[redacted] May 12, 1979 Relativity 892



Maxwell's equations using differential forms:
The idea is that the six components of $E, B$ are the components of a 2-form on space-time, in fact the 2 form is the curvature of a connection on a line bundle.

Maxwell's equations are

$$\begin{cases} \nabla\cdot B = 0 & \nabla\cdot E = \rho \\ [\text{redacted}]\ \nabla\times E = -\dfrac{\partial B}{\partial t} & \nabla\times B = \dfrac{\partial E}{\partial t} + j \end{cases}$$

The first column consists of homogeneous DE's and can be put in the form $d\omega = 0$. Try

$$\boxed{\begin{aligned}\omega = {} & B_x\, dy\, dz + B_y\, dz\, dx + B_z\, dx\, dy \\ & + E_x\, dx\, dt + E_y\, dy\, dt + E_z\, dz\, dt\end{aligned}}$$

Then

$$\begin{aligned} d\omega = {} & \frac{\partial B_x}{\partial x} dx\,dy\,dz + \frac{\partial B_y}{\partial y} dy\,dz\,dx + \frac{\partial B_z}{\partial z} dz\,dx\,dy \\ & + \frac{\partial B_x}{dt} dy\,dz\,dt + \frac{\partial B_y}{\partial t} dz\,dx\,dt + \frac{\partial B_z}{\partial t} dx\,dy\,dt \\ & [\text{redacted}] - \frac{\partial E_{\cancel{x}}}{\partial y} dx\,dy\,dt - \frac{\partial E_y}{\partial z} d\cancel{y}\,dz\,dt - \frac{\partial E_z}{\partial x} d\cancel{z}\,dx\,dt \\ & \frac{\partial E_x}{\partial z} dz\,dx\,dt + \frac{\partial E_y}{\partial x} dx\,d\cancel{y}\,dt + \frac{\partial E_z}{\partial y} dy\,dz\,dt \end{aligned}$$

Thus

$$\boxed{d\omega = 0 \iff \begin{cases} \nabla\cdot B = 0 \\ \nabla\times E + \dfrac{\partial B}{\partial t} = 0 \end{cases}}$$

Next let

$$\boxed{\eta = -\varphi\, dt + A_x dx + A_y dy + A_z dz}$$

Then

$$d\eta = -\frac{\partial\varphi}{\partial x} dx\, dt - \frac{\partial\varphi}{\partial y} dy\, dt - \frac{\partial\varphi}{\partial z} dz\, dt$$

$$+ \frac{\partial A_y}{\partial x} dx\, dy + \frac{\partial A_z}{\partial y} dy\, dz + \frac{\partial A_x}{\partial z} dz\, dx$$

$$- \frac{\partial A_x}{\partial y} dx\, dy - \frac{\partial A_y}{\partial z} dy\, dz - \frac{\partial A_z}{\partial x} dz\, dx$$

[redacted] $- \frac{\partial A_x}{\partial t} dx\, dt - \frac{\partial A_y}{\partial t} dy\, dt - \frac{\partial A_z}{\partial t} dz\, dt$

So

$$\boxed{d\eta = \omega \iff \begin{cases} \nabla \times A = B \\ -\nabla\varphi - \frac{\partial A}{\partial t} = E \end{cases}}$$

Next compute $d^*$ using the Minkowski metric:

$$ds^2 = dx^2 + dy^2 + dz^2 - dt^2$$

Then since $d = \sum e(dx_i)\, \theta\left(\frac{\partial}{\partial x_i}\right)$ one has

$$d^* = \sum \theta\left(-\frac{\partial}{\partial x_i}\right) e\left(dx_i^{dual}\right)$$

or

$$d^* = \theta\left(\frac{\partial}{\partial t}\right) i\left(\frac{\partial}{\partial t}\right) - \theta\left(\frac{\partial}{\partial x}\right) i\left(\frac{\partial}{\partial x}\right) - \theta\left(\frac{\partial}{\partial y}\right) \theta\left(\frac{\partial}{\partial y}\right) - \theta\left(\frac{\partial}{\partial z}\right) i\frac{\partial}{\partial z}$$

$$
\begin{aligned}
-d^*\omega = {} & \frac{\partial E_x}{\partial t}dx + \frac{\partial E_y}{\partial t}dy + \frac{\partial E_z}{\partial t}dz \\
& \frac{\partial E_x}{\partial x}dt + \frac{\partial E_y}{\partial y}dt + \frac{\partial E_z}{\partial z}dt \\
& -\frac{\partial B_y}{\partial x}dz - \frac{\partial B_z}{\partial y}dx - \frac{\partial B_x}{\partial z}dy \\
& \frac{\partial B_z}{\partial x}dy + \frac{\partial B_x}{\partial y}dz + \frac{\partial B_y}{\partial z}dx
\end{aligned}
$$

or

$$\boxed{-d^*\omega = (\nabla\cdot E)\,dt + \left(\frac{\partial E}{\partial t} - \nabla\times B\right)\cdot d\vec{x}}$$

So therefore

$$\boxed{+d^*\omega = -\rho\,dt + \vec{j}\cdot d\vec{x} \iff \begin{cases} \nabla\cdot E = \rho \\ \nabla\times B = \dfrac{\partial E}{\partial t} + j \end{cases}}$$

Check $-d^*(d^*\omega) = \dfrac{\partial \rho}{\partial t} + \vec{\nabla}\cdot\vec{j} = 0$ which is the equation of continuity for charge.

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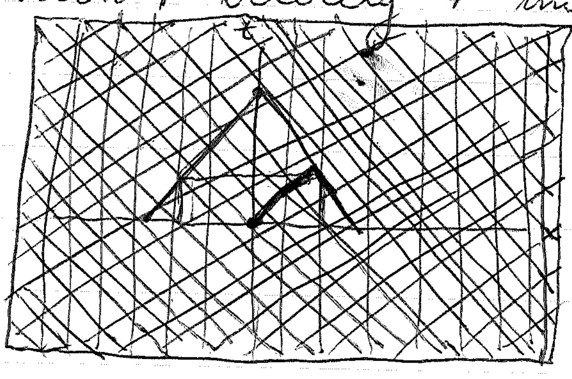
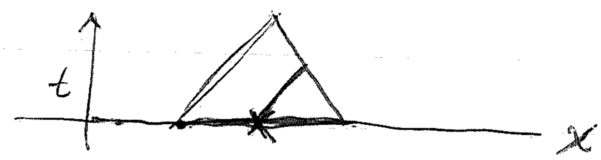
Let's now continue and understand the rest of classical electromagnetism. The basic setting is as follows: Space-time is $\mathbb{R}^4$ and we will have $N$-particles each of which has a mass and charge. A state of the system gives the positions and velocities of the particles at a fixed time, and also the initial electromagnetic field.

Notice that because Maxwell's equations are linear, if all the charges are zero, we would be able to conclude that $E = B = 0$.

Now it seems that since we know the field $F = (E, B)$ at time $0$, we know the acceleration of the charges. [redacted] It would be nice if we could use this to compute [redacted] $\frac{\partial F}{\partial t}$ at $t=0$ and so on.

Since Maxwell's equations are hyperbolic and disturbances propagate at [redacted] the speed of light, it seems as if what happens to a charge located at, say $t=0$, $\vec{x}=0$, near $(0,0)$ depends only on the initial position + velocity + initial [redacted] field $F$.






The first problem is whether one can solve this one particle problem with given initial field. The second problem is whether one can pass from a discrete set of particles to a smooth charge-current distribution, that is, to a fluid picture.

