January 4, 1979

Discrete scattering. Let U be a finite perturbation of $U_0 =$ multiplication by z on $\mathbb{C}[z,z^{-1}] \subset L^2(S^1)$. Put

$$U = U_0(I-V)$$

so that V has finite rank. Recall the LS equation

$$\psi = \varphi + G_\zeta^+ V\psi$$

where

$$G_\zeta^+ = (1-\zeta U_0^{-1})^{-1} = \sum_{k\geq 0} \zeta^k U_0^{-k}$$

extended analytically from $|\zeta|<1$ to $\zeta \neq \infty$. Here's how this equation arises. Begin with an eigenfunction $\varphi_\zeta = \sum \zeta^{-n} z^n$ for U_0:

$$U_0 \varphi_\zeta = \zeta \varphi_\zeta$$

and try to find an eigenfunction ψ_ζ for U with $\psi_\zeta(n) = \varphi_\zeta(n)$ for $n \gg 0$. Then

$$U_0(I-V)\psi_\zeta = U\psi_\zeta = \zeta\psi_\zeta$$

$$(U_0-\zeta)\psi_\zeta = U_0 V\psi_\zeta \quad \text{or}$$

$$(U_0-\zeta)(\psi_\zeta-\varphi_\zeta) = U_0 V\psi_\zeta$$

$$\text{or} \quad (1-\zeta U_0^{-1})(\psi_\zeta - \varphi_\zeta) = V\psi_\zeta$$

But $G_\zeta^+ f$ is the unique solution of

$$(1-\zeta U_0^{-1})\, u = f$$

with u vanishing to the right of Supp(f). Thus

$$\psi_\zeta - \varphi_\zeta = G_\zeta^+ V \psi_\zeta .$$

Therefore we see that

$$\psi_\zeta = \varphi_\zeta + G_\zeta^+ V \psi_\zeta \iff \begin{cases} U \psi_\zeta = \zeta \psi_\zeta \\ \psi_\zeta(n) = \varphi_\zeta(n) & n \gg 0 . \end{cases}$$

Next point: solutions of the homog. equation

$$(*) \qquad \psi_\zeta = G_\zeta^+ V \psi_\zeta$$

are the same thing as eigenfunctions for U which vanish to the right of the Support of V. (This is very reminiscent of the irreducibility of modules for sl_2.)

Consequently if U is conjugate to U_0 via an automorphism $\Theta = I +$ finite rank, then there are no non-trivial solutions of $(*)$, i.e.

$$\det(1 - G_\zeta^+ V) \neq 0 .$$

But the conjecture is that this determinant is 1.

Transitivity: Instead of having a fixed U_0 in mind let us consider all operators U on $\mathbb{C}[z, z^{-1}]$ which coincide with U_0 far out. This means that $U = U_0(I - V)$ where V is supported on $F_n \mathbb{C}[z, z^{-1}]$ for some n. The Green's function

$$\mathcal{G}^+_{u,\zeta}$$

will be defined for those ζ having no non-trivial associated eigenfunction vanishing on the right. Thus for $|\zeta|<1$ not a bound state eigenvalue for U we can write

$$\mathcal{G}^+_{U,\zeta} = (U-\zeta)^{-1} = \underbrace{(U-\zeta)^{-1}(U_0-\zeta)}_{(1-G_k^+V)^{-1}}(U_0-\zeta)^{-1}.$$

So given two of these operators U_1, U_2 we can consider the relative determinant

$$\det\left\{(U_1-\zeta)^{-1}(U_2-\zeta)\right\} = \det\left\{1+(U_1-\zeta)^{-1}(U_2-U_1)\right\}$$

defined precisely as

$$\det\left\{1+\mathcal{G}^+_{U_1,\zeta}(U_2-U_1)\right\}$$

The conjecture is that this is 1 when U_1, U_2 are conjugate via invertibles of the form I + compact.

