April · 25, 1979

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If we have a Hamiltonian which depends only on $k = \sum_{i=1}^{N} s_i$, then the partition function is

$$\sum_{k=-n}^{n} \binom{2n}{n+k} e^{-\beta H(k)}$$

where we have supposed $2n = N$ is even. For large n the binomial coefficient looks like a normal distribution. Let's use Stirling's formula

$$\begin{aligned}\log \binom{2n}{n+k} \approx\ & 2n \log 2n \cancel{-2n} + \tfrac{1}{2} \log 4\pi^2 n^2 - \tfrac{1}{2} \log \pi n \\ & -(n+k) \log (n+k) + \cancel{n+k} - \tfrac{1}{2} \log 2\pi(n+k) \\ & -(n-k) \log (n-k) + \cancel{n-k} - \tfrac{1}{2} \log 2\pi (n-k)\end{aligned}$$

Important thing to keep track of is how this depends upon k

$$\begin{aligned}\log(n+k) &= \log n + \log\left(1+\frac{k}{n}\right) \\ &= \log n + \frac{k}{n} - \frac{1}{2}\frac{k^2}{n^2} + \frac{1}{3}\frac{k^3}{n^3} - \cdots\end{aligned}$$

$$\begin{aligned}n\left[\log(n+k) + \log(n-k)\right] &= n \log(n^2 - k^2) \\ &= 2n \log n + n\left[-\frac{k^2}{n^2} - \frac{k^4}{2n^4} - \cdots\right.\end{aligned}$$

$$k\left[\log(n+k) - \log(n-k)\right] = k\left[\frac{2k}{n} + \frac{2k^3}{3n^3} + \cdots\right.$$

$$\begin{aligned}\log \binom{2n}{n+k}\frac{1}{2^n} \approx\ & n\left[\frac{k^2}{n^2} + \frac{k^4}{2n^4} + \frac{k^6}{3n^6} + \cdots\right] - k\left[\frac{2k}{n} + \frac{2k^3}{3n^3} + \cdots\right] \\ & - \frac{1}{2} \log\left(1 - \frac{k^2}{n^2}\right) - \frac{1}{2} \log \pi n\end{aligned}$$

$\frac{2}{3} - \frac{1}{2} = \frac{1}{6}$

$$\log\left(\binom{2n}{n+k}\frac{\sqrt{\pi n}}{2^n}\right) \approx -\frac{k^2}{n\text{[redacted]}} - \frac{k^4}{6n^3} + O\left(\frac{k^4}{n^4}\right) + O\left(\frac{k^6}{n^5}\right)$$

$$+\frac{1}{2}\frac{k^2}{n^2}$$

Now Barry Simon's idea to obtain a limit out of the partition function is to choose $H(k)$ so as to kill the $-\frac{k^2}{n}$ term, and then to let [redacted] $k = n^{3/4}x$.
We [redacted] have [redacted]

$$H(s) = -B\sum s_i - \frac{J}{2}\left[\left(\sum s_i\right)^2 - \sum s_i\right]$$

$$-\beta H(k) = +\beta\left(B - \frac{J}{2}\right)k - \frac{\beta J}{2}k^2$$

Simpler to write $-\beta H(k) = ak + bk^2$. So now you take $b = \frac{1}{n}$ and

$$\text{[redacted]}\binom{2n}{n+k}e^{-\beta H(k)} \approx \begin{pmatrix}\text{constant}\\ \text{dep. on } n\\ \text{ind. of } k\end{pmatrix} e^{ak - \frac{k^4}{6n^3}}$$

and this is the way he shows that

$$\int_{-\infty}^{\infty} e^{xs}e^{-x^4}\,dx$$

is the limit of Ising model partition functions.

April 26, 1979

I want to understand the Simon-Griffiths limiting process better. It would be nice to replace the use of Stirling's formula by a direct steepest descent calculation on a contour integral.

$$d\mu_N(x) = \sum_{k=0}^{N} \binom{N}{k} \frac{1}{2^N}\, \delta\left(x-\left(k-\frac{N}{2}\right)\right)$$

$$\int e^{itx}\, d\mu_N(x) = \sum_{0}^{N} \binom{N}{k} \frac{1}{2^N}\, e^{it\left(k-\frac{N}{2}\right)}$$

$$= \left(\frac{e^{it}+1}{2}\right)^N e^{-it\frac{N}{2}}$$

$$= \left(\frac{e^{it/2}+e^{-it/2}}{2}\right)^N = \left(\cos\frac{t}{2}\right)^N$$

Now we want to multiply the measure $d\mu_N$ by $f(x) = e^{\frac{a}{2}x^2}$. Formally the new F.T. is a convolution

$$\int e^{itx} f(x)\, d\mu_N(x) = \int e^{itx} \int e^{-i\xi x} \hat{f}(\xi) \frac{d\xi}{2\pi}\, d\mu_N(x)$$

$$= \int \hat{f}(\xi) \cos\left(\frac{t-\xi}{2}\right)^N \frac{d\xi}{2\pi}$$

$$= \int \hat{f}(t-\xi) \left(\cos\frac{\xi}{2}\right)^N \frac{d\xi}{2\pi}.$$

There is a slight problem in that $e^{\frac{a}{2}x^2}$ maybe doesn't have a Fourier transform for $a > 0$.