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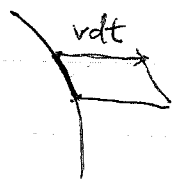
Fluid flow: Let $\rho$ be the density and $\vec{v}$ the velocity. Conservation of mass: Fix a volume $V$:

$$\iiint_V \rho\, dV = \text{mass in } V$$

$$\iiint_V \frac{\partial \rho}{\partial t}\, dV = \text{rate of change of mass in } V$$




$$= -(\text{rate of outflow of mass})$$

$$= -\iint_{\partial V} \rho\, \vec{v}\cdot d\vec{\sigma}$$

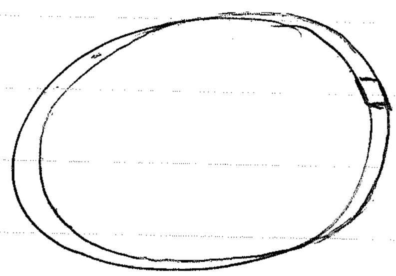
Using divergence thm, etc., you get

$$\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho\vec{v}) = 0$$

Next apply Newton's law to the momentum in $V$

$$\iiint_V \rho v_x\, dV = \text{total momentum of } V \text{ in } x \text{ direction}$$

In a small time interval $dt$ $V$ flows to a close by region. The momentum increase will be the sum of the internal increase + increase in the shell:



$$\left(\iiint_V \frac{\partial}{\partial t}(\rho v_x)\, dV + \iint_{\partial V} \rho v_x\, \vec{v}\cdot d\vec{\sigma}\right) dt$$

The force on $V$ comes from the pressure $P$, assuming no viscous force. So

$$-\iint_{\partial V} P\, \underset{\substack{\uparrow \\ \text{want } x \text{ comp.}}}{\hat{\imath}}\cdot d\vec{\sigma} = \iiint_V \frac{\partial}{\partial t}(\rho v_x)\, dV + \iint_{\partial V} \rho v_x\, \vec{v}\cdot d\vec{\sigma}$$

which gives us

$$\frac{\partial P}{\partial x} + \frac{\partial}{\partial t}(\rho v_x) + \underbrace{\nabla\cdot(\rho v_x \vec{v})}_{\nabla v_x \cdot \rho\vec{v} \;+\; v_x \underbrace{\vec{\nabla}\cdot(\rho v)}_{-\frac{\partial\rho}{\partial t}}} = 0 .$$

so

$$\frac{\partial P}{\partial x} + \rho\frac{\partial v_x}{\partial t} + \rho\vec{v}\cdot\nabla v_x = 0$$

or

$$\frac{1}{\rho}\frac{\partial P}{\partial x} + \frac{\partial v_x}{\partial t} + (\vec{v}\cdot\nabla)v_x = 0$$

or

$$\boxed{\frac{1}{\rho}\vec{\nabla}P + \frac{\partial\vec{v}}{\partial t} + (\vec{v}\cdot\nabla)\vec{v} = 0}$$

Curious thing. If $R(t)$ is a trajectory, then

$$\frac{dR(t)}{dt} = v(R(t),t)$$

so the acceleration is

$$\frac{d^2R(t)}{dt^2} = \frac{\partial v}{\partial x}\frac{dx}{dt} + \cdots + \frac{\partial v}{\partial z}\frac{dz}{dt} + \frac{\partial v}{\partial t}$$

$$= (v\cdot\nabla)\vec{v} + \frac{\partial\vec{v}}{\partial t}$$

Thus one has

$$-\nabla P = \rho\cdot\text{acceleration}$$

and this is surprising when $\rho$ is time-dependent.

To what extent can I think of a fluid flow as a bundle of particle trajectories? In other words can I describe a fluid as the limit of a many body system as the number of particles increases.

Let $M$ be the phase space for a single particle. Then $M^N$ is the phase space for $N$ particles. Suppose the particles interact with each other. Then the Hamiltonian for our $N$-body problem is

$$\sum_{i=1}^{N}\left(\frac{p_i^2}{2m} + \varepsilon_1 V(q_i)\right) + \varepsilon_2 \sum_{i<j}' V(q_i, q_j)$$

What I want to do now is to let $N \to \infty$ in a suitable way, and to consider only the part of $M^N$ for which we can see a limiting density $\rho$ and velocity field $v$. How can I make sense of this? Near any point of configuration space $X$ I want all the particles to have roughly the same velocity. Here $M = T^*(X)$.

So it seems that the limit will be a section of $M = T^*(X)$ over $X$. This should be done more carefully.

Let's think of particles as being sprinkled over $X$. In order that one has a well-defined density, as more particles are put in, their masses must decrease. But it is clear that the velocity of the fluid is the same as the velocity of its particles.

*Example*: Consider the particles moving in a gravitational field where the force on a particle is proportional to its mass, and hence the acceleration of the particle depends only on position:

$$\ddot{x} = -\nabla\phi(x)$$

[redacted] I want to make up a fluid of particles obeying this law of motion. [redacted] It's clear that we get the same motion no matter how heavy the particles are, so it must be possible to write the DE satisfied by the velocity field $\vec{v}(x,t)$:

$$\frac{\partial\vec{v}}{\partial t} + (\vec{v}\cdot\nabla)\vec{v} = -\nabla\phi(x)$$

This is a first order PDE for $v(x,t)$ which we can solve given $v(x,0)$. Once we have $v(x,t)$ then we can find $\rho$ using the equation of continuity:

$$\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho v) = 0$$

and the initial density $\rho(x,0)$.

So at this point I think I understand how to deal with independent particles. In general suppose the single particles are governed by

$$\ddot{x} = F(t, x, \dot{x})$$

Then the velocity field $v(x,t)$ can be obtained as follows. Suppose given $v(x,0)$ and let $X(t,x)$ be the solution of this DE with initial position $x$ and velocity $v(x,0)$. One assumes for each $t$ that $x \mapsto X(t,x)$ is a diffeomorphism, that is, one can follow backwards the particle having position $x$ at time $t$. Thus given $x_0, t_0$ one gets a trajectory $x(t) = X(t, x_1)$ with $x(t_0) = x_0$, and we can define

$$v(x_0, t_0) = \frac{d}{dt} x(t) \Big|_{t=t_0}$$

It follows that

$$v(x(t), t) = \frac{d}{dt} x(t)$$

whence

$$\frac{d^2}{dt^2} x(t) = \frac{\partial v}{\partial t}(x(t), t) + (\vec{\nabla} v)(x(t),t) \cdot v(x(t), t)$$

$$\parallel$$

$$F(t, x(t), v(x(t),t))$$

Thus $v(x,t)$ satisfies

$$\frac{\partial v}{\partial t} + (v \cdot \nabla) v = F(t, x, v).$$

So it seems to work, but it's messy.

If I think of a fluid as made of independent particles and assume [redacted] the force on a particle proportional to its mass, then the equation of motion of a single particle is

$$\ddot{x} = F(t, x, \dot{x})$$

This leads to the partial DE's for the density $\rho$ and velocity:

$$\frac{\partial v}{\partial t} + (v \cdot \Delta) v = F(t, x, v)$$

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho v) = 0$$

Now let's look at a fluid made up of charged particles with $q/m$ fixed, say $q = m$. The force on a charge $q$ with velocity $v$ is

$$F = q(E + v \times B)$$

and it is proportional to the mass. Thus the velocity field satisfies

$$\frac{\partial v}{\partial t} + (v \cdot \Delta) v = E + v \times B$$

Suppose [redacted] next we want both positive and negative charges. Then we treat the particles independently, but each affects and is affected by the electromagnetic field. So we want $\rho_+, v_+$ for the positive particles and $\rho_-, v_-$ for the negative ones. The equations governing the motion of the particles [redacted] are

$$\frac{\partial v_\pm}{\partial t} + (v_\pm \cdot \Delta)\, v_\pm = \pm(E + v_\pm \times B) \qquad \frac{\partial \rho_\pm}{\partial t} + \nabla\cdot(\rho_\pm v_\pm) = 0$$

