More Schweninger. Consider a harmonic oscillator with "source" term:

\[ H = \frac{p^2}{2} + \frac{1}{2} \omega^2 q^2 - J(t)q. \]

and let's rapidly review the formula for \( \langle 0 | s | 0 \rangle \).

Use actual time so that

\[ \frac{\partial}{\partial t} \langle s | u(t, t') \rangle = -i \left[ H(t) \langle s | u(t, t') \rangle + \delta H(t) \langle u(t, t') | 0 \rangle \right]. \]

One has

\[ \delta \log \langle 0 | s | 0 \rangle = -i \int_{t_m}^{t_f} \frac{\langle 0 | u(t, t') \rangle \delta H(t) \langle u(t, t') | 0 \rangle}{\langle 0 | u(t, t') | 0 \rangle} dt \]

\[ = +i \int_{t_m}^{t_f} \langle s | q(t) \rangle dt \]

where

\[ \frac{d^2}{dt^2} \langle q(t) \rangle = -\omega^2 \langle q(t) \rangle + J(t) \]

so that

\[ \langle q(t) \rangle = \int \overline{G_0(t, t')} J(t') dt' e^{-i\omega |t-t'|} \]

\[ = -\frac{1}{2i\omega} \]

Hence

\[ \log \langle 0 | s | 0 \rangle = \frac{1}{2} i \int dt \int dt' J(t) G_0(t, t') J(t') \]

Let's consider now \( J(t) = c S(t) \). Recall that to solve for \( u(0^+, 0^-) \) one spreads time out around \( 0 \).

\[ dy = -i \left( H dt \right) y = -i (H_0 - c S(t)q) dt y \]

Use new parameter \( s \) with \( ds = S(t) dt \) for \( t \) between
and so we see that
\[ U(0^+, 0^-) = e^{i\gamma} \]

Check: For \( J(t) = c \delta(t) \) the above formula for \( \langle 0 | s | 0 \rangle \) gives
\[ \log \langle 0 | s | 0 \rangle = \frac{1}{2} i \gamma^2 \frac{1}{-2i\omega} = -\frac{\gamma^2}{4\omega}. \]

But also,
\[ \langle 0 | s | 0 \rangle = \langle 0 | e^{i\gamma} | 0 \rangle = \frac{\int e^{i\gamma - \frac{1}{2} \alpha^2 \gamma^2} d\gamma}{\int e^{-\frac{1}{2} \alpha^2 \gamma^2} d\gamma} = e^{-\frac{\gamma^2}{4\omega}}. \]

Schwinger uses this as follows. He wants to find the S-matrix \( \langle n | S | n' \rangle \) in the occupation number representation \( H \) in the case of a general source \( J \). One has the above formula on page 98 for \( \langle 0 | s | 0 \rangle \) in terms of \( J \). Suppose \( J \) supported inside \((t_{in}, t_f)\) one can add \( \delta \) function sources located at \( t = t_{in} \) and \( t_f \). Let
\[ \tilde{J} = J + c \delta(t-t_f) + c' \delta(t-t_{in}) \]

Then \( \langle 0 | \tilde{s} | 0 \rangle \) will essentially be \( \langle \psi | s | \psi' \rangle \) where \( \psi, \psi' \) are states of the form
\[ e^{i\gamma} | 0 \rangle = e^{i\gamma} e^{-\frac{1}{2} \alpha^2 \gamma^2} \]

Notice that \( \sqrt{2\omega} a = \left( \frac{d}{d\gamma} + \omega \gamma \right) \) applied to this gives
\[ e^{-\frac{1}{2} \omega q^{2}} e^{\left(\frac{1}{\hbar} \omega q^{2} \frac{d}{dq} + \omega q\right)} e^{-\frac{1}{2} \omega q^{2}} e^{icq} = ic \left( e^{icq} e^{-\omega q^{2}} \right) \]

Thus \( e^{icq} \) is an eigenvector for \( a \) with eigenvalue \( \frac{ic}{\sqrt{2\omega}} \), hence it is a so-called coherent state. In the polynomial repn. \( \alpha^* = \alpha, \alpha = \frac{d}{dq} \), the eigenvectors for \( a \) are

\[ e^{\lambda z} = \sum_{n \geq 0} \frac{\lambda^n z^n}{n!} = \sum_{n \geq 0} \frac{\lambda^n}{\sqrt{n!}} |n\rangle. \]

Consequently by the use of \( S \)-functions: sources at the initial and final times are computed the \( S \) matrix elements between coherent states and then use this as a generating function for the \( S \) matrix elements between occupation number states.

So

\[ \tilde{S} = e^{i[t_f - t_i]H_0} + \tilde{F}(t) + e^{i[t_f - t_i]H_0} \]

\[ \tilde{U}(t_f, t_i) = e^{icq} U(t_f, t_i) e^{icq} \]

\[ S = e^{i[t_f - t_i]H_0} \tilde{U}(t_f, t_i) e^{-i[t_f - t_i]H_0} \]

\[ \tilde{S} = e^{i[t_f - t_i]H_0} \tilde{U}(t_f, t_i) e^{-i[t_f - t_i]H_0} \]

\[ = e^{i[t_f - t_i]H_0} e^{icq} e^{-i[t_f - t_i]H_0} S e^{icq} e^{-i[t_f - t_i]H_0} \]

So I want to compute
\[ e^{i t H_0} e^{i c \phi} e^{-i t H_0} |0\rangle = e^{i t (H_0 - E_0)} e^{i c \phi} |0\rangle \]

\[ \frac{d}{dz} \sum_{n_\infty} \frac{\lambda^n}{n!} e^{i n \omega z} = \text{const.} \sum_{n_\infty} \frac{\lambda^n}{n!} e^{i n \omega z} \]

\[ = \text{const.} e^{(e^{i c t} z)} = \text{const.} e^{i (e^{i c t} z)^{\alpha}} |0\rangle \]

To determine the constants one can proceed as follows. \[ e^{i c \phi} |0\rangle = e^{i c \phi} e^{-\frac{1}{2} \omega z^2} / \sqrt{2\pi \omega} \]

so \[ \langle 0 | e^{i c \phi} |0\rangle = e^{-c^2 / 4\omega} \]

as we saw above.

But also we have \[ e^{i c \phi} |0\rangle = C e^{\lambda z} |0\rangle \]

\[ \lambda = \frac{ic}{2\omega} \]

\[ \langle 0 | e^{i c \phi} |0\rangle = C \langle 0 | e^{\lambda z} |0\rangle \]

\[ C = e^{-c^2 / 4\omega} \]

\[ e^{i c \phi} |0\rangle = e^{-c^2 / 4\omega} \frac{ic}{2\omega} a^\dagger |0\rangle \]

Thus \[ e^{i c \phi} |0\rangle = e^{-c^2 / 4\omega} \frac{ic}{2\omega} a^\dagger |0\rangle \]

and so \[ e^{i t H_0} e^{i c \phi} e^{-i t H_0} |0\rangle = C e^{i e^{i c t} c \phi} |0\rangle \]

\[ e^{-c^2 / 4\omega} = C \cdot e^{-2it c^2 / 4\omega} \]

\[ C = e^{-\frac{1}{4\omega} (c^2 - e^{2it c^2})} \]

\[ e^{i t H_0} e^{i c \phi} e^{-i t H_0} |0\rangle = e^{-\frac{1}{4\omega} (c^2 - e^{2it c^2})} e^{i e^{i c t} c \phi} |0\rangle \]
\[ \langle 0 | \tilde{s} | 0 \rangle = \langle e^{i \frac{t}{\hbar} \hat{H}_0} e^{-i \frac{t}{\hbar} \hat{H}_0} e^{-i \frac{t}{\hbar} \hat{H}_0} | S | e^{i \frac{t}{\hbar} \hat{H}_0} e^{i \frac{t}{\hbar} \hat{H}_0} e^{-i \frac{t}{\hbar} \hat{H}_0} \rangle \]

\[ = e^{-\frac{1}{4\omega} (\tilde{c}^2 - e^{-2i\frac{t}{\hbar} \tilde{c}^2})} e^{-i e^{it} \tilde{c}^2 | 0 \rangle \langle 0 | e^{i e^{it} \tilde{c}^2} \]
Consider a forced harmonic oscillator
\[ H = \frac{p^2}{2} + \frac{\omega^2 q^2}{2} - J(t)q \]
where \( J(t) \) is periodic, say \( J(t+1) = J(t) \). Let \( U(t,t') \) be the propagator for the quantum-mechanical motion:
\[
i \frac{\partial}{\partial t} U(t,t') = H(t) U(t,t')
\]
\[ U(t,t') = I \]
It follows that \( U(t+1, t'+1) = U(t,t') \) and hence
\[ U(t+1,0) = U(t+1,1) U(1,0) = U(t,0) U(1,0) \]
\[ U(t+n,0) = U(t,0)^n U(1,0)^n \]

\( U(1,0) \) is a so-called Floquet matrix. Its eigenvectors give rise to quasi-periodic solutions
\[ \psi(t+1) = \mathcal{F} \psi(t) \]
(Check \( \psi(t+1) = U(t+1,0) \psi(0) = U(t,0) U(1,0) \psi(0) \))
\[ \psi(t+1) = \mathcal{F} \psi(t) \]
where \( |\mathcal{F}| = 1 \). These are the analogues of constant energy states.