~~This leads to the question~~ Recall

$$f(x) = e^{-\frac{b}{2}x^2} \implies \hat{f} = \int e^{itx - \frac{b}{2}x^2}\,dx = \sqrt{\frac{2\pi}{b}}\, e^{-\frac{t^2}{2b}}$$

so the question is whether there is a contour such that the formula

$$\int_{-\infty}^{\infty} e^{itx}\, e^{\frac{a}{2}x^2}\, d\mu_N(x) = \int \frac{1}{\sqrt{2\pi(-a)}}\, e^{\frac{(t-\xi)^2}{2a}} \left(\cos\frac{\xi}{2}\right)^N d\xi$$

is valid. Enough to have

$$e^{itk}\, e^{\frac{a}{2}k^2} = \int \frac{1}{\sqrt{2\pi}\,\sqrt{-a}}\, e^{\frac{(t-\xi)^2}{2a}}\, e^{ik\xi}\, d\xi$$

$$\xi \longmapsto \xi + t$$

$$e^{\frac{a}{2}k^2} = \int \frac{1}{\sqrt{2\pi}\,\sqrt{-a}}\, e^{\frac{\xi^2}{2a} + ik\xi}\, d\xi$$

So consider $\qquad\qquad$ Put $\xi = it$

$$\int_{-i\infty}^{i\infty} \frac{1}{\sqrt{2\pi}\,\sqrt{-a}}\, e^{\frac{\xi^2}{2a} + ik\xi}\, d\xi$$

$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}}\,\frac{1}{\sqrt{-a}}\, e^{-\frac{t^2}{2a} - kt}\, i\,dt = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}}\,\frac{1}{\sqrt{-a}}\, e^{-t^2 - k\sqrt{2a}\,t}\, i\sqrt{2a}\,dt$$

$$= e^{\frac{a}{2}k^2}\,\frac{i\sqrt{a}}{\sqrt{-a}} = e^{\frac{a}{2}k^2}$$

Thus we get the correct answer by integrating vertically and putting

$$\sqrt{-a} = i\sqrt{a}$$

Thus the desired formula is

$$\int_{-\infty}^{\infty} e^{itx}\, e^{\frac{a}{2}x^2}\, d\mu_N(x) = \int_{-i\infty}^{i\infty} \sqrt{\frac{2\pi}{a}}\, e^{\frac{(t-\xi)^2}{2a}} \left(\cos\frac{\xi}{2}\right)^N \frac{d\xi}{2\pi i}$$

Now we should be able to carry out the Simon–Griffiths limit process. Last integral =

$$\int_{-\infty}^{\infty} \sqrt{\frac{2\pi}{a}}\, e^{\frac{(t+i\eta)^2}{2a}} \left(\cosh\frac{\eta}{2}\right)^N \frac{d\eta}{2\pi} \qquad \xi = i\eta$$

The idea is to let $N \to \infty$ and rescale so that this converges.

$$\cosh\eta = 1 + \frac{\eta^2}{2} + \frac{\eta^4}{24} + O(\eta^6)$$

$$\log\cosh\eta = \frac{\eta^2}{2} + \frac{\eta^4}{24} - \frac{1}{2}\left(\frac{\eta^2}{2}\right)^2 + O(\eta^6)$$

$$= \frac{\eta^2}{2} - \frac{\eta^4}{12} + O(\eta^6)$$

$$\int_{-\infty}^{\infty} e^{itx}\, e^{\frac{a}{2}x^2}\, d\mu_N(x) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi a}}\, e^{+\frac{t^2}{2a} + \frac{2it\eta}{a} - \frac{4\eta^2}{2a}}$$

$$\times e^{N\log\cosh\eta} \qquad \frac{2d\eta\, N^{1/4}}{N^{1/4}}$$

First arrange $\frac{4\eta^2}{2a} = \frac{N\eta^2}{2}$ or $a = \frac{4}{N}$. Then you want

$$N\eta^4 = y^4.$$

The exponent is:

$$\frac{(t+i2\eta)^2}{2a} + N\log\cosh\eta = \frac{t^2}{2a} + \frac{2i\eta t}{a} - \frac{2\eta^2}{a} + \frac{N\eta^2}{2} - \frac{N\eta^4}{12} +$$

[redacted] $-\frac{2}{a} + \frac{N}{2} = 0 \qquad a = \frac{4}{N}$

$$\frac{2i\eta t}{a} = \frac{2i\;\eta N^{1/4}\;N^{-1/4}t}{4N^{-1}} =$$ [redacted] $$i\,\underbrace{(\eta N^{1/4})}_{y}\,\underbrace{(N^{3/4}t\tfrac{1}{2})}_{u}$$

$$y = \eta N^{1/4} \qquad t = 2N^{-3/4}u$$

$$\frac{t^2}{2a} = \frac{4N^{-3/2}u^2}{2\cdot 4/N} = \frac{1}{2}N^{-1/2}u^2 \longrightarrow 0$$

The right side

$$\sim \frac{2}{\sqrt{2\pi}\sqrt{\frac{4}{N}}}N^{-1/4}\int_{-\infty}^{\infty} e^{iyu}\,e^{-y^4/12}\,dy$$

and the left side is

$$2N^{-3/4}x = y$$

$$\int_{-\infty}^{\infty} e^{i(2N^{-3/4}u)x + \frac{2}{N}x^2}\,d\mu_N(x)$$

$$= \int_{-\infty}^{\infty} e^{iuy}\;e^{\frac{2}{N}\frac{y^2}{4N^{-3/2}}}\,d\mu_N\left(\frac{y}{2N^{-3/4}}\right)$$

Modulo some details we see therefore that if we take the binomial distribution $d\mu_N(x)$ and multiply by its normal factor $e^{+\frac{2}{N}x^2}$ and then rescale by [redacted] $x = N^{3/4}y$, then we get a distribution

$$e^{-(\text{const})y^4}\,dy.$$

April 27, 1979

Entropy and free energy. Let us be given a quantum system with energy levels E_n :.

$$H\psi_n = E_n\psi_n$$

I've already been through the ensembles arguments which shows that the equilibrium state at temperature T is described by the MBG distribution, which gives the probability of the state n as

$$P_n = \frac{e^{-\beta E_n}}{Q} \quad \text{where} \quad Q = \sum e^{-\beta E_n} \qquad \beta = \frac{1}{kT}$$

The entropy is defined by

$$S = -k\sum P_n \log P_n ,$$

The Helmholtz free energy by

$$e^{-\beta F} = \sum e^{-\beta E_n} \qquad \text{or} \qquad F = -\frac{1}{\beta}\log Q .$$