(these are 8 equations in the 8 unknowns $\rho_+, \vec{v}_+, \rho_-, \vec{v}_-$). Here $E, B$ satisfy Maxwell's equations:

$$\nabla\cdot B = 0 \qquad \nabla\times E = -\frac{\partial B}{\partial t}$$

$$\nabla\cdot E = \rho \qquad \nabla\times B = \frac{\partial E}{\partial t} + j$$

where $\rho = \rho_+ - \rho_-$, $j = \rho_+ v_+ - \rho_- v_-$. Now Maxwell's equations are 8 equations in 6 unknowns. But one relation comes from

$$0 = \nabla\cdot(\nabla\times E) = -\nabla\cdot\frac{\partial B}{\partial t} = -\frac{\partial}{\partial t}(\nabla\cdot B)$$

and the other from

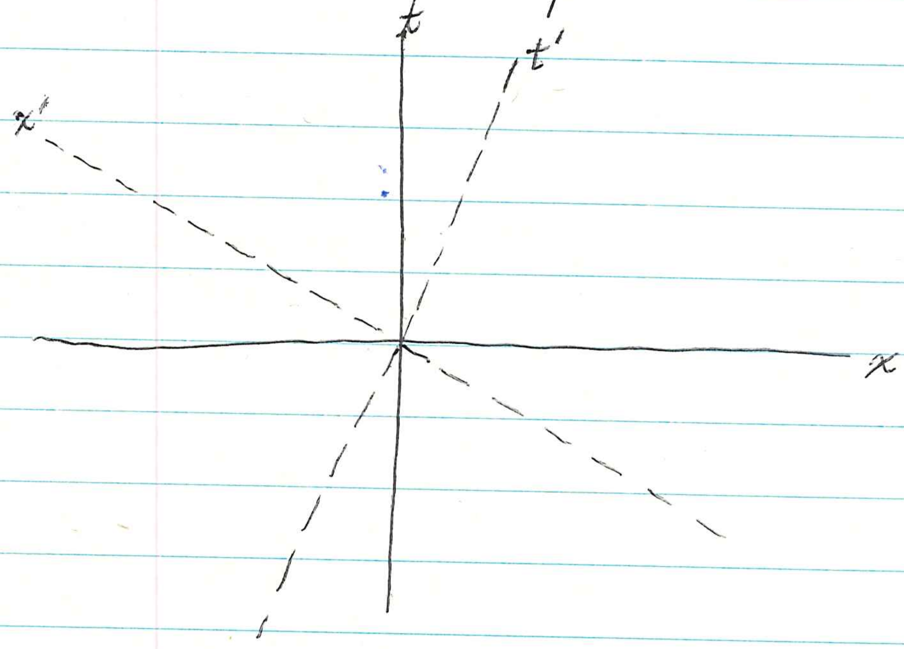
$$\frac{\partial}{\partial t}(\nabla\cdot E - \rho) = \nabla\cdot(\nabla\times B - j) - \frac{\partial \rho}{\partial t} = -\nabla\cdot j - \frac{\partial \rho}{\partial t} = 0$$

Thus if the first column of Maxwell's equations are satisfied at $t = 0$, then they remain true for all time.

Question: It seems that the above separation of $\rho, j$ into positive and negative parts is forced if one wants to have a charge to mass ratio, so as to be able to compute acceleration of charges. Is this true, or is there some way of writing an equation of motion for $\rho, j$?

The preceding is non-relativistic. The next step would be the relativistic equations.

Relativity: Space-time is a four-dimensional real vector space equipped with a quadratic form of signature $+---$. Let's work with 2-dimensional space-time. An observer has a clock and ruler which give coordinates $(t,x)$ on the vector space $X$ of space-time. The quadratic form is $t^2-x^2$. Another observer moving at constant velocity to the first observer gives another set of coordinates $(t',x')$. Einstein thought of the 2nd observer as being in a train. Suppose $(t,x)=(0,0)$ is the same point as $(t',x')=(0,0)$. Then the coordinate axes in space-time look as follows:




The line $x'=0$, that is, the $t'$ axis is the path of the point of the train with $x=0$, at time $t=0$. The slope of the $t'$ axis essentially gives the velocity $v$ of the train. Thus

$$x'=0 \iff x=vt$$

Let

$$\begin{pmatrix} x' \\ t' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ t \end{pmatrix}$$

$$(x')^2-(t')^2 = (ax+bt)^2-(cx+dt)^2 = (a^2-c^2)x^2+(2ab-2cd)xt+(b^2-d^2)t^2$$

so $a^2 - c^2 = 1$. Also $x' = 0 \iff x = vt$

$ab = cd$ says $av + b = 0$

$b^2 - d^2 = -1$

$b = -av$ $\quad c = \frac{1}{d} ab = \frac{1}{d}(-a^2 v)$ $\quad d^2 = b^2 + 1 = a^2v^2 + 1$

$$a^2 - \frac{1}{d^2} a^4 v^2 = 1$$

$$a^2 - \frac{1}{a^2v^2+1} a^4 v^2 = 1$$

$$\frac{a^2}{a^2v^2+1} = 1 \qquad v^2 + \frac{1}{a^2} = 1 \qquad a^2 = \frac{1}{1-v^2}$$

So $a = \frac{1}{\sqrt{1-v^2}}$ $\quad d^2 = \frac{v^2}{1-v^2} + 1 = \frac{1}{1-v^2}$ $\quad d = \frac{1}{\sqrt{1-v^2}}$

$b = -\frac{v}{\sqrt{1-v^2}}$ $\quad c = b.$

So we get the Lorentz transformation

$$\begin{pmatrix} x' \\ t' \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{1-v^2}} & \frac{-v}{\sqrt{1-v^2}} \\ \frac{-v}{\sqrt{1-v^2}} & \frac{1}{\sqrt{1-v^2}} \end{pmatrix} \begin{pmatrix} x \\ t \end{pmatrix}$$

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Special Relativity: [redacted] Space-time is a four-dimensional vector space $X$ with quadratic form having signature $(+\,-\,-\,-)$. [redacted] We fix coordinates $(x_0 = t, x_1, x_2, x_3)$. The standard notation is $x_\mu = (t, \vec{x})$. The scalar product of two vectors $a = (a_\mu) = (a_0, \vec{a})$ and $b = (b_\mu) = (b_0, \vec{b})$ is

$$a_\mu b_\mu = a_0 b_0 - \vec{a}\cdot\vec{b}$$

A motion of a particle through space-time is decribed by a curve $x_\mu(t) = (t, \vec{x}(t))$ in $X$ whose tangent vector is "time-like", i.e. [redacted] points more in the $t$ direction, or precisely

$$\frac{dx_\mu^2}{dt} = 1 - \left|\frac{d\vec{x}}{dt}\right|^2 > 0.$$

Along such a curve we can introduce arclength

$$ds^2 = dx_\mu^2 = dt^2 - dx_1^2 - dx_2^2 - dx_3^2$$

and define the unit tangent vector

$$\frac{dx_\mu}{ds} = \left(\frac{1}{\sqrt{1-v^2}}, \frac{\vec{v}}{\sqrt{1-v^2}}\right) \qquad \vec{v} = \frac{d\vec{x}}{dt} \qquad v = |\vec{v}|$$

Since

$$\left(\frac{dx_\mu}{ds}\right)^2 = 1$$

we have $\dfrac{dx_\mu}{ds}\dfrac{d^2x_\mu}{ds^2} = 0$. Thus $\dfrac{d^2x_\mu}{ds^2}$ is a vector [redacted] orthogonal to $\dfrac{dx_\mu}{ds}$. A natural question is whether $\dfrac{d^2x_\mu}{ds^2}$ is time-like in general. If so one could define unit normal vector and curvature. However if $\vec{v} = 0$, then $\frac{dx_\mu}{ds} = (1, 0)$ so that $\frac{d^2x_\mu}{ds^2}$ is space-like. More generally it is clear that the vectors orthogonal to a time-like vector form a space-like hyperplane, hence in general

$$\frac{d^2x_\mu}{ds^2} \quad \text{is space-like}$$

so if it is non-zero, one does gets a unit normal vector. There is a clear curvature in any case.

