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"in" and "out" operators. Suppose we have a Hamiltonian

$$H_0 = \frac{p^2}{2} + \frac{1}{2}\omega^2 q^2 \qquad H' = \varphi(t,q), \qquad \varphi \text{ poly. in } q$$

with time-development operator $U(t,t')$:

$$i\frac{d}{dt}U(t,t') = H(t)\,U(t,t')$$
$$U(t',t') = id.$$

To get the Heisenberg picture, one identifies a Schroedinger trajectory $\psi(t) = U(t,0)\psi(0)$ with its value at $t=0$, and translates all operators accordingly. Thus

$$U(t,0)\,q_H(t)\,\psi(0) = q\,U(t,0)\psi(0)$$

or

$$q_H(t) = U(0,t)\,q\,U(t,0)$$

Then

$$\frac{d}{dt}q_H(t) = i\,U(0,t)\left[\frac{p^2}{2} + \frac{1}{2}\omega^2 q^2 + \varphi(t,q), q\right]U(t,0)$$
$$= p_H(t)$$

$$\frac{d}{dt}p_H(t) = U(0,t)\underbrace{\left[\frac{p^2}{2} + \frac{1}{2}\omega^2 q^2 + \varphi(t,q), ip\right]}_{-\omega^2 q - \frac{\partial\varphi}{\partial q}(t,q)}U(t,0)$$

$$= -\omega^2 q_H(t) - \frac{\partial\varphi}{\partial q}(t, q_H(t))$$

so

$$\left(\frac{d^2}{dt^2} + \omega^2\right)q_H(t) + \frac{\partial\varphi}{\partial q}(t, q_H(t)) = 0.$$

The simplest interesting case is $\varphi(t,q) = -J(t)q$

whence $$\left(\frac{d^2}{dt^2}+\omega^2\right) q_H(t) = J(t).$$

Notice that this is an operator equation, where $J(t)$ is a scalar.

Next define $q_{in}(t)$ and $q_{out}(t)$ to be be the solutions of

$$\left(\frac{d^2}{dt^2}+\omega^2\right) q(t) = 0$$

which agree with $q_H(t)$ for $t \ll 0$ and $t \gg 0$ respectively. One has

$$q_H(t) = q_{in}(t) + \int G^R(t,t')\, J(t')\, dt'$$

$$q_H(t) = q_{out}(t) + \int G^A(t,t')\, J(t')\, dt'$$

where G^R is the retarded Green's function (supported for $t' \leq t$):

$$G^R(t,t') = \begin{cases} \frac{\sin \omega(t-t')}{\omega} & t > t' \\ 0 & t < t' \end{cases}$$

also

$$G^A(t,t') = \begin{cases} 0 & t > t' \\ -\frac{\sin \omega(t-t')}{\omega} & t < t' \end{cases}$$

$$q_{out}(t) = q_{in}(t) + \int (G^R - G^A)(t,t')\, J(t')\, dt'$$

$$(G^R - G^A)(t) = \theta(t)\frac{\sin \omega t}{\omega} + \theta(-t)\frac{\sin \omega t}{\omega} = \frac{\sin \omega t}{\omega}$$

What does all this mean? Notice that if the interaction $\varphi'(t,q)$ has support in $[0,T]$,

then
$$q_H(t) = q_{in}(t) = e^{iH_0t} q e^{-iH_0t} \qquad \text{for } t \le 0$$
$$= \frac{1}{\sqrt{2\omega}}\left(e^{-i\omega t}a + e^{i\omega t}a^*\right)$$

Somehow it might not be [redacted] very meaningful to talk about [redacted] a, a^* in the Heisenberg picture. Notice that we can decompose q_{in} (also q_{out}) into positive and negative frequency components.

$$q_{in}(t) = \frac{1}{\sqrt{2\omega}}\left(e^{i\omega t}a^*_{in} + e^{-i\omega t}a_{[redacted]\,in}\right)$$

Also there are definite Heisenberg states $|0\rangle_{in}$, $|0\rangle_{out}$ which for large t are the ground states. $\therefore$

$$U(t,0)|0\rangle_{in} \text{ [redacted] spans } \operatorname{Ker} a_{in} \quad \text{for } t \ll 0$$

[redacted]

[redacted] Review: [redacted] Think of quantum-mechanics as being a bundle of Hilbert spaces over the time axis. It comes equipped with position and momentum operators $q(t)$, $p(t)$ in each fibre satisfying CCR. Time evolution is given by a connection in this Hilbert bundle. To obtain the Heisenberg picture we trivialize the bundle via the connection; to obtain the Schroedinger picture we trivialize the bundle via the Stone-von Neumann thm.

Let's work in the Schroedinger picture. A Heisenberg state [redacted] can be identified with a trajectory for the Schroedinger equation: $\psi_S(t)$. The Heisenberg operator $q_H(t)$ acting on ψ_S gives the trajectory passing thru $q\psi_S(t)$ at time t. Maybe we should think of our Heisenberg state as a section $t \mapsto \psi(t)$ of the Hilbert bundle, and $\psi_S(t)$ as the image of $\psi(t)$ under the Stone-von-Neumann trivialization.

Notation: $\psi_S(t) = U(t,0)\psi_S(0)$, so if we think of a Heisenberg state vector as a Schroedinger trajectory evaluated at time 0: $\psi = \psi_S(0)$, $\psi_S(t) = U(t,0)\psi$

$$q_H(t)\psi = U(0,t)\, q\, \psi_S(t) = U(0,t)\, q\, U(t,0)\psi$$

so $$q_H = U(0,t)\, q\, U(t,0).$$

Now another thing one can do is use the fact that

$$e^{iH_0 t}\psi_S(t)$$

is constant for t large. Hence we can define

$$\psi_{in} = \lim_{t\to -\infty} e^{iH_0 t}\psi_S(t) = \underbrace{\lim_{t\to -\infty} e^{iH_0 t}\, U(t,0)}_{\Omega^+}\psi$$

and similarly

$$\psi_{out} = \lim_{t\to +\infty} e^{iH_0 t}\, U(t,0)\psi = \Omega^-\psi$$

The S-~~matrix~~ operator

$$S = \Omega^-(\Omega^+)^* = \lim_{\substack{t\to +\infty \\ t'\to -\infty}} e^{iH_0 t}\, U(t,t')\, e^{-iH_0 t'}$$

then satisfies

$$S\psi_{in} = \psi_{out}.$$

Notice that we can also realize the Heisenberg viewpoint by associating to $\psi_S(t)$ the vector $\psi_{in} = \Omega^+\psi_S(0)$. The operator $q_H(t)$ is then

$$\Omega^+ U(0,t)\, q\, U(t,0)(\Omega^+)^* = e^{iH_0 t'}\, U(t',t)\, q\, U(t,t')\, e^{-iH_0 t'}$$

for $t' \ll 0$. In this description one has

$$q_H(t) = e^{iH_0 t}\, q\, e^{-iH_0 t} \qquad \text{for} \quad t \ll 0.$$

and we [redacted] see that the right side in $q_{in}(t)$.

So we see that if I identify the Heisenberg state vector space with the Schroedinger [redacted] Hillbert space via $\psi_S(t) \longmapsto \psi_{in} = \lim_{t\to-\infty} e^{iH_0 t}\psi_S(t)$, then $q_{in}(t) = q_I(t)$ (interaction picture). What are the formulas in general?