**Question:** Is the spectrum of \( U(1,0) \) discrete? Example: If \( J = 0 \), then
\[ U(1,0) = e^{-iH_0} \]
has a discrete spectrum, since \( H \) does. The eigenvalues of \( H \) are \((n+\frac{1}{2})\omega\), \( n \geq 0 \), so \( U(1,0) \) has the eigenvalues \( e^{-i(n+\frac{1}{2})\omega} \). The same
example holds if $T$ is constant, because this amounts to a different origin for the oscillator.

Actually one can ask whether $U(t,0)$ has discrete spectrum for any source $T$. It seems reasonable especially since $U(0,0)$ is supposed to be of trace class for $\beta$ in the block direction.

Let's review yesterday's calculations:

$$a = \frac{1}{\sqrt{2\omega}} (ip + \omega q)$$

I want to compute the matrix element

$$\langle e_\alpha | U(t,0) | e_\alpha \rangle$$

where $e_\alpha$ denotes the coherent states:

$$e_\alpha = \sum_{n=0}^{\infty} \frac{x^n e^{i n \theta}}{\sqrt{n!}} = e^{\lambda \alpha^*} |0\rangle$$

$$a e_\alpha = \lambda e_\alpha \quad (\alpha = \frac{d}{dx}, \alpha^* = \bar{z}).$$

Let's use the Schwinger method changing $T$ by $\delta T$.

$$\delta \log \langle e_\alpha | U(t,0) | e_\alpha \rangle = i \int_0^t \frac{\langle e_\alpha | U(t,1) \delta T(t) U(t,0) | e_\alpha \rangle}{\langle e_\alpha | U(t,0) | e_\alpha \rangle} \, dt,$$

$$= i \int_0^t \delta T(t_1) \langle g(t_1) \rangle \, dt,$$

The point is that $\langle g(t) \rangle$ satisfies the same DE

$$\left( \frac{d^2}{dt^2} + \omega^2 \right) \langle g(t) \rangle = \delta(t_1) \quad 0 \leq t_1 \leq t$$

except the boundary conditions are different:

$$\frac{d}{dt} \langle g(t_1) \rangle = \langle p(t_1) \rangle$$
\[
\frac{1}{\sqrt{2\omega}} \left( i \frac{d}{dt} + \omega \right) \langle \varphi(t_1) \rangle = \langle \left( \frac{ip + \omega q}{\sqrt{2\omega}} \right)(t_1) \rangle \\
= \frac{\langle e_{\alpha'} \mid U(t_1, 0) a U(t_1, 0) \mid e_{\alpha} \rangle}{\langle e_{\alpha'} \mid U(t_1, 0) \mid e_{\alpha} \rangle} = \lambda \\
\text{at } t_1 = 0
\]

Similarly,
\[
\frac{1}{\sqrt{2\omega}} \left( -i \frac{d}{dt} + \omega \right) \langle \varphi(t_1) \rangle \bigg|_{t_1 = t} = \tilde{\lambda}
\]

solve the DE for \( \langle \varphi(t_1) \rangle \) first for \( T = 0 \).

\[
\langle \varphi(t_1) \rangle = Ae^{-i\omega t_1} + Be^{-i\omega t_1}
\]

\[
\left. \frac{1}{\sqrt{2\omega}} \left( i \frac{d}{dt} + \omega \right) \langle \varphi(t_1) \rangle \right|_{t_1 = 0} = B \frac{1}{\sqrt{2\omega}} \left( i(-i\omega) + \omega \right) e^0 = \sqrt{2\omega} B
\]

\[
\left. \frac{1}{\sqrt{2\omega}} \left( -i \frac{d}{dt} + \omega \right) \langle \varphi(t_1) \rangle \right|_{t_1 = t} = A \frac{1}{\sqrt{2\omega}} \left( -i(\omega + \omega) \right) e^{i\omega t} = \sqrt{2\omega} e^{-i\omega t} A
\]

so
\[
\langle \varphi(t_1) \rangle = \frac{1}{\sqrt{2\omega}} \left( \bar{\lambda} e^{-i\omega t + i\omega t_1} + \lambda e^{-i\omega t_1} \right)
\]

In general we have
\[
\langle \varphi(t_1) \rangle = \frac{1}{\sqrt{2\omega}} \left( \bar{\lambda} e^{-i\omega(t - t_1)} + \lambda e^{-i\omega t_1} \right)
\]

\[
+ \int_0^t G(t, t') J(t') dt'
\]

\[
= \frac{e^{-i\omega |t_1 - t'|}}{2i\omega}
\]

Now multiply by \( iST(t_1) dt \), and integrate; then integrate \( S \) and you get
\[
\log \frac{\langle e_\lambda \mid u(t_0, 0) \mid e_\lambda \rangle}{\langle e_\lambda \mid e_\lambda \rangle} = \frac{i}{2} \int_0^t \int_0^t dt_1 dt_2 \ J(t_1) \ G(t_1, t_2) \ J(t_2)
\]

\[
\psi_0(t, 0) = e^{-it\omega} + \frac{i \lambda}{\sqrt{2\omega}} \int_0^t \ J(t_1) e^{-i\omega t_1} \ dt_1
\]

\[
+ \frac{i \lambda'}{\sqrt{2\omega}} \int_0^t \ J(t_1) e^{-i\omega (t-t_1)} \ dt_1
\]

Let's check this result by taking \( J(x) = \cos(x) \) and \( t = 0^+ \). We saw that

\[
\psi(t, 0) = e^{icq}
\]

so we want to compute \( \langle e_\lambda \mid e^{icq} \mid e_\lambda \rangle \). Recall

\[
e^{A+B} = e^A e^B e^{-\frac{1}{2}[A,B]}
\]

where \([A,B] \) commutes with \( A, B \).

\[
icq = \frac{i c}{\sqrt{2\omega}} (a + a^*)
\]

\[
\langle e_\lambda \mid e^{icq} \mid e_\lambda \rangle = \langle e_\lambda \mid e^{iax} e^{ia} \mid e_\lambda \rangle
\]

\[
= e^{-\frac{1}{2} s^2 [a^2, a]} \langle e_\lambda \mid e^{ia\lambda} e^{ia} \mid e_\lambda \rangle
\]

\[
= \frac{1}{\langle e_\lambda \mid e_\lambda \rangle} \langle e_\lambda \mid e^{ia\lambda} e^{ia} \mid e_\lambda \rangle
\]

So

\[
\frac{\langle e_\lambda \mid e^{icq} \mid e_\lambda \rangle}{\langle e_\lambda \mid e_\lambda \rangle} = e^{\frac{ic}{\sqrt{2\omega}} \left( \frac{q^2}{2} + x(a + \lambda) \right)}
\]

which agrees with the above.

We want to be able to use the formula at the top of the page:

\[
\frac{1}{2} \frac{q^2}{\omega} = \frac{c^2}{4\omega}
\]

\[
= \frac{i}{2} \frac{c^2}{2 \sqrt{2\omega}}
\]

\[
= \sqrt{\frac{1}{2}} \frac{c^2}{2 \sqrt{2\omega}}
\]
in order to compute \( U(t,0) \), and see its spectrum.

\[
e^{-itH_0} \psi = e^{-itH_0} \sum \frac{\lambda^n (\alpha^n)}{n!} |\psi\rangle
\]

\[
= \sum \frac{\lambda^n}{n!} e^{-it\omega} (\alpha^n) |\psi\rangle
\]

\[
= e^{\lambda e^{-it\omega}}
\]

hence

\[
\langle \psi | U(t,0) | \psi \rangle = \langle \psi | e^{-itH_0} | \psi \rangle
\]

\[
= e^{-it\omega} \overline{\lambda}
\]

Hence we find

\[
\log \langle \psi | U(t,0) | \psi \rangle = \lambda \overline{\lambda} e^{-it\omega} + \lambda \alpha
\]

\[
- \overline{\lambda} e^{-it\omega} \overline{\lambda} + \beta
\]

where

\[
\alpha = \frac{i}{\sqrt{2\omega}} \int_{-\infty}^{t} J(t_1) e^{-i\omega t_1} dt_1
\]

\[
\beta = \frac{i}{2} \int_{-\infty}^{t} dt_1 \int_{-\infty}^{t} dt_2 J(t_1) G(t_1,t_2) J(t_2) dt_2
\]

are constants.
Yesterday we found a formula for $\langle e^\chi | e^{-iH_0} | e^\chi \rangle$. Notice that $e^{iH_0} e^\chi = e^{iH_0} \sum \frac{\lambda^n}{n!} z^n = \sum \frac{\lambda^n}{n!} e^{i\lambda t} z^n = e^{i\lambda t} e^\chi$, where $\lambda = \omega Q^* e^\chi$. Hence

$$\langle e^\chi | e^{-iH_0} | e^\chi \rangle = \langle e^{iH_0} e^\chi | e^\chi \rangle$$

$$= \langle e^\chi e^{i\lambda t} | e^\chi \rangle = e^{i\lambda \chi} e^{-i\lambda t}$$

The formula becomes simpler if $e^{i\lambda t}$ is replaced by $e^{i\lambda t}$. Then

$$\frac{\langle e^\chi | e^{-iH_0} U(t, \phi) | e^\chi \rangle}{\langle e^\chi | e^{-iH_0} | e^\chi \rangle} = \frac{\langle e^\chi | e^{-iH_0} e^{iH_0} U(t, \phi) | e^\chi \rangle}{\langle e^\chi | e^{-iH_0} | e^\chi \rangle} = \frac{\langle e^\chi | S | e^\chi \rangle}{\langle e^\chi | e^\chi \rangle}$$

$$S = e^{iH_0} U(t, \phi)$$

Thus we find

$$\langle e^\chi | S | e^\chi \rangle = \exp \left\{ i\beta + \frac{i}{\lambda^2} \overline{\lambda^2} + \frac{i}{\lambda^2} \overline{\lambda^2} \right\}$$

$$\alpha = \frac{i}{\sqrt{2\omega}} \int_{0}^{t} J(t) e^{-i\omega t} dt$$

$$\beta = \frac{i}{2} \int_{0}^{t} dt_{1} \int_{0}^{t} dt_{2} J(t_{1}) G(t_{1}, t_{2}) J(t_{2})$$

We ought to see if $\beta$ and $\alpha$ are connected in some way in order that a unitary transformation $S$ can be defined by (x).

