quantum field for $\partial_t^2 - \partial_x^2 - V(x)$
adiabatic thm: 220, 229 Puiseux 231
S-matrix for oscillator with $J(t)q$ perturb. 247



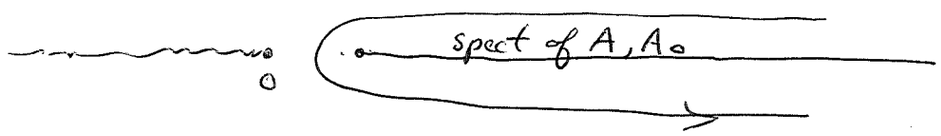
Put $A_0 = -\frac{d^2}{dx^2} + m^2$, $A = -\frac{d^2}{dx^2} + m^2 + \varepsilon(x)$

where $\varepsilon \in C_0^\infty(R)$ is small enough so that $A > 0$.

Then

$$\mathrm{tr}\left(A^{-s} - A_0^{-s}\right) = \frac{1}{2\pi i} \oint \lambda^{-s} \mathrm{tr}\left\{\frac{1}{\lambda - A} - \frac{1}{\lambda - A_0}\right\} d\lambda$$

where the contour is




Now

$$\mathrm{tr}\left\{\frac{1}{\lambda - A} - \frac{1}{\lambda - A_0}\right\} = \frac{d}{d\lambda} \mathrm{tr}\left\{\log(\lambda - A) - \log(\lambda - A_0)\right\}$$

$$= \frac{d}{d\lambda} \log\det\left(\frac{\lambda - A}{\lambda - A_0}\right)$$

where $\frac{\lambda - A}{\lambda - A_0} = 1 - G_0 \varepsilon$ $\qquad G_0 = \frac{1}{\lambda - A_0} = \frac{1}{\lambda - m^2 + \frac{d^2}{dx^2}}$

So if we put $\lambda - m^2 = k^2$, then $G_0 = \frac{1}{k^2 + \frac{d^2}{dx^2}}$ is the usual Green's function, and one has

$$\mathrm{tr}\left(A^{-s} - A_0^{-s}\right) = \frac{1}{2\pi i} \int (k^2 + m^2)^{-s} \, d \log\det\left(1 - G_0(k)\varepsilon\right)$$


$$= \sum_{\substack{\text{bound} \\ \text{states}}} (E_i)^{-s} + \frac{1}{2\pi i} \int_0^\infty (k^2 + m^2)^{-s} \, d \log\det\left(\frac{1 - G_0^-(k)\varepsilon}{1 - G_0^+(k)\varepsilon}\right)$$

Now recall that $\det\left(\frac{1 - G_0^-(k)\varepsilon}{1 - G_0^+(k)\varepsilon}\right)$ is the determinant of the scattering matrix for the potential $\varepsilon$, hence it is essentially $\frac{T(k)}{\overline{T(k)}}$ where $T(k)$ is the transmission

coefficient. For $\varepsilon$ smooth one knows that $R(k)$ $O(\frac{1}{k^N})$ for all $N$, but this doesn't imply that $T(k) \to 1$ rapidly, only that $|T(k)| \to 1$.

Review scattering on the line:

$$\left(-\frac{d^2}{dx^2} + V\right)\psi = k^2\psi \qquad V \in C_0^\infty(\mathbb{R}).$$

Define $A(k)$, $B(k)$ by

$$e^{-ikx} \longleftrightarrow A(k)e^{-ikx} + B(k)e^{ikx}$$

$e^{-ikx}$: small soln as $x \to -\infty$ for $\operatorname{Im} k > 0$

$A(k)e^{-ikx}$: a large soln as $x \to +\infty$

so that $A(k)$ is going to be defined & analytic in the UHP even when $V$ has inf. support; in any case bound states are given by the zeroes of $A(k)$ in the UHP.

$$\underbrace{\frac{1}{A(k)}}_{T(k)} e^{-ikx} \longleftrightarrow e^{-ikx} + \underbrace{\frac{B(k)}{A(k)}}_{R(k)} e^{ikx}$$

Call this solution $\psi_{-k}^+$ because it satisfies the integral equation

$$\psi_{-k}^+(x) = e^{-ikx} + \overbrace{\int \underbrace{\frac{e^{ik|x-x'|}}{2ik}}_{G_0^+} V(x')\psi_{-k}^+(x')\,dx'}^{\text{in } L^2 \text{ for } k \text{ in UHP}}$$

If we iterate this

$$\psi_{-k}^+(x) = e^{-ikx} + G_0^+ V e^{-ikx} + G_0^+ V G_0^+ V e^{-ikx} + \cdots$$

$$= e^{-ikx} + \frac{1}{2ik}\int e^{ik|x-x'|} V e^{-ikx'}\,dx' + \underbrace{\frac{1}{(2ik)^2}\cdots}_{O(\frac{1}{k^2}) \quad k \text{ real}}$$

From this we get

$$T(k) = 1 + \frac{1}{2ik}\int V(x')dx' + \cdots$$

$$R(k) = \frac{1}{2ik}\int V(x')e^{-2ikx'}dx' + \cdots\cdots$$

So for $k \to +\infty$ we have

$$\log\frac{T(k)}{\overline{T}(k)} = \frac{1}{ik}\int\varepsilon + O\left(\frac{1}{k^2}\right)$$

so for $s=\frac{1}{2}$

$$\operatorname{tr}(A^{-s}-A_0^{-s}) \sim \frac{1}{2\pi i}\int_0^\infty \sqrt{k^2+m^2}\,\frac{1}{i}\int\varepsilon\; d\left(\frac{1}{k}\right) = \frac{1}{2\pi}\int\varepsilon\cdot\underbrace{\int_0^\infty \frac{\sqrt{k^2+m^2}}{k^2}dk}_{\text{diverges like } \log(k).}$$

On the other hand from the box case we get

$$2\Delta E = \int_{-\infty}^{\infty}\operatorname{tr}\left(A^{1/2}-A_0^{1/2}\right) \sim \sum_{k\in\frac{2\pi}{L}\mathbb{Z}}\frac{\varepsilon}{2\sqrt{k^2+m^2}}$$

$$\sim \int_0^\infty \frac{\varepsilon}{2\sqrt{k^2+m^2}}\,\frac{L}{2\pi}dk = \frac{L\varepsilon}{2\pi}\cdot\underbrace{\int_0^\infty \frac{dk}{\sqrt{k^2+m^2}}}_{\text{diverges like } \log(k)}$$

Hence one gets the same kind of divergence, namely

$$\frac{1}{2\pi}\int\varepsilon\cdot\log(k)$$

Therefore it appears that the level shift $\Delta E$ will be finite provided we assume $\int\varepsilon = 0$.

Let's summarize: We are considering the quantum field theory obtained from the KG equation in 1-space dimension:

$$\frac{\partial^2\phi}{\partial t^2} = -\left(-\frac{d^2}{dx^2}+m^2\right)\phi.$$

This gets quantized as a (generalized) harmonic oscillator,

in fact we get 1 [redacted] independent oscillator of frequency $\omega_k = \sqrt{k^2+m^2}$ for each wave vector $k$. The relevant formulas are

$$H_0 = [redacted] \sum \omega_k a_k^* a_k$$

$\wedge$ means operator

$$\hat{\phi}(x) = \frac{1}{\sqrt{V}} \sum e^{ikx} \hat{\phi}_k = \frac{1}{\sqrt{V}} \sum e^{ikx} \left( \frac{a_k + (a_{-k})^*}{\sqrt{2\omega_k}} \right)$$

except this is for a box with periodic boundary conditions.

Now the above is the unperturbed theory. The perturbed field equation is

$$(*) \qquad \frac{\partial^2 \phi}{\partial t^2} = -\left( -\frac{d^2}{dx^2} + m^2 + \varepsilon \right) \phi$$

where $\varepsilon \in C_0^\infty(\mathbb{R})$ is small enough so the operator on the right remains $> 0$. Hence we are perturbing our oscillators by changing the spring constants. The perturbed Hamiltonian is still a (generalized) harmonic oscillator, unlike an anharmonic perturbation such as $\varepsilon \phi^3$. The quantum field theory belonging [redacted] to the perturbed equation will be described [redacted] as a system of oscillators belonging to the points of the spectrum of the operator $-\frac{d^2}{dx^2} + m^2 + \varepsilon$. Scattering theory tells us that the operators $-\frac{d^2}{dx^2} + m^2$ and $-\frac{d^2}{dx^2} + m^2 + \varepsilon$ are unitarily equivalent in 2 ways, assuming that there are no bound states. So therefore the question is whether we can set up an isomorphism of the quantum field theories. Actually this is probably true. What we really want is some version of scattering [redacted] theory in the Hilbert space of the KG equation.

