^{-\beta E}}{e^{-\beta E_0}}
\]
so that we get a connected diagram interpretation of the ground state energy shift. This is Goldstone's theorem.

November 5, 1979

This time instead of the partition function $\text{tr}(e^{-H})$ let us work with the amplitude

$$\langle x | e^{-TH} | x' \rangle = \int dx(t) e^{-\int_{\frac{T}{2}}^{T} x(t)^2 + V(x(t)) dt} \quad \frac{x(T)}{2} = x' \quad \frac{x(0)}{2} = x$$

Again we expand

$$V(x) = \frac{1}{2} \omega^2 x^2 + W(x)$$

and then we have

$$\langle x | e^{-TH} | x' \rangle = e^{-\int W \left( \frac{\partial}{\partial x} \frac{1}{2} \frac{\partial}{\partial x^*(t)} \right) dt} \int dx(t) e^{-\int_0^T \left( \frac{1}{2} x^2 + \omega^2 x^2 \right) dt + i \int_0^T x(t) dx(t)}$$

set $J = 0$

To simplify suppose $x = x' = 0$, so that the latter integral is a Gaussian which we can evaluate as

$$e^{-\frac{1}{2} \int_0^T G(t,t') \int_0^T \left( \frac{1}{2} x^2 + \omega^2 x^2 \right) dt + i \int_0^T x(t) dx(t)}$$

where $G$ is the Green's function for $-\partial_t^2 + \omega^2$ on $[-\frac{T}{2}, \frac{T}{2}]$ with 0 endpoint conditions.

Now the idea is to let $T \to \infty$ in which case

$$G \sim e^{-\omega |t-t'|}$$

and

$$\sqrt{\frac{\omega}{2\pi \sinh(\omega T)}} \approx \sqrt{\frac{\omega}{\pi}} e^{-\frac{\omega}{2} T}$$

To see what's happening we should take an example
\[ W = \frac{1}{2} \dot{x}^2. \] Then we get something like
\[ e^{-\int W(t) dt} e^{-\frac{1}{2} \int \frac{\partial S}{\partial \delta} \delta dt} \Bigg|_{t=0} = \sum \text{diagrams} \]

Return to computing \( \text{tr} (e^{-\beta H}) \), only let's work out the diagrams in energy coordinates, rather than time. So the idea here is to describe periodic functions \( x(t) \) of period \( \beta \) as Fourier series
\[ x(t) = \sum \alpha_k e^{ikt} \quad k \epsilon \frac{2\pi}{\beta} \mathbb{Z} \]
\[ \alpha_k = \alpha_{-k} \]
\[ \int_0^\beta \frac{1}{2} \dot{x}^2 + \frac{1}{2} \omega^2 x^2 \, dt = \frac{1}{2} \int_0^\beta x(- \dot{x}^2 + \omega^2) \, dt \]
\[ = \beta \frac{1}{2} \sum_k (k^2 + \omega^2) |\alpha_k|^2 \]
\[ \int_0^\beta x^4 \, dt = \beta \sum_{k_1, k_2, k_3, k_4} a_{k_1} a_{k_2} a_{k_3} a_{k_4} \delta (k_1 + \cdots + k_4) \]
so our path integral is
\[ \int [da_0, [a_k, da_k]] \, e^{-\beta \left( \frac{1}{2} \sum_k (k^2 + \omega^2) |\alpha_k|^2 + \beta \sum a_{k_1}^* a_{k_2} a_{k_3} a_{k_4} \delta (k_1 + \cdots + k_4) \right)} \]
As before we reduce this to a Gaussian integral
\[ \int [da] \, e^{-\beta \left( \frac{1}{2} \sum_k (k^2 + \omega^2) |\alpha_k|^2 \right) + \frac{i}{\beta} \sum_k \partial_{\delta k} \alpha_k} \]
being acted on by \( e^{-\beta \sum \left( \frac{\partial}{\partial \delta k_1} \cdots \frac{\partial}{\partial \delta k_4} \right) \delta (k_1 + \cdots + k_4)} \)
The problem is to understand the connected generating functional. This arises as follows. Suppose we have a Hamiltonian \( H = \frac{p^2}{2} + V(x) \) and we want to compute the amplitude
\[
\langle x | e^{-TH} | x' \rangle = \int Dx(t) e^{-\int_0^T \left( \frac{1}{2} \dot{x}^2 + V(x) \right) dt}
\]
subject to \( x(0) = x' \) and \( x(T) = x \).

We do a stationary phase approximation on the path integral. To simplify suppose that \( x = x' = 0 \) and that \( V \) has an absolute minimum at \( x = 0 \), say
\[
V(x) = \frac{1}{2} \omega^2 x^2 + O(x^3)
\]
Then
\[
\langle 0 | e^{-TH} | 0 \rangle = \int Dx(t) e^{-\int_0^T \left( \frac{1}{2} \dot{x}^2 + \frac{1}{2} \omega^2 x^2 \right) dt} e^{-\int V(x) dt}
\]
subject to \( x(0) = x(T) = 0 \).

The last exponential factor is expanded out in a series; what we get is a series of integrals of the form \( \int x^2 \, dx \), \( dx \) is a Gaussian measure which can be evaluated by Wick's theorem. This leads to a big series for \( \langle 0 | e^{-TH} | 0 \rangle \) whose terms are described by Feynman diagrams. Moreover
\[
\log \left( \frac{\langle 0 | e^{-TH} | 0 \rangle}{\langle 0 | e^{-TH} | 0 \rangle} \right) = e^W
\]
where \( W \) is a sum of connected diagrams.

Let's consider a simplified situation in which the space of paths is replaced by \( \mathbb{R} \), and where the
The path integral is
\[ \int e^{-f(x)} \, dx \]
where \( f(x) \) has an absolute minimum at \( x = 0 \), and say
\[ f(x) = \frac{1}{2} \omega^2 x^2 + \frac{g_3}{3!} x^3 + \frac{g_4}{4!} x^4 + \ldots \]
Then
\[ \int e^{-f(x)} \, dx = \int e^{-\frac{1}{2} \omega^2 x^2} \, dx \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \left( \frac{g_3}{3!} x^3 + \frac{g_4}{4!} x^4 + \ldots \right)^n \]
Let's take a typical term when the exponential series is expanded out, for example the term involving \( g_3 \) which is
\[ \frac{(-1)^2}{2!} \cdot 2 \frac{g_3 g_5}{3! 5!} \int e^{-\frac{1}{2} \omega^2 x^2} x^3 x^5 \, dx \]
According to Wick's theorem the integral is the sum over all possible pairwise contractions. We can represent these ways of contracting by drawing edges

\[ \text{Diagram} \]

The total number of contractions possibilities is \( \frac{8}{2^4} = 1 \cdot \frac{3 \cdot 5 \cdot 7}{4!} \). Somehow one has to count the symmetries of the graph. The goal is collect together the different contraction possibilities leading to the same graph. In the above example we can permute the lines issuing from the vertices. This gives us a group of order \( 3! \cdot 5! \) acting on the contraction possibilities. This \( 3! \cdot 5! \) cancels the denominator, so the net effect is that each graph is counted with a factor
of the order of its symmetry group.