Let us identify Heisenberg and Schroedinger pictures at $t=0$, whence

$$q_H(t) = U(0,t)\, q\, U(t,0)$$

Then for $t \ll 0$ we have [redacted]

$$q_H(t) = \underbrace{U(0,t)\, e^{-iH_0 t}}_{(\Omega^+)^*}\, q_I(t)\, \underbrace{e^{iH_0 t}\, U(t,0)}_{\Omega^+}$$

So in general

$$q_{in}(t) = (\Omega^+)^*\, q_I(t)\, \Omega^+ \qquad ?$$

$$q_{out}(t) = (\Omega^-)^*\, q_I(t)\, \Omega^-$$

September 19, 1979

Consider $H = \frac{p^2}{2} + \frac{1}{2}\omega^2 q^2 + \varepsilon(t)\frac{q^2}{2}$ and work with the Heisenberg operator

$$q_H(t) = U(0,t)\, q\, U(t,0)$$

One has
$$\frac{d}{dt} q_H(t) = U(0,t)\,[iH(t), q]\,U(t,0) = p_H(t)$$
$$\frac{d}{dt} p_H(t) = -(\omega^2 + \varepsilon(t))\, q_H(t)$$
in other words the Heisenberg operators satisfy the classical equations of motion:

$$\left(\frac{d^2}{dt^2} + \omega^2 + \varepsilon(t)\right) q_H(t) = 0$$

Hence the matrix elements between Heisenberg states

$$\langle b \,|\, q_H(t) \,|\, a \rangle$$

also satisfy the above DE.

Now assume ε has compact support so that we have propagation

$$e^{i\omega t} \xleftrightarrow{\ \varphi\ } A(\omega)e^{-i\omega t} + B(\omega)e^{-i\omega t}$$
$$e^{-i\omega t} \xleftrightarrow{\ \bar{\varphi}\ } \bar{B}(\omega)e^{i\omega t} + \bar{A}(\omega)e^{-i\omega t}$$

Here think of ω as being approached from the LHP, so that $e^{-i\omega t} = e^{i(\omega_r - i\eta)t} = e^{\eta t}e^{i\omega_r t}$ is the small solution at $t \to -\infty$ and the large soln. at $t \to +\infty$, hence $A(\omega)$ is analytic nicely in the LHP and its zeroes in the LHP are bound states.

We can write

$$q_H(t) = \frac{1}{\sqrt{2\omega}}\left(e^{i\omega t} a_{in}^{*} + e^{-i\omega t} a_{in}\right) \qquad t \ll 0$$

where $\quad [a_{in}, a_{in}^*] = 1$. In effect put

$$a_{in}(t) = \frac{1}{\sqrt{2\omega}}\left(i p_H(t) + \omega q_H(t)\right) \qquad t \ll 0$$

Then
$$\frac{d}{dt} a_{in}(t) = \frac{1}{\sqrt{2\omega}}\left(i\left(-\omega^2 q_H(t)\right) + \omega p_H(t)\right)$$
$$= -i\omega \cdot \frac{1}{\sqrt{2\omega}}\left(i p_H(t) + \omega q_H(t)\right)$$
$$= -i\omega \, a_{in}(t)$$

whence
$$a_{in}(t) = e^{-i\omega t} a_{in}$$

for some constant-in-time operator a_{in}.

Recall that $q_{in}(t)$ is defined as the operator function of time satisfying
$$\left(\frac{d^2}{dt^2} + \omega^2\right) q_{in}(t) = 0$$

which agrees with $q_H(t)$ for $t \ll 0$. Thus
$$q_{in}(t) = \frac{1}{\sqrt{2\omega}}\left(e^{i\omega t} a_{in}^* + e^{-i\omega t} a_{in}\right) \qquad \text{for all } t.$$

Next let's introduce φ = the (scalar) solution of
$$\left(\frac{d^2}{dt^2} + \omega^2 + \varepsilon(t)\right)\varphi = 0$$

such that $\quad \varphi = e^{i\omega t} \quad$ for $\quad t \ll 0$. Then we have
$$q_H(t) = \frac{1}{\sqrt{2\omega}}\left(\varphi(t) a_{in}^* + \bar{\varphi}(t) a_{in}\right)$$

and so for $t \gg 0$ we have
$$q_H(t) = \frac{1}{\sqrt{2\omega}}\left(\left(A e^{i\omega t} + B e^{-i\omega t}\right) a_{in}^* + \left(\bar{B} e^{i\omega t} + \bar{A} e^{-i\omega t}\right) a_{in}\right)$$

$$= \frac{1}{\sqrt{2\omega}}\left(e^{i\omega t}\left(A a_{in}^* + \bar{B} a_{in}\right) + e^{-i\omega t}\left(B a_{in}^* + \bar{A} a_{in}\right)\right)$$

whence

$$\begin{pmatrix} a_{out}^* \\ a_{out} \end{pmatrix} = \begin{pmatrix} A & \bar{B} \\ B & \bar{A} \end{pmatrix}\begin{pmatrix} a_{in}^* \\ a_{in} \end{pmatrix}$$

Now in the Heisenberg picture the S-matrix is an operator on the state space which satisfies

$$S^* q_{in}(t) S = q_{out}(t). \qquad \text{or} \qquad S^* | \,\rangle_{in} = | \,\rangle_{out}$$

This is the same as requiring

$$S^* \begin{pmatrix} a_{in}^* \\ a_{in} \end{pmatrix} S = \begin{pmatrix} a_{out}^* \\ a_{out} \end{pmatrix} = \begin{pmatrix} A & \bar{B} \\ B & \bar{A} \end{pmatrix}\begin{pmatrix} a_{in}^* \\ a_{in} \end{pmatrix}$$

So the problem is to construct an operator S which conjugates in the above way, and it should be possible to give this operator as a suitable normal product gadget in the operators a_{in}, a_{in}^*.

Let us assume that $\varepsilon(t)$ is supported in $[0,T]$ so that for $t \leq 0$

$$q_H(t) = q_I(t) = \frac{1}{\sqrt{2\omega}}\left(e^{i\omega t} a^* + e^{-i\omega t} a\right)$$

whence $\quad q(t) = q_{in}(t)$, , $\quad a_{in}^* = a_{in}$ etc.

Furthermore we have for all t

$$q_H(t) = \frac{1}{\sqrt{2\omega}}\left(\varphi(t) a^* + \bar{\varphi}(t) a\right)$$

Still after the S-matrix which we know can be expressed as a time-ordered product

$$S = T\left\{ \exp^{-i\int \varepsilon(t)\,\frac{q(t)^2}{2}\,dt} \right\}$$

and hence S is built up out of operators in the form

$$e^{-ic\left(e^{2i\omega t}a^{*2} + a^*a + aa^* + e^{-2i\omega t}a^2\right)}$$

Notice that the operators $\frac{1}{2}a^{*2}$, $\frac{1}{2}(a^*a + aa^*)$, $\frac{1}{2}a^2)$ span a Lie algebra:

$$[a^2, a^{*2}] = a[a, a^{*2}] + [a, a^{*2}]a$$

$$= a\,2a^* + 2a^*a = 2(aa^* + a^*a)$$

or

$$\left[\frac{a^2}{2}, \frac{a^{*2}}{2}\right] = \frac{1}{2}(aa^* + a^*a) = a^*a + \frac{1}{2}$$

[redacted]

$$\left[a^*a + \frac{1}{2}, \frac{a^2}{2}\right] = -2\,\frac{a^2}{2}$$

$$\left[a^*a + \frac{1}{2}, \frac{a^{*2}}{2}\right] = 2\,\frac{a^2}{2}$$

This is [redacted] the SL_2 Lie algebra

$$X^+ = \begin{pmatrix} 0 & +1 \\ 0 & 0 \end{pmatrix} \longleftrightarrow \frac{a^{*2}}{2}$$

$$X^- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \longleftrightarrow \frac{a^2}{2}$$

this corresp. is not quite correct, see below

$$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \longleftrightarrow \frac{1}{2}(a^*a + aa^*)$$

Since one has $[H, X^\pm] = \pm 2X^\pm$

$$[X^+, X^-] = \left[\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\right] = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = H$$

Now we know from the Baker-Hausdorff formula that for two small elements α, β of a Lie algebra we have

$$e^{\alpha} e^{\beta} = e^{\gamma}$$

where

$$\gamma = \alpha + \beta + \tfrac{1}{2}[\alpha, \beta] + \cdots$$

Consequently we can conclude, at least for small ε, that the S-matrix is in the form

$$e^{c_1 \frac{a^{*2}}{2} + c_2\left(\frac{aa^* + a^*a}{2}\right) + c_3 \frac{a^2}{2}}$$

for certain constants c_1, c_2, c_3.