Review quantizing the KG field. Instead of $\phi(x)$ for the field value at the point $x$, let me write $q_x$, (think of the $x$-th coordinate); also write $p_x$ for the $x$-th momentum. When we quantize these become operators satisfying

$$[q_x, q_{x'}] = [p_x, p_{x'}] = 0 \qquad i[p_x, q_{x'}] = \delta(x-x')$$

Next make Fourier transform

$$q_x = \frac{1}{\sqrt{2\pi}}\int dk\; e^{ikx} Q_k \qquad \text{or} \qquad Q_k = \frac{1}{\sqrt{2\pi}}\int dx\; q_x e^{-ikx}$$

$$p_x = \frac{1}{\sqrt{2\pi}}\int dk\; e^{-ikx} P_k \qquad \text{or} \qquad P_k = \frac{1}{\sqrt{2\pi}}\int dx\; p_x e^{ikx}$$

where we changed sign in the exponential so that

$$[iP_k, Q_{k'}] = \frac{1}{2\pi}\int dx \int dx' \underbrace{[ip_x, q_{x'}]}_{\delta(x-x')} e^{i(kx-k'x')}$$

$$= \frac{1}{2\pi}\int dx\; e^{i(k-k')x} = \delta(k-k').$$

Now for the oscillator given by $\left(-\frac{d^2}{dx^2} + m^2\right)$, where the $k$-th mode has frequency $\omega_k = \sqrt{k^2+m^2}$, we introduce creation operators $a_k^*$ and ann. ops. $a_k$ by

$$a_k = \frac{1}{\sqrt{2\omega_k}}\left(iP_{-k} + \omega_k Q_k\right)$$

$$a_k^* = \frac{1}{\sqrt{2\omega_k}}\left(-iP_k + \omega_k Q_{-k}\right)$$

and these satisfy the commutation relns.

$$[a_k, a_{k'}] = 0 = [a_k^*, a_{k'}^*] \qquad [a_k, a_{k'}^*] = \delta(k-k')$$

Then one has

$$Q_k = \frac{a_k + a_{-k}^*}{\sqrt{2\omega_k}} \qquad P_k = -i\sqrt{\frac{\omega_k}{2}}\left(a_{-k} - a_k^*\right)$$

or finally the following formulas

$$\begin{cases} q_x = \frac{1}{\sqrt{2\pi}} \int \frac{dk}{\sqrt{2\omega_k}} \left(e^{ikx} a_k + e^{-ikx} a_k^*\right) \\ p_x = \frac{1}{\sqrt{2\pi}} \int dk \sqrt{\frac{\omega_k}{2}} \left(-i e^{ikx} a_k + i e^{-ikx} a_k^*\right) \end{cases}$$

In practice one starts with Fock space with the operator $a_k, a_k^*$ and then one defines the operators $q_x$ (denoted $\phi(x)$) and $p_x$ (denoted $\pi(x)$) by the above. One can compute the Hamiltonian

$$\frac{1}{2}\int \left(\pi(x)^2 + \phi(x)(-\Delta + m^2)\phi(x)\right) dx = \int dk\, \omega_k \left(\frac{a_k^* a_k + a_k a_k^*}{2}\right)$$

$$= \int dk\, \omega_k a_k^* a_k + \underbrace{\frac{1}{2}\int dk\, \omega_k}_{\text{infinite constant}}$$

The infinite ground state energy $\frac{1}{2}\int dk\, \omega_k$ is removed and one looks at the Hamiltonian

$$H_0 = \int dk\, \omega_k a_k^* a_k .$$

We want next to perturb the operator $A_0 = -\Delta + m^2$ into the operator $A = -\Delta + m^2 + \varepsilon$, which means we add to the Hamiltonian the perturbation

$$H' = \frac{1}{2}\int \varepsilon(x)\phi(x)^2 dx = \frac{1}{2}\int dx\, \varepsilon(x) \left[\int \frac{dk}{\sqrt{2\pi}\sqrt{2\omega_k}} \left(e^{ikx} a_k + e^{-ikx} a_k^*\right)\right]^2$$

$$= \frac{1}{2}\frac{1}{2\pi}\int dx\, \varepsilon(x) \int \frac{dk}{\sqrt{2\omega_k}} e^{ikx}\left(a_k + a^*_{-k}\right) \int \frac{dk'}{\sqrt{2\omega_{k'}}} e^{ik'x}\left(a_{k'} + a^*_{-k'}\right)$$

$$= \frac{1}{2}\frac{1}{2\pi}\iint \frac{dk}{\sqrt{2\omega_k}}\frac{dk'}{\sqrt{2\omega_{k'}}}\, \hat{\varepsilon}(k+k')\left[a_k a_{k'} + a^*_{-k} a^*_{-k'} + a^*_{-k} a_{k'} + \underbrace{a_k a^*_{-k'}}_{[a_k, a^*_{-k'}] + a^*_{-k'} a_k}\right]$$

$[a_k, a^*_{-k'}] = \delta(k+k')$

so if we wrote $N(\phi(x)^2)$ instead of $\phi(x)^2$ we have the extra term

$$\frac{1}{2}\frac{1}{2\pi}\int \frac{dk}{2\omega_k}\, \hat{\varepsilon}(0) \qquad \hat{\varepsilon}(k) = \int \varepsilon(x) e^{ikx} dx$$

$$\text{so } \hat{\varepsilon}(0) = \int \varepsilon$$

which will be infinite unless we assume $\int \varepsilon = 0$.

This perhaps is why one writes

$$H' = \frac{1}{2}\int \varepsilon(x) \underbrace{:\phi(x)^2:}_{N(\phi(x)^2)}\, dx .$$

Hence

$$\phi(x)^2 = :\phi(x)^2: + \underbrace{\frac{1}{2\pi}\int \frac{dk}{2\omega_k}}_{\text{infinite constant independent of } x.}$$

Review the program: I start with the KG equation in 1 space dimension

$$\ddot{\phi} = -(-\Delta + m^2)\phi$$

which can be quantized as a (generalized) harmonic oscillator. The I want to consider a perturbed equation

$$\ddot{\phi} = -(-\Delta + m^2 + \varepsilon)\phi$$

where $\varepsilon \in C_0^\infty(\mathbb{R})$ is small enough so that the operator $-\Delta + m^2 + \varepsilon$ is $> 0$. The perturbed equation can be also quantized as a harmonic oscillator, and because the operators $-\Delta + m^2$ and $-\Delta + m^2 + \varepsilon$ are unitarily equivalent (assuming no bound states for the latter), we know these two quantizations are unitarily equivalent. However I want to carry out the equivalence using perturbation theory within the Hilbert space associated the KG equation, since this is the model one uses with more complicated non-linear perturbations.

So the first problem is how to construct the new ground state, and calculate the new ground state energy. Then you want to construct the new 1-particle states, always within the old Hilbert space. Now a model one can work with is furnished by lattice phonons, treated in Mattuck's book, or Donlach-Sondheimer. So it would appear that

the calculations can be done in a discrete finite-dimensional situation.

Note: For an oscillator $H_0 = \frac{p^2}{2} + \frac{1}{2}(\omega q)^2$ one obtain the 1-particle states by acting with "linear" operators $c_1 p + c_2 q$ on the ground state. In fact for a general oscillator with coordinate operators $q_i$ we get a basis for the 1-particle states in the form $q_i|0\rangle$, although this is not an orthonormal basis. (Review the formulas: Define

$$i G_0(t)_{ij} = \langle 0 | T[q_i(t) q_j] | 0 \rangle$$

Then

$$\left(\frac{d^2}{dt^2} + \omega^2\right) i G_0(t) = \frac{1}{i}\delta(t)$$

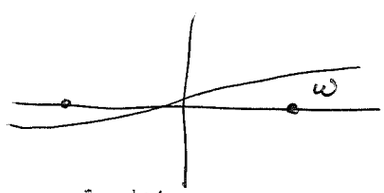
since

$$\frac{d}{dt} i G_0(t)_{ij}\Big]_{0-}^{0+} = \langle 0 | p_i q_j - q_j p_i | 0 \rangle = \frac{1}{i}\delta_{ij}.$$

so

$$\left(-k^2 + \omega^2\right) i\hat{G}_0(k) = \frac{1}{i} \qquad \hat{G}_0(k) = \frac{1}{k^2 - \omega^2 + i\eta}$$

$$G_0(t) = \int \frac{dk}{2\pi} \frac{e^{-ikt}}{k^2 - \omega^2 + i\eta}$$




$$= \frac{-2\pi i}{2\pi} \frac{e^{-i\omega|t|}}{2\omega} = \frac{e^{-i\omega|t|}}{2i\omega}$$

Hence

$$\langle 0 | q_i q_j | 0 \rangle = \left(\frac{1}{2\omega}\right)_{ij} = \frac{1}{2}\left(\omega^{-1}\right)_{ij}$$

gives the inner product between the 1-particle states $q_i|0\rangle$.)