So the general rule is as follows. Suppose we want the coefficient of \( g_3 g_4 \cdots g_n \). Then we consider all graphs with \( a \)-vertices of mult. 3, \( b \)-vertices of mult. 4, etc. We let \( \Sigma_a \times \Sigma_b \times \cdots \) act as well as \( \Sigma_3 \) on each 3-fold vertex, \( \Sigma_4 \) on each 4-fold vertex, etc. and identify equivalent graphs. Finally we have for each inequivalent graph a term

\[
\frac{(1)^n}{\text{Aut}(\text{graph})} \left( \int e^{-\frac{1}{2} \omega^2 x^2} e^{\frac{x^2}{2} \omega^2} \right)^n \sqrt{\frac{\pi}{2}} \frac{1}{\omega^3} \quad \text{not quite correct.}
\]

Correction:

\[
\int e^{-\frac{1}{2} \omega^2 x^2} x^{2m} \, dx = \left( \frac{d}{dx} \right)^{2m} \int e^{-\frac{1}{2} \omega^2 x^2 + i J x} \, dx
\]

\[
= (1)^m \left( \frac{d}{dJ} \right)^{2m} \left( e^{-\frac{1}{2} \omega^2 x^2 \sqrt{2\pi}} \right) \bigg|_{J=0}
\]

\[
= (1)^m \left( \frac{d}{dJ} \right)^{2m} \sum_{m} (1)^m \frac{J^{2m}}{m!} \bigg|_{J=0} = \frac{\sqrt{2\pi}}{\omega^{2m}} \frac{(2m)!}{2^m m!}
\]

\[
= \frac{\sqrt{2\pi}}{\omega^{2m}} \frac{1}{\omega^{2m}} \frac{(2m-1)(2m-3) \cdots 3 \cdot 1}{2^m m!}
\]

number of possible contractions in \( x^{2m} \).

Hence the contribution of a given graph is

\[
\prod g_j \frac{(1)^n}{\text{Aut}(\text{graph})} \frac{1}{\omega^m} \frac{\sqrt{2\pi}}{\omega}
\]

\( n = \) number of vertices

\( m = \) number of edges

\( \prod \) product over the edges

\( \frac{1}{\text{Aut}(\text{graph})} \) normalizer factor for the

\( \frac{\sqrt{2\pi}}{\omega} \) Gaussian integral which is 1 for \( \frac{1}{\omega^m} \).
November 9, 1979

We are trying to understand the diagram expansion for an integral of the form

\[ \int d\mu \ e^{\sum \frac{g}{\lambda^4} x^x} \]

where \( d\mu \) is a Gaussian measure on the space with coordinates \( x_i \). Let's first work out the contractions

\[ \int e^{-\frac{1}{2}(x, Ax)} \ e^{iJx} \ dx = (2\pi)^{n/2} (\det A)^{-1/2} \ e^{-\frac{1}{2}(J, A^{-1}J)} \]

\[ \int \ e^{-\frac{1}{2}(x, Ax)} \ \frac{dx}{(2\pi)^{n/2} (\det A)^{1/2}} \ e^{Jx} = e^{\frac{1}{2}(J, A^{-1}J)} \]

Hence

\[ \int d\mu \ x_i x_j = \frac{\partial^2}{\partial J_i \partial J_j} \ e^{\frac{1}{2}(J, A^{-1}J)} \bigg|_{J=0} = (A^{-1})_{ij} \]

Wick's theorem says that if \( I = (i_1, \ldots, i_n) \) and \( x_I = x_{i_1} \cdots x_{i_n} \), then

\[ \int d\mu \ x_I = \text{sum over all possible pairwise contractions of } I \text{ of the products of the contracted factors. The number of these is } (2n-1)!! = 1 \cdot 3 \cdot 5 \cdots (2n-1). \]

Yesterday I found it advantageous to write

\[ \sum_{x} \ \frac{g}{\lambda^4} x^x = \sum_{I} \frac{g_I}{|I|!!} x_I \]

where \( I \) runs over all finite sequences \( i_1, \ldots, i_n \) in the variables. Then the exponential to be
\[
\sum_{\text{all } I_i} \frac{g_{I_1} \cdots g_{I_n}}{|I_1|! \cdots |I_n|!} x_{I_1} \cdots x_{I_n} = \sum_n \frac{1}{n!} \left( \sum \frac{g_{I_1}}{|I_1|!} x_{I_1} \right)^n
\]

This means that the integral \( \int dq e^{\sum_{\text{all } I_i} \frac{g_{I_1} \cdots g_{I_n}}{|I_1|! \cdots |I_n|!} x_{I_1} \cdots x_{I_n}} \) is a sum of terms each described by a finite sequence \((I_1, \ldots, I_n)\) of finite sequences of variables together with all ways of pairwise contracting all the variables. Such a term can be drawn as a diagram

![Diagram](image)

Such a diagram receives a weight of \( \frac{1}{n!} \frac{1}{|I_1|!} \cdots \frac{1}{|I_n|!} \) but now to take the sum over all such labeled diagrams is very inefficient, because in computing the q-factors and the contraction factors one doesn't care about the ordering of \( I_1, I_2, \ldots, I_n \), nor the ordering of the variables inside each \( I_j \). Consequently we group together equivalent diagrams. Two graphs are equivalent when there is a one-one correspondence between vertices and between edges which is compatible with the variables attached to the ends of each edge. The problem is to count the number of labeled graphs in each equivalence class.
Start again. A point you're missing is that one draws a graph first then adds all terms resulting from labelling the edges issuing from a vertex. This amounts to selecting \( n \), the number of vertices, and \( p_1, p_2, \ldots, p_n \), the multiplicity of each vertex, and then one of the \((2n-1)!!\) ways of contracting. Once this data is given you then sum over all possible ways of putting a variable at the end of each edge. So now each of these sums is weighted by \( \frac{1}{n!} \frac{1}{p_1} \cdots \frac{1}{p_n} \).