To get the correct correspondence between the operators $\frac{a^2}{2}$, $\frac{a^{*2}}{2}$, $a^*a + \frac{1}{2}$ and 2×2 matrices let us make them act on span of a^*, a.

$$\left[\left(a^*a + \tfrac{1}{2}\right), \begin{cases} a^* \\ a \end{cases}\right] = \begin{cases} a^* \\ -a \end{cases} \qquad \text{matrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\left[\frac{a^{*2}}{2}, \begin{cases} a^* \\ a \end{cases}\right] = \begin{cases} 0 \\ -a^* \end{cases} \qquad \text{matrix} \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix}$$

$$\left[\frac{a^2}{2}, \begin{cases} a^* \\ a \end{cases}\right] = \begin{cases} a \\ 0 \end{cases} \qquad \text{matrix} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

So the correspendence is

$$X^+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \longleftrightarrow -\frac{a^{*2}}{2}$$

$$X^- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \longleftrightarrow \frac{a^2}{2}$$

$$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \longleftrightarrow a^*a + \frac{1}{2}$$

Check $\quad [X^+, X^-] = \left[-\frac{a^{*2}}{2}, \frac{a^2}{2}\right] = \left[\frac{a^2}{2}, \frac{a^{*2}}{2}\right] = a^*a + \frac{1}{2} = H$

Next note that a self-adjoint linear combination

$$c\,\frac{a^{*2}}{2} + b\left(a^*a+\tfrac{1}{2}\right) + \bar{c}\,\frac{a^2}{2} \qquad b \text{ real}$$

gives rise to the matrix

$$\begin{pmatrix} b & -c \\ \bar{c} & -b \end{pmatrix}$$

which when multiplied by i and exponentiated gives rise to a matrix in $SU(1,1)$.

If [redacted] we want to compute the S-matrix in a normal product form, then we are reduced to converting

$$e^{c_1 \frac{a^{*2}}{2} + c_2\left(a^*a+\frac{1}{2}\right) + c_3 \frac{a^2}{2}}$$

to normal product form. Is there a convenient way to do this, e.g. even for $e^{iT\frac{q^2}{2}}$?

September 21, 1979

Let us consider $H = \frac{p^2}{2} + \frac{\omega^2 q^2}{2} + \frac{\varepsilon(t) q^2}{2}$ again and put $q_H(t) = U(0,t)\, q\, U(t,0)$ so that

$$\left\{ \frac{d^2}{dt^2} + \omega^2 + \varepsilon(t) \right\} q_H(t) = 0$$

Recall that $q_{in}(t)$ is the solution of

$$(*) \qquad \left(\frac{d^2}{dt^2} + \omega^2 \right) q_{in}(t) = 0$$

such that $q_{in}(t) = q_H(t)$ for $t \ll 0$. Put

$$q_I(t) = e^{iH_0 t} q\, e^{-iH_0 t} \qquad H_0 = \frac{p^2}{2} + \frac{1}{2}\omega^2 q^2.$$

Then clearly $A^{-1} q_I(t) A$ satisfies $(*)$, hence we see that

$$q_{in}(t) = A^{-1} q_I(t) A$$

where

$$e^{-iH_0 t} A = U(t,0) \qquad t \ll 0$$

or

$$A = e^{iH_0 t}\, U(t,0) \qquad t \ll 0$$

is the operator denoted Ω^+ or $\tilde{U}(-\infty, 0)$. Similarly

$$q_{out}(t) = (\Omega^-)^{-1} q_I(t)\, \Omega^- \qquad \Omega^- = \tilde{U}(\infty, 0)$$

$$q_{in}(t) = (\Omega^+)^{-1} q_I(t)\, \Omega^+$$

and so we have $\theta^{-1} q_{in}(t)\, \theta = q_{out}(t)$

where $\theta^{-1} = (\Omega^-)^{-1} \Omega^+ \qquad \theta = (\Omega^+)^{-1} \Omega^-$

Note that θ is not the usual S-matrix $\tilde{U}(\infty, -\infty)$

$$S = \hat{U}(\infty,-\infty) = \Omega^{-}(\Omega^{+})^{-1}$$

but rather $\theta = (\Omega^{+})^{-1}\Omega^{-} = (\Omega^{+})^{-1} S\, \Omega^{+}$.

This explains some of my earlier confusions.

The safe thing to do is to assume $\varepsilon = 0$ for $t \leq 0$ whence $\Omega^{+} = id$.

Let's return to the problem of computing the S matrix in the interaction picture

$$S = T\left\{ e^{-i\int \varepsilon(t)\frac{q(t)^2}{2}\,dt} \right\}.$$

The idea I had is to work in the Lie group whose Lie algebra is spanned by skew-adjoint linear combinations of the operators $\frac{a^{*2}}{2}$, $\frac{a^2}{2}$, $\frac{aa^*+a^*a}{2} = a^*a+\frac{1}{2}$. This ought to be the metaplectic group. More precisely the group of operators generated by $\exp(X)$ where X is one of these skew-adjoint operators is the metaplectic group.

The idea is that S ought to be representable in the form

$$S = e^{ic_1\left(a^*a+\frac{1}{2}\right) + c_2\left(\frac{a^{*2}}{2}\right) - \bar{c}_2\left(\frac{a^2}{2}\right)}$$

where $c_1 \in \mathbb{R}$, $c_2 \in \mathbb{C}$. It should be possible to figure out what c_1, c_2 are, up to a ± 1 ambiguity, by seeing what S does to the operators a^*, a.

Idea: Try to write S in the form

doesn't work

$$S = e^{-\bar{c}_2 \frac{a^2}{2}} e^{ic_1 (a^* a + \frac{1}{2})} e^{c_2 \frac{a^{*2}}{2}}$$

for suitable $c_1 \in \mathbb{R}$, $c_2 \in \mathbb{C}$. This should be analogous to the big cell in the Bruhat decomposition.