Then one has

$$TS = -\frac{1}{\beta}\sum P_n\left[[illegible] -\beta E_n - \log Q\right]$$

$$= U - F \qquad\qquad U = \sum P_n E_n = \text{average energy.}$$

or

$$U = F + TS$$

April 29, 1979 BCS

RH for curves 856
Ising 859

Return to the magnet model where one looks at spin waves. Here the set of sites is a lattice (say $I = \mathbb{Z}$) and the interaction between two sites m, n is described by a multiple of

$$\frac{1}{2}(\vec{\sigma}_m \cdot \vec{\sigma}_n) = \frac{1}{2}(\sigma_m^x \sigma_n^x + \sigma_m^y \sigma_n^y + \sigma_m^z \sigma_n^z)$$

This is an operator on $V_m \otimes V_n$ which has the basis $|\pm\rangle \otimes |\pm\rangle$ which we arrange in the order

$$|++\rangle, |+-\rangle, |-+\rangle, |--\rangle.$$

Then

$$\sigma^x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \text{ rel to } |+\rangle, |-\rangle \quad \text{so}$$

$$\sigma^x \otimes \sigma^x = \left(\begin{array}{cc|cc} & & & 1 \\ & & 1 & \\ \hline & 1 & & \\ 1 & & & \end{array}\right)$$

$$\sigma^y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \Rightarrow \sigma^y \otimes \sigma^y = \left(\begin{array}{cc|cc} & & & -1 \\ & & 1 & \\ \hline & 1 & & \\ -1 & & & \end{array}\right)$$

$$\sigma^z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \Longrightarrow \sigma^z \otimes \sigma^z = \left(\begin{array}{cc|cc} 1 & & & \\ & -1 & & \\ \hline & & -1 & \\ & & & 1 \end{array}\right)$$

So

$$\frac{1}{2}(\vec{\sigma} \otimes \vec{\sigma}) = \begin{pmatrix} 1/2 & & & \\ & -1/2 & 1 & \\ & 1 & -1/2 & \\ & & & 1/2 \end{pmatrix}$$

and so

$$\frac{1}{2}(\vec{\sigma}\otimes\vec{\sigma})+\frac{1}{2}=\begin{pmatrix}1&&&\\&0&1&\\&1&0&\\&&&1\end{pmatrix}=$$ Pauli spin exchange operator P.

Thus

$$\frac{1}{2}(\vec{\sigma}_m\cdot\vec{\sigma}_n)=P_{m,n}-\frac{1}{2}$$

Next we interpret the Hilbert space

$$\mathcal{H}=\bigotimes_{I}V$$

as an exterior algebra generated by the vectors

$$e_n=\cdots|+\rangle\otimes|+\rangle\otimes\cdots\otimes\underset{\substack{\uparrow\\ n\text{th position}}}{|-\rangle}\otimes|+\rangle\otimes\cdots$$

having one spin down in the nth position. Thus V is interpreted as an exterior algebra with one generator $|-\rangle$ and unit element $|+\rangle$. Then

$$\sigma^+=\frac{1}{2}(\sigma_x+i\sigma_y)=\begin{pmatrix}0&1\\0&0\end{pmatrix}=a \quad \text{destruction op.}$$

$$\sigma^-=\frac{1}{2}(\sigma_x-i\sigma_y)=\begin{pmatrix}0&0\\1&0\end{pmatrix}=a^* \quad \text{creation op.}$$

and

$$\sigma_x=(\sigma^++\sigma^-)\quad,\quad\sigma_y=\frac{1}{i}(\sigma^+-\sigma^-)$$

$$[\sigma^+,\sigma^-]=\sigma_z$$

Thus $\sigma_x=a+a^*$, $\sigma_y=\frac{1}{i}(a-a^*)$, $\sigma_z=aa^*-a^*a=1-2a^*a$.

$$\sigma^x\otimes\sigma^x=(a+a^*)\otimes(a+a^*)=a\otimes a+a^*\otimes a+a\otimes a^*+a^*\otimes a^*$$
$$\sigma^y\otimes\sigma^y=-(a-a^*)\otimes(a-a^*)=-a\otimes a+a^*\otimes a+a\otimes a^*-a^*\otimes a^*$$

$$\sigma^z_{\text{[redacted]}} \otimes \sigma^z_{\text{[redacted]}} = (1-2a^*a)\otimes(1-2a^*a) = 1\otimes 1 - 2(a^*a\otimes 1) - 2(1\otimes a^*a)$$
$$+\ 4\, a^*a\otimes a^*a$$

$$\frac{1}{2}\vec{\sigma}\otimes\vec{\sigma} = \frac{1}{2}(1\otimes 1) - a^*a\otimes 1 - 1\otimes a^*a + a^*\otimes a + a\otimes a^*$$
$$+\ 2\, a^*a\otimes a^*a$$

$$= \frac{1}{2}(1\otimes 1) - (a^*\otimes 1 - 1\otimes a^*)(a\otimes 1 - 1\otimes a) + 2\, a^*a\otimes a^*a$$

So the part of the Hamiltonian from the [redacted] interaction between the m, n sites is

$$H_{mn} = \left[\frac{1}{2} - (a_m^* - a_n^*)(a_m - a_n)\right] + 2\, a_m^* a_n^* a_m a_n \qquad ?$$

The next step will be to interpret the last term as a perturbation of the part in brackets.

I think the above computation is wrong. It's not true that under the identification

$$\bigotimes_I V \simeq \Lambda\Big(\bigoplus_I \mathbb{C}\Big)$$

that the destruction operator a_n is σ_n^+ because $\sigma_n^+ = \sigma^+$ on the nth copy of V together with identity on the rest, whereas

$$a_n(\omega\cdot u) = \pm\,\omega\cdot a_n(u) \qquad u \in n\text{-th space.}$$

May 3, 1979

BCS model. V = 2 diml repn. of SU_2 = spin space. Form

$$\mathcal{H}_N = \bigotimes_{i=1}^{N} V$$

and let Σ_N act in the obvious way. [redacted] It is well-known that the autos of [redacted] $V^{\otimes N}$ coming from [redacted] SU_2 span the space of endos of $V^{\otimes N}$ commuting with Σ_N. Recall the proof. One has

$$\mathrm{End}(V^{\otimes N}) = (\mathrm{End}\, V)^{\otimes N}$$

hence

$$\mathrm{End}(V^{\otimes N})^{\Sigma_N} = \Gamma_N(\mathrm{End}\, V)$$

and the latter is spanned by [redacted] the image of the canonical map

$$\gamma_n : \mathrm{End}\, V \longrightarrow \Gamma_N(\mathrm{End}\, V)$$

[redacted] Finally $GL(V)$ is Zariski-dense in $\mathrm{End}(V)$ and U_2 is Zariski-dense in $GL(V)$. This should imply the image of

$$U_2 \longrightarrow \mathrm{End}_{\Sigma_N}(V^{\otimes N})$$

spans the latter. In effect, if not, then a linear functional [redacted] vanishing on the image, will give a non-zero polynomial on End V, which will then be non-zero on $GL(V)$, and then this can't vanish on U_2, because U_2 = real points of $GL(V)$.