Consequently if we have a perturbation

$$H = \frac{1}{2}p^2 + \frac{1}{2}(\omega q)^2 + \frac{1}{2} q \cdot \varepsilon q$$

Then once the new ground state $\Psi_0$ has been found, all we have to do is to operate on it by the coordinate operators: $q_i \Psi_0$, so as to get the 1-particle states.

My notation of the old oscillator potential energy $\omega^2$ is going to lead to confusion. It would be better maybe to use

$$H_0 = \tfrac{1}{2} p^2 + \tfrac{1}{2}(\omega_0 q)^2$$

or maybe $\frac{1}{2}(\Omega_0 q)^2$, so as to leave $\omega$ to be a frequency variable. Then

$$\hat{G}_0(\omega) = \frac{1}{\omega^2 - \Omega_0^2 + i\eta} \qquad \hat{G}(\omega) = \frac{1}{\omega^2 - \Omega_0^2 - \Sigma + i\eta}$$

In practice $\Omega_0$ is a diagnal matrix (Einstein oscillators.)

Recall that the poles of $\hat{G}(\omega)$ give the excitation frequencies, or better, the energy levels of the new particles.

---

It seems that to compute the Green's functions is easier to do than to compute the actual states. Somehow Green's functions carry all the good information, or at least most of it. At the moment I don't really understand why one can interpret $\psi^*(x)|\Psi_0\rangle$ as the state obtained from the ground state by adding a particle at $x$.

The program now will be to compute the new ground state for the perturbed oscillator by perturbation theory. Go over this more generally armed with what I know about Goldstone's thm. Review first order business:

$$H = H_0 + V$$

$$H_0 \varphi_a = E_a \varphi_a \qquad \text{assume non-deg.}$$

$$\psi_a = \varphi_a + \psi_a^{(1)} + \cdots\cdots$$

$$E_a' = E_a + E_a^{(1)} + \cdots\cdots$$

$$0 = (H_0 - E_a')\psi_a = (V - E_a^{(1)})\varphi_a + (H_0 - E_a)\psi_a^{(1)} + \cdots$$

$$\therefore \quad E_a^{(1)} = \langle \varphi_a | V | \varphi_a \rangle \qquad \text{assuming } \langle \varphi_a | \varphi_a \rangle = 1$$

and $\psi_a^{(1)} = (E_a - H_0)^{-1} Q V \varphi_a$

$Q$: orth projection on $\varphi_a^{\perp}$

Gell-Mann Low gives:

$$\psi_a = \lim_{\eta \to 0+} \frac{\varphi_a + \frac{1}{i\eta + E_a - H_0} V\varphi_a + \frac{1}{2i\eta + E_a - H_0} V \frac{1}{i\eta + E_a - H_0} V\varphi_a + \cdots}{1 + \frac{\langle \varphi_a | V | \varphi_a \rangle}{i\eta} + \frac{\langle \varphi_a | V \frac{1}{i\eta + E_a - H_0} V | \varphi_a \rangle}{2i\eta} + \cdots}$$

To first order this is

$$\varphi_a + \underbrace{\lim_{\eta \to 0+} \left\{ \frac{1}{i\eta + E_a - H_0} V\varphi_a - \frac{\langle \varphi_a | V | \varphi_a \rangle}{i\eta} \varphi_a \right\}}_{(E_a - H_0)^{-1} Q V \varphi_a}$$

Instead of the above series

$$\varphi_a + \frac{1}{i\eta + E_a - H_0} V\varphi_a + \frac{1}{2i\eta + E_a - H_0} V \frac{1}{i\eta + E_a - H_0} V + \cdots$$

which comes from the adiabatic method, and which converges, so it seems in the finite dim'l situation, one might use the series

$$\psi = \varphi_a + \frac{1}{i\eta + E_a - H_0} V\varphi_a + \left(\frac{1}{i\eta + E_a - H_0} V\right)^2 \varphi_a + \cdots$$

Clearly $$(i\eta + E_a - H_0)\psi = (i\eta + E_a - H_0)\varphi_a + V\psi$$

or $$(i\eta + E_a - H)\psi = i\eta\,\varphi_a$$

Now divide by $\langle \varphi_a | \psi \rangle$ to get

$$(i\eta + E_a - H)\frac{\psi}{\langle \varphi_a | \psi \rangle} = i\eta \frac{\varphi_a}{\langle \varphi_a | \psi \rangle}$$

If one assumes that $\dfrac{\psi}{\langle \varphi_a | \psi \rangle}$ converges to $\Psi$ as $\eta \to 0+$, then as

$$\langle \varphi_a | \psi \rangle = 1 + \frac{1}{i\eta} (\varphi_a | V | \varphi_a) + \frac{1}{i\eta} \langle \varphi_a | V \frac{1}{i\eta + E_a - H_0} V | \varphi_a \rangle + \cdots$$

one expects $\langle \varphi_a | \psi \rangle$ either gets infinite or stays away from 0, hence we have in the limit

$$(E_a - H)\Psi = 0.$$

So there is no ground energy shift under these conditions. Actually we have assuming $i\eta + E_a$ is

not an eigenvalue of $H$

$$\psi = \left(1 - \frac{1}{i\eta + E_a - H_0} V\right)^{-1} \varphi_a = (i\eta + E_a - H)^{-1}(i\eta + E_a - H_0)\varphi_a$$

and if $E_a$ [redacted] is also not an eigenvalue of $H$, then as $\eta \to 0$, this converges to

$$(E_a - H)^{-1}(E_a - H_0)\varphi_a = 0. \qquad \therefore \langle \varphi_a | \psi \rangle \to 0$$

Thus the above "geometric" $\psi$ is not as good as the "adiabatic" $\psi$.

This raises the question of whether for the adiabatic $\psi$, one has that [redacted].

$$\lim_{\eta \to 0} \frac{\psi}{(\varphi_a | \psi)}$$

exists, even for $H_0, V$ finite-dimensional operators. (The existence of this limit seems to be implicit in the proofs of Gell-Mann-Low thm. presented in Schweber and Fetter-Walecka.)

The problem of whether adiabatic switching works for finite-dim'l matrices $H_0, V$ seems to be very interesting. The idea is that somehow one is looking at the way $\varphi_a$ evolves into an eigenvector for $H$. Now I believe that if we have $H = H_0 + \lambda V$, then [redacted] as soon as one hits a value of $\lambda$ for which [redacted] two eigenvalues coincide one runs into problems.

The problem now is to see if the adiabatic approach works for finite-dimensional operators. The first thing to do is to see that it works when the perturbation $V$ is sufficiently small that the given non-degenerate eigenvalue $E_a$ for $H_0$ moves into a non-degenerate eigenvalue for $H = H_0 + V$.

---

Let $H_0, V$ be $n \times n$ matrices. The equation

$$\det(\mu - (H_0 + \lambda V)) = 0$$

defines an affine curve in $\mathbb{C}^2$ of degree $n$ over the $\lambda$-line. Assume that for generic $\lambda$, the matrix $H_0 + \lambda V$ has distinct eigenvalues. Then the curve is smooth over points in the $\lambda$-line where the eigenvalues remain distinct, and moreover, one has a canonical sub-line bundle given by the eigenspace corresponding to a point $(\mu, \lambda)$ on the curves. Suppose $\lambda = \lambda_0$ is a point where eigenvalues coincide, say that $\mu_0$ is an eigenvalue of multiplicity $\rho$ of $H_0 + \lambda_0 V$. Then as $\lambda$ describes a loop around $\lambda_0$, there is a monodromy transformation on the set of $\rho$ eigenvalues $\mu$ over $\lambda$ which collapse to $\mu_0$. This monodromy transformation is a permutation of degree $\rho$. An interesting question is how to compute this monodromy permutation.

Question: Suppose $H_0, V$ hermitian and $\lambda_0$ is real.

Is there any chance that the monodromy permutation around $\lambda_0$ is trivial?

[redacted] This seems to follow by conjugation: Hermitianness implies that $\mu_0$ and the eigenvalues $\{\mu\}$ associated to it over a real $\lambda$ near $\lambda_0$ are real. So if we take $\mu$ over $\lambda$ with $\lambda < \lambda_0$ and let $\lambda_t$ run clockwise around $\lambda_0$ and lift this semi-circle to $\mu_t$ over $\lambda_t$, we want to see if at the end [redacted] (where $\lambda_1 = \lambda_0 + (\lambda_0 - \lambda)$) the value $\mu_1$ is the same as the value you get by going counter-clockwise. But $\overline{\lambda_t}$, $\overline{\mu_t}$ give the counter-clockwise path, and $\overline{\mu_1} = \mu_1$, because we know $\mu_1$ is real. Thus the monodromy is trivial.