The program will be to define force, [redacted] momentum and energy [redacted] so as to be compatible with this intrinsic acceleration $\frac{d^2x_\mu}{ds^2}$. The only quantity that can intrinsically be attached to our curve $x_\mu(t)$ at a point depending only on its first order behavior is a multiple of its unit tangent vector:

$$m_0 u_\mu = m_0 \frac{dx_\mu}{ds} = \left(\frac{m_0}{\sqrt{1-v^2}}, \frac{m_0\vec{v}}{\sqrt{1-v^2}}\right)$$

For small velocities one has

$$\frac{m_0}{\sqrt{1-v^2}} = m_0 + \tfrac{1}{2}m_0v^2 + O(v^4)$$

$$\frac{m_0\vec{v}}{\sqrt{1-v^2}} = m_0\vec{v} + O(v^3).$$

Einstein concluded that $\vec{p} = \frac{m_0\vec{v}}{\sqrt{1-v^2}}$ is the relativistic analogue of momentum, and $p_0 = \frac{m_0}{\sqrt{1-v^2}}$ is the relativistic version of energy. Thus in relativity one has the energy-momentum 4-vector

$$p_\mu = m_0 u_\mu.$$

Now we can check conservation of energy-momentum. Let us define the "Minkowski" force on a particle of rest mass $m_0$ so that the Newton Law holds:

$$\frac{d}{ds}(p_\mu) = \frac{d}{ds}(m_0 u_\mu) = K_\mu$$

Suppose you have two particles interacting with each other so the forces are equal and opposite. It seems one gets trouble from using the $\frac{d}{ds}$ related to each particle? ?

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If $p_\mu = (E, \vec{p})$ is defined to $m_0 u_\mu$, then

$$p_\mu^2 = E^2 - p^2 = m_0^2 \quad \text{or}$$

$$\boxed{E^2 = m_0^2 + p^2}$$

$$E = \sqrt{m_0^2 + p^2} = m_0 + \frac{p^2}{2m_0} + \cdots\cdots$$

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The way to handle forces is to define force to be time-derivative of momentum:

$$\vec{F} = \frac{d}{dt}(\vec{p}).$$

We can check this is consistent with the idea that force acting through distance equals energy:

$$\vec{F}\cdot\vec{v} = \frac{d\vec{p}}{dt}\cdot\vec{v}$$

$$\frac{d}{dt}E = \frac{d}{dt}\sqrt{m_0^2+p^2} = \frac{1}{2\sqrt{m_0^2+p^2}}\,2\vec{p}\cdot\frac{d\vec{p}}{dt}$$

$$\therefore \quad \frac{dE}{dt} = \frac{d\vec{p}}{dt} \cdot \frac{\vec{p}}{E} \qquad \vec{p} = \frac{m_0 \vec{v}}{\sqrt{1-v^2}}\,, \quad E = \frac{m_0}{\sqrt{1-v^2}}$$

$$\therefore \quad \frac{dE}{dt} = \frac{d\vec{p}}{dt} \cdot \vec{v} = \vec{F} \cdot \vec{v}$$

It's also clear from $\vec{F} = \frac{d\vec{p}}{dt}$ that given a bunch of particles interacting on each other by equal and opposite forces, that the total momentum is conserved. If we define the <u>4-vector force</u> by

$$F_\mu = \frac{dp_\mu}{dt} = m_0 \frac{du_\mu}{dt}$$

then we know

$$F_\mu u_\mu = 0$$

so that

$$F_0 \cdot 1 - \vec{F} \cdot \vec{v} = 0.$$

Thus

$$F_0 = \vec{F} \cdot \vec{v} = \frac{dE}{dt}$$

So we see the time component of the 4-force is the rate of energy change.

It follows that if particles interact with equal and opposite 4-forces, then the total energy-momentum 4-vector is conserved. Question: Suppose you have two particles interacting with equal and opposite $\vec{F}$'s; do the $F_0$'s have to cancel? It seems not for

$$F_0 = \vec{F} \cdot \vec{v}$$

depends on the velocity. (This has nothing to do with relativity: In an inelastic collision momentum is conserved but energy isn't.)

Next project is to understand the force on a charged particle due to the electromagnetic field:

$$\vec{F} = q(\vec{E} + \vec{v} \times \vec{B})$$

The idea will be to show that the equation of motion

$$\frac{d\vec{p}}{dt} = q(\vec{E} + \vec{v} \times \vec{B})$$

is relativistically invariant. Here $\vec{p} = \dfrac{m\vec{v}}{\sqrt{1-|v|^2}}$ where $m = m_0$ is the rest mass. Also

$$ds = \sqrt{1-v^2}\, dt$$

so this can be written

$$\frac{d\vec{p}}{ds} = q\left(\frac{1}{\sqrt{1-v^2}}\vec{E} + \frac{\vec{v}}{\sqrt{1-v^2}} \times \vec{B}\right)$$

I want to replace $\vec{E}, \vec{B}$ by a 2-form $\omega$ on space-time, so that Maxwell's equations say $d\omega = 0$ and $d^*\omega$ = charge current density. Conventions: The coordinate frame is $dt, dx, dy, dz$ and the metric is

$$ds^2 = dt^2 - dx^2 - dy^2 - dz^2$$

Consequently the form $df = \frac{\partial f}{\partial t} dt + \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy + \frac{\partial f}{\partial z} dz$ is identified with the vector

$$\nabla_\mu f = \left(\frac{\partial}{\partial t}, -\frac{\partial}{\partial x}, -\frac{\partial}{\partial y}, -\frac{\partial}{\partial z}\right) f$$

Recall from 873-875 that $E, B$ are given in terms of the vector potential $A_\mu = (\phi, \vec{A})$ by putting

$$-\eta = \varphi\, dt - A_x\, dx - A_y\, dx - A_z\, dz = \text{form belonging to } A_\mu.$$

Then
$$d\eta = F_{\mu\nu}\, dx_\mu\, dx_\nu = B_x\, dy\, dz + B_y\, dz\, dx + B_z\, dx\, dy$$
$$- E_x\, dt\, dx - E_y\, dt\, dy - E_z\, dt\, dz$$

Next find $i(u_\mu)\, d\eta$. $\qquad u_\mu = \left(\frac{1}{\gamma}, \frac{\vec{v}}{\gamma}\right)$

$$i(u_\mu)\, d\eta = \frac{1}{\gamma}\Big(- E_x\, dx - E_y\, dy - E_z\, dz + v_x(-B_y\, dz + B_z\, dy)$$
$$+ v_y(B_x\, dz - B_z\, dx) + v_z(-B_x\, dy + B_y\, dx)\Big)$$
$$+ (v_x E_x + v_y E_y + v_z E_z)\, dt$$

$\therefore$
$$i(u_\mu)\, d\eta = \frac{1}{\sqrt{1-v^2}}\Big(-(E + v \times B)_x\, dx - \cdots$$
$$+ (v \cdot E)\, dt \Big)$$

which converts nicely to the vector

$$\frac{1}{\sqrt{1-v^2}} \left( (v \cdot E), \ \vec{E} + \vec{v} \times \vec{B} \right)$$

Conclusion with $d\eta$ the 2-form describing the EM field one has the equation of motion for a charged particle.

$$m_0 \frac{d^2 x}{ds^2} = q\, i\!\left(\frac{dx}{ds}\right) \cdot d\eta .$$

Next thing to try to do is to get charge-current density as a fluid.

**Review:** Space-time is a four-dimensional vector space $X$ with [redacted] quadratic form of signature $(+,-,-,-)$. An "observer" provides coordinates $x = (t, \vec{x})$ on $X$ such that the form is $x_0^2 - |\vec{x}|^2$. The world-line of a particle is a curve whose tangent vector is time-like, hence the curve is $x(t) = (t, \vec{x}(t))$ where

$$\left(\frac{ds}{dt}\right)^2 = 1 - \left|\frac{d\vec{x}}{dt}\right|^2 > 0$$

and hence the velocity $\vec{v} = \frac{d\vec{x}}{dt}$ is [redacted] always of magnitude $< 1$. The curve has unit tangent vector

$$u = \frac{dx}{ds} = \left(\frac{1}{\sqrt{1-v^2}}, \frac{\vec{v}}{\sqrt{1-v^2}}\right)$$

and a natural arc-length, which one calls the proper time of the particle. Note that $\vec{v}$ depends on the coordinates.