Now U_2 [redacted] $= SU_2 \cdot S^1$, where the diagonal S^1 acts as scalars on $V^{\otimes N}$. Consequently the span of SU_2 in $\mathrm{End}(V^{\otimes N})$ is the same as the span of U_2 which we've seen is $\mathrm{End}_{\Sigma_N}(V^{\otimes N})$.

So now one uses complete reducibility for the representations of a finite group. Let [redacted] $C = \mathrm{End}(V^{\otimes N})$ and let $A = \mathrm{Im}\{\mathbb{C}[\Sigma_N] \longrightarrow C\}$, $B =$ span of SU_2. Then we know that $B =$ commutant of A by the above. The general theory tells me that A is the commutant of B, and that $V^{\otimes N}$ as a representation of $\Sigma_N \times SU_2$ is of multiplicity one, and [redacted] sets up [redacted] some sort of correspondence between the irred. repns. of Σ_N and SU_2.

Let $K_1, \ldots, K_n$ be the different irreducible repns of Σ_N. Then

$$V^{\otimes N} = \bigoplus_i K_i \otimes \underbrace{\mathrm{Hom}_{\Sigma_N}(K_i, V^{\otimes N})}_{\text{denote this } L_i}$$

Then

$$B = \mathrm{End}_{\Sigma_N}(V^{\otimes N}) = \prod_i \mathrm{End}(L_i)$$

and L_i is an irreducible B module, hence an irreducible SU_2-module. Also L_i, L_j are distinct over B, hence over SU_2.

So therefore $V^{\otimes N} = \bigoplus K_i \otimes L_i$, where K_i is a family of inequivalent repns of Σ_N and L_i ——————————— SU_2.

Philosophy about density matrices: Universe is divided into the system under study and the outside. This decomposes the underlying Hilbert space into a tensor product

$$\mathcal{H} = \mathcal{H}_{sys} \otimes \mathcal{H}_{outside} .$$

If ψ is the state vector for the universe, and φ_i is an orthonormal basis for $\mathcal{H}_{sys}$, we have

$$\psi = \sum \varphi_i \otimes \psi_i \;, \quad 1 = |\psi|^2 = \sum |\psi_i|^2$$

Let A be an operator on $\mathcal{H}_{sys}$, extended to $\mathcal{H}$ as $A \otimes 1$. The expected value for A is

$$\begin{aligned} \langle \psi | A\psi \rangle &= \left\langle \sum \varphi_i \otimes \psi_i \,\middle|\, \sum A\varphi_i \otimes \psi_i \right\rangle \\ &= \sum_{i',i} \langle \varphi_{i'} | A\varphi_i \rangle \langle \psi_{i'} | \psi_i \rangle \\ &= \operatorname{tr}(A\rho) \end{aligned}$$

where ρ is the [redacted] operator [redacted] on $\mathcal{H}_{sys}$ with

$$\langle \varphi_{i'} | \rho \varphi_i \rangle = \langle \psi_i | \psi_{i'} \rangle$$

Notice that one has

$$\rho \geq 0 \qquad \operatorname{tr} \rho = 1$$

since

$$\begin{aligned} \sum \bar{a}_{i'} \langle \varphi_{i'} | \rho \varphi_i \rangle a_i &= \sum \bar{a}_{i'} \langle \psi_i | \psi_{i'} \rangle a_i \\ &= \left\langle \sum \bar{a}_i \psi_i \,\middle|\, \sum \bar{a}_{i'} \psi_{i'} \right\rangle \geq 0 \end{aligned}$$

and $\sum \langle \varphi_i | \rho \varphi_i \rangle = \sum \langle \psi_i | \psi_i \rangle = 1$ by the above.

Such a ρ is called a <u>density matrix</u>. It is the quantum analogue of a probability distribution on phase space. The statistical mechanical density matrix (which gives expectation values when the system is in thermal equilibrium at temperature T) is

$$\rho = \frac{e^{-\beta H}}{Q} \qquad Q = \operatorname{tr} e^{-\beta H} \qquad \beta = \frac{1}{kT}.$$

(Even when the universe is not a pure state, one gets a density matrix.)

Now return to the BCS model which is made up of N spin sites, so that the underlying Hilbert space is $V^{\otimes N}$, and where the Hamiltonian is given by an element of the enveloping algebra of SU_2. The idea now is to fix one of the sites, call it the system under consideration, so that $\mathcal{H}_{sys} = V$, and to call the rest of the sites the outside, so $\mathcal{H}_{out} = V^{\otimes(N-1)}$. The idea will be to take the thermal equilibrium "state" for $\mathcal{H} = V^{\otimes N}$ and to compute the induced state on $\mathcal{H}_{sys}$.