First review the way the $e^\chi$ are the joint-evaluators for the holomorphic representation. Recall
This representation consists of entire functions $f(z)$ with finite norm

$$
|f|^2 = \int \frac{|f(z)|^2}{|z|^2} \frac{dxdy}{\pi}
$$

and

$$
|0\rangle = 1, \quad \alpha = \frac{d}{dz}, \quad a^* = z
$$

Then

$$
f(\omega) = \sum_{n=0}^{\infty} \frac{1}{n!} f^{(n)}(0) \omega^n = \sum_{n=0}^{\infty} \left( a^n f, 1 \right) \omega^n
$$

$$
= \sum \left( f, \frac{1}{n!} \omega^n \bar{z}^n \right) = \left( f, e^{\bar{z} \omega} \right)
$$

so we see that $e^z$ is the point evaluator at $\bar{z}$. Moreover, we have (interchanging $\omega, z$)

$$
f(z) = \int f(\omega) e^{\bar{z} \omega} e^{-|\omega|^2} \frac{dwd\omega}{2\pi i}
$$

or

$$
f = \int e^{\bar{z} \omega} f(\omega) e^{-|\omega|^2} \frac{dwd\omega}{2\pi i}
$$

which expresses $f$ in terms of the $e^z$.

Suppose we want to define a linear operator $S$ by giving its effect $Se_\lambda$ on the coherent states. Clearly we want $(Se_\lambda)(z)$ to be analytic in both $\lambda$ and $z$. Since

$$
f = \int e^{\bar{z} \lambda} f(\lambda) d\lambda,
$$

$d\lambda = Gaussian$

we must have

$$
Sf = \int S e^{\bar{z} \lambda} f(\lambda) d\lambda.
$$

In order to use this to define $Sf$ we need to know that

$$
Se_\lambda = \int Se_{\bar{\lambda}} e^{\lambda \bar{z}} d\lambda.
$$
which will be the case if \((S_{\lambda})(z)\) as a function of \(\lambda\) is in the Hilbert space. So it's clear that we need to know \((S_{\lambda})(w)\) is analytic in \(\lambda, w\) and separately for each variable with the other one fixed in the holomorphic function Hilbert space. So the formula

\[
\langle e_{\lambda''} | S | e_{\lambda'} \rangle = e^{i\lambda} \left\{ c_1 + c_2 \lambda + c_3 \overline{\lambda'} + c_4 \lambda' \overline{\lambda''} \right\}
\]

with arbitrary constants will define an operator in the holomorphic Hilbert space.

Our next problem will be to understand when we get a unitary operator. It should be that we get the transformations coming from the metaplectic representation.

**Example:** Translation \(f(z) \rightarrow f(z+a)\) can be made into a unitary operator:

\[
\|f\|^2 = \int |f(z+a)|^2 e^{-|z|^2} \, dV = \int |f(z+a)|^2 e^{-z\overline{z} - 2\overline{a}z - a\overline{a} - a\overline{a}} \, dV
\]

\[
= \int |f(z+a) e^{-\overline{a}z - \frac{1}{2}|a|^2}|^2 e^{-|z|^2} \, dV = \|T_a f\|^2
\]

where

\[
(T_a f)(z) = e^{-\overline{a}z - \frac{1}{2}|a|^2} f(z+a)
\]

Then

\[
(T_a e_{\lambda})(z) = e^{\lambda z + \overline{\lambda} a - \overline{a}z - \frac{1}{2}|a|^2}
\]

\[
\langle e_{\lambda'} | T_a | e_{\lambda''} \rangle = e^{-\frac{1}{2}|a|^2 + a\lambda - \overline{a}\lambda' + a\overline{\lambda'}}
\]
So from this formula it is clear that the transformation \( S = e^{i \theta \hbar} U(t, 0) \) is a scalar of modulus 1 times \( T_0 \) where

\[
\alpha = \frac{i}{\sqrt{2 \omega}} \int_0^t dt_1 e^{-i \omega t_1} J(t_1)
\]

We should next see what this scalar is, i.e. compare \( i \beta \) with \(-\frac{1}{2} |\alpha|^2\).

\[
+ \alpha \bar{\alpha} = \frac{1}{2 \omega} \int dt_1 \int dt_2 J(t_1) e^{-i \omega t_1} J(t_2) e^{i \omega t_2}
\]

\[-2i \beta = -2 \frac{1}{2} \int dt_1 \int dt_2 J(t_1) e^{-i \omega |t_1 - t_2|} J(t_2) \]

\[
= \frac{1}{2 \omega} \int dt_1 \int dt_2 J(t_1) e^{i \omega |t_1 - t_2|} J(t_2)
\]

Clearly both have the same real part, but \( i \beta \) has a possibly non-trivial imaginary part.

\[\text{Im}(i \beta) = + \frac{1}{2} \frac{1}{2 \omega} \int dt_1 \int dt_2 J(t_1) J(t_2) \sin \omega |t_1 - t_2|\]

We should now be in a position to determine if

\[U(t, 0) = e^{-i \theta \hbar} S\]

has discrete spectrum by using the explicit formulas we have in the holomorphic representation. We ignore the scalar factor and replace \( S \) by \( T_0 \). We know

\[(e^{-i \theta \hbar} f)(z) = f(e^{-i \theta t} z)\]
\((U(t,0)f)(z) = (T_f)(e^{-i\omega t}z)\)
\[= f(e^{-i\omega t}z + \alpha) e^{-\alpha e^{-i\omega t}z - \frac{1}{2} |\alpha|^2}\]

Put \(f = e^{-i\omega t}\) and look for eigenfunctions for \(U(t,0)\):
\[f(z + \alpha) e^{-\frac{1}{2} |\alpha|^2} = \mu f(z)\]

Look at the fixed point \(z + \alpha = z \implies z = \frac{\alpha}{1-\xi}\) (assume \(\xi \neq 1\)).

Simpler way to proceed: Take
\[U(t,0) = e^{-i\theta H_0 T_d}\]

and conjugate with \(T_\beta\)
\[T_\beta U(t,0) T^{-1}_\beta = e^{-i\theta H_0} e^{i\theta H_0} T_\beta e^{-i\theta H_0} T_{\beta} T^{-1}_{\beta}\]

Now
\[(e^{i\theta H_0} T_\beta e^{-i\theta H_0} f)(z) = (T_\beta f)(e^{i\omega t}z)\]
\[= (e^{-i\theta H_0}f)(e^{i\omega t}z + \beta) e^{-\beta e^{i\omega t}z - \frac{1}{2} |\beta|^2}\]
\[= f(z + e^{-i\omega t} \beta) e^{-\beta e^{i\omega t}z - \frac{1}{2} |\beta|^2}\]
\[= (T_{e^{-i\omega t} \beta} f)(z)\]

\[[(T_{\theta} T_{\theta}' f)](z) = (T_{\theta}' f)(z + \alpha) e^{-\alpha z - \frac{1}{2} |\alpha|^2}\]
\[= f(z + \alpha + \theta) e^{-\theta(\theta + \alpha) - \frac{1}{2} |\theta|^2} e^{-\theta z - \frac{1}{2} |\theta|^2}\]
\[
= f(z + x + \delta) e^{-i(x+\delta)z - \frac{1}{2} |x|^2} e^{\frac{1}{2} (x\overline{\delta} + \overline{x}\delta)} - zJ
\]

\[
= e^{\frac{1}{2} (x\overline{\delta} - \overline{x}\delta)} (T_{\alpha + \delta} f)(z)
\]

And \(\frac{1}{2} (x\overline{\delta} - \overline{x}\delta) = i \text{Im}(x\overline{\delta})\) so it vanishes when \(Re = Re\). In particular

\[
T_{\beta}^{-1} = T_{-\beta}
\]

and we see that

\[
e^{-i\lambda_{\alpha}} T_{\beta} e^{-i\lambda_{\alpha}} T_{\alpha} T_{\beta}^{-1} = \text{scalar}. \quad T_{e^{-i\omega t} = \alpha - \beta}
\]

If we choose \(\beta\) so that

\[
e^{-i\omega t} \beta + \alpha - \beta = 0 \quad \Rightarrow \beta = \frac{\alpha}{1 - e^{-i\omega t}}
\]

it follows that

\[
T_{\beta} U(t, 0) T_{\beta}^{-1} = e^{-i\lambda_{\alpha}}. \text{ scalar}
\]

and hence the spectrum of \(U(t, 0)\) is discrete. All this assumes that

\[
e^{i\omega t} \neq 1.
\]

Notice that the eigenvalues of \(U(t, 0)\) are those of \(e^{-i\lambda_{\alpha}}\) shifted around the unit circle by a fixed scalar of modulus 1.