A particle is supposed to be given a "rest" mass $m$. One defines [redacted] a four-vector

$$p = (p_0, \vec{p}) = mu$$

which is called the energy-momentum four-vector of the particle. One justification for this is that

$$p_0 = \frac{m}{\sqrt{1-v^2}} = m + \tfrac{1}{2}mv^2 + O(v^4)$$

$$\vec{p} = \frac{m\vec{v}}{\sqrt{1-v^2}} = m\vec{v} + O(v^3)$$

Another justification is that if we define the four-vector force by [redacted]

$$F = \frac{dp}{dt}$$

[redacted] or

$$F_0 = \frac{dp_0}{dt}, \quad \vec{F} = \frac{d\vec{p}}{dt}$$

then from the fact $u$ is a unit vector we have

$$u^2 = 1 \implies u \cdot \frac{du}{dt} = 0 \implies u \cdot \frac{dp}{dt} = 0$$

so

$$F_0 = \vec{F} \cdot \vec{v}.$$

[redacted] If $\vec{F} \cdot \vec{v}$ is interpreted as the rate of work done on the particle, then $F_0 = \frac{dp_0}{dt}$ is the rate of energy increase. Hence $p_0$ should be the energy. If we put $E = p_0$, then

$$m^2 = p \cdot p = E^2 - |\vec{p}|^2$$

so

$$E = \sqrt{m^2 + |\vec{p}|^2} = m + \frac{1}{2m}|\vec{p}|^2 + \cdots .$$

The electromagnetic field can be interpreted as a linear map on tangent vectors which is skew-symmetric with respect to scalar product. At each point [redacted] of space-time one has the 2-form $\sum F_{\mu\nu}\, dx^\mu dx^\nu$ which gives a skew-symmetric bilinear form

$$a_\mu, b_\mu \longmapsto \sum_{\mu\nu} F_{\mu\nu} a^\mu b^\nu$$

where $a^\mu = (a_0, -a_1, -a_2, -a_3)$. We interpret this as a skew-symmetric transformation

$$a \longmapsto a_\mu F_{\mu\cdot}$$

of the tangent space.

This construction is unfamiliar. For example, I could consider a Riemann manifold, equipped with a 2-form. The 2-form I interpret as a skew-symmetric map on tangent vectors. Then consider the flow on the unit sphere bundle which is geodesic flow twisted by this 2-form, that is, whose integral curves [redacted] are the lifts of curves whose acceleration are what the 2-form gives when applied to the tangent vector to the curves.

---

May 20, 1979 [redacted]

Yesterday we came across the notion of "geodesic flow twisted by a 2-form" in connection with the motion of a charged particle through the electromagnetic field. Interpret the 2 form as given at each point $x$ a skew-symmetric map [redacted] $F(x)$ on the tangent space at $x$ with respect to the metric. Then one [redacted] gets a 2nd order DE

$$\frac{d^2x}{d\tau^2} = F(x)\frac{dx}{d\tau}$$

which one can view as a flow on the tangent bundle:

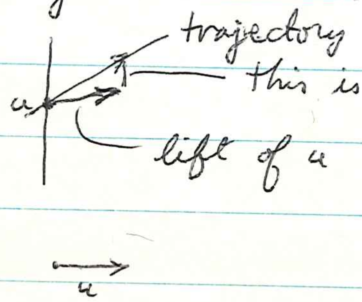
$$\begin{cases} \dfrac{dx}{d\tau} = u \\[2ex] \dfrac{du}{d\tau} = F(x)u \end{cases}$$

Since $F$ is skew-symmetric

$$\frac{d}{d\tau}(u\cdot u) = 2u\cdot\frac{du}{d\tau} = 2u\cdot F(x)u = 0$$

so one gets a flow on the unit sphere bundle.

To be precise, one should say that $\frac{du}{d\tau}$ uses the connection on the tangent bundle given by the metric. If one is given $x, u$ in the tangent bundle, then the connection gives us a [redacted] tangent vector at $u$ over $u$




this is a tangent vector the the fiber which since the fibre is a vector space is [redacted] a point of the tangent space and the actual trajectory differs from this lift by $F(x)u$.

Now that we have a flow on the unit tangent bundle we can talk about a fluid of independent particles. This means [redacted] that at each $x$ [redacted] I assign a unit tangent vector $u(x)$ [redacted] such that if I follow the trajectory $x(\tau)$ starting at $(x, u(x))$, then

$$u(x(\tau)) = \frac{d}{d\tau} x(\tau)$$

Thus

$$\frac{d^2x(\tau)}{d\tau^2} = \frac{d}{d\tau} u(x(\tau)) = \nabla u \cdot \frac{dx}{d\tau}$$

and so $u(x)$ satisfies the PDE

$$\boxed{(u \cdot \nabla) u = F(x) u}$$

I can check this in the following way: Recall derivation

$$\underbrace{\frac{q}{m}}_{=1} \iiint \rho F_x \, dV = \frac{d}{dt} \iiint \rho v_x \, dV = \iiint \frac{\partial(\rho v_x)}{\partial t} dV + \iint \rho v_x \, \vec{v} \cdot d\vec{S}$$

$$\rho F_x = \frac{\partial}{\partial t}(\rho v_x) + \nabla \cdot (\rho v_x \vec{v})$$

$$= \cancel{\frac{\partial \rho}{\partial t} v_x} + \rho \frac{\partial v_x}{\partial t} + \cancel{v_x \nabla \cdot (\rho \vec{v})} + \vec{\nabla} v_x \cdot \rho \vec{v}$$

or

$$\frac{\partial v}{\partial t} + (\vec{v} \cdot \nabla) v = F = E + v \times B$$

Relativistically how does this change? In calculating momentum you replace $\rho v_x$ by $\frac{\rho v_x}{\sqrt{1-v^2}}$, but the force law doesn't change, so you get

$$\left(\frac{\partial}{\partial t} + \vec{v} \cdot \nabla\right) \left(\frac{\vec{v}}{\sqrt{1-v^2}}\right) = E + v \times B$$

which agrees with the above boxed equation.

Summary: If one has a fluid made up of non-interacting charged particles with $\frac{q}{m} = 1$, then the unit tangent vector field $u$ to this flow satisfies

$$(u \cdot \nabla) u = F(x) u$$

where $F$ is the electromagnetic field.

What remains is to determine the charge-current density associated to the flow of these charged particles. [redacted] Notice that [redacted] once $F$ is given the flow lines of the fluid do not depend on the density of charge, because we assume the charge-mass ratio is fixed $= 1$. The actual charge density is a function $f(x)$ constant along the lines of the flow, i.e.

$$(u \cdot \nabla) f = 0$$

The charge-current 4-vector should be [redacted] $fu$. I should check its 4-divergence is zero. Since

$$\nabla \cdot (fu) = f(\nabla \cdot u) + \underbrace{u \cdot \nabla f}_{= \, 0}$$

it is enough to see $\nabla \cdot u = 0$. [redacted] ??

Example in $R^2$: Consider the radial flow of unit speed:

$$u = \frac{x\hat{\imath} + y\hat{\jmath}}{r}$$

Then $u \cdot \nabla = \frac{1}{r}\left(x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y}\right) = \frac{\partial}{\partial r}$ so $(u \cdot \nabla) u = 0$

But $\nabla \cdot u \neq 0$ obviously since flux through [redacted] $r = a$ is $2\pi a$

So what I missed is that once $u$ is found then the charge-current 4-vector is a multiple $fu$ of $u$; which satisfies

$$\boxed{\nabla\cdot(fu) = 0}$$

$f$ is rigged so that if the flow diverges then $f$ goes down. Notice that if

$$(\rho, \vec{J}) = fu = \left(\frac{f}{\sqrt{1-v^2}}, \frac{f\vec{v}}{\sqrt{1-v^2}}\right)$$

then $\vec{J} = \rho\vec{v}$ as it should and the above equation takes the form

$$\frac{\partial}{\partial t}\rho + \vec{\nabla}\cdot\vec{J} = 0 .$$

Next I ought to be able to write down the complete equations. First suppose we only have positively charged particles with $q = m$. Then matter obeys the equations

$$\begin{cases} (u\cdot\nabla)\, u = Fu \\ \nabla\cdot(fu) = 0 . \end{cases}$$

and the electromagnetic field $F$ satisfies Maxwell's equations:

$$\begin{cases} dF = 0 \\ d^*F = fu \end{cases}$$

The case of both positive and negative particles should be analogous to p. 883.