May 4, 1979

Let's return to BCS situation described by Dubin. V = standard repn of $G = SU_2$, on $V^{\otimes N}$ we have a Hamiltonian H given by an element of $U(\mathfrak{g})$, and this gives rise to a Gibbs state

$$\mathrm{End}(V)^{\otimes N} \ni A \longmapsto tr(\rho A) \qquad \rho = \frac{e^{-\beta H}}{Q}$$

Since one has $tr(\rho A) = tr(\rho \cdot \mathcal{S}A)$, where $\mathcal{S}$ = average over Σ_N, the state is determined by its restriction to

$$\mathrm{End}(V^{\otimes N})^{\Sigma_N} = \Gamma_N(\mathrm{End}\, V)$$

which is spanned by elements $g^{\otimes N}$ with $g \in SU_2$. Thus I want to describe the function on SU_2 given by

$$g \longmapsto tr(\rho\, g^{\otimes N}).$$

We have to decompose $V^{\otimes N}$ under SU_2. Recall that the irreducible repn. of SU_2 are $S_n V$, $n = 0, 1, 2, \ldots$ The generic diagonal element of SU_2 is

$$\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}$$

and its character on $S_n V$ is

$$t^n + t^{n-2} + \cdots + t^{-n} = \frac{t^{n+1} - t^{-n-1}}{t - t^{-1}}$$

Thus

$$V^{\otimes n} = \sum_{\ell=0}^{[\frac{n}{2}]} m_{n\ell}\, S_{n-2\ell}(V)$$

where [redacted]

$$(t+t^{-1})^n = \sum_{l=0}^{[\frac{n}{2}]} m_{nl} \frac{t^{n-2l+1} - t^{-(n-2l+1)}}{t - t^{-1}}$$

or

$$(t-t^{-1}) \sum_{k=0}^{n} \binom{n}{k} t^{n-k} t^{-k} = \sum_{l=0}^{[\frac{n}{2}]} m_{nl} \left(t^{n-2l+1} - t^{-n+2l-1}\right)$$

$$= \sum_{0}^{n} \binom{n}{k} t^{n-2k+1} - \sum_{0}^{n} \binom{n}{k} t^{n-2k-1}$$

$$= \sum_{k=0}^{n+1} \left\{ \binom{n}{k} - \binom{n}{k-1} \right\} t^{n-2k+1} = \sum_{l=0}^{[\frac{n}{2}]} m_{nl} \left(t^{n-2l+1} - t^{-n+2l-1}\right)$$

so

$$m_{nl} = \binom{n}{l} - \binom{n}{l-1} \qquad \text{for} \quad l \le [\tfrac{n}{2}]$$

$$= \frac{n!}{l!\,(n-l)!} - \frac{n!}{(l-1)!\,(n-l+1)!}$$

$$= \frac{n!}{l!\,(n-l+1)!} \left[n-l+1-l \right]$$

$$= \frac{n!\,(n-2l+1)}{l!\,(n-l+1)!}$$

so

$$V^{\otimes n} \; [redacted] = \sum_{j=0}^{n} \tilde{m}_{nj}\, S_j V \qquad \begin{aligned} j &= n-2l \\ l &= \frac{n-j}{2} \end{aligned}$$

or

$$V^{\otimes n} = \sum_{\substack{j \equiv n\,(2) \\ 0 \le j \le n}} \frac{n!\,(j+1)}{\left(\frac{n-j}{2}\right)!\left(\frac{n+j}{2}+1\right)!} S_j V$$

Next we have to understand the Hamiltonian H which is a linear combination of σ^z and $\sigma^+\sigma^-$

where $\sigma^z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ $\sigma^+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ $\sigma^- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$

One has

$$[\sigma^z, \sigma^+\sigma^-] = [\sigma^z, \sigma^+]\sigma^- + \sigma^+[\sigma^z, \sigma^-]$$
$$= 2\sigma^+\sigma^- + \sigma^+(-2\sigma^-) = 0$$

hence the Hamiltonian commutes with σ^z, and hence in S_jV, the standard basis diagonalizing σ^z also diagonalizes H.

Note that [redacted] the Casimir operator for $\mathfrak{g}$ is

$$\Delta = (\sigma^z)^2 - 2\sigma^z + 4\sigma^+\sigma^-$$

In effect it commutes with σ^z, and with σ^+:

$$[\Delta, \sigma^+] = [(\sigma^z - 1)^2 + 4\sigma^+\sigma^-, \sigma^+]$$
$$= [\sigma^z, \sigma^+](\sigma^z - 1) + (\sigma^z - 1)[\sigma^z, \sigma^+] + 4\sigma^+[\sigma^-, \sigma^+]$$
$$= 2\sigma^+(\sigma^z - 1) + 2(\sigma^z - 1)\sigma^+ + 4\sigma^+(-\sigma^z)$$
$$= -4\sigma^+ - 2\sigma^+\sigma^z + 2\sigma^z\sigma^+$$
$$= 2[\sigma^z, \sigma^+] - 4\sigma^+ = 4\sigma^+ - 4\sigma^+ = 0$$

Taking adjoints gives $[\sigma^-, \Delta] = 0$. Finally on S_nV, Δ has the value which we can find by evaluating [redacted] on $|-\rangle^n$.

$$\Delta|-\rangle^n = ((\sigma^z)^2 - 2\sigma^z)|-\rangle^n = (n^2 + 2n)|-\rangle^n.$$

Thus Δ on S_nV $= n^2 + 2n = (n+1)^2 - 1^2$

which has the familiar $(\lambda + \rho)^2 - \rho^2$ form.

So now we want to make the partition function have a nice limit as $N \to \infty$. First we compute $\mathrm{tr}(e^{-\beta H})$ on $S_n V$.

Suppose

$$H = a\sigma^z + b\sigma^+\sigma^-$$

operates on $S_n V$ which has the basis

$$x^{\frac{n+k}{2}} y^{\frac{n-k}{2}} \qquad k = -n, -n+2, \dots, n$$

where $x = |+\rangle$, $y = |-\rangle$. Then $\sigma^- |+\rangle = |-\rangle$, so

$$\sigma^+\sigma^-\left(x^{\frac{n+k}{2}} y^{\frac{n-k}{2}}\right) = \sigma^+ \frac{n+k}{2} x^{\frac{n+k}{2}-1} y^{\frac{n-k}{2}+1}$$

$$= \left(\frac{n+k}{2}\right)\left(\frac{n-k}{2}+1\right) x^{\frac{n+k}{2}} y^{\frac{n-k}{2}}$$

$$\sigma^z\left(x^{\frac{n+k}{2}} y^{\frac{n-k}{2}}\right) = \underbrace{\left(\frac{n+k}{2} - \frac{n-k}{2}\right)}_{k} x^{\frac{n+k}{2}} y^{\frac{n-k}{2}}$$