January 5, 1978:

$U_0 =$ mult. by z on $\mathbb{C}[z,z^{-1}]$
$U =$ (finite) perturbation of U_0.
We've seen how to make sense of

$$\det\left((U_0-\zeta)^{-1}(U-\zeta)\right)$$

by analytic continuation from $|\zeta| < 1$. Moreover this determinant is a polynomial in ζ. Let's review this. We write

$$(U_0-\zeta)^{-1}(U-\zeta) = I + (U_0-\zeta)^{-1}(U-U_0)$$
$$= I + (1-\zeta U_0^{-1})^{-1}(U_0^{-1}U - I)$$

and realize $(1-\zeta U_0^{-1})^{-1}$ as the operator

$$f \longmapsto \Big(n \mapsto \sum_{m\geq 0} \zeta^m f(n+m)\Big)$$

defined on sequences vanishing to the right. Thus we interpret

$$(U_0-\zeta)^{-1}(U-\zeta) \quad \text{as} \quad I - G_\zeta^+\underbrace{(I-U_0^{-1}U)}_{V}$$

and this has a determinant because V is a finite support matrix. The matrix for G_ζ^+ is

$$\begin{pmatrix} 1 & \zeta & \cdots & & & \\ & 1 & \zeta & \zeta^2 & \cdots & \\ & & 1 & \zeta & \zeta^2 & \cdots \\ & & & 1 & \zeta & \cdots \\ & 0 & & & 1 & \cdots \\ & & & & & 1 \end{pmatrix}$$

So $G_\zeta V$ is a matrix with finitely many columns and poly entries in ζ, hence $\det(1-G_\zeta^+V)$ is a poly in ζ.

I claim that if $U = \Theta U_0 \Theta^{-1}$ with Θ a finite perturbation of I, then

$$\det\left((U_0-\zeta)^{-1}(U-\zeta)\right) = 1.$$

It's enough to show this for $|\zeta|<1$ where $(U_0-\zeta)^{-1}$ is defined, at least, on ℓ^2. Then

$$(U_0-\zeta)^{-1}(U-\zeta) = \left\{(U_0-\zeta)^{-1}\Theta(U_0-\zeta)\right\}\cdot\Theta^{-1}$$

where both factors on the left are finite perturbations of I, and hence have determinants. Now it is clear that

$$\det\left\{(U_0-\zeta)^{-1}\Theta(U_0-\zeta)\right\} = \det(\Theta)$$

hence one wins.

Q: In what sense might the polynomial

$$\det(1-G_\zeta^+V)$$

be some sort of characteristic polynomial?

Q: If $\det(1-G_\zeta^+V)=1$ does it follow that U, U_0 are conjugate?

Suppose that $U_0 =$ [redacted] mult. by z on $\mathbb{C}[z,z^{-1}]$, [redacted] U is a finite perturbation of U_0, $\mathcal{D} = \mathbb{C}[z]$, $\mathcal{D}' = z^N \mathcal{D}$, and that $U = U_0$ on $\mathcal{D}'$ and $\mathcal{D}^\perp$:

$$\mathbb{C}[z,z^{-1}] \supset \mathcal{D} \supset \mathcal{D}'$$

$$\cdots \quad z^{-2} \quad z^{-1} \quad \text{[redacted]} 1 \quad \Big(z \quad z^2 \quad \Big) \quad z^N \quad z^{N+1}$$

Let T be the [redacted] operator on $\mathcal{D}/\mathcal{D}'$ induced by U.

Let ψ be an eigenvector for U:

$$U\psi = \lambda\psi$$

which vanishes to the left. Then $\psi \in \hat{\mathcal{D}}$ where $\wedge$ denotes [redacted] the completion giving formal power series. Then ψ mod $\mathcal{D}'$ is an eigenvector for T.