The next point is to use the Puiseux theory. Recall that this tells that the algebraic closure of the field of convergent Laurent series $\mathbb{C}\{z\}[z^{-1}]$ is the union of the Laurent series in $z^{1/n}$ for all $n$. We use it as follows. We have a plane curve

$$f(\mu, \lambda) = \det(\mu - (H_0 + \lambda V)) = 0$$

which we want to understand near $(\mu, \lambda) = (0, 0)$. By Weierstass preparation, one has in $\mathbb{C}\{\mu, \lambda\}$

$$f(\mu, \lambda) = \text{unit} \cdot (\mu^p + a_1(\lambda)\mu^{p-1} + \cdots + a_p(\lambda))$$

where $a_i(\lambda)$ vanish at $\lambda$. Here $p$ is the mult. of the 0 eigenvalue; near [redacted] $\lambda = 0$ we get a vector bundle of rank $p$ which is stable under $H_0 + \lambda V$, and $\mu^p + \cdots + a_p(\lambda)$ is its characteristic polyn.

Now take the equation

$$\mu^p + a_1(\lambda)\mu^{p-1} + \cdots + a_p(\lambda) = 0$$

and apply the Puiseux theorem which says that it factors completely over $\bigcup \mathbb{C}\{z^{1/n}\}$. Moreover the Galois group [redacted] of $\mathbb{C}\{z\}[z^{-1}]$ is $\hat{\mathbb{Z}}$. So it's now clear that our plane curve locally breaks up into branches according to the irreducible factors, etc. [redacted] and the degree of the irreducible factors determined the monodromy. Hence if the monodromy is trivial all the branches are of degree 1. So it seems we can prove:

Prop: Let $H_0, V$ be hermitian matrices, and [redacted] suppose $H_0$ has distinct eigenvalues, [redacted] call them $\mu_1(0), \cdots, \mu_n(0)$. Then there are unique real analytic functions $\mu_i(\lambda)$ starting with these eigenvalues such that $\{\mu_i(\lambda)\}$ are the eigenvalues counted with multiplicity of $H_0 + \lambda V$. Moreover we can choose eigen-lines $L_i(\lambda)$ in a real-analytic fashion belonging to the eigenvalues $\mu_i(\lambda)$.

The last statement comes from the fact that a branch $\mu = \mu_i(\lambda)$ is a non-singular curve, so that if we have a map to complete variety defined for $\lambda \neq 0$, then it extends by the valuative criterion for properness. It's not clear these lines are independent for all $\lambda$.

Conclude: The possible obstruction to the "adiabatic" process working is not there.

To understand effect of a single impurity atom on an electron gas. Think of a bunch of electrons moving independently and freely in a box of volume V. These can be described individually by wave functions (ignore spin)

$$|p\rangle = \frac{1}{\sqrt{V}} e^{ipx}$$

and the Hamiltonian $-\frac{\nabla^2}{2m}$. Collectively we works in Fock space with

$$H_0 = \sum_p \varepsilon_p a_p^* a_p \qquad \varepsilon_p = \frac{p^2}{2m}$$

Now put in an impurity which acts on the electrons with a potential $U(x)$. The new single particle Hamiltonian is

$$H = -\frac{\nabla^2}{2m} + U(x)$$

and it extends on Fock space to

$$H = \sum_{p,p'} \langle p'|H|p\rangle a_{p'}^* a_p = \sum_p \varepsilon_p a_p^* a_p + \sum_{p',p} U(p'-p) a_{p'}^* a_p$$

$$\langle p'|U|p\rangle = \frac{1}{V}\int e^{-ip'x} U(x) e^{ipx} dx \overset{\text{defn.}}{=} U(p'-p)$$

Let's compute the single electron Green's function (ordinary Schroedinger propagator):

$$iG_{pp'}(t) = \langle 0|T[a_p(t) a_{p'}^*]|0\rangle$$

$$= \theta(t)\langle p|e^{-iHt}|p'\rangle$$

Then

$$i\frac{d}{dt}G_{pp'} = \delta(t)\delta_{pp'} + \sum_{p_1}\langle p|H|p_1\rangle G_{p_1p'}$$

$$\left(i\frac{d}{dt} - \varepsilon_p\right)G_{pp'} = \delta(t)\delta_{pp'} + \sum_{p_1} u(p-p_1)G_{p_1p}$$

Take Fourier transform in time

$$\int G_{pp'}(t)\, e^{i\omega t}\, dt = G_{pp'}(\omega) \qquad \text{anal. for } \omega \text{ in U HP.}$$

$$\boxed{(\omega - \varepsilon_p + i\eta)\, G_{pp'}(\omega) = \delta_{pp'} + \sum_{p_1} u(p-p_1)G_{p_1p'}(\omega)}$$

This last equation is of the standard form

$$G = G_0 + G_0 V G$$

which one can iterate.

Now we look at the case of a $\delta$-function potential:

$$u(x) = c\,\delta(x)$$

$$u(q) = \frac{c}{V} \quad \text{is ind. of } q.$$

so the basic equation becomes

$$G_{pp'}(\omega) = \frac{\delta_{pp'}}{\omega - \varepsilon_p + i\eta} + \frac{1}{\omega - \varepsilon_p + i\eta}\,\frac{c}{V}\sum_{p_1} G_{p_1p'}(\omega)$$

This can be solved by summing

$$\left(\frac{1}{V}\sum_p G_{pp'}(\omega)\right) = \frac{1}{\omega - \varepsilon_{p'} + i\eta} + \underbrace{\left(\frac{1}{V}\sum_p \frac{1}{\omega - \varepsilon_p + i\eta}\right)}_{G^0(\omega)} c\left(\frac{1}{V}\sum_p G_{pp'}(\omega)\right)$$

yielding $$\frac{1}{V}\sum_p G_{pp'}(\omega) = \frac{\frac{1}{\omega-\varepsilon_{p'}+i\omega}}{1-c\,\overline{G^0(\omega)}}$$

and then one can put this back to obtain $G_{pp'}(\omega)$.

Perhaps the way to look at this is to note that the potential operator $\tilde{V}=U(x)$ is of rank 1, so the equation

$$G = G_0 + G_0\tilde{V}G_0$$

is easy to solve. One can put

$$T = \tilde{V} + \tilde{V}G_0\tilde{V} + \tilde{V}G_0\tilde{V}G_0\tilde{V} + \cdots = \tilde{V} + \tilde{V}G_0T$$

$$T_{pp'} = \frac{c}{V} + \frac{c}{V}\sum_{p_1,p_2} G^0_{p_1,p_2}$$  $$\frac{c}{V} + \cdots$$

$$\sum_{p_1,p_2}' \frac{\delta_{p_1,p_2}}{\omega-\varepsilon_{p_1}+i\eta}$$

$$\therefore\quad T_{pp'} = \frac{c}{V} + c\,\overline{G^0(\omega)}\,\frac{c}{V} + \left(c\,\overline{G^0(\omega)}\right)^2\frac{c}{V} + \cdots$$

$$\boxed{T_{pp'}(\omega) \quad \frac{c/V}{1-c\overline{G^0(\omega)}}}$$

$$G_{pp'}(\omega) = \frac{\delta_{pp'}}{\omega-\varepsilon_p+i\eta} + \frac{1}{\omega-\varepsilon_p+i\eta}\,\frac{c/V}{1-cG^0(\omega)}\,\frac{1}{\omega-\varepsilon_{p'}+i\eta}$$

Now

$$\overline{G^0(\omega)} = \frac{1}{V}\sum_p \frac{1}{\omega-\varepsilon_p+i\eta}$$

diverges for $\varepsilon_p = \frac{p^2}{2m}$ in three dimensions although it does converge in dimension 1. The way one

gets around this is to argue that in a real metal one wouldn't be using the plane waves $\frac{1}{\sqrt{V}} e^{ipx}$ but rather Bloch wave functions $u_p$ with energies in a band. Already using $U(x) = c\delta(x)$ is an approximation; let's now just look at a $U$ with $\langle u_{p'} | U | u_p \rangle = \frac{c}{V}$ constant for $p$ in the band.  The same formulas for $G$, $T$ etc. hold,  except that

$$\overline{G^\circ(\omega)} = \frac{1}{V} \sum_p \frac{1}{\omega - \varepsilon_p + i\eta}$$

$$= \int \frac{\rho^\circ(\varepsilon)\, d\varepsilon}{\omega - \varepsilon}$$

where

$$\rho^\circ(\varepsilon) = \frac{1}{V} \sum_{p \text{ in the band}} \delta(\varepsilon - \varepsilon_p)$$