Recall that we are assuming $\varepsilon(t) = 0$ for $t \leq 0$ so that

$$q_H(t) = \frac{1}{\sqrt{2\omega}} \left(e^{i\omega t} a^* + e^{-i\omega t} a \right) \qquad t \leq 0$$

$$= \frac{1}{\sqrt{2\omega}} \left((A e^{i\omega t} + B e^{-i\omega t}) a^* + (\bar{B} e^{i\omega t} + \bar{A} e^{-i\omega t}) a \right)$$

$$= \frac{1}{\sqrt{2\omega}} \left(e^{i\omega t} (A a^* + \bar{B} a) + e^{-i\omega t} (B a^* + \bar{A} a) \right).$$

for $t \gg 0$.

and that we want

$$q_{out}(t) = S^{-1} q_{in}(t) S \qquad S = e^{iH_0 T} U(T, 0)$$

or

$$\boxed{S^{-1} \begin{pmatrix} a^* \\ a \end{pmatrix} S = \begin{pmatrix} A & \bar{B} \\ B & \bar{A} \end{pmatrix} \begin{pmatrix} a^* \\ a \end{pmatrix}}$$

$$e^{t \frac{a^2}{2}} \begin{pmatrix} a^* \\ a \end{pmatrix} e^{-t \frac{a^2}{2}}$$

$$\frac{d}{dt} e^{t \frac{a^2}{2}} a^* e^{-t \frac{a^2}{2}} = e^{t \frac{a^2}{2}} \left(\frac{a^2}{2} a^* - a^* \frac{a^2}{2} \right) e^{-t \frac{a^2}{2}}$$

$$= e^{t \frac{a^2}{2}} \underbrace{\left[\frac{a^2}{2}, a^* \right]}_{a} e^{-t \frac{a^2}{2}} = a$$

$$e^{t \frac{a^2}{2}} \begin{pmatrix} a^* \\ a \end{pmatrix} e^{-t \frac{a^2}{2}} = \begin{pmatrix} a^* + t a \\ a \end{pmatrix} = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a^* \\ a \end{pmatrix}$$

$$\frac{d}{dt}\, e^{t\frac{a^{*2}}{2}}\, a\, e^{-t\frac{a^{*2}}{2}} = e^{t\frac{a^{*2}}{2}} \underbrace{\left[\tfrac{a^{*2}}{2}, a\right]}_{-a^*} e^{-t\frac{a^{*2}}{2}} = -a^*$$

$$\therefore\quad e^{t\frac{a^{*2}}{2}} \begin{pmatrix} a^* \\ a \end{pmatrix} e^{-t\frac{a^{*2}}{2}} = \begin{pmatrix} a^* \\ a - ta^* \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -t & 1 \end{pmatrix} \begin{pmatrix} a^* \\ a \end{pmatrix}$$

$$e^{t(a^*a+\frac{1}{2})} \begin{pmatrix} a^* \\ a \end{pmatrix} e^{-t(a^*a+\frac{1}{2})} = \begin{pmatrix} e^t a^* \\ e^{-t} a \end{pmatrix} = \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix} \begin{pmatrix} a^* \\ a \end{pmatrix}$$

So therefore if

$$S = e^{-\bar{c}_2 \frac{a^2}{2}}\, e^{-ic_1(a^*a+\frac{1}{2})}\, e^{c_2 \frac{a^{*2}}{2}}$$

we have

$$S^{-1} \begin{pmatrix} a^* \\ a \end{pmatrix} S = \begin{pmatrix} 1 & \bar{c}_2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} e^{ic_1} & 0 \\ 0 & e^{-ic_1} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ c_2 & 1 \end{pmatrix} \begin{pmatrix} a^* \\ a \end{pmatrix}$$

Unfortunately I see now that S is not unitary. ? ?

Question: We've seen that the operators

$$X^+ = -\frac{a^{*2}}{2}, \quad X^- = \frac{a^2}{2}, \quad H = a^*a + \frac{1}{2}$$

satisfy the bracket relations for sl_2. Do we get an action of $SL_2(\mathbb{C})$ on the underlying Hilbert space for the oscillator?

It seems the answer has to be NO because [redacted] $SL_2(\mathbb{C})$ is simply-connected, [redacted] and we know that from the operators X^+, X^-, H we can construct the generators of the metaplectic repn.

For example, in $SL_2(\mathbb{C})$ we have

$$\exp it\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} = \begin{pmatrix}e^{it} & 0\\ 0 & e^{-it}\end{pmatrix} = I \quad \text{if } t = 2\pi$$

yet

$$e^{it(a^*a+\frac{1}{2})}|0\rangle = e^{\frac{1}{2}it}|0\rangle \neq |0\rangle \text{ if } t = 2\pi.$$

Actually it seems only for self-adjoint linear combinations Y of [redacted] X^+, X^-, H that the operator e^{itY}, t real, makes sense. The idea is that if this operator were defined, then its effect on the subspace of linear operators $\mathbb{C}a^* + \mathbb{C}a$ would be given by an element of $SL_2(\mathbb{C})$. But we know that the "coherent" states $e^{i\tau\frac{x^2}{2}}$ are characterized as the [redacted] kernels of linear operators, hence it ought to follow that e^{itY} preserves the UHP. So therefore I see that operators like

$$e^{t\frac{a^{*2}}{2}}$$

don't make any <u>sense</u>, except possibly formally.

Recall for future reference: $SU(1,1)$ consists of $\begin{pmatrix}A & \bar{B}\\ B & \bar{A}\end{pmatrix}$ of det. 1 and its Lie algebra consists of $\begin{pmatrix}ia & \bar{b}\\ b & -ia\end{pmatrix}$ with $a \in \mathbb{R}$, $b \in \mathbb{C}$.

September 22, 1979

Again we consider $\quad H = \frac{p^2}{2} + \frac{1}{2}\omega^2 q^2 + \frac{1}{2}\varepsilon(t)\frac{q^2}{2}$

with $\varepsilon(t)=0$ for $t \leq 0$, and let

$$q_H(t) = U(0,t)\, q\, U(t,0)$$

so that
$$\left[\frac{d^2}{dt} + \omega^2 + \varepsilon(t)\right] q_H(t) = 0$$

$$q_H(t) = q_{in}(t) = q_I(t) = \frac{1}{\sqrt{2\omega}}\left(e^{i\omega t}a^* + e^{-i\omega t}a\right)$$
for $t \leq 0$.

~~[illegible]~~ For $t \gg 0$ we have

$$q_H(t) = q_{out}(t) = S^{-1} q_{in}(t)\, S$$

where
$$S = e^{iH_0 t}\, U(t,0) \qquad t \gg 0$$
is the S-operator.

Now we want to perturb H by a source:

H^J[redacted] $= H - J$[redacted]$(t)\, q$

say $J(t)$ supported in $[0,T]$. Then one has

$$\frac{d}{dt} U(0,t)\, U^J(t,t') = U(0,t)\underbrace{\left[iH(t) - iH^J(t)\right]}_{iJ(t)q} U^J(t,t')$$

$$= iJ(t)\, q_H(t)\; U(0,t)\, U^J(t,t')$$

so
$$U(0,t)\, U^J(t,t')\, U(t',0) = T\left\{e^{i\int_{t'}^{t} J(t)q_H(t)\,dt}\right\}$$

$$\underbrace{e^{iH_0 t}\, U^J(t,0)}_{S^J} = \underbrace{e^{iH_0 t}\, U(t,0)}_{S}\; T\left\{e^{i\int J(t)q_H(t)\,dt}\right\}$$

So we end up with the formula:

$$S^J = S\, T\{J\} \qquad T\{J\} = T\{e^{-i\int J(t)\, q_H(t)\, dt}\}$$

The problem is to compute the S-operator knowing $\langle 0|S^J|0\rangle$. The formula to understand is

$$S = :e^{\int q(t)\, K_t \frac{\delta}{\delta J(t)}\, dt}: \langle 0|S^J|0\rangle \Big|_{J=0}$$

Now

$$:e^{\int q(t) f(t)\, dt}: = e^{\int \underbrace{q^-(t)}_{\frac{e^{i\omega t} a^*}{\sqrt{2\omega}}} f(t)\, dt} \cdot e^{\int \underbrace{q^+(t)}_{\frac{e^{-i\omega t} a}{\sqrt{2\omega}}} f(t)\, dt}$$

$$= \sum_{p,q} \frac{1}{p!\, q!} \left(\frac{1}{\sqrt{2\omega}}\right)^{p+q} a^{*p} a^q \left(\int e^{i\omega t} f(t)\right)^p \left(\int e^{-i\omega t} f(t)\right)^q$$