Conversely let $v \in \mathcal{D}$ be such that v mod $\mathcal{D}'$ is an eigenvector for T: $Tv = \lambda v$, and let f [redacted] $= (U - \lambda) v \in \mathcal{D}'$. We can solve

$$\text{[redacted]} \quad (U_0 - \lambda) w = f$$

with $w \in \widehat{\mathcal{D}'}$, namely

$$w = (U_0 - \lambda)^{-1} (U - \lambda) v$$
$$= \left(1 - \frac{U_0}{\lambda}\right)^{-1} \left(1 - \frac{U}{\lambda}\right) v$$

(This works except for $\lambda = 0$). Another expression for

w is
$$w = (1 - G_\lambda^- V)v$$
where G_λ^- is the analytic continuation of $(U_0-\lambda)^{-1}$ from $|\lambda| > 1$. (Note: G_λ^- as a kernel is defined for $\lambda \neq 0$.)
It follows that
$$(U-\lambda)(v-w) = 0$$
hence $v - w \in \hat{\mathcal{D}}$ is an eigenvector for U vanishing to the left.

So in this way we see that we can identify eigenvectors for T with eigenvectors for U which vanish to the left, with $\lambda = 0$ excepted.

The next thing is to prove $\det(1 - G_\zeta^- V)$ is essentially the same as $\det(\zeta - T)$. Because $G_\zeta^- V$ has its range in $\mathcal{D}$ and its kernel kills $\mathcal{D}'$ we have
$$\det(1 - G_\zeta^- V) = \det\left((1 - G_\zeta^- V)\big|\mathcal{D}\right)$$
$$= \det\left((1 - G_\zeta^- V)\big|(\mathcal{D}/\mathcal{D}')\right)$$
But one has
$$(U_0 - \zeta)(1 - G_\zeta^- V) = U - \zeta$$
as transformations of $\mathcal{D}/\mathcal{D}'$ which is finite-dimensional, hence
$$\det\left((1 - G_\zeta^- V) \text{ on } \mathcal{D}/\mathcal{D}'\right) = \frac{\det\left((U-\zeta) \text{ on } \mathcal{D}/\mathcal{D}'\right)}{\det\left((U_0-\zeta) \text{ on } \mathcal{D}/\mathcal{D}'\right)}$$

$$= \frac{\det(\xi - T)}{\det(\xi - T_0)}$$

$T_0 = u_0$ on $\mathcal{D}/\mathcal{D}'$ which is nilpotent

$$= \det(1 - \xi^{-1} T)$$

Picture: Shift to $+$ side. Then

$$\det(1 - G_\xi^+ V) = \det\left((1 - G_\xi^+ V) \,|\, \mathcal{D}_- / \mathcal{D}_-'\right)$$

where $\mathcal{D}_- \supset \mathcal{D}_-'$ are $\mathbb{C}[z^{-1}]$ lattices between which the perturbation takes place.

$$(1 - \xi u_0^{-1})(1 - G_\xi^+ V) = u_0^{-1}(u - \xi)$$

Hence

$$\det(1 - G_\xi^+ V) = \det\left(u_0^{-1}(u - \xi) \text{ on } \mathcal{D}_- / \mathcal{D}_-'\right).$$

January 7, 1979

We seek a continuous analogue of the interpretation of $\det(1-G_j^- V)$ as a characteristic polynomial.

So consider the self-adjoint operator

$$H_0 = \frac{1}{i}\frac{d}{dx}$$

on $L^2(\mathbb{R})$, and let H be a perturbation of it:

$$H = \frac{1}{i}\frac{d}{dx} - \frac{1}{i}V$$

Then the eigenvector equation for H becomes

$$H\psi = k\psi$$

$$\left(\frac{d}{dx} - V\right)\psi = ik\psi$$

$$\left(\frac{d}{dx} - ik\right)\psi = V\psi$$

$$\psi(x) = \varphi(x) + \int_{-\infty}^{x} e^{ik(x-x')}(V\psi)(x')\,dx'$$

We assume $V\psi$ has support in $[a,b]$ and it depends only on ψ in $[a,b]$. The integral vanishes for $x<a$, so $\psi = \varphi$ for $x \ll 0$. Note that φ is a multiple of e^{ikx}. The last equation can be written in the Lippmann-Schwinger form

$$\psi = \varphi + G_k^+\psi$$

where

$$G_k^+(x,x') = \begin{cases} e^{ik(x-x')} & x > x' \\ 0 & x < x' \end{cases}$$

is the analytic continuation of the kernel for $\left(\frac{d}{dx} - ik\right)^{-1}$ on L^2 from the UHP.