When \(e^{i\omega t} = 1\), then \(e^{-i\lambda_{\alpha}} = I\) and so \(U(t, 0) = T_\alpha\). In this case the spectrum is continuous, in fact I think that \(T_\alpha\) is equivalent to a shift on \(L^2(\mathbb{R})\).
Recall for the forced oscillator

$$H = \frac{1}{2} p^2 + \frac{1}{2} \omega^2 q^2 - J(t) q$$

one has two formulas for the ground-ground amplitude

$$\langle 0 | s | 0 \rangle = \exp \left( \frac{i}{2} \int_0^t J(t') G(t', t') J(t') \, dt' \right)$$

From the latter it follows that

$$\langle 0 | s (0) \rangle = i^n \langle T(q(t_1), \ldots, q(t_n)) \rangle,$$

But notice that if you wanted to find the coefficient of $x_1 \cdots x_n$ in the Taylor series expansion of

$$e^{\frac{1}{2} \sum a_{ij} x_i x_j},$$

you can write

$$e^{\frac{1}{2} \sum a_{ij} x_i x_j} = \prod_{i < j} e^{a_{ij} x_i x_j} \prod_{i} e^{\frac{1}{2} a_{ii} x_i^2}.$$

Now it is crystal clear that to get a product $x_1 \cdots x_n$ where these are assumed distinct, you have to partition $1, \ldots, n$ into pairs (hence $n$ must be even) and then take the product of the $a_{ij}$ for each pair, then add up over all partitions. This is Wick's sum over all possible pairwise contractions, and it obviously works even for thermal averages.

I want to understand the corresponding situation.
for fermions, like the Dirac field. Let's consider
the simpler boson situation:

\[ H = \omega a^* a + \tilde{T}a + \tilde{T}a^* \]

where \( \tilde{T} \) is to have compact support. Then

\[ \delta \log \langle 0 | s | 0 \rangle = +i \int \langle \delta \tilde{T}(t) a(t) + \delta \tilde{T}(t) a^*(t) \rangle dt \]

\[ = +i \int \left[ \delta \tilde{T}(t) \langle a(t) \rangle + \delta \tilde{T}(t) \langle a^*(t) \rangle \right] dt \]

\[ \frac{d}{dt} \langle a(t) \rangle = \langle [iH, a]\rangle(t) \]

\[ [H, a] = [\omega a^* a + \tilde{T}^2, a] = \omega [a^* a] + \tilde{T} [a^* a] \]

\[ = -\omega a^* \tilde{T} \]

\[ [H, a^*] = [\omega a^* a + \tilde{T}^2, a^*] = \omega a^* \tilde{T} \]

Thus we have

\[ \left( \frac{d}{dt} + i\omega \right) \langle a(t) \rangle = +i \tilde{T} \]

\[ \left( \frac{d}{dt} - i\omega \right) \langle a^*(t) \rangle = -i \tilde{T} \]

Since \( \langle a(t) \rangle = 0 \) for \( t < 0 \), \( \langle a^*(t) \rangle = 0 \) for \( t > 0 \) we have

\[ \langle a(t) \rangle = \int_{-\infty}^{+\infty} e^{-i\omega(t-t')} (i \tilde{T}(t')) dt' \]

\[ \langle a^*(t) \rangle = \int_{-\infty}^{+\infty} e^{-i\omega(t-t')} (i \tilde{T}(t')) dt' \]

So changing the signs doesn't help anything.

Reestablish notation:
\[ H = \omega a^* a + \tilde{\omega} a + \tilde{\omega} a^* \]

\[ 8 \log \langle 0 | s | 0 \rangle = -i \int \left[ 8 \tilde{\omega} \langle a(t) \rangle + 8 \tilde{\omega} \langle a^*(t) \rangle \right] dt \]

\[ \langle a(t) \rangle = \int e^{-i\omega(t-t')} (i\tilde{\omega}(t')) dt' \]

\[ \langle a^*(t) \rangle = \int e^{i\omega(t-t')} (-i\tilde{\omega}(t')) dt' \]

So

\[ 8 \log \langle 0 | s | 0 \rangle = (-1) \int dt \left[ 8 \tilde{\omega}(t) \int e^{-i\omega(t-t')} \tilde{\omega}(t') dt' \right] + \]

\[ (-1) \int dt \left[ 8 \tilde{\omega}(t) \int e^{i\omega(t-t')} \tilde{\omega}(t') dt' \right] \]

In the second integral reverse the order of integration

\[ \int dt \int dt' = \int dt' \int dt \]

Then interchange \( t, t' \) and you get

\[ 8 \log \langle 0 | s | 0 \rangle = \int dt \int dt' \left[ 8 \tilde{\omega}(t) e^{-i\omega(t-t')} \tilde{\omega}(t') + \tilde{\omega}(t) e^{i\omega(t-t')} \tilde{\omega}(t') \right] \]

or integrating out the \( \delta \)

\[ \log \langle 0 | s | 0 \rangle = \int dt \int dt' e^{-i\omega(t-t')} \tilde{\omega}(t) \tilde{\omega}(t') \]

Check: Put \( \tilde{\omega} = \tilde{\omega} = \frac{-i\tilde{\omega}}{\sqrt{2\omega}} \) so that \( a + a^* \tilde{\omega} = -i\tilde{\omega} \). You get

\[ \log \langle 0 | s | 0 \rangle = \frac{i}{2} \int \int e^{-i\omega(t-t')} \tilde{\omega}(t) \tilde{\omega}(t') \]

which agrees with our earlier result.
Let's return to the Dyson expansion
\[
\langle 0 | s | 0 \rangle = 1 - i \int < H(t) > dt_1 + \frac{(i)^2}{2!} \int \int < TH_2(t_1) H_2(t_2) > dt_1 dt_2
\]
where
\[
H_2 = \tilde{F}a + F a^\dagger
\]
This is a big expansion, think of it as a power series expansion in the variables \(J(t), \tilde{F}(t)\) and we can ask for the coefficient of the monomial
\[
J(t_1) \cdots J(t_p) \tilde{F}(t_{p+1}) \cdots \tilde{F}(t_n)
\]
where \(t_1, \ldots, t_n\) are assumed distinct. This means you have to go to the \(n\)-th term in the Dyson expansion which is
\[
\frac{(i)^n}{n!} \int \int \int < TH_2(t_1) \cdots H_2(t_n) > dt_1 \cdots dt_n
\]
Let us order times so that \(t_1, \ldots, t_n\) occurs in order. In other words the above integral can be taken over any "chambre", so let's use the chambre where the given \(t_1, \ldots, t_n\) are in order. Then it is clear that the coefficient is
\[
\frac{(i)^n}{n!} \langle T a^\dagger(t_1) \cdots a^\dagger(t_p) a(t_{p+1}) \cdots a(t_n) \rangle
\]
or in other words
\[
\delta_{n} \frac{\partial^n}{\partial J(t_1) \cdots \partial J(t_p) \partial \tilde{F}(t_{p+1}) \cdots \partial \tilde{F}(t_n)} \langle 0 | s | 0 \rangle =
\]
But we've seen that \(\langle 0 | s | 0 \rangle = \exp \int \int J(t') G(t,t') \tilde{F}(t)\)
where \(G(t,t') = -e^{-i \omega(t-t)}\)
Now if you want the coefficient of $x_i \cdots x_n y_1 \cdots y_n$ in
\[
\sum_{ij} x_i a_{ij} y_j = \prod_{ij} e^{x_i a_{ij} y_j}
\]
it is the sum over all ways ($n!$ in all) of attaching each $x$ variable to a $y$ variable and you multiply the corresponding $a_{ij}$. So again one sees how Wick's theorem holds in this case.