The problem is to decide when two sums are the same. For example

obviously contribute the same. The thing to do is look at isomorphic graphs and to figure out how many times a graph type contributes, and use this to sum once over each graph type with a modified symmetry factor. So we want to look at "contraction diagrams" leading to the same graph as above. One has an obvious group of symmetries: one can permute the lines issuing from a vertex, and one can permute vertices of the same multiplicity to arrange vertices in order of increasing multiplicities \( p_1 < p_2 < \cdots < p_n \). This means we write

\[
\sum e^{\sum \frac{g_i}{i!}} x^i \prod_{p} \frac{1}{p!} \sum_{l=0}^{p_i} \frac{g_i^l}{l!} x_i^l
\]

\[
= \sum_{k_1, k_2, \ldots} \frac{1}{k_1!} \frac{1}{k_2!} \cdots \left( \frac{1}{1!} \sum_{l=1}^{g_1} x_i^l \right)^{k_1} \left( \frac{1}{2!} \sum_{l=2}^{g_2} x_i^l \right)^{k_2} \cdots
\]
Once we agree that graphs have the multiplicity of their vertices fixed in order, then we have the wreath product \( \Sigma_k \times \Sigma_p \) acting on the vertices of multiplicity \( p \). In the expansion

\[
\sum_{i=1}^{\infty} \frac{g^i}{i!} = \sum_{k_1, k_2, \ldots} \frac{1}{k_1! (k_1)!} \frac{1}{k_2! (k_2)!} \cdots \left( \sum_{i=1}^{k_1} \frac{g^i}{i!} \right) \left( \sum_{i=1}^{k_2} \frac{g^i}{i!} \right) \cdots
\]

Each way of contracting is weighted by dividing by the order of the symmetry group

\[
\left( \sum_{k_1} \cdots \sum_{k_n} \right) \times \left( \sum_{k_1} \cdots \sum_{k_n} \right) \times \cdots
\]

associated to vertex multiplicities. So it follows we can sum inequivalent graphs provided we divide by the order of the symmetry group of the graph.

**Example:** Take 1 variable and compute

\[
\int d\mu \, e^{g \frac{x^4}{4!}}
\]

where \( d\mu = e^{-\frac{x^2}{2}} \frac{dx}{\sqrt{2\pi}} \)

\[
= \int d\mu \, \sum \frac{g^n}{n! (4!)^n} = \sum \frac{g^n}{n!} \frac{(4n-1)!!}{(4!)^n} \left( \frac{1}{\omega^2} \right)^{2n} \frac{1}{(4!)^n}
\]

**First order graphs:** There's only one which has

\( 2 \cdot 2 \cdot 2 = 2^3 \) symmetries.

\[
\frac{9}{8} \left( \frac{1}{\omega^2} \right)^2 \quad \frac{9}{1!} \cdot \frac{1}{2} \cdot \frac{1}{3} \cdot \frac{1}{4!} \left( \frac{1}{\omega^2} \right)^{2} \frac{1}{4!} = \frac{9}{2^4} \left( \frac{1}{\omega^2} \right)^{2}
\]

**2nd order graphs**

\[
2 \cdot 2 \cdot 2 = 2^3 \\
2^4 \\
2 \cdot (2^3)^2 = 2^7
\]
\[ g^2 \left( \frac{1}{2^{1/3}} + \frac{1}{2^{1/4}} + \frac{1}{2^{1/2}} \right) = \frac{g^2}{2^{7/3}} \left( \frac{2^3 + 3^3 + 3}{2^1 + 4^1 + 3} \right) = \frac{g^2}{2^{7/3}} \frac{5 \cdot 7}{2^{7/3}} \]

Other formulas:
\[ \frac{g^2}{2^1}, \frac{1.3 \cdot 5 \cdot 7}{\omega^8}, \frac{1}{(4!)^2}, \frac{\frac{5 \cdot 7}{2^{7/3}}}{2^2 \cdot 3^2 \cdot 2^1} \]

so it checks.

Important: these graphs as well as their symmetry factors are completely independent of the number of variables. The variables enter when you label the ends of the edges and compute the contraction factors.

Next consider the example of finding the ground state for
\[ H = \frac{p^2}{2} + \frac{1}{2} \omega^2 q^2 + q^4 \frac{\dot{q}}{} \]

We want to compute the behavior of
\[ \langle x(t) \rangle = \int dx(t) e^{-iHt} \]

as \( T \to \infty \) where the path integral is taken over periodic paths \( x(t) \) of period \( T \). Expand in Fourier series
\[ x(t) = \sum_{k} \alpha_{k} e^{ikt} \quad k \in \frac{2\pi}{T} \mathbb{Z} \]

The fact \( x \) is real-valued signifies that \( \alpha_{k} = \alpha_{-k} \).

Now one has:
\[ \int \left( \frac{1}{2} \ddot{x}^2 + \frac{1}{2} \omega^2 x^2 \right) dt = \sum_{k} \frac{1}{2} (k^2 + \omega^3) |\alpha_{k}|^2 \cdot T \]

\[ \int \frac{x(t)^4}{4!} dt = \sum_{k_1, k_2, k_3, k_4} \frac{1}{4!} \alpha_{k_1} \alpha_{k_2} \alpha_{k_3} \alpha_{k_4} \delta(k_1 + \cdots + k_4) \cdot T \]
When we expand \( e^{-g \sum_{k} \frac{1}{k^2 + \omega^2}} \) in powers of \( g \) we get a sum of terms involving integrals,

\[
\int d\mu \ a_k \ldots a_n
\]

where \( d\mu \) is an appropriate measure, essentially

\[
e^{-\frac{1}{2} \sum_{k} (k^2 + \omega^2) |a_k|^2} \frac{d\omega_0}{\sqrt{2\pi}} \prod_{k=1}^{\infty} \frac{d\omega_k \ d\tilde{\omega}_k}{2\pi}
\]

Let's use the generating function

\[
\int e^{-\frac{1}{2} \sum_{k} (k^2 + \omega^2) |a_k|^2 + \sum \tilde{J}_k a_k} \frac{d\omega_0}{\sqrt{2\pi}} \prod_{k=1}^{\infty} \frac{d\omega_k \ d\tilde{\omega}_k}{2\pi}
\]

where the \( \tilde{J}_k \) are independent variables. I need

\[
\int e^{-\frac{1}{2} x^2 |z|^2 + \tilde{J} z + \tilde{\omega} \bar{z}} \frac{dx \ dz \ d\tilde{z}}{2\pi}
\]

\[
= \int e^{-\frac{1}{2} x^2 (x^2 + y^2) + (\tilde{J} + \tilde{\omega}) x + \tilde{z} (\bar{J} - \bar{\omega})} \frac{dx \ dy \ d\tilde{x} \ d\tilde{y}}{2\pi}
\]

\[
= e^{\frac{1}{2} \frac{1}{2x^2} \left\{ (\tilde{J} + \tilde{\omega})^2 - (\bar{J} - \bar{\omega})^2 \right\} + 2\tilde{z} \tilde{J} + 2\tilde{\omega} \bar{\omega}}
\]