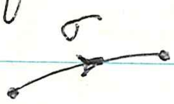
Idea: To understand electromagnetism [redacted] using the analogy with electrostatics, but with the indefinite Lorentz metric.

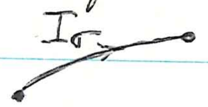
Review electrostatics from the viewpoint of a resistive network. Imagine space filled with a resistive material, or better, go back to your freshman physics experiment using resistive paper to understand electrostatic problems.

Discrete model: One has a graph [redacted] whose edges are resistances. [redacted] The voltage $V$ is a function on the vertices, i.e. a 0-cochain. The current is a [redacted] 1-chain, which one can identify with a 1-cochain using the duality provided by the energy form

$$\text{Energy}(I) [redacted] = \frac{1}{2}\sum_\sigma R_\sigma I_\sigma^2 .$$

[redacted] Specifically if $I_\sigma$ is the current [redacted] thru the edge $\sigma$, then the one-cochain [redacted] assigns to $\sigma$ the voltage drop across $\sigma$, which by Ohm's law is $R_\sigma I_\sigma$.

If $\sigma$ is 

, then for $I_\sigma > 0$ the current flows 


so the voltage drop across $\sigma$ is $R_\sigma I_\sigma$, so that the voltage at the end of $\sigma$ is [redacted] lower. Thus the equations are:

Ohm's Law: $\qquad -dV = I$

Kirchoff's law: $\partial I = 0$. if no current sources.

Change notation to be more suitable with the continuous situation. Instead of $I$ we have the electric field $E$, assuming the conductivity is 1. The equations are then

$$-\nabla\phi = E$$

$$\nabla\cdot E = \rho$$

where $\rho$ is the charge density. Then the energy is

$$\text{Energy} = \frac{1}{2}\iiint |E|^2\, dV$$

What are the problems one tries to solve?

1) Fix $\rho$ and solve for $\phi$. Like a current source in a network.

Example: Put point charges at various places and compute the resulting electric field.

2) For a network one specifies the voltages at certain vertices and asks about the resulting current flow assuming no current sources. The continuous analogue is to specify $\phi$ on the "boundary" and solve the Dirichlet problem.

<u>May 26, 1979</u>

Electrostatics is described by the equations

$$\nabla \times E = 0$$

$$\nabla \cdot E = \rho$$

or in differential form notation

$$dE = 0$$

$$\delta E = \rho \qquad\qquad \delta = -d^*$$

Because $dE = 0$, there is a potential function, $\phi$ unique up to an additive constant with

$$-d\phi = E$$

so the equations become the Poisson equation

$$\Delta \phi = -\rho .$$

Physicists like variational principles behind equations. To solve Poisson's equation in a region one can look at the functional

$$F : \phi \longmapsto \int \left(\tfrac{1}{2}|\nabla\phi|^2 - \rho\phi\right) d^3x$$

The first variation is

$$\delta F = \int \left(\nabla\phi \cdot \nabla(\delta\phi) - \rho\,\delta\phi\right) d^3x$$

$$= -\int (\Delta\phi + \rho)\cdot\delta\phi \; d^3x \quad + \text{ boundary term.}$$

The Lagrangian density involved here is

$$\mathcal{L}(\phi, \nabla\phi) = \tfrac{1}{2}|\nabla\phi|^2 - \rho\phi$$

Does this have anything to do with electrostatic energy?

Approaches to electrostatic energy. First viewpoint is that the electrostatic energy = total work done to assemble the charges. The field of a point charge $q$ is given by solving

$$\Delta\phi = -q\,\delta(\vec{x}).$$

The preferred solution in $\mathbb{R}^3$ is the Coulomb potential

$$\phi = \frac{q}{4\pi r} \qquad r = |\vec{x}| .$$

(This is not $L^2$, so why this is the only relevant possibility in space is not completely clear, although it is the only radially symmetric soln.)

Thus two charges $q_1, q_2$ separated by distance $r_{12}$ require the energy

$$\frac{q_1 q_2}{4\pi r_{12}}$$

to be assembled. Thus if one has a charge distribution $\rho$, the total electrostatic energy involved to assemble it is

$$\text{En.} = \frac{1}{2}\int d^3x \int d^3x' \, \frac{\rho(x)\,\rho(x')}{4\pi|x-x'|}$$

The potential belonging to $\rho$ is

$$\phi(x) = \int \frac{\rho(x')}{4\pi|x-x'|}\, d^3x'$$

so the total electrostatic energy involved in a charge distribution $\rho$ is

$$\text{En.} = \frac{1}{2}\int \rho\phi \, d^3x$$

Now if we use

$$\nabla\cdot(\phi\nabla\phi) = |\underbrace{\nabla\phi}_{E}|^2 + \phi \underbrace{\Delta\phi}_{-\rho}$$

we find

$$\int \tfrac{1}{2}|E|^2 d^3x - \int \tfrac{1}{2}\rho\phi \, d^3x = \tfrac{1}{2}\int_{\text{sphere at } \infty} \phi \cdot \vec{E}\cdot\hat{n} \, ds = 0$$

because if all charges are in a finite sphere, then $\phi \sim \frac{1}{R}$, $E \sim \frac{1}{R^2}$ far out. So we see that

$$\text{En.} = \frac{1}{2}\int \rho\phi \, d^3x = \frac{1}{2}\int |E|^2 d^3x$$

which allows us to interpret $\frac{1}{2}|E|^2$ as the energy density of the electrostatic field.

---

Magnetostatics; better terminology - steady currents:

$$\begin{cases} \nabla\cdot B = 0 \\ \nabla\times B = j \end{cases} \qquad \text{where } \nabla\cdot\vec{j} = 0$$

In differential form notation $B$ is a closed 2-form:

$$dB = 0$$

$$\pm\, d^*B = j \qquad \text{(sign to be worked out later)}$$

Since $B$ is closed one can find $A$ with $\nabla\times A = B$.

Moreover one can arrange $\nabla\cdot A = 0$, whence

$$\nabla\times(\nabla\times A) = \nabla(\nabla\cdot A) - \nabla^2 A = \vec{j}$$

Thus we get $\boxed{\nabla^2 A = -j}$ analogous to the Poisson equation

$$\nabla^2\phi = -\rho .$$

So the fundamental solution for $\Delta$ we have gives an integral formula for $A$ in terms of $\vec{j}$:

$$\vec{A}(x) = \int \frac{\vec{j}(x')}{4\pi|x-x'|}\,d^3x' = \int \frac{\vec{j}(x+u)}{4\pi|\vec{u}|}\,d^3u$$

From the last formula it is clear that $\nabla\cdot\vec{j} = 0 \Rightarrow \nabla\cdot A = 0$.

Next we want a variational principle.