If we are interested in the matrix elt. $\langle 0|a S a^*|0\rangle$ then only the terms $p=q=1$ and $p=q=0$ matter. For $p=q=1$ we get

$$\frac{1}{2\omega}\, a^* a \int dt_1\, e^{i\omega t_1} K_{t_1} \frac{\delta}{\delta J(t_1)} \int dt_2\, e^{-i\omega t_2} K_{t_2} \frac{\delta}{\delta J(t_2)} \langle 0|S^J|0\rangle \Big|_{J=0}$$

or

$$\boxed{a^* a\, \frac{-1}{2\omega} \int dt_1 dt_2\, e^{i\omega(t_1 - t_2)} K_{t_1} K_{t_2} \langle 0|T[q(t_1) q(t_2) S]|0\rangle}$$

Recall that $$iG(t_1, t_2) = \frac{\langle 0|T[q(t_1)\, q(t_2)\, S]|0\rangle}{\langle 0|S|0\rangle}.$$

For $t_1 \gg 0$ and $t_2 \ll 0$ we have

$$iG(t_1, t_2) = \frac{\langle 0|q(t_1)\, S\, q(t_2)|0\rangle}{\langle 0|S|0\rangle}$$

$$= \frac{1}{2\omega}\, e^{-i\omega(t_1 - t_2)}\, \frac{\langle 0|a\, S\, a^*|0\rangle}{\langle 0|S|0\rangle}$$

September 23, 1979

$$\int_a^b f \frac{d^2}{dt^2} g \, dt = \left[f g' - f' g \right]_a^b + \int f'' g \, dt$$

$$\int_a^b f \underbrace{\left(\frac{d^2}{dt^2} + \omega^2 \right)}_{K_t} g \, dt = W(f,g) \Big]_a^b + \int_a^b \left(\frac{d^2}{dt^2} + \omega^2 \right) f \; g \, dt$$

$$\int dt_2 \, e^{-i\omega t_2} K_{t_2} \langle 0 | T[q(t_1) q(t_2) S] | 0 \rangle$$

$$= W\Big(e^{-i\omega t_2} , \underbrace{\langle 0 | T[q(t_1) q(t_2)] S | 0 \rangle}_{\text{propert. to } e^{-i\omega t_2} \text{ for } t_2 \gg 0, \; e^{i\omega t_2} \text{ for } t_2 \ll 0} \Big) \Big]_{-\infty}^{\infty}$$

$$= -2i\omega \, e^{-i\omega t_2} \langle 0 | T[q(t_1) q(t_2) S] | 0 \rangle \qquad t_2 \ll 0$$

$$\int dt_1 \, e^{i\omega t_1} K_{t_1} \int dt_2 \, e^{-i\omega t_2} K_{t_2} \langle \quad \rangle = (-2i\omega)^2 e^{i\omega(t_1 - t_2)} \langle \quad \rangle$$

for $t_1 \gg 0 \quad t_2 \ll 0$

$$= (-2i\omega)^2 \frac{1}{2\omega} \langle 0 | a S a^* | 0 \rangle.$$

Thus the $a^* a$ term in S is

$$a^* a \left(-\frac{1}{2\omega} \right) (-2i\omega)^2 \frac{1}{2\omega} \langle 0 | a S a^* | 0 \rangle$$

$$= a^* a \langle 0 | a S a^* | 0 \rangle.$$

But we have made a slight mistake in doing the second integral $\int dt_1 \, e^{i\omega t_1} K_{t_1}$.

$$\int dt_1 \, e^{i\omega t_1} K_{t_1} \quad e^{-i\omega t_2} \langle 0 | T[q(t_1) S] q(t_2) | 0 \rangle$$

$$= e^{i\omega t_1} e^{-i\omega t_2} \langle 0 | q(t_1) S q(t_2) | 0 \rangle$$

— lower limit which involves

$$e^{-i\omega t_2} W\left(e^{i\omega t_1}, \langle 0| S\, q(t_1) q(t_2)|0\rangle\right)$$

where $0 \gg t_1 \gg t_2$. The other order $q(t_2)q(t_1)$ gives 0 and the commutation $[q(t_1), q(t_2)] = \frac{1}{2\omega}\left(e^{-i\omega t_1} e^{i\omega t_2} - e^{i\omega t_1} e^{-i\omega t_2}\right)$ which gives another term. The net effect is that the $a^* a$ term is

$$a^* a\left(\langle 0|a S a^*|0\rangle - \langle 0|S|0\rangle\right)$$

It seems desirable to leave this formula for a while. Evidently the basic formula

$$S = \sum_{n \geq 0} \frac{i^n}{n!} \int dt_1 \ldots dt_n \; K_{t_1} \ldots K_{t_n} \langle 0| T[q(t_1) \ldots q(t_n) S]|0\rangle$$
$$: q_{in}(t_1) \ldots\ldots q_{in}(t_n):$$

is due to LSZ (see Schweber §18b (195)).

September 24, 1979

I want to understand diagrams in energy-momentum notation. Let's again consider the oscillator model

$$H = \frac{p^2}{2} + \frac{1}{2}\nu^2 q^2 + \frac{1}{2}\varepsilon(t) q^2 .$$

The notation is changed from ω to ν so that I can use ω as frequency variable. Introduce the Green's function

$$i\,G(t,t') = \frac{\langle 0|T[q(t)q(t')S]|0\rangle}{\langle 0|S|0\rangle}$$

which one can calculate via Dyson, Wick in terms of Feynman diagrams. Let's go over this carefully, at least to the first order in ε.

$$S = T\left\{ e^{-i\int \frac{\varepsilon(t)}{2} q(t)^2 dt} \right\}$$

$$\langle 0|T[q(t)q(t')S]|0\rangle = \langle 0|T[q(t)q(t')]|0\rangle$$

$$+ (-i)\int dt_1 \frac{\varepsilon(t_1)}{2} \langle 0|T[q(t)q(t')\, q(t_1)^2]|0\rangle$$

$$+ \frac{(-i)^2}{2!}\int dt_1 dt_2 \frac{\varepsilon(t_1)}{2}\frac{\varepsilon(t_2)}{2} \langle 0|T[q(t)q(t')\,q(t_1)^2 q(t_2)^2]|0\rangle$$

$$+ \cdots$$

Put $\Delta(t,t') = \langle 0|T[q(t)q(t')]|0\rangle = \dfrac{e^{-i\nu|t-t'|}}{2\nu}$

According to Wick's thm.

$$\langle 0|T[q(t)\,q(t')\,q(t_1)q(t_2)]|0\rangle = \Delta(t,t')\Delta(t_1,t_2) + \Delta(t,t_1)\Delta(t',t_2)$$

$$+ \Delta(t,t_2)\,\Delta(t',t_1)$$

$$\langle 0 | T[q(t)\, q(t')\, q(t_1)^2] | 0 \rangle = \Delta(t,t')\,\Delta(t_1,t_1) + 2\Delta(t,t_1)\,\Delta(t',t_1)$$

and so the first order term in the numerator for iG is

$$(-i)\int dt_1\, \frac{\varepsilon(t_1)}{2}\left[\Delta(t,t')\,\Delta(t_1,t_1) + 2\Delta(t,t_1)\Delta(t',t_1)\right]$$

$$= \Delta(t,t')\left\{ -i\int dt_1\, \frac{\varepsilon(t_1)}{2}\Delta(t_1,t_1)\right\} + (-i)\int dt_1$$ [redacted] $$\Delta(t,t_1)\,\varepsilon(t_1)\,\Delta(t_1,t')$$