Now what I want to do is to interpret

$$\det\left(1 - G_k^+ V\right)$$

as a characteristic polynomial. Let us suppose that V operates inside $(0, b)$. Put $\mathcal{D} = L^2(0,\infty)$, $\mathcal{D}' = L^2(b,\infty)$.

We first compute the operator on $\mathcal{D}/\mathcal{D}'$ induced by H_0. Recall that

$$\left(e^{-iH_0 t} f\right)(x) = \left(e^{-t\frac{d}{dx}} f\right)(x) = f(x-t)$$

and that $\mathcal{D}, \mathcal{D}'$ are stable under the shift $e^{-iH_0 t}$ for $t \geq 0$. Hence we get a contraction semi-group $T(t) = e^{-tA_0}$ on $\mathcal{D}/\mathcal{D}' \cong L^2(0,b)$ induced by $e^{-iH_0 t}$.

$$\left(e^{-tA_0} f\right)(x) = f(x-t) \text{ projected } \perp \text{ to } \mathcal{D}'$$

This has a derivative wrt t when f is absolutely continuous with L^2-derivative on $[0,b]$ and $f(0)=0$. Then

$$\left(-A_0 f\right)(x) = \left.\frac{d}{dt} f(x-t)\right|_{t=0} = -f'(x)$$

So we conclude $A_0 = \frac{d}{dx}$ on $L^2(0,b)$ with domain restricted to the f with $f(0)=0$. Note that $A_0 > 0$ in the sense that for $f \in \mathcal{D}_{A_0}$

$$(A_0 f, f) + (f, A_0 f) = \int_0^b \left(\frac{df}{dx}\bar{f} + f\frac{d\bar{f}}{dx}\right) dx = |f|^2\Big]_0^b = |f(b)|^2 > 0.$$

Notice that A_0^{-1} exists on L^2 and is given by

$$\left(A_0^{-1} g\right)(x) = \int_0^x g(t)\,dt.$$

similarly $(A_0 - ik)^{-1}$ [redacted] exists on $L^2(0,b)$ for any k

$$((A_0 - ik)^{-1} g)(x) = \int_0^x e^{ik(x-x')} g(x')\,dx'$$

confirming the feeling A_0 is a sort of a nilpotent operator.

Similarly [redacted] iH will induce on [redacted] $\mathcal{D}/\mathcal{D}'$ $\cong L^2(0,b)$ an operator A, and from

$$iH = iH_0 - V$$

we get

$$A = A_0 - V$$

$$A - ik = A_0 - ik - V$$

Now $1 - G_k^+ V$ restricted to $\mathcal{D}/\mathcal{D}'$ coincides with the operator

$$1 - (A_0 - ik)^{-1} V = (A_0 - ik)^{-1}(A - ik)$$

by the above formula for $(A_0 - ik)^{-1}$. Thus

$$\det{}'(1 - G_k^+ V) = \det(1 - G_k^+ V \text{ on } \mathcal{D}/\mathcal{D}')$$

$$= \det\left((A_0 - ik)^{-1}(A - ik)\right)$$

$$= \det\left((1 - ikA_0^{-1})^{-1}(A_0^{-1}A - ikA_0^{-1})\right)$$

$$= \det [redacted] (A_0^{-1}A - ikA_0^{-1})$$

where we know $\det(1 - ikA_0^{-1}) = 1$ because A_0^{-1} is a Volterra operator. The last determinant involves an operator in which k appears linearly, so it is comes close to being the characteristic poly of the operator $-iA =$ operator induces by H on $\mathcal{D}/\mathcal{D}'$.

In fact if A^{-1} [redacted] exists on L^2, [redacted] then we get

$$\det(1-G_k^+V) = \det(A_0^{-1}A)\,\det(1-ikA^{-1}).$$

(There might be some problem with domains for these operators.)