For later reference the formulas are
\[
\langle T a(t) \bar{a}(t') \rangle = \begin{cases} e^{-i \omega (t-t')} & t > t' \\ 0 & t < t' \end{cases}
\]
\[
= \Theta(t-t') e^{-i \omega (t-t')}
\]
July 29, 1972

I still haven't deciphered what Schwinger is doing with sources in the fermion situation. Again consider a space $W$ (finite-diml complex Hilb space) in which we have $H_0$: $H_0 \varphi_k = E_k \varphi_k$ where the $\varphi_k$ are orthonormal. Extend $H_0$ to $\Lambda W$ where

$$H_0 = \sum E_k \varphi_k^* \varphi_k$$

with $\varphi_k = \varphi(\varphi^*_k)$, $\varphi_k^* = \varphi(\varphi_k)$. The ground state for $H_0$ on $\Lambda W$ is $|0\rangle = \varphi_1 \cdots \varphi_p$ where $E_1, \ldots, E_p < 0$ and the rest are $>0$. For simplicity let us take $|0\rangle = 1$, i.e., assume all $E_i > 0$. In this case the Green's function for the operator $\frac{\partial}{\partial t} + i H_0$ on $W$

is

$$G(t,t') = \begin{cases} e^{-iH_0(t-t')} & t > t' \\ 0 & t < t' \end{cases}$$

We use the Green's function with positive frequencies for positive times and negatives frequencies for negative times. Our problem is to interpret Schwinger's formula

$$\langle 0 | S | 0 \rangle = \exp \left\{ i \int_{-\infty}^{\infty} \eta(t) G(t,t') \eta(t') \ dt \ dt' \right\}$$

that is, to find $H = H_0 + H_1$ which gives this formula for the ground-ground amplitude. My guess is that $\eta(t)$ should be an element of $W$ and that $\overline{\eta}(t) \in W^*$ and

$$H_1 = e(\eta) + i(\overline{\eta})$$
so that if $\bar{\eta} = \eta^\dagger$ then $H_1$ is self-adjoint.

Let's try computing the Dyson series

$$
\langle 0 | s | 0 \rangle = 1 - i \int \langle (e^\eta + i \bar{\eta}) (t) \rangle + \frac{(-1)^2}{2!} \int \langle (e^\eta + i \bar{\eta}) (t_1) \rangle \langle \bar{\eta} (t_2) \rangle
$$

Look at the second order term

$$
\sum_{k,l} \langle (\eta_K (t_1) a_k^\dagger + \bar{\eta}_K (t_1) a_k) e^{-i H_0 t_1} e^{-i H_0 t_2} (\eta_K (t_2) a_l^\dagger + \bar{\eta}_K (t_2) a_l) \rangle
$$

$$
= \sum_{k,l} \eta_K (t_1) \eta_K (t_2) \langle 0 | a_k^\dagger e^{-i E_k t_1} e^{-i E_k t_2} a_k^\dagger | 0 \rangle
$$

$$
= \sum_{k} \eta_K (t_1) \eta_K (t_2) e^{-i E_k (t_1 - t_2)}
$$

Look at fourth order. \[ \hat{H}_1 (t) = \sum_k \left( \eta_K (t) e^{i E_k t} a_k^\dagger + \bar{\eta}_K (t) e^{-i E_k t} a_k \right) \]

To get something $\neq 0$ in fourth order

$$
a_k^\dagger a_k a_m a_m^\dagger a_n
$$

There are three possibilities:

$$
\langle 0 | a_k^\dagger a_k a_m a_m^\dagger 10 \rangle = 1
$$

$$
\langle 0 | a_k^\dagger a_e a_e a_k^\dagger | 0 \rangle = 1 \quad l \neq k
$$

$$
\langle 0 | a_k^\dagger a_e a_e a_k^\dagger | 0 \rangle = -1 \quad l \neq k
$$

which give the following

$$
\sum_{k,j,m} \eta_K (t_1) \eta_K (t_2) \eta_j (t_3) \eta_j (t_4) e^{-i E_k (t_1 - t_2)} \eta_m (t_3) \eta_m (t_4) e^{-i E_m (t_3 - t_4)}
$$

$$
+ \sum_{k+l} \eta_K (t_1) \eta_K (t_2) \eta_k (t_3) \eta_k (t_4) e^{-i E_k t_1 - i E_k t_2 + i E_k t_3 + i E_k t_4}
$$

$$
- \sum_{k+l} \eta_K (t_1) \eta_K (t_2) \eta_k (t_3) \eta_k (t_4) e^{-i E_k t_1 - i E_k t_2 + i E_k t_3 + i E_k t_4}
$$
which can be written
\[ F(t_1, t_2) F(t_3, t_4) + F(t_1, t_4) F(t_2, t_3) - F(t_1, t_3) F(t_2, t_4) \]

where
\[ F(t_1, t_2) = \sum_k \eta_k(t_1) \eta_k(t_2) e^{-iE_k(t_1-t_2)} \]

\[ = \eta(t_1) G(t_1, t_2) \eta(t_2) \]

It seems that the - sign on the last term fouls things up. Compute the 2nd order term in \( \exp(i \int \bar{\eta}(t_1) G(t_1, t_2) \eta(t_2)) \) you get (-1) times

\[ \frac{1}{2!} \int \int \int \int F(t_1, t_2) F(t_3, t_4) = \frac{1}{2!} \left( \int + \int \right) \]

We have the possibilities six in all:

\[
\begin{array}{cccc}
1 & 2 & 3 & 4 \\
1 & 3 & 2 & 4 \\
1 & 3 & 4 & 2 \\
3 & 1 & 4 & 2 \\
3 & 1 & 2 & 4 \\
3 & 4 & 1 & 2 \\
\end{array}
\]

Now
\[ \int F(t_3, t_4) F(t_1, t_2) = \int F(t_3, t_4) F(t_1, t_2) \]
\[ t_3 > t_1 > t_4 > t_2 \]
\[ t_1 > t_3 > t_2 > t_4 \]

so interchanging \( 1 \leftrightarrow 3, 2 \leftrightarrow 4 \) reduces us to 3 possibilities:

\[
\begin{align*}
\int F(t_1, t_2) F(t_3, t_4) &+ \int F(t_1, t_2) F(t_3, t_4) + \int F(t_1, t_2) F(t_3, t_4) \\
&= t_1 > t_2 > t_3 > t_4 & t_1 > t_3 > t_2 > t_4 & t_1 > t_3 > t_4 > t_2
\end{align*}
\]
which differs from the expression at the top of the preceding page by a = signs.

So we see that we have to do something else in order to interpret Schrödinger's source. Try the following. Let's take the basic space \( W \) and enlarge it by adjoining some extra basis elements to \( W \oplus W' \). Then

\[
\Lambda(W \oplus W) \cong \Lambda W' \otimes \Lambda W
\]

can be interpreted as enlarging our number system from \( \mathbb{C} \) to \( \Lambda W' \). Now we consider the Hamiltonian

\[
H = \sum E_k a_k^* a_k + \sum (a_k^* \gamma_k + \tilde{\gamma}_k a_k)
\]

where \( \gamma_k(t), \tilde{\gamma}_k(t) \) are functions with values in \( W' \) interpreted as exterior multiplication operators. Now let's compute \( \langle 0 | S | 0 \rangle \) by variation

\[
\delta \log \langle 0 | S | 0 \rangle = -i \int \langle 0 | (\delta H(t)) | 0 \rangle dt.
\]

I should be more careful:

\[
\frac{\partial}{\partial t} U(t,t') = -i H(t) U(t,t')
\]

\[
\frac{\partial}{\partial t} (e^{iH_0 t} U(t,t')) = e^{iH_0 t} (e^{iH_0 t} U(t,t')) H(t) U(t,t')
\]
\[-ie^{-iH_0 t} \frac{\hat{H}_1(t) e^{-iH_0 t}}{\hat{H}_1(t)} e^{iH_0 t} U(t, t')\]

There should be no problem with the scattering formalism, because there is nothing unusual with the Hamiltonian \(H\). So

\[\delta \log \langle 0 | S | 0 \rangle = -i \int \sum_k \langle \delta \eta_k a_k + a_k^* \delta \eta_k \rangle(t) \, dt\]

\[\frac{d}{dt} \langle \delta \eta_k a_k \rangle(t) = i \langle [H_3 \delta \eta_k, a_k] \rangle(t)\]

\[[H_3 \delta \eta_k a_k] = [H_3 \delta \eta_k] a_k + \delta \eta_k [H_3 a_k]\]

I claim that \([H_3 \delta \eta_k] = 0\). Check:

\[[a_k^* \eta \delta \eta_k] = a_k^* \{ \eta, \delta \eta_k \} - \{ \delta \eta_k, \eta \} \eta_k = 0\]

e tc. Also

\[[H_3 a_k] = \sum_k (\varepsilon_k [a_k^* a_k] a_k + [a_k^* \eta + \eta a_k] a_k) + [\varepsilon_k a_k^* + a_k \eta] a_k]\]

\[= -\varepsilon_k a_k^* a_k - \eta_k\]

There is a problem with interpreting \(\langle a_k \rangle\). What one wants to do is to use the basis for \(\mathcal{W} \oplus \mathcal{W}'\) as a module over \(\Lambda \mathcal{W}'\) and take the 1-1 matrix element. If we do this it is clear that

\[\langle (\delta \eta_k a_k + a_k^* \delta \eta_k) \rangle(t) \]

\[= \delta \eta_k \langle a_k(t) \rangle + \langle a_k^*(t) \rangle \delta \eta_k(t)\]

because we've seen that \([H_3 \delta \eta_k] = 0\), so \(\delta \eta_k, \delta \eta_k\) remain...
\[ \frac{d}{dt} \langle a_k(t) \rangle = -i E_k \langle a_k(t) \rangle - i \gamma_k \]

\[ \langle a_k(t) \rangle = 0 \quad \text{for} \quad t \ll 0 \]

So

\[ \langle a_k(t) \rangle = \int_{-\infty}^{t} e^{-iE_k(t-t')} \gamma_k(t') \, dt' \]

Hence just as in the boson case we should get

\[ \langle 0 \mid s \mid 0 \rangle = \exp \left\{ -\sum_k \int_{t' < t} \gamma_k(t') e^{-iE_k(t-t')} \gamma_k(t) \right\} \]
July 30, 1979

Let's review path integrals. In the case of 1-dimensional motion with Hamiltonian

\[ H = \frac{p^2}{2} + U(q) \]

we saw that the propagator is expressed as a path integral

\[ \langle \bar{q}' | U(t, 0) | \bar{q} \rangle = \int [d\bar{q}] e^{i\bar{q}L} \]

The path integral is taken over all paths \( \bar{q} : [0, t] \rightarrow \mathbb{R} \) with \( \bar{q}(0) = \bar{q} \), \( \bar{q}(t) = \bar{q}' \), and it represents the average of the amplitude \( e^{i\bar{q}L} \) where \( L = \frac{1}{2} \bar{q}^2 - U(\bar{q}) \) is the Lagrangian.