The first term is [redacted] cancelled by the first order denominator term, so we get to first order

[redacted] $$iG(t,t') = \Delta(t,t') + (-i)\int dt_1\, \Delta(t,t_1)\,\varepsilon(t_1)\,\Delta(t_1,t')$$

or

$$G(t,t') = G_0(t,t') + \int dt_1\, G_0(t,t_1)\,\varepsilon(t_1)\, G_0(t_1,t) + \cdots$$

where

$$G_0(t,t') = \frac{e^{-i\nu|t-t'|}}{2i\nu} = +\int \frac{d\omega}{2\pi}\, \frac{e^{-i\omega(t-t')}}{\omega^2 - \nu^2 + i\eta}$$

$$\int dt_1\, G_0(t,t_1)\,\varepsilon(t_1)\, G_0(t_1,t')$$

$$= \int \frac{d\omega_1}{2\pi}\frac{d\omega_2}{2\pi}\int dt_1\, \varepsilon(t_1)\, \frac{e^{-i\omega_1(t-t_1)}\, e^{-i\omega_2(t_1-t')}}{(\omega_1^2 - \nu^2 + i\eta)(\omega_2^2 - \nu^2 + i\eta)}$$

$$= \int \frac{d\omega_1}{2\pi}\frac{d\omega_2}{2\pi}\, e^{-i\omega_1 t}\, \frac{1}{\omega_1^2 - \nu^2 + i\eta}\, \hat{\varepsilon}(\omega_1 - \omega_2)\, \frac{1}{\omega_2^2 - \nu^2 + i\eta}\, e^{i\omega_2 t'}$$

It might be better to Fourier transform t, t'. So [redacted] put

$$G_0(t,t') = \int \frac{d\omega}{2\pi}\frac{d\omega'}{2\pi}\, \hat{G}(\omega,\omega')\, e^{-i\omega t + i\omega' t'}$$

so that when $\hat{G}$ has $\delta(\omega - \omega')$ as a factor, then G is a function of $t - t'$. Thus

$$\hat{G}_0(\omega,\omega') = \frac{2\pi\delta(\omega-\omega')}{\omega^2-\nu^2+i\eta}$$

Thus we get the formula

$$\hat{G}(\omega,\omega') = \hat{G}_0(\omega,\omega') + \frac{1}{\omega^2-\nu^2+i\eta}\,\hat{\varepsilon}(\omega-\omega')\,\frac{1}{\omega'^2-\nu+i\eta} + \cdots$$

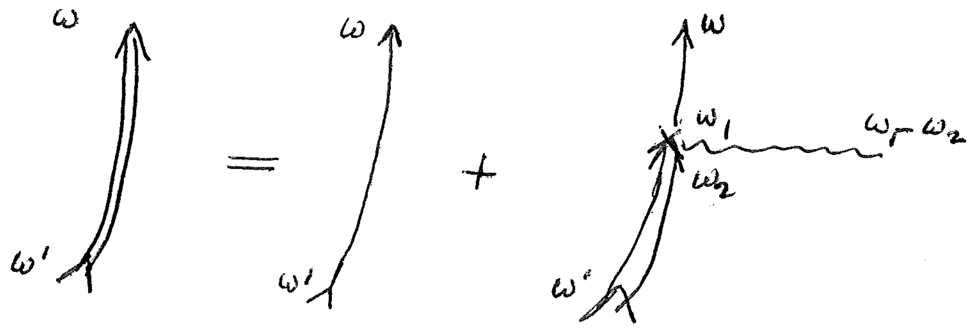
or better

$$\hat{G}(\omega,\omega') = \hat{G}_0(\omega,\omega') + \int\frac{d\omega_1}{2\pi}\frac{d\omega_2}{2\pi}\,G_0(\omega,\omega_1)\,\hat{\varepsilon}(\omega_1-\omega_2)\,G_0(\omega_2,\omega') + \cdots$$

The Dyson equation in this form is

$$\hat{G}(\omega,\omega') = \hat{G}_0(\omega,\omega') + \int\frac{d\omega_1\,d\omega_2}{(2\pi)^2}\,\hat{G}_0(\omega,\omega_1)\,\hat{\varepsilon}(\omega_1-\omega_2)\,\hat{G}(\omega_2,\omega')$$

which can be pictured in the form



Sept 28, 1979

Melrose's problem: Propagation of singularities for
$$\partial_t^2 u = (\partial_x^2 + x^2 \partial_y^2)\, u$$

First compute the bicharacteristic flow. The characteristic variety is given in the cotangent bundle of $\mathbb{R}^3$, that is $(t, x, y, \omega, \xi, \eta)$ space, by
$$\omega^2 = \xi^2 + x^2\eta^2 .$$

Perhaps it's natural to look at the flow in the forward direction, that is, to look at the hyper-surface
$$\omega = \sqrt{\xi^2 + x^2\eta^2} .$$

The bicharacteristic flow is the Hamiltonian flow belonging to the Hamiltonian
$$H(x, y, \xi, \eta) = \sqrt{\xi^2 + x^2\eta^2}$$

(Recall from your study of mechanics that the characteristic hypersurface determines bicharacteristic curves, but not a time parameter along them. One gets an actual flow by choosing a function $\tilde{H}$ such that the char. hypersurface is a level surface. When a time coordinate t is given on space time we can choose the big Hamiltonian $\tilde{H}$ so that ω appears linearly:
$$\tilde{H} = \omega + H(t, x, \xi) \qquad \text{so that} \quad \dot{t} = \frac{\partial \tilde{H}}{\partial \omega} = 1 .)$$

Anyway the bicharacteristic flow is
$$\dot{x} = \frac{\partial H}{\partial \xi} = (\xi^2 + x^2\eta^2)^{-1/2}\,\xi , \quad \dot{y} = (\xi^2 + x^2\eta^2)^{-1/2}\, x^2\eta$$

$$\dot{\xi} = -\frac{\partial H}{\partial x} = -(\xi^2 + x^2\eta^2)^{-1/2} x\eta^2 \qquad \dot{\eta} = -\frac{\partial H}{\partial y} = 0.$$

Since H is time-independent it is a constant of motion

$$\omega = \sqrt{\xi^2 + x^2\eta^2}$$

as well as η.

So

$$\dot{x} = \omega^{-1/2}\xi$$
$$\dot{\xi} = -\omega^{-1/2}\eta^2 x$$

or

$$\ddot{x} = -\frac{\eta^2}{\omega} x.$$

So

$$x = A\sin\sqrt{\frac{\eta^2}{\omega}}\,t \qquad A > 0$$

$$\xi = A|\eta|\cos\sqrt{\frac{\eta^2}{\omega}}\,t$$

$$\omega = \sqrt{\xi^2 + \eta^2 x^2} = \sqrt{\eta^2 A^2} = A|\eta|$$

So

$$\boxed{\begin{aligned} x &= \frac{\omega}{|\eta|}\sin\frac{|\eta|}{\sqrt{\omega}}t = \frac{\omega}{\eta}\sin\left(\frac{\eta}{\sqrt{\omega}}t\right) \\ \xi &= \omega\cos\left(\frac{\eta}{\sqrt{\omega}}t\right) \end{aligned}}$$

$$\dot{y} = \frac{\eta}{\sqrt{\omega}}x^2 = \frac{\eta}{\sqrt{\omega}}\,\frac{\omega^2}{\eta^2}\sin^2\left(\frac{\eta}{\sqrt{\omega}}t\right)$$

$$= \frac{\omega^{3/2}}{\eta}\,\frac{1}{2}\left(1 - \cos\left(2\frac{\eta}{\sqrt{\omega}}t\right)\right)$$

$$\boxed{y = \frac{\omega^{3/2}}{2\eta}t - \frac{\omega^2}{4\eta^2}\sin\left(2\frac{\eta}{\sqrt{\omega}}t\right) + y_0}$$