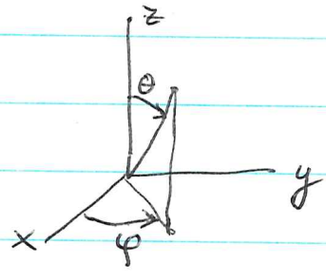
September 6, 1979

Scattering by a spherical potential revisited:

Recall the formulas:

$$\Delta = \frac{1}{r^2}\frac{\partial}{\partial r} r^2 \frac{\partial}{\partial r} + \frac{1}{r^2}\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta} + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}$$

Use physics notation:




Legendre polys:

$$\frac{1}{(1-2rx+r^2)^{1/2}} = \sum_{\ell\geq 0} r^\ell P_\ell(x) \qquad P_\ell(x) = \frac{(2\ell-1)!}{\ell!}x^\ell + O(x^{\ell-2}) \quad x\to 0$$

$$P_\ell(1) = 1, \quad P_\ell(-1) = (-1)^\ell \qquad \int_{-1}^{1} P_\ell P_{\ell'}\,dx = \frac{2}{2\ell+1}\delta_{\ell\ell'}$$

Spherical Bessel functions:

$$e^{ir\cos\theta} = \sum_{\ell\geq 0} i^\ell (2\ell+1) j_\ell(r) P_\ell(\cos\theta)$$

$$j_\ell(r) = \frac{i^{-\ell}}{2}\int_{-1}^{1} e^{-irx} P_\ell(x)\,dx \sim \frac{i^\ell r^\ell}{(2\ell+1)!} \quad \text{as } r\to 0$$

$$\sim \frac{i^{-\ell}}{2ir}\left(e^{ir} - (-1)^\ell e^{-ir}\right) \quad \text{as } r\to\infty$$

$$= \frac{\sin(r-\ell\pi/2)}{r}$$

$$\left[\frac{1}{r^2}\frac{d}{dr}r^2\frac{d}{dr} - \frac{\ell(\ell+1)}{r^2} + 1\right] j_\ell(r) = 0.$$

$$j_0(r) = \frac{\sin r}{r}$$

The problem is to understand scattering by a spherically-symmetric potential $V(r)$. Although you will probably never really understand this without going into the Radon transform, one can try to calculate the answer. The answer consists of giving phase shifts $\delta_\ell(E)$ for each integer $\ell \geq 0$ and real number $E > 0$; $\ell$ is an angular momentum quantum no. and $E$ is the energy. Also you might give the bound state energies.

These phase shifts arise as follows. One starts with an eigenfunction $\varphi$ for the free Hamiltonian $-\Delta$, say

$$-\Delta\varphi = E\varphi$$

for example $\varphi_{\underline{k}}(x) = e^{i\underline{k}\cdot\underline{x}}$ where $E = |\underline{k}|^2$. Then one "switches on" the perturbation $V(r)$ so as to obtain eigenfunctions $\psi^\pm$ for $-\Delta + V$ satisfying the Lippmann-Schwinger equations

$$\psi^\pm = \varphi + G_E^{0\pm} V \psi^\pm$$

$$G_E^{0\pm} = \frac{1}{E + \Delta \pm i\eta}$$

From $\varphi_{\underline{k}} = e^{i\underline{k}\cdot\underline{x}}$ one obtains $\psi_{\underline{k}}^\pm$. The $\varphi_{\underline{k}}$ form a basis of free eigenfunctions, hence assuming no bound states one obtains two bases $\{\psi_{\underline{k}}^+\}$, $\{\psi_{\underline{k}}^-\}$ of eigenfunctions for $-\Delta + V$.

But when $V(r)$ is spherically symmetric a

nicer way to describe things is to use eigenfunctions geared to rotational symmetry. Let $\underline{k} = (0,0,k)$ where $k > 0$, so $\psi_{\underline{k}} = e^{ikr\cos\theta}$ is invariant under rotation around the z-axis. Clearly $\psi^{\pm}_{\underline{k}}$ are also invariant under z-axis rotation, so they have expansions

$$\psi^{\pm}_{\underline{k}} = \sum_{\ell \geq 0} (2\ell+1) \psi^{\pm}_{\ell,k}(r) P_\ell(\cos\theta)$$

where the $(2\ell+1)$ is a convenient constant which arises as

$$\psi_{\underline{k}} = e^{ikr\cos\theta} = \sum_{\ell \geq 0} (2\ell+1) j_\ell(kr) P_\ell(\cos\theta)$$

The radial (wave functions) $\psi^{\pm}_{\ell,k}(r)$ satisfy

$$\left\{ -\frac{1}{\hbar^2}\frac{d}{dr} r^2 \frac{d}{dr} + \frac{\ell(\ell+1)}{r^2} + V(r) - k^2 \right\} \psi^{\pm}_{\ell,k}(r) = 0$$

and they are "well-behaved" as $r \to 0$, assuming $V(r)$ does blow up too fast. Hence one has

$$\psi^{-}_{\ell,k}(r) = S_\ell(k) \psi^{+}_{\ell,k}(r)$$

where

$$S_\ell(k) = e^{2i\delta_\ell(k)}$$

is of modulus 1. Here $\delta_\ell(k)$ which is determined modulo $\pi$ is the phase shift for the angular mom. $\ell$ and the energy $k^2$.

Assuming $V(r) = 0$ for $r >> 0$ we know that $\psi^{+}_{\ell,k}(r)$ for large is a solution of

$$(*) \qquad \left\{ -\frac{1}{r^2}\frac{d}{dr}r^2\frac{d}{dr} + \frac{\ell(\ell+1)}{r^2} + V(r) - k^2 \right\}\psi = 0$$

which differs from

$$\varphi_{\ell,k}(r) = j_\ell(kr) \sim \frac{1}{2ikr}\left(e^{ikr} - (-1)^\ell e^{-ikr}\right)$$

by outgoing spherical waves. From the known asymptotic behavior of solns. of (*) one has

$$\psi \sim \frac{1}{2ikr}\left(c_1 e^{ikr} + c_2 e^{-ikr}\right)$$

and hence

$$\psi^+_{\ell,k} \sim \frac{1}{2ikr}\left(e^{2i\delta_\ell(k)} e^{ikr} - (-1)^\ell e^{-ikr}\right)$$

$$\psi^-_{\ell,k} \sim \frac{1}{2ikr}\left(e^{ikr} - (-1)^\ell e^{-2i\delta_\ell(k)} e^{-ikr}\right)$$

Let us now consider the example of a square-well potential:

$$V(r) = \begin{cases} b & r < a \\ 0 & r > a \end{cases}$$

and consider only s-wave scattering (i.e. $\ell = 0$). Here $j_0(kr) = \dfrac{\sin kr}{kr}$ is the free eigenfunction, so inside $r = a$ we have

$$\psi = \text{const} \cdot \frac{\sin\sqrt{k^2-b}\, r}{\sqrt{k^2-b^2}\, r} \qquad r < a.$$

Suppose $\psi$ outside $r = a$ is real:

$$\psi = \sin(kr + \delta_0(k)) / kr = \frac{e^{i(kr+\delta_0)} - e^{-i(kr+\delta_0)}}{2ikr}$$

Continuity requires

$$\text{Const.}\ \frac{\sin\sqrt{k^2-b}\,a}{\sqrt{k^2-b}\,a} = \frac{\sin(ka+\delta_0)}{ka}$$

or

or

$$\sqrt{k^2-b}\,a = ka+\delta_0 \pmod{\pi}$$

$$\delta_0(k) = \sqrt{k^2-b}\,a - ka$$

Now my idea was to see what happens as $V(r) \to c\delta(r)$, i.e. we let $a \to 0$ with

$$\int V(r)\,d^3r = b\cdot\tfrac{4}{3}\pi a^3 = C$$

and continuity of the first derivative requires

$$\text{Const.}\ \cos(\sqrt{k^2-b}\,a) = \cos(ka+\delta_0)$$

(multiply [redacted] both expressions for $\psi$ by [redacted]$r$, apply $\frac{d}{dr}$, and set $r=a$.) Thus we get

$$(*)\qquad \frac{\tan(\sqrt{k^2-b}\,a)}{\sqrt{k^2-b}} = \tan(ka+\delta_0(k))$$

Now my idea was to see what happens as $V(r) \to c\delta(r)$, i.e. $a \to 0$ with

$$\int V(r) = b\,\tfrac{4}{3}\pi a^3 = C$$

Suppose $c>0$, so that the potential is repulsive. Now

$$\sqrt{b}\,a = \text{const}\ a^{-3/2}\cdot a \to \infty$$

and in an infinite imaginary direction $\tan \to \mp i$, so that as $a \to 0$ $(*)$ goes to 0, hence $\delta_0(k) \to 0$.