Let us now consider a perturbation situation

\[ H = H_0 + V(q, t) \quad \text{e.g.} \quad V = -Jq \]

Then

\[ \langle \bar{q}' | U(t, 0) | \bar{q} \rangle = \int [d\bar{q}] e^{i\bar{q}L_0} e^{-i\int V} \]

Think of this as being the integral with respect to the measure \( [d\bar{q}] e^{i\bar{q}L_0} \) of the function

\[ q(t) \mapsto e^{-i\int V(q(t), t) dt} \]

In the case of \( V = -Jq \), it is just the Fourier transform of the measure \([d\bar{q}] e^{i\bar{q}L_0}\), where one thinks of \( J \) as being an element of the dual space to the space of paths.

Now if \( dq \) is a measure on \( \mathbb{R} \) say we have
\[
\int x^n d\mu = \left(\frac{d}{dt}\right)^n \int e^{i\lambda x} d\mu \bigg|_{\lambda = 0}
\]

and more generally for any polynomial

\[
\int f(x) d\mu = f\left(\frac{i}{\lambda} \frac{d}{dt}\right) \int e^{i\lambda x} d\mu \bigg|_{\lambda = 0}.
\]

So one has

\[
\langle q' | U(t, 0) | q \rangle = \int [d\gamma] e^{i S_{\gamma} - i \int L(q(t), \dot{q}) dt}
\]

\[
= \exp\left\{-i \int_0^t V \left(\frac{1}{\lambda} \frac{\delta}{\delta q(t)}, \dot{q} \right) dt\right\} \cdot \int [d\gamma] e^{i S_{\gamma} - i \int L_{\gamma}} \bigg|_{\lambda = 0}
\]
Recall that \( H = \frac{p^2}{2} + \frac{(\omega q)^2}{2} - J^2 \)

\[
\langle 0 | S^J | 0 \rangle = \exp \left\{ \frac{i}{2} \int \int J(t) \, G(t,t') J(t') \, dt \, dt' \right\}
\]

where \( G(t,t') = e^{-i\omega|t-t'|} \). Notice that the quadratic form

\[
(1) \quad J \mapsto \int \int J(t) \, G(t,t') J(t') \, dt \, dt'
\]

is symmetric and that its imaginary part is positive semi-definite. This is because for \( J \) real we know \( S^J \) is unitary, hence \( \langle 0 | S^J | 0 \rangle \leq 1 \). But we can also see this directly:

\[
\text{Re} \left\{ J(t) \, i \, G(t,t') \, J(t') \right\} = \text{Re} \left\{ J(t) \, \frac{e^{-i\omega|t-t'|}}{-2\omega} \, J(t') \right\}
\]

\[
= \text{Re} \left\{ \frac{-1}{2\omega} \, J(t) \, e^{-i\omega|t-t'|} \, J(t') \right\}
\]

\[
= -\frac{1}{2\omega} \text{Re} \left( J(t) \, e^{-i\omega t} \, J(t') e^{-i\omega t'} \right)
\]

Hence

\[
\int \int \text{Re} \left( J(t) \, i \, G(t,t') \, J(t') \right) \, dt \, dt'
\]

\[
= -\frac{1}{2\omega} \left( \int J(t) \, e^{-i\omega t} \, dt \right)^2
\]

Here \( J \) is a real function with compact support and

\[
J \mapsto \int J(t) \, e^{-i\omega t} \, dt
\]

is a complex linear functional; it follows that the real part of the quadratic form (1) has rank 2. Let's look at the Euclidean version. Here the
propagator for the Bloch equation is the path integral

\[ \langle q' | U(t, 0) | q \rangle = \int [dq] e^{-\left(\frac{i}{\hbar} \dot{q}^2 + \frac{1}{2} \omega^2 q^2\right)} e^{i \int \dot{q} \cdot \gamma} \]

Better compute \( \langle 0 | s | 0 \rangle = 1 + \int \langle 0 | e^{i \omega t \cdot \gamma} e^{-i \omega t \cdot \gamma} | 0 \rangle + \ldots \)

\[ \text{Slog} \langle 0 | s | 0 \rangle = \int \mathbb{F} [J(t)] \langle q(t) \rangle \, dt \]

\[ \frac{d}{dt} \langle q(t) \rangle = \langle [H, q](t) \rangle = \frac{i}{\hbar} \langle p(t) \rangle \]

\[ \frac{d}{dt} \langle p(t) \rangle = \frac{i}{\hbar} \langle \left[ \frac{1}{2} \omega^2 q^2 - \int \dot{q} \cdot \gamma, p \right](t) \rangle \]

\[ = \omega^2 \langle q(t) \rangle - J(t) \]

so

\[ \langle q(t) \rangle = -\int e^{-\omega |t-t'|} \frac{\mathbb{F} [J(t) \, e^{-i \omega |t-t'|}]}{2\omega} \, dt' \]

so

\[ \log \langle 0 | s | 0 \rangle = \frac{1}{2} \int \mathbb{F} [J(t)] \frac{\mathbb{F} [J(t) \, e^{-i \omega |t-t'|}]}{2\omega} \, dt \, dt' \]

Endeavour case

Now we know that on \( L^2 \)

\[ \left(-\frac{d^2}{dt^2} + \omega^2\right)^{-1} \text{ has kernel } e^{-\omega |t-t'|} \]

so therefore the quadratic form

\[ J \mapsto \mathbb{F} \int \mathbb{F} [J(t) \, e^{-i \omega |t-t'|} / 2\omega] \mathbb{F} [J(t')] \, dt \, dt' \]

is positive-definite. Notice that if \( J \) is replaced by \( iJ \) it becomes negative-definite.
From the path integral theory we get for
\[ H = \frac{P^2}{2} + \frac{1}{2} \omega^2 q^2 + V \]
we get the formula

\[ \langle 0 | S | 0 \rangle = \exp \left\{ -i \int V(\frac{\partial}{\partial \tilde{T}(t)}, t) dt \right\} \exp \left\{ \int \frac{i}{2} \left[ \tilde{T}(t) \tilde{S}(t) \right] dt \right\} \]

which is the basis for the perturbation expansion, Feynman diagrams, etc.

I want to take the quadratic case \( V = \frac{1}{2} \epsilon(t) q^2 \)
in which case I get

\[ \langle 0 | S | 0 \rangle = \exp \left\{ \int \frac{i}{2} \epsilon(t) \frac{\partial^2}{\partial \tilde{T}(t)^2} \right\} \exp \left\{ \int \frac{i}{2} \tilde{T}(t) \tilde{S}(t) \right\} \]

Let us look at a finite-dimensional analogue of this

\[ \exp \left\{ \frac{i}{2} \sum m \frac{\partial^2}{\partial x_m^2} \right\} e^{-\frac{1}{2} \sum a_{mn} x_m x_n} \bigg|_{x=0} \]

Consider the simplest possible case

\[ e^{aD^2} e^{bx^2} \bigg|_{x=0} \]

\[ = \sum_1 \frac{a^m D^{2m}}{m!} \sum_1 \frac{b^n x^{2n}}{n!} \bigg|_{x=0} = \sum_1 \frac{a^m b^n}{m! n!} (2m)! \]

\[ = \sum_0 \frac{1 \cdot 3 \cdots 2m-1}{m!} (2ab)^m \]

Now \( (1 - u)^{-1/2} = \sum_0 \frac{1}{m!} \left( \frac{1}{2} \right) \left( \frac{3}{2} \right) \cdots \left( \frac{2m-1}{2} \right) u^m = \sum_0 \frac{1 \cdot 3 \cdots 2m-1}{m!} \left( \frac{u}{2} \right)^m \)
\[ e^{aD^2}e^{bx^2} \bigg|_{x=0} = (1-4ab)^{-1/2} \]

and furthermore the perturbation series converges only for \( |4ab| < 1 \).

**General case:** To evaluate

\[ e^{\frac{1}{2}D^tPD} e^{\frac{1}{2}x^tQx} \bigg|_{x=0} \]

where \( x = (x_i) \) \( D = (D_i) \) are column vectors with \( D x^t = I \).