January 9, 1979.

$$H = \frac{1}{i}\left(\frac{d}{dx}-V\right) = H_0 + iV \qquad \text{operator on } L^2(\mathbb{R})$$

where V is a perturbation of compact support, say supported in (a,b). Look for eigenfunctions:

$$H\psi = k\psi$$

$$\left(\frac{d}{dx}-V\right)\psi = ik\psi$$

$$\left(\frac{d}{dx}-ik\right)\psi = V\psi$$

$$\psi = \varphi + \int_{-\infty}^{x} e^{-ik(x-x')}(V\psi)(x')\,dx'$$

where $\left(\frac{d}{dx}-ik\right)\varphi = 0$. This equation can be put in the form

$$(1-G_k^+V)\psi = \varphi.$$

We are trying to understand $\det(1-G_k^+V)$ by interpreting it as a characteristic poly.

Notice G_k^+V has its image in $L^2(a,\infty) = \mathcal{D}$ and that on $\mathcal{D}$, G_k^+ is the inverse of the operator $\left(\frac{d}{dx}-ik\right)$ restricted to functions in the domain of d/dx which vanish at $x=a$. Let A_0 be this restriction of $\frac{d}{dx}$ on $L^2(a,\infty)$.

Then

$$(A_0 - ik)(1 - G_k^+ V) = (\underbrace{A_0 - V}_{A} - ik)$$

where $A = \frac{d}{dx} - V$ restricted in the same way to $L^2(a,\infty)$.

So

$$(1 - ikA_0^{-1})(1 - G_k^+ V) = A_0^{-1}(A - ik)$$

and as A_0^{-1} is a Volterra operator we get

$$\det(1 - G_k^+ V) = \det(A_0^{-1}(A - ik))$$

$$= \det(A_0^{-1}A)\det(1 - ikA^{-1})$$

assuming 0 not an eigenvalue for A.

The point of this calculation is that the determinant we are after is essentially $\det(1 - k(iA^{-1}))$ where $\frac{1}{i}A$ is the operator on $L^2(a,\infty)$ induced by H, but with the boundary condition $f(a) = 0$ imposed. In other words once you find D and impose the bdry. condition, then you compute $\det(1 - kH^{-1})$.

Let's next consider Dirac scattering on $\mathbb{R}$:

$$\frac{d}{dx}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} ik & h \\ \bar{h} & -ik \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$$

where h is of compact support. What we do is to choose (a,b) to contain $\operatorname{Supp}(h)$, and then compute the Green's matrix for the above DE with the boundary conditions $u_1(a) = 0$, $u_2(b) = 0$.

To be more precise, write the DE in the form

$$k\begin{pmatrix}u_1\\u_2\end{pmatrix} = \left\{\begin{pmatrix}\frac{1}{i}\frac{d}{dx} & 0\\ 0 & -\frac{1}{i}\frac{d}{dx}\end{pmatrix} - \begin{pmatrix}0 & \frac{1}{i}h\\ i\bar{h} & 0\end{pmatrix}\right\}\begin{pmatrix}u_1\\u_2\end{pmatrix}$$

i.e.

$$k\psi = H\psi$$

and then compute the Green's matrix for: $H\mathcal{G} = I$. Then the determinant you want is $\det(1-k\mathcal{G})$.

The point of all this is that $\mathcal{G}$ is a known quantity, a pseudo-differential operator whose symbol can be readily computed. It has order order $\underline{1}$; and [redacted] we can restrict to (a,b). We should check that it has the appropriate traces, since this matrix has <u>discontinuities</u> along the diagonal.