Hence the generating function is

\[
\frac{1}{2} \omega_0 \tilde{J}_0^2 + \frac{1}{2} \sum_{k} \frac{1}{k^2 + \omega^2} \tilde{J}_k \tilde{J}_{-k}
\]

\[N e \]
November 10, 1979

Review: I want to compute the first few terms of the perturbation expansion of
\[ \int D\mathbf{x} \ e^{i S[\mathbf{x}]} \]
where the integral is taken over periodic paths of period \( T \). We describe such paths by Fourier series
\[ x(t) = \sum a_k e^{i k t} \quad k \in \frac{2\pi}{T} \]
where \( x \) real means \( \overline{a_k} = a_{-k} \). We use this formula to change variables in the path integral. One has
\[ \int \frac{1}{2} x^2 + \frac{1}{2} \omega^2 x^2 \ dt = \frac{1}{2} \sum_k (k^2 + \omega^2) |a_k|^2 \]
hence up to a normalization factor \( N \)
\[ D\mathbf{x} e^{-\frac{1}{2} \int x^2 + \frac{1}{2} \omega^2 x^2} \longrightarrow N e^{-\frac{i}{2} \sum (k^2 + \omega^2) |a_k|^2} \frac{d\omega}{2\pi} \prod_{k > 0} da_k \]
We need
\[ \int e^{-x^2/2 l^2} \ dx + J \int e^{-y^2/2 l^2} \ dy = \int e^{-x^2/2 l^2} (x^2 + y^2) \ dx \ dy = \frac{(\sqrt{2\pi})^2}{(\sqrt{2\pi})^2} e^{-\frac{1}{2} l^2 \left((J + \overline{J})^2 - (J - \overline{J})^2\right)} = \frac{\pi}{l^2} e^{-\frac{1}{2} l^2 J \overline{J}} \]
It follows that
\[ \int d\omega \prod_{k > 0} da_k \ d\overline{a}_k e^{-\frac{i}{2} \sum (k^2 + \omega^2) |a_k|^2} + \sum J_k a_k \]
\[ = N \cdot e^{-\frac{1}{2} \sum \frac{1}{(k^2 + \omega^2)^T} J_k J_k} \]
Compute the ground energy for the anharmonic oscillator

\[ H = \frac{p^2}{2} + \frac{1}{2} \omega^2 q^2 + \frac{g}{4!} q^4 \]

through the second order in \( q \). Recall for \( H = H_0 + V \)
with \( H_0 \phi_n = E_n \phi_n \), \( \phi_n \) an orthonormal basis, then
the ground energy for \( H \) is

\[ E = \lambda_0 + \lambda_1 + \lambda_2 + \ldots \]

\[ E_0 = \frac{1}{2} \omega \]

\[ \lambda_1 = \langle \phi_0 | V | \phi_0 \rangle \]

\[ \lambda_2 = \sum_{n \neq 0} \frac{1}{E_n - E_0} |\langle \phi_n | V | \phi_0 \rangle|^2 \]

\[ E_n = (n + \frac{1}{2}) \omega \]

In the case \( V = \frac{g}{4!} q^4 \) we have

\[ \lambda_1 = \frac{g}{4!} \langle 0 | q^2 | 0 \rangle = \frac{g}{4!} \left( \frac{1}{2 \omega} \right)^2 = \frac{g}{8 \omega^2} \]

since

\[ \langle 0 | q^2 | 0 \rangle = \langle 0 | \left( \frac{(a + a^*)^2}{2i \omega} \right) | 0 \rangle = \frac{1}{2 \omega} \langle 0 | a a^* + a^* a | 0 \rangle = \frac{1}{2 \omega} \]

Now for the harmonic oscillator \( \phi_n = \frac{1}{\sqrt{n!}} (a^*)^n | 0 \rangle \), hence

\[ \langle \phi_0 | V | \phi_n \rangle = \frac{1}{\sqrt{n!} \ 4!} \frac{g}{(2 \omega)^2} \langle 0 | (a + a^*)^4 a^n | 0 \rangle \]

When \((a + a^*)^4 \) is put into normal product form we get terms

\[ a^4 + (4) a^3 a^* + (6) a^2 a^2 + 4 a a^3 + a^4 \]

\[ + (\ ) a^2 + (\ ) a^* + (\ ) a^2 + (\ ) \]

hence we only get contributions for \( n = 4 \) from the \( a^4 \)
and \( n = 2 \) from the \( 4 a^3 a^* \) and \( (\ ) a^2 \) terms. So

\[ \langle \phi_0 | V | \phi_4 \rangle = \frac{1}{4!} \frac{g}{4! (2 \omega)^2} \]

\[ = \frac{g}{4! (2 \omega)^2} \]
\[ \langle \varphi_0 | V | \varphi_2 \rangle = \frac{1}{\sqrt{2}} \frac{g}{4!} \frac{1}{(2\omega)^2} \langle 0 \left| \left\{ \begin{array}{c} a_{a_1}^{\dagger} a_{a_2}^{\dagger} a_{a_2} a_{a_1} \\ a_{a_2}^{\dagger} a_{a_1}^{\dagger} a_{a_1} a_{a_2} \\ a_{a_1}^{\dagger} a_{a_2}^{\dagger} a_{a_2} a_{a_1} \\ + a_{a_1} a_{a_2} a_{a_2}^{\dagger} a_{a_1}^{\dagger} \\ + a_{a_2} a_{a_1} a_{a_1}^{\dagger} a_{a_2}^{\dagger} \end{array} \right\} | 0 \rangle \]

\[ (3! + 2\cdot2 + 2) = 12 \]

Hence,

\[ \lambda_2 = \left( \frac{g}{\sqrt{3}} \frac{1}{(2\omega)^2} \right)^2 \left[ \frac{1}{-4\omega} \right] + \left( \frac{g}{\sqrt{2}} \frac{1}{(2\omega)^2} \right)^2 \left[ \frac{1}{-2\omega} \right] \]

\[ = - \frac{g^2}{2(2\omega)^5} \left( \frac{1}{4\cdot2} + \frac{1}{8\cdot6} \right) = \frac{-g^2}{2^8 \omega^5 \cdot \frac{7}{6}} \]

Now let's attack the same problem by Feynman diagrams. We start with the path integral expression for the partition function.

\[ \chi \left( e^{-\frac{T}{\hbar} H} \right) = \int D\Phi \ e^{-\frac{T}{4\hbar} \int \left( \frac{1}{2} \dot{x}^2 + \frac{1}{2} \omega^2 x^2 \right) dt} - \frac{g^2}{2\hbar^2} \int x^4 dt \]

where the paths are periodic of period T. Then one uses Fourier series to change variables in the path integral.

\[ x(t) = \frac{1}{T} \sum a_k e^{ikt} \quad k \in \frac{2\pi}{T} \mathbb{Z} \]

We use this so that as \( T \to \infty \) one has

\[ x(t) = \int a_k e^{ikt} \frac{dk}{2\pi} \]

so

\[ \int \left( \frac{1}{2} \dot{x}^2 + \frac{1}{2} \omega^2 x^2 \right) dt = \frac{1}{2} \sum \left( \frac{k^2 + \omega^2}{T} \right) |a_k|^2 \]

\[ \int x^4 dt = \frac{1}{T^3} \sum a_{k_1} a_{k_2} a_{k_3} a_{k_4} \delta(k_1 + k_2 + k_3 + k_4) \]

\[ \to \int \frac{dk_1 dk_2 dk_3 dk_4}{(2\pi)^3} a_{k_1} a_{k_2} a_{k_3} a_{k_4} \delta(k_1 + k_2 + k_3 + k_4) \]
Our partition function becomes
\[ N \int \left( d\varphi_0 \prod d\varphi_k d\bar{\varphi}_k \right) e^{-\frac{1}{2} \sum_{k} \frac{\hbar^2 \omega^2}{k^2 + \omega^2} / \alpha_k^2} e^{-\frac{\hbar T}{\hbar^2 + \omega^2} \sum \varphi_k \bar{\varphi}_k} \delta(k_1 + \cdots + k_4) \]

by yesterday's analysis we know that the non-zero contractions occur between the variables \( \varphi_k, \bar{\varphi}_k \) and lead to the "propagator" factor \( \frac{\hbar T}{k^2 + \omega^2} \).