Special cases: $\eta = 0$, .

$$x = \sqrt{\omega}\,t \qquad y = \text{const.}$$

$$\xi = \omega$$

The real problem is to construct the Green's fn. for $\partial_t^2 - \partial_x^2 - x^2\partial_y^2$ with a δ fn singularity at $t=0$ $x = x_0$, $y = 0$, and then analyze its singularities. There are two Green's functions (forward or retarded, and advanced or backward), and we concentrate on the forward one.

$$\left(\partial_t^2 - \partial_x^2 - x^2\partial_y^2\right) G(t,x,y) = \delta(t)\,\delta(x-x_0)\,\delta(y)$$

$$G(t,x,y) = 0 \qquad t<0$$

It seems that we want to look at the operator $A = -\partial_x^2 - x^2\partial_y^2$ on $L^2(\mathbb{R}^2)$ and solve the equation

$$\partial_t^2 u = -Au$$

This is an oscillator, so the retarded Green's function is

$$\begin{cases} \dfrac{\sin\sqrt{A}\,t}{\sqrt{A}} & t \geq 0 \\ 0 & t \leq 0 \end{cases}$$

Sept 29, 1979

To construct the Green's function for $-\partial_x^2 + \eta^2 x^2$. First take the case $\eta = 1$.

$$(-\partial_x + x)(\partial_x + x)u = (-\partial_x^2 + x^2 - 1)u = 0$$

is satisfied by $u = e^{-x^2/2}$. Put $v = e^{-x^2/2} u$

$$e^{x^2/2}(-\partial_x + x)(\partial_x + x)e^{-x^2/2} = (-\partial_x + 2x)\partial_x$$

so the eigenvalue problem becomes

$$(-\partial_x^2 + 2x\partial_x) v = \lambda v$$

Try $\displaystyle v = \int_C e^{xt}\phi(t)\,dt$

where C is chosen so as to make integration by parts possible

$$x\partial_x v = \int \underbrace{xe^{xt}}_{\frac{d}{dt}(e^{xt})} t\phi\,dt = -\int e^{xt}\frac{d}{dt}(t\phi)\,dt$$

ϕ should satisfy

$$-t^2\phi - 2\frac{d}{dt}(t\phi) = \lambda\phi$$

$$\frac{t}{2}(t\phi) + \frac{d}{dt}(t\phi) + \frac{\lambda}{2t}t\phi = 0$$

$$\frac{t^2}{4} + \ln(t\phi) + \frac{\lambda}{2}\ln(t) = C$$

so

$$\phi = e^{-\frac{t^2}{4}} t^{-\frac{\lambda}{2}-1}$$ up to a scalar mult.

$$\therefore\quad v = \int_C e^{-\frac{t^2}{4}+xt} t^{-\frac{\lambda}{2}-1}\,dt = \text{const}\int_C e^{-t^2+2xt} t^{-\frac{\lambda}{2}-1}\,dt$$

We will take our basic solution of

$$(-\partial_x^2 + x^2)u = \lambda u$$

to be

(*) $$u_s(x) = e^{-x^2/2} \frac{1}{\Gamma(s)} \int_0^\infty e^{-t^2+2xt} t^s \frac{dt}{t}$$

where $\lambda = -2s+1$ The above integral converges only for $Re(s) >> 0$, but it has the analytic continuation

$$\frac{1}{\Gamma(s)} \int_0^\infty = \frac{1}{\Gamma(s)} \frac{1}{e^{2\pi i s} - 1} \int = \frac{e^{-i\pi s}}{2\pi i} \Gamma(1-s) \int e^{-t^2+2xt} t^s \frac{dt}{t}$$

as in the theory of the Γ-function. Also u_s is an entire function of s since the poles of $\Gamma(1-s)$ at $s = 1, 2, \dots$ coincide with zeroes of the integral.

$$u_s(0) = \frac{1}{\Gamma(s)} \int_0^\infty e^{-t^2} t^s \frac{dt}{t} = \frac{1}{\Gamma(s)} \int_0^\infty e^{-t} t^{s/2} \frac{dt}{2t}$$

$$= \frac{\frac{1}{2}\Gamma(\frac{s}{2})}{\Gamma(s)} \quad \text{entire with zeroes at } s = -1, -2, \dots$$

$$u_s'(0) = \frac{1}{\Gamma(s)} 2\int_0^\infty e^{-t^s} t^{s+1} \frac{dt}{t} = \frac{\Gamma(\frac{s+1}{2})}{\Gamma(s)} \quad \text{entire with zeroes at } s = 0, -2, -4, \dots$$

Now

$$u_s(x) \quad \text{decays as} \quad x \to -\infty$$

This is clear from (*) if $Re(s) >> 0$; the same formula holds for general s with t^s interpreted as a distribution, and this means the integral grows like a poly in x as $x \to -\infty$, so it's still OK.

$u_s^-(x) = u_s(-x)$ is the solution decaying as $x \to +\infty$. and its values are

$$u_s^-(0) = u_s(0)$$

$$u_s^{-\prime}(0) = -u_s'(0)$$

hence $$W(u_s, u_s^-) = u_s(0)u_s'(0)\begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} = -2\,\frac{\frac{1}{2}\Gamma(\frac{s}{2})}{\Gamma(s)}\,\frac{\Gamma(\frac{s+1}{2})}{\Gamma(s)}$$

$$= -\frac{\Gamma(\frac{s}{2})\Gamma(\frac{s+1}{2})}{\Gamma(s)^2}$$

But we have the duplication formula

$$2^{s-1}\Gamma(\tfrac{s}{2})\Gamma(\tfrac{s+1}{2}) = \sqrt{\pi}\,\Gamma(s)$$

so $$W(u_s, u_s^-) = -\frac{2^{-s+1}\sqrt{\pi}}{\Gamma(s)}$$

entire fn.
vanishing at $s = 0, -1, -2, \ldots$
corresp. to eigenvalues
$\lambda = -2s+1 = 1, 3, 5, \cdots$
for $-\partial_x^2 + x^2$.

So the Green's function for $-\partial_x^2 + x^2$ is

$$G_\lambda(x,x') = -\frac{u_s(x_<)\,u_s(-x_>)}{W(u_s, u_s^-)}$$

$$= \frac{\Gamma(s)}{2^{1-s}\sqrt{\pi}}\,u_s(x_<)\,u_s^-(x_>)$$

Check: G is the kernel for $(-\partial_x^2 + x^2 - \lambda)^{-1}$ so its residue at $\lambda_n = 2n+1$ should be $-\varphi_n(x)\varphi_n(x')$ where φ_n is normalized to 1. Take λ_1 or $s=0$.

$$\lim_{\lambda\to 1}(1-\lambda)\,G_\lambda(x,x') = \lim_{s\to 0} 2s\,G_\lambda(x,x')$$

$$= \frac{2}{2\sqrt{\pi}}\,u_0(x)\,u_0(x')$$

where $$u_0(x) = e^{-x^2/2}\,\frac{1}{2\pi i}\oint e^{-t^2+2xt}\,\frac{dt}{t} = e^{-x^2/2}$$

and $\int (e^{-x^2/2})^2\,dx = \int e^{-x^2}dx = \sqrt{\pi}$, so it works.