Consider the functional

$$F(A) = \int\left(\frac{1}{2}|\nabla\times A|^2 - j\cdot A\right)d^3x$$

Then

$$\delta F = \int\left[(\nabla\times A)\cdot(\nabla\times\delta A) - j\cdot\delta A\right]d^3x$$

From the identity $\quad \nabla\cdot(A\times B) = (\nabla\times A)\cdot B - A\cdot(\nabla\times B)$

we get

$$\nabla\cdot\big((\nabla\times A)\times\delta A\big) = \underbrace{\nabla\times(\nabla\times A)}_{\nabla(\nabla\cdot A)-\Delta A}\cdot B - (\nabla\times A)\cdot(\nabla\times\delta A)$$

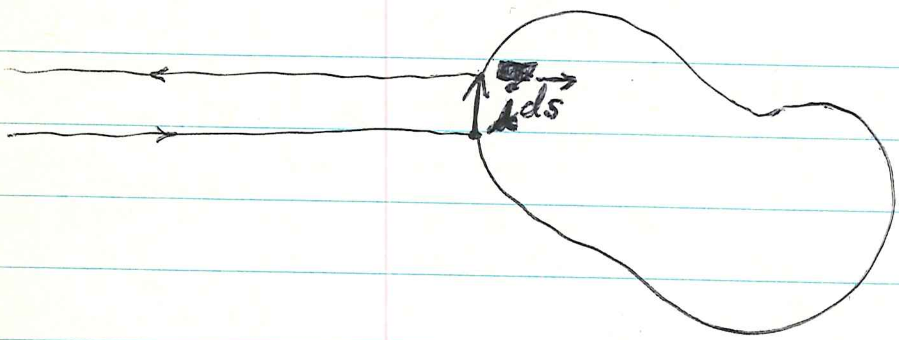
and hence for $\delta A$ of compact support, and $\nabla\cdot A = 0$ we have

$$\delta F = \int -(\Delta A + j)\cdot\delta A \;\; d^3x$$

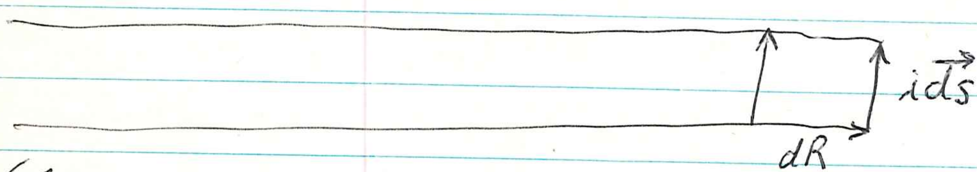
yielding the equation $\Delta A = -j$. So the Lagrangian density in this situation is

$$\mathcal{L}(A, \nabla\times A) = \frac{1}{2}|\nabla\times A|^2 - j\cdot A$$

*Magnetic energy:* Let us compute the energy required to assemble a current distribution. The idea is to build up $j$ by infinitesimal amounts $\delta j$ and then add up the resulting work done. Suppose we want to add a closed current loop with infinitesimal current to our system. We can do this by combining little pieces $\vec{i}\,ds$ joined to $\infty$ by wires which when combined cancel out.




So we only have to worry about the work done in bringing a little current element in from $\infty$:




(The work done in the horizontal direction is 2nd order infinitesimal)
The work done in moving $i\,ds$ thru $dR$ is

$$\vec{F}\cdot d\vec{R} \quad \text{where} \quad \vec{F} = i\vec{ds}\times\vec{B} \qquad \text{(Lorentz force)}$$

So the total work done is $\int F\cdot dR = \int -(B\times i\vec{ds})\cdot dR = \int B\cdot d\vec{R}\times i\vec{ds}$

$$= \int i B \cdot d\vec{\sigma} = \int \nabla \times A \cdot d\vec{\sigma} = \int i\, A \cdot d\vec{s}$$ [redacted]

Adding this pieces all up and cancel[redacted]ing the wires to $\infty$ we see that the work done in bringing the loop is

$$\oint [redacted]\ i\, d\vec{s} \cdot \vec{A}$$

This makes it reasonable to conclude that the work done in changing the current [redacted] density $\vec{j}$ by an infinitesimal amount is

$$\int \delta \vec{j} \cdot \vec{A}\ d^3x .$$

For some reason by electrostatic analogy (Feynman Lectures Vol. II, 15-3), one ought to be able to see that the magnetic energy is

$$U = \frac{1}{2} \int \vec{j} \cdot \vec{A}\ d^3x .$$

Because of the identity

$$\nabla \cdot \Big[ \underbrace{(\nabla \times A)}_{B} \times A \Big] = \underbrace{\nabla \times (\nabla \times A)}_{j} \cdot A - (\nabla \times A) \cdot (\nabla \times A)$$

one has

$$\iint_{\text{boundary}} (B \times A)\, d\vec{S} = \int \vec{j} \cdot \vec{A}\ d^3x - \int |B|^2 d^3x$$

The boundary term should be zero in the limit for

$$A(x) = \int \frac{j\, d^3x'}{4\pi\, |x - x'|} \sim \frac{1}{R} \quad , \quad B \sim \frac{1}{R^2} \quad \text{far out}$$

so we see that

$$\frac{1}{2}\int \vec{J}\cdot\vec{A}\, d^3x = \int \frac{1}{2}|B|^2 d^3x$$

which allows one to interpret $\frac{1}{2}|B|^2$ as the energy density of the magnetic field.

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May 27, 1979

Let's setup magnetostatics using differential forms.

$$\beta = B_x\, dy\, dz + \cdots \qquad \text{cyclic perm.}$$

$$d\beta = \left(\frac{\partial B_x}{\partial x} + \cdots\right) dx\, dy\, dz = (\nabla\cdot B)\, d^3x$$

Recall the $*$ operator on forms on an oriented Riem. manifold is such that

$$\alpha \wedge *\alpha = |\alpha|^2 \cdot \text{volume}$$

In $\mathbb{R}^n$

$$*(dx_1 \cdots dx_p) = dx_{p+1} \cdots dx_n$$

$$*(dx_{p+1} \cdots dx_n) = dx_1 \cdots dx_p\, (-1)^{p(n-p)}$$

so

$$*^2 = (-1)^{p(n-p)} \quad \text{on } p\text{-forms}$$

Thus $*^2 = 1$ on an odd-diml manifold.

$$*(\beta) = B_x\, dx + \cdots$$

$$d*\beta = -\frac{\partial B_x}{\partial y} dx\, dy + \frac{\partial B_x}{\partial z} dz\, dx + \cdots$$

$$= \left(\frac{\partial B_z}{\partial y} - \frac{\partial B_y}{\partial z}\right) dy\, dz + \cdots = (\nabla\times B) \text{ interpreted as a 2 form}$$

$$*d*\beta = (\nabla\times B)_x\, dx + \cdots$$

Idea: B interpreted as a 2-form is just $B\cdot\hat{n}\,dS$. Thus

$$B\cdot\hat{n}\,dS = B_x\,dy\,dz + \cdots$$

in the same way that

$$B\cdot dR = B_x\,dx + B_y\,dy + B_z\,dz$$

With this terminology one has

$$*(B\cdot dR) = B\cdot\hat{n}\,dS$$

$$d(B\cdot dR) = (\nabla\times B)\cdot\hat{n}\,dS$$

(Possible better notation is $d\vec{s}$, $d\vec{S}$ for the line and surface element.) If

$$\delta = *d*$$

then for $\alpha$ a 1-form, $\beta$ a 2-form

$$(d\alpha,\beta) = \int d\alpha\wedge *\beta = \int \alpha\wedge d*\beta = (\alpha,\delta\beta)$$

$$d(\alpha\wedge *\beta) = d\alpha\wedge *\beta - \alpha\wedge d*\beta$$

so $\delta$ is the adjoint of $d$. Also

$$\delta(B\cdot\vec{dS}) = *\,d(B\cdot d\vec{s}) = *(\nabla\times B)\cdot d\vec{S} = (\nabla\times B)\cdot d\vec{s}.$$

Consequently the basic equations are

$$\begin{cases} d(B\cdot d\vec{S}) = 0 & \Longleftrightarrow \quad \nabla\cdot B = 0 \\ \delta(B\cdot d\vec{S}) = j\cdot ds & \Longleftrightarrow \quad \nabla\times B = \vec{j} \end{cases}$$

$$d(A\cdot d\vec{s}) = B\cdot d\vec{S} \quad \Longleftrightarrow \quad \nabla\times A = B$$