First order diagrams: There's only one:

and its contribution to the partition function is

\[-\frac{g^2 \hbar}{8 T^3} \sum_{k_1, k_2} \frac{\hbar T}{k_1^2 + \omega^2} \frac{\hbar T}{k_2^2 + \omega^2} \]

\[= -T \left( \frac{g^2 \hbar}{8} \sum \left( \frac{1}{k_1^2 + \omega^2 \frac{1}{T}} \right) \left( \frac{1}{k_2^2 + \omega^2 \frac{1}{T}} \right) \right) \]

\[\xrightarrow{T \to \infty} \frac{g^2 \hbar}{8} \left( \int \frac{dk}{(k^2 + \omega^2) 2\pi} \right)^2 \]

\[\left[ \frac{1}{2\pi} \frac{1}{\omega \arctan \frac{k}{\omega} \left]^{+\infty}_{-\infty}} = \frac{1}{2\omega} \right. \]

so the first order correction to the ground energy is

\[ \lambda_1 = \frac{g^2 \hbar}{8 \left( \frac{1}{2\omega} \right)^2} \]

There are 2 second order diagrams

\[2.4! \]

The former gives the contribution

\[ \frac{1}{2.4!} \left( -\frac{g^2 \hbar}{\hbar^2 + \omega^2} \right)^2 \sum_{k_1, \cdots, k_4} \left( \frac{1}{\pi} \frac{\hbar T}{k_i^2 + \omega^2} \right) \delta(k_1 + \cdots + k_4) \]
\[
T \frac{1}{2.4!} g^2 h^2 \sum_{k_1, \ldots, k_4} \prod_{k_1, \ldots, k_4} \frac{1}{k_1^2 + \omega^2} \delta(k_1 + \ldots + k_4) \frac{1}{T^3}
\]

\[
\sim T \left( \frac{1}{2.4!} g^2 h^2 \int \frac{2 \pi \delta(k_1 + \ldots + k_4)}{(k_1^2 + \omega^2) \cdots (k_4^2 + \omega^2)} \frac{dk_1 \ldots dk_4}{(2\pi)^4} \right)
\]

The latter gives the contribution to the partition function
\[
\frac{1}{16} \left( -\frac{g}{hT^3} \right)^2 \sum \frac{hT}{k_1^2 + \omega^2} \frac{\delta(k_3 + k_4)}{(k_1^2 + \omega^2)(k_2^2 + \omega^2)} \frac{hT}{k_2^2 + \omega^2}
\]

\[
= T \left( \frac{1}{16} g^2 h^2 \sum \frac{1}{(k_1^2 + \omega^2)(k_2^2 + \omega^2)(k_3^2 + \omega^2)^2} \frac{1}{T^3} \right)
\]

\[
\sim T \frac{1}{16} g^2 h^2 \int \frac{1}{(k_1^2 + \omega^2)(k_2^2 + \omega^2)(k_3^2 + \omega^2)^2} \frac{dk_1 \ldots dk_3}{(2\pi)^3}
\]

But
\[
\int_{-\infty}^{\infty} \frac{dk}{(k^2 + \omega^2)^2} = \int_{-\infty}^{\infty} \frac{d\omega \tan \Theta}{\omega^4 \sec^2 \Theta} = \frac{1}{2\pi \omega^3} \int_{-\pi/2}^{\pi/2} \cos^2 \Theta = \frac{1}{2\pi \omega^3} \frac{\pi}{2} = \frac{1}{4\omega^3}
\]

So the contribution is
\[
\sim T \frac{1}{16} g^2 h^2 \frac{1}{4\omega^3} \frac{1}{(2\omega)^2} = T \left( \frac{g^2 h^2}{8(2\omega)^5} \right)
\]

It remains to evaluate
\[
\int \frac{2 \pi \delta(k_1 + \ldots + k_4)}{(k_1^2 + \omega^2)} \prod_{k_1, \ldots, k_4} \frac{dk_1}{2\pi}
\]

\[
\int \frac{dk_3/2\pi}{(k_3^2 + \omega^2)((k_3 + k_1 + k_2)^2 + \omega^2)} = \int \frac{dx}{2\pi} \frac{1}{(x^2 + \omega^2)((x-b)^2 + \omega^2)}
\]

\[
= 2\pi i \left[ \text{Res}_{i\omega} + \text{Res}_{b+i\omega} \right]
\]

\[
= i \left[ \frac{1}{2i\omega((\omega-b)^2 + \omega^2)} + \frac{1}{(b+i\omega)^2 + \omega^2)(2i\omega)} \right]
\]

\[
= \frac{1}{2\omega^2 - 2i\omega b + b^2}
\]
\[
\frac{1}{2\omega} \left( \frac{1}{b^2 - 2i\omega b} + \frac{1}{b^2 + 2i\omega b} \right) = \frac{1}{2\omega} \left( \frac{1}{b - 2i\omega} + \frac{1}{b + 2i\omega} \right)
\]
\[
= \frac{1}{2\omega b} \frac{2b}{b^2 + 4\omega^2} = \frac{1}{\omega} \frac{1}{b^2 + 4\omega^2} = \frac{1}{\omega (b^2 + 4\omega^2)}
\]

It seems that the integrations are easier if done over time. So go back to
\[
\int \frac{dx}{\sqrt{x^2 + \frac{1}{4} \omega^2 x^2}} = \frac{2}{\sqrt{8}} \int \frac{dx}{\sqrt{x^2 + \frac{1}{4} \omega^2}}
\]
and do the diagrams directly thinking of there being one variable \(x(t)\) for each \(t\). Recall that once the diagram is given we then add up over all ways of assigning variables to the ends of the edges. So in 2nd order we have

\[
\begin{align*}
\frac{1}{2\cdot 4!} \left( \frac{-i}{\hbar} \right)^2 & \int \left[ \frac{h G(t_1, t_2)}{h} \right]^4 dt_1 dt_2 \\
\frac{1}{16} \left( \frac{-i}{\hbar} \right)^2 & \int \left[ \frac{h G(t_1, t_1) h G(t_1, t_2)}{h} \right]^2 h G(t_2, t_3) dt_1 dt_2
\end{align*}
\]