Consider now a Schroedinger DE

$$\left(\frac{d^2}{dx^2} - V + k^2\right)\psi = 0$$

[redacted] on $\mathbb{R}$, with V [redacted] of compact support. Here we [redacted] have

$$\psi(x) = \varphi(x) + \int_{-\infty}^{\infty} \frac{e^{ik|x-x'|}}{2ik}(V\psi)(x')\,dx'$$

so $\det(1-G_k^+V)$ has [redacted] in general a pole at $k=0$. In fact recall

$$\det(1-G_k^+V) = \frac{W(Ae^{-ikx}+Be^{ikx}, e^{ikx})}{W(e^{-ikx}, e^{ikx})} = \frac{A(k)}{[redacted]}$$

where $A(k)$ is entire when there is a tunnel soln, but otherwise has a simple pole at $k=0$.

January 10, 1979:

Problem: Interpret $\det(1-G_k^+V)$ as a characteristic poly in the case of the Schroedinger DE.

We know

$$\det(1-G_k^+V) = \frac{W(Ae^{-ikx}+Be^{ikx}, e^{ikx})}{W(e^{-ikx}, e^{ikx})} = A(k)$$

where $A(k)$ has a simple pole at $k=0$ when there is no tunnel solution. It would be nice if we could interpret precisely the formulas:

$$\text{"}\det(\Delta - V + k^2)\text{"} = W(Ae^{-ikx}+Be^{ikx}, e^{ikx}) = 2ikA(k)$$

$$\text{"}\det(\Delta + k^2)\text{"} = W(e^{-ikx}, e^{ikx}) = 2ik.$$

Here the operators on the left should incorporate the outgoing boundary conditions which depend on k.

I haven't been able to make the preceding work. So let us consider the wave equation interpretation as a possible approach. Here you assume there are no bound states and you consider solutions of

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} - Vu$$

with finite energy norm:

$$E(u) = \frac{1}{2}\int\left\{\left|\frac{\partial u}{\partial t}\right|^2 + \left|\frac{\partial u}{\partial x}\right|^2 + V|u|^2\right\}dx$$

Actually you start with $u(x,t)$ which are smooth of

Compact support in x and then you complete with respect to the energy norm.

One introduces incoming and outgoing spaces as follows. Suppose $\operatorname{Supp} V \subseteq [a,b]$. Then the incoming space $\mathcal{D}^-$ is spanned by

$$f(x-t) + g(x+t)$$

with $\operatorname{Supp} f \subset (-\infty, a)$, $\operatorname{Supp} g \subset (b, \infty)$. The outgoing space $\mathcal{D}^+$ is spanned by those solutions with

$$\operatorname{Supp} f \subset (b, \infty) \qquad \operatorname{Supp} g \subset (-\infty, a)$$

[redacted] Now time evolution determines a 1-parameter contraction semi-group $T(t)$ on $(\mathcal{D}^-)^\perp \ominus \mathcal{D}^+$, and it is the eigenfunctions for T which correspond to scattering states. [redacted]

The problem is how to describe $\mathcal{D} = (\mathcal{D}^-)^\perp$ which consists of those solutions of the wave equation $u(x,t)$ which for $t \leq 0$ [redacted] are supported in $[a+t, b-t]$. Note that $\mathcal{D}^-$ consists of solutions supported in $(-\infty, a+t) \cup (b-t, \infty)$ and orthogonality for energy norm with $V=0$ implies derivatives are 0, hence the function is zero by compact support perhaps.

Thus $\mathcal{D}$ consists of solutions of the wave equation [redacted] such that for $t \leq 0$ they are supported in $[a+t, b-t]$, whereas $\mathcal{D}^+$ consists of solutions such that for $t \geq 0$ they are supported in $(-\infty, a-t) \cup (b+t, \infty)$. ?

January 11, 1979:

Let us consider a Schröd DE on $\mathbb{R}$

(1[redacted]) $$\left(\frac{d^2}{dx^2} - V + k^2\right)\psi = 0$$

and the associated wave equation

(2) $$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} - Vu$$

Using the F.T.

$$u(x,t) = \int e^{-ikt}\,\psi(x,k)\,dk/2\pi$$

global solutions u of (2) [redacted] correspond to solutions $\psi(x,k)$ of (1) defined for all k, that is to sections of a rank 2 bundle over the k-plane. (Proceed heuristically).