In this notation the basic integration by parts identity is

$$\boxed{d\alpha \wedge {*d\alpha} - \alpha \wedge d{*d\alpha} = d(\alpha \wedge {*d\alpha})}$$ $\alpha = A \cdot d\vec{r}$

or $$B \cdot B \, d^3x - A \cdot j \, d^3x = \nabla \cdot (A \times B) \, d^3x$$

---

Let's now try to understand [redacted] connections and curvature for line bundles. Begin with a complex line bundle $L$ over a manifold $X$. Let $\{U_\alpha\}$ be an open covering over which $L$ is trivial, and choose a trivialization

$$L|U_\alpha \xrightarrow{\sim} U_\alpha \times \mathbb{C}$$

$$s(x) \longmapsto (x, z_\alpha(s(x)))$$

$s = z_\alpha(s) \cdot s_\alpha$

$s_\alpha$ = section of $L$ over $U_\alpha$ with $z_\alpha s_\alpha = 1$

Then we have transition functions

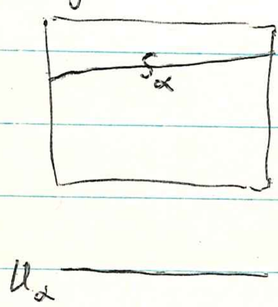
$$g_{\alpha\beta} = \frac{z_\alpha}{z_\beta}$$

$$z_\alpha = g_{\alpha\beta} z_\beta$$

$$s_\alpha = g_{\beta\alpha} s_\beta$$

[redacted]

A connection on $L$ gives [redacted] a notion of parallel transport along curves, i.e. a way of lifting tangent vectors. The connection tells us how $s_\alpha$ deviates from being horizontal




$$Ds_\alpha \in \Gamma_{U_\alpha}(\mathrm{Hom}(T, L)) = \Gamma_{U_\alpha}(L \otimes T^*)$$

$$Ds_\alpha = \theta_\alpha s_\alpha$$

Then $\theta_\alpha g_{\beta\alpha} s_\beta = \theta_\alpha s_\alpha = D s_\alpha = D(g_{\beta\alpha} s_\beta) = dg_{\beta\alpha} s_\beta + g_{\beta\alpha}\theta_\beta s_\beta$

so

$$\theta_\alpha - \theta_\beta = \frac{dg_{\beta\alpha}}{g_{\beta\alpha}}$$

Up on the principle bundle $L' = L - 0$ we have

$$\frac{dz_\alpha}{z_\alpha} = \frac{dg_{\alpha\beta}}{g_{\alpha\beta}} + \frac{dz_\beta}{z_\beta}$$

It follows that $\eta = \theta_\alpha + \frac{dz_\alpha}{z_\alpha}$ is a global 1-form on $L'$ whose restriction to a fibre is the invariant form $\frac{dz}{z}$. The curvature of the connection is the form

$$\omega = d\theta_\alpha \quad \text{on } U_\alpha$$

One gets the same result if one looks at $D^2 : L \to \Lambda^2 T^* \otimes L$:

$$D^2(s_\alpha) = D(\theta_\alpha s_\alpha) = \cancel{\theta_\alpha \theta_\alpha s_\alpha} + d\theta_\alpha \cdot s_\alpha$$

Finally notice that the good form on the fibre $\mathbb{C}^*$ is $\frac{dz}{2\pi i z} = \frac{d\theta}{2\pi}$ since it is an integral class. Thus

$$\frac{1}{2\pi i}\,\omega$$

is an integral class on $X$.

So we see that a line bundle over $X$ equipped with a connection gives rise to an element $c$ of $H^2(X,\mathbb{Z})$ together with a closed two-form $\omega$ whose class is the image of $c$ in $H^2(X,\mathbb{R})$. Conversely given $c$ and $\omega$, construct $L$ with $c_1(L) = c$ and then put a connection on $L$. This gives a connection form $\omega' = d\theta_\alpha$ cohomologous to $\omega$, i.e. $d\theta_\alpha = \omega + d\eta$ for a global 1-form

$\eta$. But then changing $\theta_\alpha$ to $\theta_\alpha - \eta$ gives a different connection on $L$

$$(\theta_\alpha - \eta) - (\theta_\beta - \eta) = \frac{dg_{\beta\alpha}}{g_{\beta\alpha}}$$

whose curvature is $\omega$. (The difference between connections is a 1-form, in fact the bundle of connections is a torsor for $T^*$.) If $\eta$ is a closed 1-form, then changing the connection $\theta_\alpha$ to $\theta_\alpha - \eta$ doesn't change the curvature form.

Suppose one has two line bundles with connection having the same $(c_1, \omega)$. Then the bundle $\underline{Hom}(L_1, L_2)$ is trivial and it has a flat connection, hence if $X$ is simply-connected, the connection gives a flat trivialization.

Let's return to magnetostatics where we have

$$d(A \cdot \vec{ds}) = B \cdot \vec{dS}$$

If one has a charged particle in a magnetic field, moving along a classical path, the amplitude associated to this path is

$$e^{\frac{i}{\hbar} q \int_{path} A \cdot \vec{ds}}$$

The idea here seems to be that we have the trivial bundle over $\mathbb{R}^3$ and the connection form $\theta$ is $A \cdot \vec{ds}$. Changing $A$ by a gradient corresponds to moving the connection around by an automorphism of the bundle, i.e. by a gauge transformation.

Yang-Mills: Consider line bundles over $\mathbb{R}^3$. These are trivial so the connection is given by a global form $A\cdot d\vec{s}$, which we write simply $A$. The curvature is

$$d(\underbrace{A\cdot d\vec{s}}_{\alpha}) = (\nabla\times A)\cdot d\vec{S}$$

so the Yang-Mills functional is

$$F(A) = \int |\nabla\times A|^2\, d^3x \qquad = (d\alpha, d\alpha)$$

Taking the first variation leads to (conflict in $\delta$ notation)

$$\delta F = (d\alpha, d\delta\alpha) = (d^*d\alpha, \delta\alpha)$$

$$= \int (\nabla\times(\nabla\times A))\cdot \delta A\ d^3x$$

which yields the equations

$$\nabla\times(\nabla\times A) = 0.$$

This is a linear equation in $A$.

Next consider the trivial line bundle over a Riemann surface; the metric is given. Again a connection is a 1-form $\theta$ which is purely imaginary when the connection preserves the metric in the trivial bundle. The Yang-Mills functional is

$$F(\theta) = \int d\theta \wedge {*d\theta} \quad = (d\theta, d\theta)$$

and it leads to the equation

$$d^*d\theta = 0$$

This is not the same as requiring $\theta$ to be harmonic, because we have not yet made $\Theta$ orthogonal to exact forms. This last step consists in making $\Theta$ orthogonal to the effects of gauge transformations.

Next example will be for $SU_2$-bundles over Euclidean space. Now $SU_2$ is the double covering of $SO_3$ and it has the same Lie algebra. Now we know that a 1-parameter subgroup of $SO_3$ is a rotation ~~motion~~ about an axis with a given angular speed $\omega$. If $\vec{\omega}$ is the vector of length $\omega$ with direction the axis of rotation, then the velocity vector field for the rotation is

$$\vec{v} = \vec{\omega} \times \vec{R} \qquad \vec{R} = \text{position vector}.$$

For example, consider rotating thru $\Theta = \omega t$ about the $z$ axis:

$$\begin{pmatrix} \cos\omega t & -\sin\omega t & 0 \\ \sin\omega t & \cos\omega t & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

The velocity field is given by the transf. obtaining by $\frac{d}{dt}$ at $t=0$:

$$\vec{v} = \begin{pmatrix} 0 & -\omega & 0 \\ \omega & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

This is the same as $\vec{v} = \omega\hat{k} \times \vec{R}$.

Therefore we see that elements of $Lie(SO_3)$ can be identified with vectors, so that the action of a vector $\vec{\omega}$ is the transformation $X \mapsto \vec{\omega} \times X$. Now let's check the bracket:

$$A\times(B\times X) = (A\cdot X)B-(A\cdot B)X$$
$$B\times(A\times X) = (B\cdot X)A-(B\cdot A)X$$

$$\therefore\quad A\times(B\times X)-B\times(A\times X) = (A\cdot X)B-(B\cdot X)A = X\times(B\times A)$$
$$= (A\times B)\times X.$$

Thus indeed
$$[A,B] = A\times B$$

The Killing form is $\quad tr(\mathrm{ad}A\ \mathrm{ad}B)\quad$ or

$$tr(X\mapsto A\times(B\times X)) = tr(X\mapsto(A\cdot X)B) - tr(X\mapsto(A\cdot B)X)$$
$$= A\cdot B - 3(A\cdot B) = -2(A\cdot B).$$

which is nicely negative-definite.