Sept 9, 1979

Consider $H = \frac{1}{2}p^2 + \frac{1}{2}(\omega q)^2 - Jq$ where $J$ is supported inside $[0,T]$. One wants to compute the transformation function

$$\langle q' | U_J(T,0) | q \rangle$$

where $U_J(t,t')$ is the propagator for the Schroed. equation

$$i\frac{\partial \psi}{\partial t} = H\psi.$$

Consider Schwinger's method of using an infinitesimal change $\delta J$.

$$\delta \langle q' | U_J(T,0) | q \rangle = \int_0^T dt\, (-i)(-\delta J(t)) \langle q' | U_J(T,t)\, q\, U_J(t,0) | q \rangle$$

$$\delta \log \langle q' | U_J(T) | q \rangle = i \int_0^T dt\, \delta J(t) \langle q(t) \rangle$$

where

$$\langle q(t) \rangle = \frac{\langle q' | U_J(T,t)\, q\, U_J(t) | q \rangle}{\langle q' | U_J(T) | q \rangle}$$

Then one has

$$\frac{d^2}{dt^2}\langle q(t) \rangle = -\omega^2 \langle q(t) \rangle + J(t)$$

$$\langle q(t) \rangle = \begin{cases} q & t = 0 \\ q' & t = T \end{cases}$$

This is a two-point boundary-value problem that can be solved uniquely unless $T$ is a period for solutions of the homogeneous DE, i.e. $e^{-i\omega T} = 1$

or $T \in \mathbb{Z}\frac{2\pi}{\omega}$. This seems to means that $\langle q'|U_J(T)|q\rangle$ is a multiple of $\delta(q-q')$ for such $T$. Not quite: $\langle q'|U_J(T)|q\rangle$ will be a multiple of $\delta(q'-\tilde{q})$, where $\tilde{q}$ is the common value at $T$ of any soln. of the DE ~~starting~~ at $q$ when $t=0$. This makes sense with the fact that we know

$$e^{iTH_0} U_J(T) = e^{i\frac{\omega}{2}T} U_J(T)$$

is a generalized translation operator.

Anyway let's assume $T$ not a period. Then we can solve the DE.

$$\langle q(t)\rangle = \int_0^T G(t,t')\, J(t')\, dt' + q_{class}(t)$$

where

$$G(t,t') = \frac{\sin(\omega t_<)\,\sin(\omega(T-t_>))}{W}$$

is the Green's function for $\frac{d^2}{dt^2}+\omega^2$ with 0 endpt conditions on $[0,T]$, and $q_{class}(t)$ is the classical motion between $q$ at $t=0$ and $q'$ at $t=T$.

Then integrating out wrt $\delta J$ we get

$$\log\langle q'|U_J(T)|q\rangle = \log\langle q'|U(T)|q\rangle + \frac{i}{2}\iint JGJ + i\int_0^T J\, q_{class}\, dt$$

It should be possible to make the above clearer using path integrals. Recall that the transformation fn.

is given by a path integral

$$\langle q' | U_J(T) | q \rangle = \int_{\substack{q(0)=q \\ q(T)=q'}} \mathcal{D}q \; e^{i\int_0^T L_0 + i\int_0^T Jq}$$

where

$$L_0(q,\dot{q}) = \frac{1}{2}\dot{q}^2 - \frac{1}{2}\omega^2 q^2$$

We can evaluate this as follows when $T$ is not a period. Let $q_{cl}(t)$ denote the classical motion from $(0,q)$ to $(T,q')$ and write

$$q = q_{cl} + \tilde{q} \qquad \text{and } \tilde{q}(0) = \tilde{q}(T) = 0$$

so that $\mathcal{D}q = \mathcal{D}\tilde{q}$. Since $L_0$ is quadratic

$$L_0(q) = \frac{1}{2}(\dot{q}_{cl} + \dot{\tilde{q}})^2 - \frac{1}{2}\omega^2(q_{cl}+\tilde{q})^2$$

$$= L_0(q_{cl}) + \dot{q}_{cl}\dot{\tilde{q}} - \omega^2 q_{cl}\tilde{q} + L_0(\tilde{q})$$

and by integration by parts the linear term in $\tilde{q}$ is

$$\int_0^T (\dot{q}_{cl}\dot{\tilde{q}} - \omega^2 q_{cl}\tilde{q})\,dt = -\int_0^T (\ddot{q}_{cl} + \omega^2 q_{cl})\tilde{q}\,dt = 0$$

(Hamilton's principle says the first variation around $q_{cl}$ is 0).

Hence

$$\langle q' | U_J(T) | q \rangle = e^{i\int_0^T (L_0(q_{cl}) + Jq_{cl})dt} \int_{\tilde{q}(0)=\tilde{q}(T)=0} \mathcal{D}\tilde{q}\; e^{i\int L_0(\tilde{q}) + i\int J\tilde{q}}$$

Hence by the translation $q = q_{cl} + \tilde{q}$ we reduce the path integral to one over a vector space (paths beginning and ending at 0), where the Lagrangian is quadratic centered at $\tilde{q} = 0$. ∎

Review pages 168-170

$$e^{a\frac{D^2}{2}} e^{b\frac{x^2}{2}} = \frac{1}{\sqrt{1-ab}} e^{\frac{b}{1-ab}\frac{x^2}{2}}$$

From this we got the following impression: That the path integral

$$\langle 0|U_J(T)|0\rangle = \int_{q(0)=q(T)=0} \mathcal{D}q \; e^{i\int L_0 + i\int Jq}$$

is the Fourier transform of the Fresnel-Gaussian fn. belonging to the quadratic form

$$q(t) \longmapsto \frac{i}{2}\int_0^T (\dot{q}^2 - \omega^2 q^2)\,dt = -\frac{i}{2}\int_0^T q\left(\frac{d^2}{dt^2}+\omega^2\right)q\,dt$$

and hence this path integral should be

$$\frac{1}{(2\pi)^{\infty/2}\det\left(\frac{d^2}{dt^2}+\omega^2\right)^{1/2}} e^{\frac{i}{2}\iint J\,G\,J}$$

where $G$ is the inverse of $\frac{d^2}{dt^2}+\omega^2$ on $[0,T]$ with bdry conditions $=0$ at ends. The factor in front is effectively dropped in defining the path integral and replaced by $1/\langle 0|U(T)|0\rangle$.

Furthermore the path integral

$$\langle \psi_0|U_J(T)|\psi_0\rangle = \int e^{-\frac{1}{2}\omega q(0)^2 - \frac{1}{2}\omega q(T)^2}\mathcal{D}q\; e^{i\int L_0 + i\int Jq}$$

is

$$e^{\frac{i}{2}\iint J\,G\,J}\langle \psi_0|U(T)|\psi_0\rangle$$

where $G$ is the Green's function appropriate to the "Feynman" bdry conditions

So the impression is that a path integral

$$\int \mathcal{D}q\, e^{i[L_0(q)+Jq}$$

$L_0$ quadratic non-degenerate

on a family of path defined by homogeneous linear endpt conditions, should essentially be given by a Gaussian function with covariance matrix inverse to $L_0$ on the path space.

Consider a simple harmonic oscillator $H_0 = \frac{p^2}{2} + \frac{1}{2}\omega^2 q^2$ with perturbation $H' = -J(t)q$ where $J$ has compact support. Recall the formula

$$\langle 0|S|0\rangle = \exp\left\{\frac{i}{2}\iint J(t)\underbrace{G_0(t,t')}_{\frac{e^{-i\omega|t-t'|}}{-2i\omega}}J(t')\,dt\,dt'\right\}$$

A nice derivation which gives the whole $S$-matrix goes as follows: First you consider $J = c\delta(t)$. Then $S = U(0+,0-)$ and since the perturbation occurs instantaneously, in the Sch. eqn.

$$\frac{\partial\psi}{\partial t} = (-iH_0 + iJq)\psi$$

we can ignore the $H_0$ in computing $U(0+,0-)$. We get

$$S = U(0+,0-) = e^{i\int_{0-}^{0+} J(t)dt\, q} = e^{icq}$$

Now consider $J = c\delta(t-\tau)$. Then

$$S = e^{iH_0\tau}e^{icq}e^{-iH_0\tau} = e^{icq(\tau)}$$

where

$$q(\tau) = e^{iH_0\tau}qe^{-iH_0\tau} = \frac{1}{\sqrt{2\omega}}\left(e^{-i\omega\tau}a + e^{i\omega\tau}a^*\right).$$