Notice that with periodic boundary conditions on \([0, T]\)

\[
G(t_1, t_2) = \sum_k \frac{e^{ik(t_1 - t_2)}}{k^2 + \omega^2} \frac{1}{T} \rightarrow \int \frac{e^{ik(t_1 - t_2)}}{k^2 + \omega^2} \frac{1}{2\pi} \frac{e^{-i\omega(t_1 + t_2)}}{+2\omega}
\]

is a function of \(t_1 - t_2\) and hence
\[ \int \int G(t_1, t_2) \, dt_1 \, dt_2 = T \int G(t, 0) \, dt \]

\[ \sim T \frac{1}{(2\omega)^4} \int e^{-4\omega |t|} \, dt \]

\[ = T \frac{1}{(2\omega)^4} \frac{2}{4\omega} = T \frac{1}{(2\omega)^5} \]

So the first diagram contributes to the partition function

\[ \sim T \left( \frac{1}{2 \cdot 4!} \frac{g^2 h^2}{(2\omega)^5} \right) \]

and the second contributes

\[ \sim T \left( \frac{1}{16} \frac{g^2 h^2}{(2\omega)^4} \frac{2}{2\omega} \right) = T \left( \frac{1}{8} \frac{g^2 h^2}{(2\omega)^5} \right) \]

Hence the second order correction to the ground energy is

\[ \lambda_2 = - \frac{g^2 h^2}{(2\omega)^5} \left( \frac{1}{2 \cdot 4!} + \frac{1}{8} \right) \]

which agrees with the elementary calculation.

Some basic facts. This time consider a general interaction, which means effectively that one has vertices of different multiplicities. Let's use Fourier coefficient description and check that the power of $T$ comes out correctly. Let's suppose we have a diagram with vertices of multiplicities $p_1 \leq p_2 \leq \ldots \leq p_v$ where $v$ is the number of vertices. To a vertex of mult. $p$ belongs the power

\[ \frac{1}{1 \cdot p!} \sum_{k_1 \ldots k_p} \delta \text{ factor} \]
and each edge furnishes us with \( \frac{hT}{\ell^2 + w^2} \).

\[ (*) \quad \phi_{\nu} = \frac{1}{\hbar \nu} \frac{1}{T(2\pi i - \nu)} (hT)^e \sum_{\text{indep. momenta}} \]

The number of independent momenta is found as follows: The set of all momenta is the group \( \sum \frac{2\pi i}{T} \) of oriented 1-chains on the graph. The momenta adding up to zero at each vertex is the group of 1-cycles which is \( H_1(\text{graph}) \), which has rank = no. of loops.

\[ \text{number of loops in the graph} = \text{number of independent momenta} \]

Call this \( = \) number \( L \). Note by Euler

\[ v - e = c - L \]

where \( c \) is the number of components of the graph.

Each momentum integration requires a \( \frac{1}{T} \) factor so that we can take the \( T \rightarrow 0 \) limit.

Hence (*) becomes

\[ \frac{e}{\hbar} = \frac{1}{\ell^2 + w^2} \sum_{\text{ind. momenta}} \frac{1}{T} \]

where we use \( \sum p_i = 2e \). Hence we get

\[ \hbar = \frac{L - c}{T} \]

in the partition function. This shows that the connected graph
are not exact and that if we restrict to connected graphs, then expanding in powers of $\hbar$ is the same as the loop expansion.

So far I have a sort of understanding of the ground energy. The next obstacle seems to be to understand the meaning of the Greens functions. The two-point Greens function is defined as follows. Suppose we fix an inverse temperature $\beta$. Then the Greens function is, up to a factor of $i$,

$$\langle T[e^{iHt} e^{-iH'(t')}] \rangle,$$

where $\langle \rangle$ denotes thermal average $\langle A \rangle = \text{tr}(e^{-\beta H} A)/\text{tr}(e^{-\beta H})$. 
Review Victor's lectures on Tauberian theorems:

Theorem: Suppose $a_n$ bounded below and $\sum a_n x^n$ converges for $|x| < 1$, (i.e. $\lim_{n \to \infty} |a_n x^n| = 1$). Then

$$\lim_{x \to 1} (1-x) \sum a_n x^n = A \Rightarrow \frac{1}{N} \sum_{n \in N} a_n \to A$$

Proof. Can suppose $a_n > 0$ by adding a constant. Putting $x^k$ in form $x$ gives

$$(1-x) \sum \frac{a_n x^n}{1-x} \to A$$

$$\frac{(1-x) \sum a_n x^n (x^n)^{-1}}{1-x} \to A = A \int_0^1 x^{k-1} dx$$

hence for any polynomial $P(x)$ we have

$$\lim_{x \to 1} (1-x) \sum a_n x^n P(x^n) \to A \int_0^1 P(x) dx$$

But now it is easily seen that for any piecewise continuous $f$ on $0 \leq x \leq 1$ we can find polynomials $P_1, P_2$ with $P_1 \leq f \leq P_2$ on $[0,1]$ and $\int (P_2 - P_1) dx < \epsilon$. It follows because $a_n > 0$ that these inequalities persist for the left side of \( \otimes \) with $P$ replaced by $P_1, f, P_2$, hence

$$(1-x) \sum a_n x^n f(x^n) \to A \int_0^1 f(x) dx$$

Now take $f(x) = \begin{cases} 0 & 0 \leq x < \frac{1}{2}, \\ 1/x & \frac{1}{2} \leq x \leq 1 \end{cases}$ with $\int f(x) dx = 1$ and

$$(1-x) \sum a_n x^n f(x^n) = (1-x) \sum_{n \in N} a_n$$

for $x^n > \frac{1}{e}$, $n \log x > -1$, $n \leq \frac{1}{\log(1/e)}$

where $N = \frac{1}{\log(1/e)}$, or $x = e^{-1/N}$. Thus we have

$$(1-e^{-1/N}) \sum a_n \to A$$

so

$$\frac{1}{N} \sum_{n \in N} a_n \to A$$

QED.
Reformulation: Suppose \( a_n \) is the sequence of partial sums of a series \( \sum b_n \) so that

\[
a_n = \sum_{k=1}^{n} b_k \frac{x^k}{1-x}
\]

Then

\[
\sum a_n x^n = \sum_{n=1}^\infty x^n \left( \sum_{k=1}^{n} b_k \right) = \sum_{k} b_k \sum_{n=k}^\infty x^n = \sum_{k} b_k x^k \frac{1}{1-x}
\]

so the preceding theorem implies

\[
\sum b_n = A \text{ in the sense of Abel} \Rightarrow \sum_{n=1}^\infty b_n = A \text{ in (C,1) sense}
\]

assuming that the partial sums \( a_n = \sum_{k=1}^{n} b_k \) are bounded.

The above theorem can be generalized to

\[
\lim_{s \to 0^+} s \int_0^\infty e^{-st} d\mu(t) = A \Rightarrow \lim_{s \to 0^+} s \int_0^1 e^{-st} d\mu(t) = A
\]

for any measure on \([0,\infty)\) such that \( \int e^{-st} d\mu \) converges for \( s > 0 \).
Wiener-Tauberian Thm. is a consequence of
Thm. \( f \in L^1(\mathbb{R}), \ \hat{f}(\xi) = \int e^{ix\xi} f(x) \, dx \neq 0 \) all \( \xi \in \mathbb{R} \)
\( \Rightarrow \) closed ideal in \( L^1 \) generated by \( f \) is all of \( L^1 \).