So H has the eigenvalues

$$ak + b\left(\frac{n+k}{2}\right)\left(\frac{n-k}{2}+1\right)$$

$$= ak + b\left(\frac{n^2-k^2}{4} + \frac{n+k}{2}\right)$$

$$= \frac{b}{4}(n^2+2n) + \left(a+\frac{b}{2}\right)k - \frac{b}{4}k^2$$

and so the partition function [illegible] on $S_n V$ is

$$\cancel{\sum_{\substack{k=-n \\ k\equiv n(2)}}^{n} e^{-\beta\left(\frac{b}{4}\right)(n^2+\text{[illegible]})}} \quad e^{-\beta\frac{b}{4}(n^2+2n)} \sum_{\substack{k=-n \\ k\equiv n(2)}}^{n} e^{-\beta\left[ak+\frac{b}{2}k-\frac{b}{4}k^2\right]}$$

Physics notation: $S_n V$ is spin$\left(\frac{n}{2}\right)$ space.
One should think of

$$\frac{n^2+2n}{4} \quad \text{as} \quad \left(\frac{n}{2}\right)\left(\frac{n}{2}+1\right)$$

corresponding to the fact that $-\Delta$ on S^2 has the eigenvalues $l(l+1)$.

Another point: Is

$$\vec{\sigma}\otimes\vec{\sigma} = \sigma^x\otimes\sigma^x + \sigma^y\otimes\sigma^y + \sigma^z\otimes\sigma^z$$

some kind of Casimir operator?

The partition function on $V^{\otimes N}$ is

$$\sum_{\substack{j\equiv N\ (2)\\ 0\leq j\leq N}} \frac{N!\,(j+1)}{\left(\frac{N-j}{2}\right)!\left(\frac{N+j}{2}+1\right)!} \sum_{\substack{k=-j\\ k\equiv j\ (2)}}^{j} e^{-\beta\left[ak+b\left[\left(\frac{j}{2}\right)^2+\left(\frac{j}{2}\right)-\left(\frac{k}{2}\right)^2+\frac{k}{2}\right]\right]}$$

The idea now is to extract some limiting behavior out of this as $N\to\infty$. Recall

$$\frac{N!}{\left(\frac{N-j}{2}\right)!\left(\frac{N+j}{2}\right)!}\frac{1}{2^N} \sim \frac{1}{\sqrt{2\pi N}}\,e^{-\frac{j^2}{2N}}$$

which means that for N large, one expects j's with size around $\sqrt{N}$ to matter. Thus we might be able to replace the sum over k with a Θ-function.

Put $N=2n$, divide the partition fn. by 2^N, replace j by $2j$, k by $-2k$ and then the partition function becomes

$$Q = \sum_{j=0}^{n} \frac{1}{2^{2n}} \frac{(2n)!\,(2j+1)}{(n-j)!\,(n+j+1)!} \sum_{k=-j}^{j} e^{-\beta[-a2k+b(j^2+j-k^2-k)]}$$

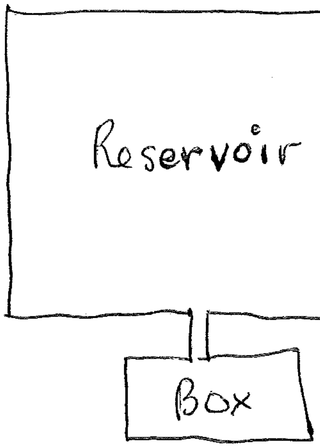
In Dubin $b = -\frac{2g}{N} = -\frac{g}{n}$, and a is constant. So the idea is to replace j by $x\sqrt{n}$. The key problem becomes the behavior of

$$\sum_{k=-x\sqrt{n}}^{x\sqrt{n}} e^{\beta a 2k - \frac{\beta g}{n}(k^2+k)}$$

as $n \longrightarrow \infty$. ??

May 5, 1979:

Bose gas. This is a gas made up of indistinguishable particles. Let's first understand the grand canonical ensemble. The idea is that the box containing the gas is connected to a large reservoir, but it requires energy μ to take a gas molecule out of the box, say a voltage drop occurs along the connecting tube. Then by adjusting μ we can arrange the average number of particles in the box to be what we want. The partition function is a sum over N = number of particles as follows.



Suppose that if N particles are in the box the partition function is

$$Q_N = \sum_\alpha e^{-\beta E_\alpha}$$

so α runs over the different N-particle states. Then when connected to the reservoir the α state has the energy $E_\alpha - N\mu$. Thus the grand partition function is

$$\sum_N \left(e^{\beta\mu}\right)^N Q_N$$

and it is the same as if we took an ensemble of closed boxes with different numbers of particles, and weighted them with the weights $\left(e^{\beta\mu}\right)^N$.

To understand thermo a little better. Let's consider a particle moving on the line, but confined to the interval $(0, V)$ by impenetrable walls. Then to get the energy levels we want to solve the Schroedinger equation

$$\frac{1}{2m}\left(\frac{\hbar}{i}\frac{\partial}{\partial x}\right)^2 \psi = E\psi$$

with the boundary conditions $\psi(0) = \psi(V) = 0$.
Eigen functions are

$$\psi_n = \sin n\frac{\pi}{V}x \qquad n = 1, 2, \dots$$

$$E_n = \frac{\hbar^2}{2m}\left(n\frac{\pi}{V}\right)^2 = n^2\frac{1}{2m}\left(\hbar\frac{\pi}{V}\right)^2$$

So the partition function is

$$Q = \sum_{n=1}^{\infty} e^{-\beta n^2 \frac{1}{2m}\left(\hbar\frac{\pi}{V}\right)^2}.$$

This is a function of β, V.