Next express $J(t)$ as a sum of $\delta$-functions. The corresponding $S$ is a kind of product integral:

$$S = \prod_t e^{i\,J(t)dt\,q(t)} = T\left\{e^{i\int J(t)q(t)dt}\right\}$$

where the $t$ factor occurs to the left of the $t'$ factor

for $t > t'$. Actually if $J$ were a sum of $\delta$-fns. then $S$ would be an actual product of the operators $e^{-icq(t)}$. Now we carry out the product using the identity

$$e^A e^B = e^{A+B} e^{\frac{1}{2}[A,B]}$$

where $[A,B]$ commutes with $A, B$. The idea is to write the product in normal form with all creators on the left of the annihilators. When we come to the term

$$e^{i J(t) q(t) dt} = e^{\frac{i J(t) dt}{\sqrt{2\omega}} (e^{i\omega t} a^*)} e^{i \frac{J(t) dt}{\sqrt{2\omega}} (e^{-i\omega t} a)}$$

we have to commute the latter factor past the creator part of what has already been obtained which is

$$e^{\left(\frac{i}{\sqrt{2\omega}} \int^t J(t') e^{i\omega t'} dt'\right) a^*}$$

So we get the term

$$e^{\frac{i}{\sqrt{2\omega}} J(t) e^{-i\omega t} dt} \; \frac{i}{\sqrt{2\omega}} \int^t J(t') e^{i\omega t'} dt' \; \underbrace{[a, a^*]}_{=1}$$

so adding this up for all $t$ we get the formula

$$S = e^{-\iint_{t>t'} J(t) \frac{e^{-i\omega(t-t')}}{2\omega} J(t') \, dt \, dt'} e^{\left(\frac{i}{\sqrt{2\omega}} \int J(t) e^{i\omega t} dt\right) a^*} e^{\left(\frac{i}{\sqrt{2\omega}} \int J(t) e^{-i\omega t}\right) a}$$

or

$$\boxed{S = e^{\frac{i}{2} \iint J(t) \frac{e^{-i\omega|t-t'|}}{-2i\omega} J(t') dt dt'} \; :e^{i \int J(t) q(t) dt}:}$$

where the colons denote putting into normal products.

The problem is to understand how to express the S-matrix in terms of time-ordered vacuum expectation values. (Amit, Ch 3). Let's consider the situation of an s.h. oscillator

$$H_0 = \tfrac{1}{2}p^2 + \tfrac{1}{2}\omega^2 q^2 - \tfrac{1}{2}\omega = \omega a^* a$$

with a perturbation (time-dependent) supported in a finite time-interval, say $[0,T]$. The scattering operator is

$$S = e^{iH_0 T}\, U(T,0)$$

If the perturbation is a function of $t, q$

$$H' = V(t,q)$$

say polynomial in $q$, then we have in the interaction picture

$$H'(t) = V(t, q(t)) \qquad q(t) = e^{iH_0 t} q\, e^{-iH_0 t}$$

and we have Dyson's formula

$$S = T\{e^{-i\int H'(t)\,dt}\}$$

$$= T \prod_t e^{-iH'(t)\,dt}$$

where the latter is like an integral but in the time-ordered product sense. Dyson's formula holds even when $H'$ depends on $p$.

The point: At first glance the S-matrix consists of the matrix elements

$$\langle m|S|n\rangle = \langle 0|\frac{a^m}{\sqrt{m!}} S \frac{a^{*n}}{\sqrt{n!}}|0\rangle.$$

However this way of describing the S-matrix contains a lot of garbage, once $m, n$ are big. Think of the above as being the transition amplitude for going from $n$ particles present to $m$ particles. Then $p$ of the particles could be left alone and the interaction would work on going from $n-p$ particles to $m-p$ particles.

Hence we see that $\langle m|S|n\rangle$ involves terms coming from $\langle m-p|S|n-p\rangle$. To see what's going on let's consider the case where

$$H' = -J(t)q \qquad J(t) = c\delta(t) \qquad c \text{ real}$$

whence we know that

$$S = e^{icq} = e^{i\frac{c}{\sqrt{2\omega}}(a^*+a)} = e^{\frac{-c^2}{4\omega}} e^{i\frac{c}{\sqrt{2\omega}}a^*} e^{i\frac{c}{\sqrt{2\omega}}a}$$

Change $c$ to $\frac{c}{\sqrt{2\omega}} = \gamma$, so that

$$S = e^{-\frac{1}{2}\gamma^2} e^{i\gamma a^*} e^{i\gamma a}$$

$$= e^{-\frac{1}{2}\gamma^2} \sum_{m,n} \frac{(i\gamma)^{m+n}}{m!\,n!} a^{*m} a^n$$

Let's now compute some matrix elements.

$$\langle 0|a^2 S a^{*2}|0\rangle = e^{-\frac{1}{2}\gamma^2} \sum_{m,n} \frac{(i\gamma)^{m+n}}{m!\,n!} \langle 0|a^2 a^{*m} a^n a^{*2}|0\rangle$$

There are 3 contributions

$(m,n) = (0,0)$ $\quad \langle 0 | a^2 a^{*2} | 0 \rangle = 2!$

$(1,1)$ $\quad (i\gamma)^2 \langle 0 | a^2 a^* a a^{*2} | 0 \rangle = (i\gamma)^2 2 \cdot 2$

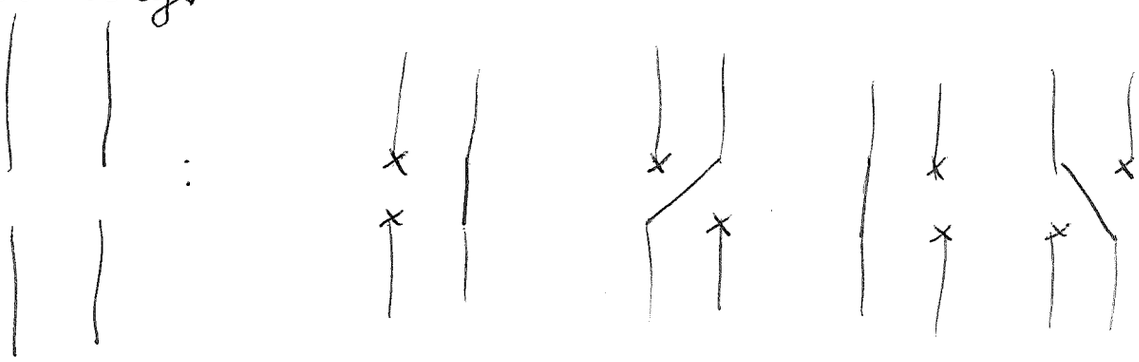
$(2,2)$ $\quad \frac{(i\gamma)^4}{2!2!} \langle 0 | a^2 a^{*2} a^2 a^{*2} | 0 \rangle = \frac{(i\gamma)^4}{2!2!}(2!)^2 = (i\gamma)^4$

which can be summarized by diagrams

fills in via | | or X

these give $m=n=0$ term

Next one can kill off one line, create another line by interaction in four ways



Finally you can kill off & create all line.

Somehow the point here is that the first two terms are disconnected graphs.

The good way to do this in general is

to write the S-matrix in the form

$$S = \sum_{m,n} S_{mn} \frac{(a^*)^m}{m!} \frac{a^n}{n!}$$

where $S_{mn}$ represents the connected processes going from $n$ particles to $m$ particles.

$$\langle 0 | a^q a^p S | a^{*p} | 0 \rangle = \sum_{m,n} S_{mn} \frac{1}{m!\,n!} \underbrace{\langle 0 | a^q a^p a^{*m} a^n a^{*p} | 0 \rangle}$$

$$a^n a^{*p} | 0 \rangle = \frac{d^n}{dz^n} z^p = \frac{p!}{(p-n)!} z^{p-n}$$

$$\frac{q!}{(q-m)!} \frac{p!}{(p-n)!} (p-n)!\, \delta_{q-m,p-n}$$

Put $q-m = p-n = \ell$ or $m = q-\ell$, $n = p-\ell$.

$$\langle 0 | a^q S a^{*p} | 0 \rangle = \sum_{\ell \geq 0} S_{q-\ell,p-\ell} \binom{q}{\ell} \binom{p}{\ell} \ell!$$

Interpretation: You select out from the group of $p$ incoming particles $\ell$ of them, and similarly $\ell$ of the $q$ outgoing particles; then you connect up these particles (this gives the $\ell!$) and allow the rest to interact by connected processes.

The next thing is to understand why the connected process amplitude is a vacuum amplitude.

---

There ought to be a simple relation between the Green's function

$$i G(t,t') = \langle \Psi_0 | T[\psi_H(t) \psi_H(t')] | \Psi_0 \rangle$$

and the S-matrix element $\langle 0|aSa^*|0\rangle$ between 1-particle states. Be careful, because Green's functions usually appear when $H$ is time-independent. $\Psi_0$ probably should be interpreted as the ground ~~state~~ in the Heisenberg picture.