Introduce the basic solutions of (1)

$$\psi^+_{in}(x,k) = e^{-ikx} \qquad x \gg 0$$

$$\psi^+_{out}(x,k) = \psi^+_{in}(x,-k)$$

$$\psi^-_{in}(x,k) = e^{+ikx} \qquad x \ll 0$$

$$\psi^-_{out}(x,k) = [redacted]\ \psi^-_{in}(x,-k)$$

and the functions $A(k)$, $B(k)$ describing propagation through the disturbance:

$$e^{-ikx} \longleftarrow\!\!\!+\!\!\!\longrightarrow A(k)e^{-ikx} + B(k)e^{ikx}$$

$$\psi^-_{out} = A(k)\,\psi^+_{in} + B(k)\,\psi^+_{out}$$

The next step is to decide about incoming and outgoing spaces. Let $\mathcal{D}'$ consist of solutions which for $t \geq 0$ are moving away from the obstacle which we take to be supported in $[0, b]$. Such a solution should be described by

$$\psi(x,k) = \alpha(k)\psi^+_{out} + \beta(k)\psi^-_{out}$$

where for $\forall t > 0, \quad b < x < b+t$

$$\int e^{-ikt}\alpha(k)e^{ikx}\,dk/2\pi = 0$$

$$\text{or} \quad \int \alpha(k)e^{ikb}e^{-ik\varepsilon}\,dk/2\pi = 0 \qquad \forall \varepsilon > 0$$

$$\text{or} \quad \alpha(k)e^{ikb} \in H^- \quad \Longrightarrow \quad \alpha(k) \in e^{-ikb}H^-$$

and for $t > 0, \quad -t < x < 0.$

$$\int e^{-ikt}\beta(k)e^{-ikx}\,dk = 0$$

$$\forall \varepsilon > 0 \qquad \int \beta(k)e^{-ik\varepsilon}\,dk = 0 \quad \Longrightarrow \quad \beta(k) \in H^-$$

$$\therefore \quad \mathcal{D}' = (e^{-ikb}H^-)\psi^+_{out} + (H^-)\psi^-_{out}$$

Now $\mathcal{D}$ should consist of solutions which for $t \leq 0$ are supported in $[t, b-t]$

$$\psi(x,k) = \alpha(k)\psi^+_{in} + \beta(k)\psi^-_{in}$$

$x > b-t$

$$\int e^{-ikt}\alpha(k)e^{-ikx}dk = \int \alpha(k)e^{-ikb}e^{-ik(x+t-b)}\,dk = 0$$

or $\quad$ [redacted] $\alpha e^{-ikb} \in H^-$ $\quad$ or $\quad \alpha \in e^{ikb} H^-$

$x<t \quad \int e^{-ikt} \beta(k) e^{ikx}\, dk = \int \beta(k) e^{-ik(t-x)}\, dk = 0 \quad \Rightarrow \beta \in H^-$

Thus

$$\mathcal{D} = \left(e^{ikb} H^-\right) \psi_{in}^+ + \left(H^-\right) \psi_{in}^{[redacted]\,-}$$

As a check note that if $V=0$ so that $\psi_{in}^+ = e^{-ikx} = \psi_{out}^-$, $\psi_{out}^+ = e^{ikx} = \psi_{in}^-$, then

$$\mathcal{D} = e^{ikb} \mathcal{D}' \qquad \text{or} \qquad \underbrace{e^{-ikb}}_{\text{time evolution thru } t=b.} \mathcal{D} = \mathcal{D}'$$

Now on $\mathcal{D}/\mathcal{D}'$ time-evolution induces [redacted] a semi-group of operators, denoted $T(t)$ = effect of multiplying by e^{-ikt}. I am interested in the eigenvalues of this semi-group.

We need a simple model to understand: Take $\mathcal{D} = H^+$, $\mathcal{D}' =$ [redacted] $m(k) H^+$ and let $T(t)$ = effect of multiplying by e^{ikt}. It should be possible to relate the eigenvalues to $m(k)$, maybe even the characteristic polynomial.