Recall that the energy (internal) in the Gibbs state is

$$U = \frac{\sum E_n e^{-\beta E_n}}{\sum e^{-\beta E_n}} = -\frac{\partial}{\partial\beta}\log Q\Big|_{V \text{ fixed}}$$

Next we want the pressure P. Notice that E_n is a function of V

$$E_n = n^2\frac{\hbar^2\pi^2}{2m}V^{-2}$$

which decreases as V increases. If V is changed by an infinitesimal amount dV, then the energy level E_n changes by $dE_n = \frac{dE_n}{dV} dV$, hence the energy of our gas changes by

$$dU = \frac{\sum dE_n e^{-\beta E_n}}{\sum' e^{-\beta E_n}}$$

(I think one has to think of the Gibbs state as an ensemble of particles with energies distributed according to the temperature, OR as a gas of independent particles with the same distribution. If one changes V by a small amount, then a particle with definite energy E_n will go to one with energy $E_n + dE_n$. Thus each particle in the gas changes its energy, and then if we have adiabatic expansion, the temperature changes so the new Gibbs state gives the new energy.)

This energy change is the work done on the gas, so we have

$$-PdV = dU = \frac{\sum dE_n e^{-\beta E_n}}{\sum e^{-\beta E_n}}$$

or

$$P = -\frac{\sum' \frac{dE_n}{dV} e^{-\beta E_n}}{\sum e^{-\beta E_n}} = \frac{1}{\beta} \frac{\partial}{\partial V} \log Q \Big|_{T \text{ fixed}}$$

For an ideal monatomic gas of N molecules of mass 1, the classical partition function is Q^N where

$$Q = \int e^{-\beta \frac{p^2}{2}} \left(\frac{dp\,dq}{2\pi}\right)^3 = V\left(\sqrt{\frac{2}{\beta}}\,\frac{\sqrt{\pi}}{2\pi}\right)^3 = (2\pi)^{-3/2}\beta^{-3/2}V$$

Thus

$$U = -\frac{\partial}{\partial\beta}\log Q^N = -N\left(-\frac{3}{2}\right)\frac{1}{\beta} = \frac{3}{2}N\beta^{-1} = \frac{3}{2}NkT$$

$$P = \frac{1}{\beta}\frac{\partial}{\partial V}\log Q^N = \frac{N}{\beta}\frac{1}{V}$$

so

$$PV = NkT = \frac{2}{3}U$$

Change the interval to $(0, \ell)$.

$$-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\psi = E\psi$$

Eigenfunctions $\psi_n = \sin n\frac{\pi}{\ell}x$, eigenvalue

$$E_n = \frac{\hbar^2}{2m}\frac{n^2\pi^2}{\ell^2} = n^2\left(\frac{\hbar^2}{2m}\frac{\pi^2}{\ell^2}\right)$$

So the partition function is

$$Q = \sum_{n=1}^{\infty} e^{-n^2\beta\left(\frac{\hbar^2\pi^2}{2m\ell^2}\right)}$$

Recall that the Θ function

$$\theta(t) = \sum_{n\in\mathbb{Z}} e^{-\pi n^2 t}$$

satisfies $\theta(t) = \frac{1}{\sqrt{t}}\theta\left(\frac{1}{t}\right)$. Clearly

$$Q = \frac{1}{2}\left(\theta\left(\beta\frac{\hbar^2}{2m}\frac{\pi}{\ell^2}\right) - 1\right)$$

Assuming $\frac{\beta\hbar^2}{2ml^2}$ small we get

$$\Theta\left(\frac{\beta\hbar^2\pi}{2ml^2}\right) \sim \frac{1}{\sqrt{\frac{\beta\hbar^2\pi}{2ml^2}}} = \sqrt{\frac{2m}{\hbar^2\pi}}\, l\beta^{-1/2}$$

so

$$Q \sim \sqrt{\frac{m}{\hbar^2 2\pi}}\, l\beta^{-1/2}$$

The usual way to get this asymptotic relation is to replace the sum in the Θ-function by an integral.

So now we want to understand a Bose gas. This means [redacted] that [redacted] instead of taking the N-fold tensor product of the one particle Hilbert space $\mathcal{H}$, we take the N-th symmetric product. Moreover we put in the chemical potential μ to make the algebra nice.

Let ε_a denote the set of energy levels for the one-particle Hamiltonian. Then the grand Hilbert space is $S(\mathcal{H})$, and its clear the grand partition function is

$$Q^{(\mu)} = \prod_a \left(1 + e^{-\beta(\varepsilon_a-\mu)} + e^{-\beta 2(\varepsilon_a-\mu)} + \cdots\right)$$

$$= \prod_a \frac{1}{1-e^{\beta\mu-\beta\varepsilon_a}}$$

Put

$$e^{-\beta g} = Q^{(\mu)} = \sum_N \left(e^{\beta\mu}\right)^N Q_N$$

Then

$$-\beta\frac{\partial g}{\partial\mu} = \sum_N N\beta e^{\beta\mu N} Q_N \Big/ \sum e^{\beta\mu N} Q_N$$

so
$$\boxed{-\frac{\partial g}{\partial \mu} = \langle N \rangle}$$

This equation relates μ to the number of particles.

$$+\beta g = \sum_a \log\left(1 - e^{\beta\mu} e^{-\beta \varepsilon_a}\right)$$

Next the idea will be let the volume V and $\langle N \rangle$ go to ∞ [redacted] such that the density

$$\rho = \frac{\langle N \rangle}{V}$$

remains fixed. We want to replace the sum over a by an integral. Recall that for a segment of length ℓ the energy levels are

$$\varepsilon_n = n^2 \frac{\hbar^2 \pi^2}{2m\ell^2} = \frac{p^2}{2m} \quad \text{where} \quad p = \frac{n\hbar\pi}{\ell}$$

hence summing for $n = 1, 2, \ldots$ [redacted] can be approximated by integrating with respect to

$$dn = \frac{\ell}{2\hbar\pi}\, dp$$

(the 2 comes from $\pm p$ giving same ε)