---

September 16, 1979:

At some point you ought to formulate Heisenberg-Schroedinger + Dirac pictures in the language of bundles over $\mathbb{R}$. But for the moment let us understand the Green's function for a system with time-dependent Hamiltonian, say a perturbed oscillator where the perturbation has compact support in time. Work in the Schroedinger picture. Then a trajectory $\psi(t)$  coincides with free trajectories as $t \to \pm\infty$. Recall that Heisenberg's view involves regarding the Schroed. trajectories as being the basic Hilbert space. Thus I can define Heisenberg states $|0\rangle_{in}$, $|0\rangle_{out}$ to be the trajectories coinciding with the free trajectory $e^{-iH_0 t}|0\rangle$ as $t \to \mp\infty$:

$$\underset{out}{U(t,0)}|0\rangle_{in} = e^{-iH_0 t}|0\rangle \qquad \begin{matrix} t \ll 0 \\ t \gg 0. \end{matrix}$$

Then I can define the Green's function as

$$iG(t,t') = \frac{\underset{out}{\langle 0|}\, T[\phi_H(t)\phi_H(t')]\, |0\rangle_{in}}{\underset{out}{\langle 0|}0\rangle_{in}}$$

Let's compute this in the interaction picture.

$$ {}_{out}\langle 0|0\rangle_{in} = {}_{out}\langle 0|\, U(0,T)\, U(T,-T)\, U(-T,0)\,|0\rangle_{in} $$

$$ = \langle 0|\, e^{-iH_0T}\, U(T,-T)\, e^{-iH_0T}\,|0\rangle $$

$$ = \langle 0|\, S\,|0\rangle $$

Also

$$ {}_{out}\langle 0|\, T[q_H(t)\, q_H(t')]\,|0\rangle_{in} \qquad t>t' $$

$$ = {}_{out}\langle 0|\, \underbrace{U(0,t)}_{U(0,T)\,U(T,t)}\, q\, U(t,t')\, q\, \underbrace{U(t',0)}_{U(t',-T)\,U(-T,0)}\,|0\rangle_{in} $$

$$ = \langle 0|\, e^{iH_0T}\, U(T,t)\, q\, U(t,t')\, q\, U(t',-T)\, e^{iH_0T}\,|0\rangle $$

$$ = \langle 0|\, U_I(T,t)\, q(t)\, U_I(t,t')\, q(t')\, U_I(t',-T)\,|0\rangle $$

$$ = \langle 0|\, T[\, q_I(t)\, q_I(t')\, S\,]\,|0\rangle . $$

Thus we do get the usual formula

$$ iG(t,t') = \frac{\langle 0|\, T[q_I(t)\, q_I(t')\, S]\,|0\rangle}{\langle 0|\, S\,|0\rangle} $$

In the case of an oscillator

$$ q_I(t) = \frac{1}{\sqrt{2\omega}}\left(e^{-i\omega t}a^* + e^{-i\omega t}a\right) $$

Hence for $t \gg 0 \gg t'$ one has

$$ iG(t,t') = \frac{\langle 0|\, q_I(t)\, S\, q_I(t')\,|0\rangle}{\langle 0|\, S\,|0\rangle} $$

$$ = \frac{1}{2\omega}\, e^{-i\omega(t-t')}\, \frac{\langle 0|\, a\, S\, a^*\,|0\rangle}{\langle 0|\, S\,|0\rangle} $$

which relates the S-matrix amplitude between 1-particle states to the large-time behavior for the Green's function. Since

$$iG_0(t,t') = \langle 0 | q_I(t)\, q_I(t') | 0 \rangle \qquad t > t'$$

$$= \frac{e^{-i\omega(t-t')}}{2\omega}$$

we have

$$\langle 0 | a S a^* | 0 \rangle = \langle 0 | S | 0 \rangle \frac{iG(t,t')}{iG_0(t,t')} \qquad \begin{matrix} t \to +\infty \\ t' \to -\infty \end{matrix}$$

Let consider the case where $H' = \varepsilon(t)\frac{q^2}{2}$.
Then I know that

$$\left[\frac{d^2}{dt^2} + \omega^2 + \varepsilon(t)\right] iG(t,t') = \frac{1}{i}\delta(t,t')$$

so that

$$-G(t,t') = \frac{\varphi(t_<)\,\psi(t_>)}{W(\varphi,\psi)}$$

where $\varphi$ (resp. $\psi$) satisfies the boundary condition on the left (resp right)

$$\varphi(t) = e^{i\omega t} \qquad t \ll 0$$

$$\psi(t) = e^{-i\omega t} \qquad t \gg 0$$

Suppose one has

$$\varphi = e^{i\omega t} \longleftrightarrow A(\omega)\, e^{i\omega t} + B e^{-i\omega t}$$

i.e. $\varphi = A\psi_{-\omega} + B\psi$

Then $W(\varphi,\psi) = A\begin{vmatrix} 1 & 1 \\ i\omega & -i\omega \end{vmatrix} = A(-2i\omega)$

so

$$-G(t,t') = \frac{\psi(t_>)\,\varphi(t_<)}{W} = \frac{e^{-i\omega(t_>-t_<)}}{A(\omega)(-2i\omega)}$$

$$\frac{iG(t,t')}{iG_0(t,t')} = \frac{1}{A(\omega)} \qquad \text{for} \quad t >> 0 >> t'$$

Now we know already that

$$\langle 0|S|0\rangle = \det(1-\varepsilon G_0)^{-1/2}$$

note change in sign

and

$$\boxed{\det(1-\varepsilon G_0) = \det\left(\frac{\frac{d^2}{dt^2}+\omega^2+\varepsilon}{\frac{d^2}{dt^2}+\omega^2}\right) = A(\omega)}$$

Hence it appears that

$$\boxed{\langle 0|aSa^*|0\rangle = \det(1-\varepsilon G_0)^{-3/2} = A(\omega)^{-3/2}}$$

We can check this to first order

$$S = 1 - \frac{i}{2}\int \varepsilon(t)\,q(t)^2\,dt \qquad + O(\varepsilon^2)$$

$$\langle 0|aSa^*|0\rangle = 1 - \frac{i}{2}\int \varepsilon(t)\langle 0|a\,\frac{(e^{i\omega t}a^* + e^{-i\omega t}a)^2}{2\omega}\,a^*|0\rangle\,dt$$

$$= 1 - \frac{i}{2}\int \varepsilon(t)\,\frac{1}{2\omega}\underbrace{\langle 0|a(a^*a+aa^*)a^*|0\rangle}_{3}\,dt$$

$$= 1 - \frac{3}{2}\int \varepsilon(t)\underbrace{\frac{1}{2i\omega}}_{G_0(t,t)}\,dt = 1 - \frac{3}{2}\,\mathrm{Tr}(\varepsilon G_0)$$

so it checks

The obvious conjecture then is that for the quadratic perturbation $H' = \varepsilon(t)\frac{q^2}{2}$ one has

$$\langle 0 | a^n S a^{*n} | 0 \rangle = \left(\det(1-\varepsilon G_0)\right)^{-n-\frac{1}{2}}$$

since $a^*a + aa^* = 2a^*a + 1 = 2n+1$ on $a^{*n}|0\rangle$. Since $q^2$ is quadratic in the $a, a^*$ operators the parity of the number of particles should be conserved hence

$$\langle 0 | a^m S a^{*n} | 0 \rangle = 0 \quad \text{for} \quad m \neq n \quad (2).$$

---

Can we find the S-matrix for a quadratic perturbation $H' = \varepsilon(t)\frac{q^2}{2} - J(t)q$ of finite duration? We should have the formula

$$\langle 0 | S_J | 0 \rangle = \det(1+\varepsilon G_0)^{-1/2} \cdot e^{+\frac{i}{2}\int JGJ}$$

where $G = \left(\frac{d^2}{dt^2} + \omega^2 + \varepsilon\right)^{-1}$, $\omega$ approaching pos. real axis from below. Review the Schwinger proof:

$$\langle 0 | S_J | 0 \rangle = \langle 0 | e^{iH_0 T} U_J(T) | 0 \rangle$$

$$\delta \log \langle 0 | S_J | 0 \rangle = \int i\,\delta J(t)\,dt \underbrace{\frac{\langle 0 | e^{iH_0 T} U_J(T,t) q\, U_J(t,0) | 0 \rangle}{\langle 0 | e^{iH_0 T} U_J(T) | 0 \rangle}}_{\langle q(t) \rangle}$$

Then

$$\left[\frac{d^2}{dt^2} + \omega^2 + \varepsilon(t)\right]\langle q(t) \rangle = J(t)$$

with the standard boundary conditions gives

$$\langle q(t) \rangle = \int G(t,t') J(t') dt'$$

etc. etc.