If we make a variable change \( x = Ax' \), then

\[ D x'^t A^t = I \quad \text{so} \quad A^tD x^t = I \]

or \( D' = A^t D \) and \( D = (A^t)^{-1} D' \)

Then

\[ D^t PD = D'^t A^{-1} P(A^t)^{-1} D' \]

\[ x^t Q x = x'^t A^t Q A x' \]

so we are allowed the transformation

\[ Q \mapsto A^t Q A \]

\[ P^{-1} \mapsto A^t P^{-1} A \]

i.e., the simultaneous transformation of quadratic forms.

The general theory here says that at least generically, we can make \( P'=I \) and \( Q' \) diagonal. In this case \((*)\) becomes

\[ \Pi (1-\delta_i)^{-1/2} = \det (I-P'Q')^{-1/2} = \det (I-PQ)^{-1/2} \]

where the \( \delta_i \) are diagonal entries of \( Q' \).
So we get the formula
\[ e^{\frac{1}{2} \text{Tr} \Phi \Phi' \Phi' \Phi} e^{\frac{1}{2} x^T Q x} \bigg|_{x=0} = \left[ \text{det} \ (1 - P Q) \right]^{-1/2} \]

This leads to the formula
\[ \langle 0 | s | 0 \rangle = \text{det} \ (1 + 2G)^{-1/2} \]

which we found on March 3.

Notice that the quartic interaction expression
\[ e^{a D^4} e^{b x^2} \bigg|_{x=0} = \sum_n \frac{a^n b^{2n}}{n! (2n)!} D^{4n} x^{4n} = \sum_n \frac{a^n b^{2n}}{n! (2n)!} (4n)! \]

diveses since if we apply the ratio test then
\[ \frac{u_{n+1}}{u_n} = \frac{a b^2 (4n+4)(4n+3)(4n+2)(4n+1)}{(n+1)(2n+1)(2n+2)} \to \infty \]

Hence some other ideas will have to be used in order to handle a quartic potential such as
\[ V(q) = \text{const} \ 8^4 \]

In the Euclidean case where we solve Block's equation
\[ \frac{\partial \psi}{\partial t} = -H \psi \]

for the time evolution we get
\[ \langle 0 | \int [d\sigma] e^{-\frac{1}{2} \omega^2 + \omega^2 \sigma^2} e^{\int \sigma d\sigma} \bigg| 0 \rangle = \exp \left\{ \frac{1}{2} \int J(t) \left[ D(t, t') J(t') \right] dtdt' \right\} \]
\[ \left[ \frac{d^2 + \omega^2}{2\alpha} \right]^{-1/2} \]

means endpts of path weighted by \( e^{-\omega |t-t'|} \).
so that if we replace $iJ$ by $iJ$ we see that the Gaussian

$$\exp \left\{ -\frac{1}{2} \int J(t) D(t, t') J(t') \, dt \, dt' \right\}$$

is the Fourier transform of the path space measure.

Let's now try to understand diagrams for the perturbation expansion of

$$H = \frac{1}{2} p^2 + \frac{1}{2} \omega^2 q^2 + \varepsilon q^4$$

where $\varepsilon(t)$ has compact support. We have

$$\langle 0 \mid s \mid 0 \rangle = 1 - i \int \varepsilon(t) \langle g(t) g \rangle \, dt + \frac{(i)^2}{2!} \int \varepsilon(t_1) \varepsilon(t_2) \langle g(t_1) g(t_2) \rangle - x \
\times \, dt_1, dt_2.$$ 

Recall

$$\langle T g(t_1) g(t_2) \rangle = -i \, G(t_1, t_2) = \frac{1}{\omega} \, e^{-i \omega |t_1 - t_2|}$$

and that $\langle T g(t_1) \cdots g(t_n) \rangle$ is the sum over all possible pairwise contractions. For $n=4$ we have 3 possible ways of contracting

```
  o
 / \o
 /  \o
```

hence

$$\langle g(t)^4 \rangle = 3 \left( -i \, G(t, t) \right)^2 = \frac{3}{4 \omega^2}$$

hence

$$\langle 0 \mid s^{(1)} \mid 0 \rangle = -i \frac{3}{4 \omega^2} \int \varepsilon(t) \, dt$$

Next consider the 2nd order term. To compute
\[ \langle T g(t_1)^4 g(t_2)^4 \rangle \text{ we make } 1 \cdot 3 \cdot 5 \cdot 7 = 105 \text{ contractions} \]

however \( \Sigma_4 \times \Sigma_4 \) acts on these leaving three types which we can represent by the diagrams

\[ t_1, t_2 \]

with multiplicities (\( = \) index of stabilizer)

\[ \frac{(4!)}{8^2} = 9 \quad \frac{(4!)}{8} = 3 \cdot 24 \quad \text{and} \quad \frac{(4!)}{4!} = 24 \]

(total \( 9 + 72 + 24 = 105 \)). Thus

\[ T \big( g(t_1)^4 g(t_2)^4 \big) = 9 \left( \frac{1}{2\omega} \right)^4 + 72 \left[ \left( \frac{1}{4\omega} \right)^2 \left( \frac{e^{-i \omega |t_1 - t_2|}}{2\omega} \right) \right]^2 + 24 \left( \frac{1}{2\omega} \right)^4 e^{-i \omega |t_1 - t_2|} \]

hence

\[ \langle 0 | S^{(2)} | 0 \rangle = \frac{-1}{2!} \frac{1}{(2\omega)^4} \int \delta(t_1) \delta(t_2) \left\{ 9 + 72 e^{-i \omega |t_1 - t_2|} + 24 e^{-i \omega |t_1 - t_2|} \right\} \]

\[ \int \delta(t_1) \delta(t_2) \]
To understand Green's functions, suppose we have an oscillator
\[ H = \frac{1}{2} p^2 + \frac{1}{2} m \omega^2 q^2 \]
or better
\[ H = \sum \frac{1}{2} p_i^2 + \sum \frac{1}{2} m_i \omega_i^2 q_i^2 \].
Then the Green's function
\[ \langle 0 | T \{ q_i(t) q_j(0) \} | 0 \rangle \]
is the probability amplitude for the system starting in the state \( q_i | 0 \rangle \) and being found at the latter time \( t \) in the state \( q_i | 0 \rangle \). (It would be nice in the case of lattice vibrations to interpret \( q_i | 0 \rangle \) as the state where the \( i \)-th atom has been excited one step above the ground state. This is perhaps reasonable.)

In the case of a general non-degenerate energy levels, it is not clear how \( q_i | 0 \rangle \) can be interpreted as an excited state.

Suppose one considers a many body problem with fermions:
\[ H = \sum \omega_k a_k^a_k + \sum \omega_m \mu_k a_m^a_k \]
Suppose that \( | 0 \rangle \) is the ground state for \( H \) in \( \Lambda^W \). What is the significance of the average
\[ \langle T \{ \psi(t) \psi(t') \} \rangle ? \]
Linear Response (Kubo):

Start with a system described by a Hamiltonian $H$. Assume it is initially in its ground state $|0\rangle$ and we perturb it by a small external field $H_{ex}$. In practice

$$H_{ex} = e \eta$$

where $\eta$ is a particle density operator (i.e., $\eta = a^\dagger a$) and $e = e(t)$ is the applied field. We want to compute the change in density $\delta \langle n(t) \rangle$ resulting from the perturbation. Here

$$\langle n(t) \rangle = \langle 0 | U(0,t) \eta U(t,0) | 0 \rangle.$$

We have to first order in $H_{ex}$

$$\delta U(t,0) = -i \int_0^t U(t,t') H_{ex}(t') U(t',0) \, dt'$$

$$\delta U(0,t) = -U(0,t) \delta U(t,0) U(0,t)$$

so

$$\delta \langle n(t) \rangle = \langle 0 | \int_0^t U(0,t') \eta U(t',0) \eta U(t,0) \, dt' - i \int_0^t U(0,t') \eta U(t,t') H_{ex}(t') U(t',0) \, dt' | 0 \rangle$$

$$+ U(0,t) \tilde{H}_{ex}(t')$$

or

$$\delta \langle n(t) \rangle = i \int_0^t \langle [\tilde{H}_{ex}(t'), \eta(t)] \rangle \, dt'$$

When $H_{ex} = e \eta$ this becomes

$$\delta \langle n(t) \rangle = i \int_0^t e(t') \langle [n(t'), n(t)] \rangle \, dt'$$
\[ \delta < n(t) > = \int_0^t -i < [n(t), n(t')] > \varepsilon(t') \, dt' \]

This expresses the linear response of the density \( < n(t) > \) to the applied field \( \varepsilon(t) \). The kernel is a so-called retarded Green's function:

\[ G^R(t, t') = -i < [n(t), n(t')] > \Theta(t-t') \]

Now the Feynman-Dyson series computes the time-ordered Green's function

\[ G^T(t, t') = -i < T n(t) n(t') >. \]

To relate \( G^T \) and \( G^R \) one uses the Lehmann representation.
Perturbation expansion of the Green’s function:
Begin with \( H = H_0 + V \). The Green’s function we want to compute is
\[
\langle T g(t') g(t) \rangle.
\]