Proof in the case where \( f \) decays fast enough at \( \infty \)
(after Kac): What we have to show is that elements of \( \mathfrak{U} \)
the form \( g \ast f \) with \( g \in L^1 \) are dense in \( L^1 \). As
\( \hat{g} \ast \hat{f} = \hat{g} \hat{f} \)
what we want to do is take an element \( h \) of \( L^1 \) and
divide \( h \) by \( f \). Recall that \( C_c^\infty \) is dense in \( \mathfrak{U} \).
This is possible if \( \hat{h} \in C_c^\infty \) and \( \hat{f} \in C_c^\infty \)
for then \( \hat{h}/f \in C_c^\infty \subset \mathfrak{U} \). Note \( \hat{f} \in C_c^\infty \)
follows if \( x^n f \) for all \( n \). The only problem is why \( C_c^\infty \) is dense in \( L^1 \).
But \( C_c^\infty \) is dense in the Schwartz space \( \mathcal{S} \) in the Schurty
topology, so because \( \mathcal{S} \approx \mathcal{S} \) is a topological isomorphism,
\( C_c^\infty \) is dense in \( \mathcal{S} \). But \( C_c^\infty \) is dense in \( L^1 \), and \( C_c^\infty \subset \mathcal{S} \subset L^1 \)
so \( \mathfrak{U} \) is dense in \( L^1 \).

Wiener-Tauberian Thm. Suppose \( f \in L^1 \) and \( \hat{f} \neq 0 \) \( \forall \xi \).
Then for any \( f' \in L^1 \) and \( h \in L^\infty \) one has
\[ \lim_{x \to \infty} (f \ast h)(x) = A \int f \Rightarrow \lim_{x \to \infty} (f' \ast h)(x) = A \int f' \]
Proof. Clear if \( f' = g \ast f \) with \( g \in L^1 \) because
\[ (f' \ast h)(x) = \int g(y) (f \ast h)(x-y) \to A \int g \int f \]
\[ \to A \int f \int f' \]
where we have used $f \cdot h$ is bounded and the dominated convergence theorem. Reason:

$$|(f \cdot h)(x)| = \left| \int f(x-y) h(y) \, dy \right| \leq \|f\|_1 \cdot \|h\|_\infty$$

Similarly, if $f', f'' \in L^1$ are close then $f' \cdot h, f'' \cdot h$ are supremum close, etc., etc.

Let's briefly review something about the $\zeta$ function and prime numbers. The starting point is Euler's proof of infinitely many primes using

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod (1 - p^{-s})^{-1}$$

Let $s > 1$, and use that $\zeta(s) \to +\infty$. Then

$$\log \zeta(s) = \sum_{p} \log (1 - p^{-s}) = \sum_{p, k \geq 1} \frac{1}{k} (p^{-s})^k$$

$$= \sum_{p} p^{-s} \sum_{k \geq 2} \frac{1}{k} \left( \sum_{p^k} p^{-k} \right)$$

and the rearrangement of summation is possible for $Re(s) > 1$ because the series converges absolutely.

Now estimates show that the second term for $\log \zeta(s)$ is analytic for $Re(s) > \frac{1}{2}$. In effect put $\zeta(s) = \sum p^{-s}$, so that

$$\log \zeta(s) = f(s) + \frac{1}{2} f(2s) + \frac{1}{3} f(3s) + \ldots$$

We have for $s$ real

$$f(s) \leq \sum_{n \geq 2} n^{-s} \leq 2^{-s} + \int_2^\infty x^{-s} \, dx$$

$$\therefore f(s) \leq 2^{-s} \left[ 1 + \frac{2}{s-1} \right] = 2^{-s} \left( \frac{s+1}{s-1} \right) \quad \frac{x^{-s+1}}{-s+1} \int_2^\infty = \frac{2^{-s+1}}{s-1}$$
Hence
\[ \sum_{k} \frac{1}{k} f(k) s < \sum_{k} \frac{1}{k} 2^{-k s} (\frac{ks+1}{ks-1}) \rightarrow 1 \text{ as } k \rightarrow \infty \]

This shows that \[ |\sum_{k \geq 2} \frac{1}{k} f(k) s| \leq C \text{ uniformly for } \Re s > \frac{1}{2} + \epsilon. \]
So therefore \( \log f(s) \) and \( f(s) \) have the same singularities for \( \Re s > \frac{1}{2} \).

Now
\[ f(s) = \int_{0}^{\infty} e^{-st} d\mu(t) \quad d\mu(t) = \sum_{p \leq t} d(t - \log p) \frac{dt}{p} \]
\[ \mu(t) = \sum_{p \leq t} \frac{1}{\log p} = \pi(e^t) \]

The prime number theorem asserts
\[ \pi(x) \sim \frac{x}{\log x} \quad \text{or} \quad \mu(t) \sim e^{\gamma t} \]

and somehow this results from the singularity \( f(s) \sim \frac{1}{s-1} \)
as \( s \rightarrow 1 \). Laplace inversion gives formally
\[ \frac{d\mu}{dt} = \frac{1}{2\pi i} \int e^{st} f(s) ds \]
but there are problems with the fact \( f(s) \) doesn't decay vertically. So instead one can do the following
\[ f(s) = \int_{0}^{\infty} e^{-st} d\mu(t) = \left[ e^{-st}\mu(t) \right]_{0}^{\infty} + s \int_{0}^{\infty} e^{-st}\mu(t) dt \]
whence
\[ \mu(t) = \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} e^{st} f(s) ds \]
and this should probably hold except where \( \mu(t) \) jumps, since \( \mu(t) \) is piecewise continuous. It probably is necessary to go to \( \infty \) symmetrically, à la Eisenstein.

Now we know \( f(s) \sim \frac{1}{s-1} \) as \( s \rightarrow 1 \), so we really
First note that this integral converges, for if we put \( s = 1 + iy \) then

\[
\int_{-\infty}^{\infty} e^{x + iy} \log(iy) \, dy \rightarrow \frac{s}{1 + iy} = \frac{1}{y} \left( \frac{1}{1 - iy} \right) = \frac{1}{y} + O\left(\frac{1}{y^2}\right)
\]

so the non-convergent part is \( e^{iy} \log y \). Since \( \log y \to 0 \) as \( y \to \infty \), this is like an alternating \( \frac{1}{y} \) series.

The next thing is to deform the contour to the branch cut for \( \log(1 - s) \), which we will take to be \( -\infty < s < 1 \) except we push it away from \( s = 0 \).