Next modify the above for $-\partial_x^2 + \eta^2 x^2$, by substituting $x \mapsto |\eta|^{1/2} x$.

$$\left(-\partial_x^2 + \eta^2 x\right) u_s(|\eta|^{1/2}x) = |\eta|\left(\left(-\frac{\partial}{\partial |\eta|^{1/2}x}\right)^2 + \left(|\eta|^{1/2}x\right)^2\right) u_s(|\eta|^{1/2}x)$$
$$= |\eta|\,(1-2s)\, u_s(|\eta|^{1/2}x)$$

so $u_s(|\eta|^{1/2}x)$ is an eigenfn. for $-\partial_x^2 + \eta^2 x^2$ with the eigenvalue $\lambda = |\eta|(1-2s)$. The only other change is when we compute the Wronskian where we pick up

$$\frac{d}{dx} u_s(|\eta|^{1/2}x)\Big|_{x=0} = |\eta|^{1/2} u_s'(0).$$

Thus

$$G_{\lambda,\eta}(x,x') = \frac{\Gamma(s)}{2^{1-s}\sqrt{\pi}}\frac{1}{|\eta|^{1/2}} u_s(|\eta|^{1/2}x_<)\, u_s^-(|\eta|^{1/2}x_>)$$

where
$$u_s(|\eta|^{1/2}x) = e^{-\frac{1}{2}|\eta|x^2}\frac{1}{\Gamma(s)}\int_0^\infty e^{-t^2+2|\eta|^{1/2}xt}\, t^s \frac{dt}{t}$$

and $$s = \frac{1}{2}\left(1 - \frac{\lambda}{|\eta|}\right).$$

Now the program is to find the forward Green's function for:

$$\left(\partial_t^2 - \partial_x^2 - x^2\partial_y^2\right) G(x,y,t) = \delta(x-x')\,\delta(y)\,\delta(t)$$

Recall that the abstract problem

$$\left(\partial_t^2 + A\right) G(t) = \delta(t) \qquad G(t) = 0 \quad t<0$$

has the solution

$$G(t) = \theta(t)\frac{\sin\sqrt{A}\,t}{\sqrt{A}} = \int \frac{d\omega}{2\pi} e^{i\omega t}\hat{G}(\omega)$$

where $$\hat{G}(\omega) = \int_0^\infty e^{-i\omega t}\frac{e^{i\sqrt{A}t} - e^{-i\sqrt{A}t}}{2\sqrt{A}\,i}\,dt = \frac{1}{2\sqrt{A}\,i}\left(\frac{1}{i\omega - i\sqrt{A}} - \frac{1}{i\omega + i\sqrt{A}}\right)$$

$$= - \frac{1}{\omega^2 \,\text{[redacted]}\, A} \qquad \text{continued analytically from LHP.}$$

[redacted] So [redacted]

$$\hat{G}(\omega) = - \frac{1}{(\omega - i\varepsilon)^2 \,\text{[redacted]}\, A} \qquad \varepsilon \text{ pos. infintesimal}$$

$$G(x, y; t) = \int \frac{d\omega}{2\pi} e^{i\omega t} \int \frac{d\eta}{2\pi} e^{i\eta y} \hat{G}(x, x'; \eta, \omega)$$

$$\left(-\omega^2 - \partial_x^2 + \eta^2 x^2\right) \hat{G} = \delta(x - x')$$

Here the eigenvalue λ is $+\omega^2$. [redacted] So we have

$$\hat{G}(x, x'; \eta, \omega) = \frac{\Gamma(s)}{2^{1-s}\sqrt{\pi}} \frac{1}{|\eta|^{1/2}} u_s(|\eta|^{1/2} x_<)\, u_s(-|\eta|^{1/2} x_>)$$

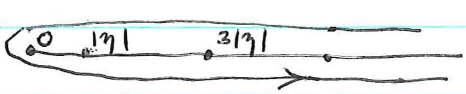
$$\text{where } 1 - 2s = \frac{\omega^2}{|\eta|}$$

$$= \sum_n \frac{\varphi_n(|\eta|^{1/2} x)\, \varphi_n(|\eta|^{1/2} x')}{\left(-\frac{\omega^2}{\text{[redacted]}} + (2n+1)|\eta|\right) |\eta|^{1/2}}$$

So therefore we [redacted] have

$$G(x, x'; y, t) = \int \frac{d\omega}{2\pi} \int \frac{d\eta}{2\pi} e^{i\omega t + i\eta y} \frac{\Gamma(s)}{2^{1-s}\sqrt{\pi}} \frac{1}{|\eta|^{1/2}} u_s(|\eta|^{1/2} x_<)\, u_s(-|\eta|^{1/2} x_>)$$

where the ω integration is done below the real axis and $1 - 2s = \frac{\omega^2}{|\eta|}$. This means that ω^2 describes a contour:



and hence $s = \frac{1}{2}\left(1-\frac{\omega^2}{|\eta|}\right)$ describes a contour

picking up the poles of $\Gamma(s)$ at $s = 0, -1, -2, \ldots\ldots\ldots$.

$$u_s\left(|\eta|^{1/2}x\right) = e^{-\frac{1}{2}|\eta|x^2}\frac{1}{\Gamma(s)}\int_0^\infty e^{-t^2+2|\eta|^{1/2}xt}\, t^s\frac{dt}{t}$$

$$= e^{-\frac{1}{2}|\eta|x^2}|\eta|^{-s/2}\frac{1}{\Gamma(s)}\int_0^\infty e^{-\frac{t^2}{|\eta|}+2xt}\, t^s\frac{dt}{t}$$

$$G(x,x';y,t) = \int\frac{d\omega}{2\pi}\int\frac{d\eta}{2\pi}\, e^{i\omega t+i\eta y}\,\frac{1}{2^{1-s}\sqrt{\pi}\,|\eta|^{1/2}}\, e^{-\frac{1}{2}|\eta|x_<^2}|\eta|^{-s/2}$$

$$\times\int_0^\infty e^{-\frac{t_1^2}{|\eta|}+2x_<t_1}\, t_1^s\frac{dt_1}{t_1}\; e^{-\frac{1}{2}|\eta|x_>^2}|\eta|^{-s/2}\int_0^\infty e^{-\frac{t_2^2}{|\eta|}-2x_>t_2}\, t_2^s\frac{dt_2}{t_2}$$

$$= \int\frac{d\omega}{2\pi}\int\frac{d\eta}{2\pi}\, e^{i\omega t+i\eta y}\,\frac{1}{2^{1-s}\sqrt{\pi}}\,|\eta|^{-s-1/2}\, e^{-\frac{1}{2}|\eta|(x^2+x'^2)}\frac{1}{\Gamma(s)}$$

$$\times\int_0^\infty\int_0^\infty\frac{dt_1}{t_1}\frac{dt_2}{t_2}\, e^{-\frac{t_1^2+t_2^2}{|\eta|}}(t_1t_2)^s\, e^{2(t_1x_<-t_2x_>)}$$

Now
$$x_> = \frac{1}{2}\left(x+x'+|x-x'|\right)$$
$$x_< = \frac{1}{2}\left(x+x'-|x-x'|\right)$$

so
$$2\left(t_1x_<-t_2x_>\right) = (t_1-t_2)(x+x') - (t_1+t_2)|x-x'|$$

Now the problem is to understand the singularities of $G(x,x';y,t)$. It's not clear how to get this out of the above integral expression.

Question: Given $f(x) = \int\frac{d^n\xi}{(2\pi)^n}\, e^{ix\cdot\xi}\hat{f}(\xi)$

what is the exact relation between singularities of f and the behavior of $\hat{f}$ at ∞?