Let us consider the temperature Green’s function, where
\[
\langle A \rangle = \frac{\text{tr} \left( e^{-\beta H A} \right)}{\text{tr} \left( e^{-\beta H} \right)} \quad g(t) = e^{iHt} g e^{-iHt}
\]
then we are after
\[
G(t) = \frac{\text{tr} \left( e^{-\beta H} e^{iHt} g e^{-iHt} g \right)}{\text{tr} \left( e^{-\beta H} \right)}
\]
We want to write this in terms of thermal averages with \( H_0 \). Now
\[
\frac{Z}{Z_0} = \frac{\text{tr} \left( e^{-\beta H} \right)}{\text{tr} \left( e^{-\beta H_0} \right)} = \frac{\text{tr} \left( e^{-\beta H_0} e^{\beta H_0} e^{-\beta H} \right)}{Z_0} = \langle U(\beta, 0) \rangle
\]
where \( U(\beta, \sigma) = e^{\beta H_0} e^{-\beta H} e^{\sigma H} e^{-\sigma H_0} \) is the propagator in the interaction picture. Recall we have
\[
U(\beta, \sigma) = I - \int_0^\beta V(\tau) \, d\tau_1 + \frac{(-1)^2}{\sigma} \int_0^\beta d\tau_1 \int_0^\tau_1 d\tau_2 \, V(\tau_1) V(\tau_2) + \ldots
\]
\[
\langle U(\beta, 0) \rangle = 1 - \int_0^\beta <V(\tau)> + \frac{(-1)^2}{\sigma} \int_0^\beta d\tau_1 \int_0^\tau_1 d\tau_2 \, <V(\tau_1) V(\tau_2)> + \ldots
\]
The numerator can be written (after dividing by $Z_0$)

\[ \frac{\text{tr} \left( e^{-\beta H} e^{iH} e^{-iH} \right)}{Z_0} \]

\[ = \frac{1}{Z_0} \text{tr} \left( e^{-\beta H_0} e^{iH_0} e^{-\beta H} e^{iH} e^{-iH_0} e^{iH_0} e^{-iH} e^{iH} \right) \]

\[ = \langle U(\beta, t) \varrho(t) U(t, 0) \varrho \rangle \]

\[ = \sum \sum (-1)^n \int dt_1 \ldots dt_n \sum (-1)^{n'} \int dt'_1 \ldots dt'_n \langle V(t_1) \ldots V(t_n) \varrho(t) \times \]

\[ V(t'_1) \ldots V(t'_n) \rangle \]

Now suppose you look at all terms involving $p$ $V$-factors; you have one for each $n + n' = p$. Given $\beta > t_1 > \ldots > t_p > 0$, it belongs to the term where there are $n - t_i$'s bigger than $t$. So it's clear we have for the degree $p$ contribution

\[ (-1)^p \int \langle TV(t_1) \ldots V(t_p) \varrho(t) \rangle \varrho \rangle dt_1 \ldots dt_p \]

\[ = \frac{(-1)^p}{p!} \int \int \langle TV(t_1) \ldots V(t_p) \varrho(t) \rangle \varrho \rangle \]

So we get the formula

\[ G(t) = \frac{\sum (-1)^p \int \langle TV(t_1) \ldots V(t_p) \varrho(t) \rangle \varrho \rangle dt_1 \ldots dt_p}{\sum (-1)^p \int \langle TV(t_1) \ldots V(t_p) \rangle \varrho \rangle dt_1 \ldots dt_p} \]

Tomorrow we want to understand why this reduces to a sum over connected diagrams.
What I am missing is a feeling for the physical significance of the 1-particle Green's function. I think in the many-body problem one is able to write the Hamiltonian

\[ H = E_g + \sum \varepsilon_k a_k^* a_k + \text{small term} \]

and somehow the Green's function tells me about the elementary excitations.

So let's consider an interacting system of fermions with

\[ H = \frac{p_i^2}{2} + \sum U(q_i) + \frac{1}{2} \sum V(q_i, q_j) \]

\[ H_0 \]

Find the eigenvectors for the 1-particle Hamiltonian

\[ H_0 \phi_k = \omega_k \phi_k \]

and form Fock space \( \Lambda = \) exterior algebra on 1-particle Hilbert space with creation and annihilation operators \( a_k^* = e(\phi_k), a_k = i(\phi_k^*) \). On \( \Lambda \) we have

\[ H_0 = \sum \omega_k a_k^* a_k \]

Now instead of the operators \( \phi_k, \phi_k^* \) it is sometimes useful to use the field operators

\[ \psi(x) = \sum \phi_k(x) a_k \]

\[ \psi(x)^* = \sum \overline{\phi_k(x)} a_k^* \]

(Here I assume the one-particle states are scalar...
functions of position. If \( \psi_k(x) = (\psi_{k\lambda}(x)) \) is a vector function, e.g. \( \lambda \) is a spin coordinate, then we have field operators \( \psi_k(x), \psi_k(x)^* \). If one thinks of Fock space as being the exterior algebra with basis \( |x\rangle = \delta(q-x) \) for different \( x \), then \( \psi(x) \) destroys a particle at \( x \) and \( \psi(x)^* \) creates a particle at \( x \). Then in the 1-particle space

\[
H_0 = \int |x\rangle \langle x| \left( -\frac{i}{2} \nabla^2 + U(x) \right) dx dx',
\]

so on Fock space

\[
H_0 = \int \psi(x)^* \langle x| \left( -\frac{i}{2} \nabla^2 + U \right) \psi(x') dx dx',
\]

Now \( \langle x| \left( -\frac{i}{2} \nabla^2 + U \right) |x'\rangle \) is sort of a diagonal matrix, which is why one sees written

\[
H_0 = \int \psi(x)^* \left( -\frac{i}{2} \nabla^2 + U(x) \right) \psi(x) dx
\]

Next let us consider the interaction

\[
H_1 = \frac{1}{2} \sum_{i \neq j} V(q_i q_j)
\]

or more generally suppose we are given a two-particle operator and we want to understand its extension to Fock space. Let's first look at the linear algebra.

Look at an operator \( V \) on \( L^2 \). Its matrix elements are

\[
\langle \phi_k | V | \phi_m \rangle
\]

so we can write \( V \) as
\[ V = \frac{1}{4} \sum_{klnm} e(\varphi_k) e(\varphi_l) \langle \varphi_k^* \varphi_l^* \mid V \mid \varphi_m \varphi_n \rangle \tilde{a}^*(\varphi_m) \tilde{a}(\varphi_n) \]

\[ = \frac{1}{4} \sum_{klnm} \langle \varphi_k^* \varphi_l^* \mid V \mid \varphi_m \varphi_n \rangle \tilde{a}_k^* \tilde{a}_l^* \tilde{a}_m \tilde{a}_n \]

at least on \( \Lambda^2 \). Now when you extend a 2-particle operator to Fock space just what do you do?

So suppose given \( V : \Lambda^2 W \rightarrow \Lambda^2 W \) where \( W \) is the 1-particle space.

Let us think of an element \( \omega \in \Lambda^n W \) as giving a function \( \psi_i \ldots \psi_n \) namely its components with respect to the basis \( \varphi_1 \wedge \ldots \wedge \varphi_n \). Thus \( \psi_i \ldots \psi_n \) is a skew-symmetric tensor, with

\[ \langle \psi_i^* \psi_j \ldots \psi_n^* \psi_k \mid \omega \rangle = \omega_{ij} \ldots \omega_{kn} \]

and

\[ \omega = \sum_{4k < i_n} \omega_{ij} \ldots \omega_{kn} \varphi_i \wedge \ldots \wedge \varphi_n \]

For example if we use the basis \( |x_1\rangle \), then \( \omega \) gives us a skew-symmetric function \( \omega(x_1, \ldots, x_n) \) of the coordinates. Moreover element \( \varphi_1 \wedge \ldots \wedge \varphi_n \) of \( \Lambda^n W \) corresponds to the function
\[
\langle x_1, \ldots, x_n | \psi_1, \ldots, \psi_n \rangle = \det (\langle \phi_i \chi_j \rangle).
\]

Now let \( V : \Lambda^2 W \rightarrow \Lambda^2 W \) be a 2-particle operator. Its effect on

\[\omega = \sum_{m<n} \omega_{mn} \psi_m \psi_n\]

is

\[V \omega = \sum_{k<l}^m \langle \psi_k | \psi_l \rangle \langle \psi_k, \psi_l | V | \psi_m, \psi_n \rangle \omega_{mn}\]

hence

\[(V \omega)_{kl} = \frac{1}{2} \sum_{m,n}^l V_{klnm} \omega_{mn} \]

Now when we extend \( V \) to \( \Lambda^N W \) we make it operate on each pair of components and then we add. So for

\[\omega = \sum_{i_1 < \cdots < i_N} \omega_{i_1, \ldots, i_N} \psi_{i_1} \cdots \psi_{i_N} = \frac{1}{N!} \sum_{i_1 < \cdots < i_N} \omega_{i_1, \ldots, i_N}\]

we have

\[V \omega = \sum_{1\leq a < b \leq N} \omega_{i_1, \ldots, i_N} \]

\[1\]