As we push the contour through 0 we get the residue at 0 which is

\[
\cot \log(0 - 1) = 1 + \pi i
\]

Define \( \log(s - 1) \) to have \( \text{Im} = -\pi \) on the bottom of the cut. It then increases to \( 2\pi i + \log(s - 1) \) on the top of the cut.
So we get

\[ +
\frac{1}{2\pi i} \int_{-\infty}^{\infty} e^{st} \frac{\log(s-1)}{s} \, ds + \frac{1}{2\pi i} \int_{-\infty}^{\infty} e^{st} \frac{\log(s-1)+2\pi i}{s} \, ds
\]

\[ = +
\frac{1}{2\pi i} \int_{-\infty}^{\infty} e^{st} \frac{\log(s)}{s} \, ds = + P \int_{-\infty}^{\infty} \frac{e^{st}}{s} \, ds
\]

where in the integral the contour passes above the pole at \( s = 0 \). For \( t > 0 \) we can change variables to get

\[ \frac{1}{2\pi i} \int_{-\infty}^{\infty} e^{st} \log(s) \, ds = + P \int_{-\infty}^{\infty} \frac{e^{s}}{s} \, ds
\]

Up to a constant this is

\[ \int_{1}^{t} \frac{e^{s}}{s} \, ds
\]

**Summary:**

\[ \frac{1}{2\pi i} \int_{-\infty}^{\infty} e^{st} \left( -\log(s) \right) \, ds = P \int_{-\infty}^{\infty} \frac{e^{s}}{s} \frac{dx}{x}
\]

The latter is the so-called exponential integral function. It has a logarithmic singularity at \( t = 0 \), and its Laplace transform is defined. Actually things might be simpler. Put

\[ E(t) = \int_{0}^{\infty} e^{-u} \frac{du}{u}. \]

Then \( E(t) \) has a log. singularity at \( t = 0 \), so its Laplace transform is defined:

\[ \int_{0}^{\infty} e^{-st} E(t) \, dt = \int_{0}^{\infty} \int_{0}^{\infty} e^{-st} e^{-u} \frac{du}{u}
\]
\[
\int_0^\infty \frac{du}{u} e^{-u} \int_0^u e^{-st} dt = \int_0^\infty \frac{du}{u} e^{-u} \left[ e^{-st} \right]_0^u \\
= \int_0^\infty \frac{du}{u} e^{-u} \left( \frac{1 - e^{-su}}{s} \right) = \frac{1}{s} \int_0^\infty \frac{e^{-u} - e^{-(s+1)u}}{u} du \\
= \frac{1}{s} \log(s+1). \quad \text{Here I have used} \\
\int_0^\infty \frac{e^{-au} - e^{-bu}}{u} du = \int_0^b \int_0^\infty e^{-su} du = \int_a^b ds = \log(b). \\
\]

More generally for \( a > 0 \):
\[
\frac{1}{2\pi i} \int e^{st} \frac{\log(s+a)}{s} ds = \log(a) - \int_0^\infty \frac{e^{st}}{s} ds \\
= \log(a) + \int_a^\infty \frac{e^{-tx}}{x} dx \\
= \log(a) + \int_0^\infty \frac{e^{-x}}{x} dx = \log(a) + \Gamma(at) \\
\]

but this results from \( a=1 \) by scaling. Suppose \( a > 0 \):
\[
\frac{1}{2\pi i} \int e^{st} \frac{\log(s)}{s} ds = \left[ \log a + \int_1^\infty \frac{e^{-x}}{x} dx + \int_0^\infty \frac{e^{-x}}{x} dx + \int_0^\infty \frac{1}{x} dx \right]_{a \to 0} \\
= \int_1^\infty \frac{e^{-x}}{x} dx + \int_0^\infty \frac{e^{-x}}{x} dx - \log t \\
\]

As a check
\[
\int_0^\infty e^{-st} t^x \frac{dt}{t} = \frac{\Gamma(x)}{s^x} \\
\int_0^\infty e^{-st} t^x \log t \frac{dt}{t} = \frac{\Gamma'(x)}{s^x} + \frac{\Gamma(x)}{s^x} (-\log s) \\
\int_0^\infty e^{-st} \log t \frac{dt}{t} = \frac{\Gamma'(1)}{s} - \frac{\log s}{s} 
\]
which shows
\[ \frac{\log s}{s} = \mathcal{L} \left\{ \Gamma'(1) - \frac{\log t}{t} \right\} \]

Finally we want to estimate
\[ F(t) = \int_{-\infty}^{t} \frac{e^x}{x} \, dx = \int_{-\infty}^{t} e^{xt} \, dx \]
as \( t \to +\infty \). We have
\[ F(t) = c + \int_{1}^{t} \frac{e^x}{x} \, dx \leq c + \int_{1}^{t} e^x \, dx \leq c - 1 + e^t \]
Also \( F \leq 1 \) for \( t > 0 \) and
\[ F(t) - F(t/2) = \int_{t/2}^{t} \frac{e^x}{x} \, dx = \int_{t/2}^{1} e^{-y} y \, dy \]
Laplace’s method gives the asymptotic behavior of the latter. Actually we should use this method directly in
\[ F(t) = \int_{-\infty}^{0} e^{xt} \, dx = \frac{1}{t} \text{ interpreted as a} \]
\[ = e^{t} \int_{0}^{1} e^{-yt} \frac{1}{1-y} \, dy \approx e^{t} \left\{ \frac{1}{t} + \frac{f(t)}{t^2} + \ldots \right\} \]
The upshot is that
\[ \frac{1}{2\pi i} \int e^{st} \frac{-\log(s-1)}{s} \, ds = \int_{-\infty}^{t} e^{xt} \, dx \sim \frac{e^t}{t} \]
which in principle yields the leading term for \( \pi(x) \)
and the prime number theorem.

It remains to understand the Tauberian business that deduces the PNT from \( f(1+iy) \neq 0 \)
How Tauberian results are used in spectral theory. Victor took $H = -\Delta + V$ on $\mathbb{R}^3$ with $V$ growing sufficiently fast and by using a parametrix for $(u + H)^{-1}$ he was able to show

$$\sum_{n} \frac{1}{(u + \lambda_n)^2} \sim \int \frac{1}{(u + x^2 + V(x))^2} \frac{dx \, d^3x}{(2\pi)^3} \quad \text{as } u \to \infty$$

$$\int \frac{1}{(u + x)^2} \, dN(x) \quad \int \frac{1}{(u + x)^2} \, dV(x)$$

where $N(x) = \text{number of } \lambda_n \leq x$.

He will use a Tauberian result to deduce

$$N(x) \sim V(x).$$

A slightly different version considers the heat operator which gives the partition function

$$\text{tr} \left( e^{-\beta H} \right) = \int e^{-\beta x^2} \, dN(x) = \int \mathcal{D}x(t) \ e^{-\int_0^\beta (\frac{x^2}{2} + V(x)) \, dt}$$

for paths of period $\beta$ and compares it with classical partition fn.

$$\int e^{-\beta H} \frac{dx \, d^3x}{(2\pi)^3} = \int e^{-\beta x^2} \, dV(x).$$

as $\beta \to 0$. This case has been studied by Abjahl and others using the fact that $\gamma_0$-theory gives an asymptotic expansion for $\langle x | e^{-\beta H} | x' \rangle$ as $\beta \to 0$. 

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