In three dimensions we replace ε_a by $\frac{p^2}{2m}$ and integrate wrt $\frac{d^3p}{(2\pi\hbar)^3}\underbrace{\ell^3}_{V}$, Thus

$$g = \frac{1}{\beta}\int \ln\left(1 - e^{\beta\mu} e^{-\frac{\beta p^2}{2m}}\right) \frac{d^3p}{(2\pi\hbar)^3} V$$

$$\rho = \frac{\langle N \rangle}{V} = -\frac{\partial g}{\partial \mu}\frac{1}{V} = \int \frac{e^{-\beta p^2/2m} e^{\beta\mu}}{1 - e^{-\beta p^2/2m} e^{\beta\mu}} \frac{d^3p}{(2\pi\hbar)^3}$$ [redacted]

Convert this to a radial integral $d^3p \longmapsto 4\pi p^2 dp$ and you get

$$\rho = \left(\frac{m}{2\pi\hbar^2}\right)^{3/2} \beta^{-3/2} \left[\underbrace{\alpha + \frac{\alpha^2}{2^{3/2}} + \frac{\alpha^3}{3^{3/2}} + \cdots}_{\zeta_{3/2}(\alpha)}\right]$$

$\alpha = e^{\beta\mu}$

where

$$\zeta_\nu(\alpha) = \sum_{n=1}^{\infty} \frac{\alpha^n}{n^\nu}$$

So given ρ and β, we can find $\alpha = e^{\beta\mu}$ by solving

$$\zeta_{3/2}(\alpha) = \left(\frac{m}{2\pi\hbar^2}\right)^{-3/2} \rho\,\beta^{3/2}$$

Now $\zeta_{3/2}(\alpha)$ is obviously increasing for $0 \leq \alpha \leq 1$, ~~but~~ and it converges for $\alpha = 1$. So if

$$\zeta_{3/2}(1) < \left(\frac{m}{2\pi\hbar^2}\right)^{-3/2} \rho\,\beta^{3/2}$$

which means β is too high, i.e. T is too low, then something breaks down. This is the phenomenon of Bose-Einstein condensation. What happens is that the sum can't be replaced by the integral, because the low energy terms are important.

?

Let X be a curve over $k = \overline{\mathbb{F}_q}$ defined over $\mathbb{F}_q$, and let $F : X \longrightarrow X$ be the Frobenius endomorphism. I want to show how the RH follows from the Hodge index theorem for curves on the surface $X \times X$.

Let $\alpha_1, \dots, \alpha_{2g}$ denote the eigenvalues of F on $H^1(X)$. By the Lefschetz formula

$$\begin{array}{l} \mathrm{Card}\left(\mathrm{Fix}\left(F^m\right)\right) = 1 - \sum_{i=1}^{2g} \alpha_i^m + q^m \\ \qquad\quad \| \\ \qquad \Delta \cdot \Gamma_{F^m} \end{array}$$

Now by considering the radius of convergence of

$$\sum_{m=0}^{\infty} \left(\sum_i \alpha_i^m \right) z^m = \frac{1}{1 - \alpha_1 z} + \dots + \frac{1}{1 - \alpha_{2g} z}$$

one sees that $|\alpha_i| \leq q^{1/2}$ for all i is equivalent to an estimate

$$\left| \Delta \cdot \Gamma_{F^m} - (1 + q^m) \right| \leq C q^{m/2} \qquad \text{as } m \longrightarrow \infty$$

It is enough to show

$$\left| \Delta \cdot \Gamma_F - (1 + q) \right| \leq 2g q^{1/2}$$

because we can then apply this inequality to F^m instead of F. Put $\Gamma = \Gamma_F$.

Now on $X \times X$ an ample line bundle is obtained by tensoring pull-backs of ample line bundles on X. Let H be the curve

$$H = \{x_0\} \times X \cup X \times \{x_0\}$$

Then $\qquad H \cdot H = 2.$ The Hodge index thm. tells us that for curves C with $H\cdot C = 0$, one has $C\cdot C \leq 0$.

Clearly
$$\Delta\cdot H = 2$$
$$\Gamma\cdot H = 1+q \quad \text{because } F^{-1}\{x_0\} \text{ has mult } q.$$

Also $\Delta\cdot\Delta$ = degree of normal bundle to Δ which is $pr_2^*(T_X)|\Delta$. Thus $\Delta\cdot\Delta = \int_X c_1(T_X) = \chi(X) = 2-2g$.

Similarly
$$\Gamma\cdot\Gamma = [redacted] \deg\left(pr_2^* T_X \,|\, \Gamma\right)$$
$$= \deg\left(F^* T_X\right) = \int F^* c_1(T_X) = q(2-2g)$$

So now one has
$$(\Delta - H)\cdot H = 0$$
$$\left(\Gamma - \frac{1+q}{2}H\right)\cdot H = 0$$

hence $\Delta - H$, $\Gamma - \frac{1+q}{2}H$ lie in the space with intersection product [redacted] ≤ 0. So one has Cauchy-Schwarz

$$\left|(\Delta-H)\cdot\left(\Gamma - \frac{1+q}{2}H\right)\right|^2 \leq \left((\Delta-H)\cdot(\Delta-H)\right)\left(\left(\Gamma-\frac{1+q}{2}H\right)\left(\Gamma-\frac{1+q}{2}H\right)\right)$$

$$\left|\Delta\cdot\Gamma - (1+q)\right|^2 \leq (2-2g-2)\left(q(2-2g) - \frac{1+q}{2}(1+q)\right)$$

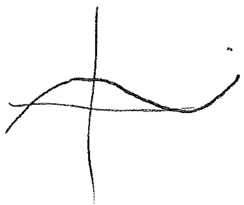
$$\leq 2g\Big(\underbrace{\frac{1+2q+q^2}{2} - \frac{4q}{2}}_{\frac{(1-q)^2}{2} \geq 0} + 2gq\Big)$$

which doesn't quite work.

The correct[redacted] thing to do is to use

[redacted] $\Gamma - X\times pt - q(pt\times X)$

i.e. you work in the ring of correspondences



Then

$$H\cdot(\Gamma - X\times pt - q\, pt\times X) = 0$$

$$(\Delta - H)\cdot(\Gamma - X\times pt - q\, pt\times X) = \Delta\cdot\Gamma - 1 - q$$

and

$$(\Gamma - X\times pt - q\, pt\times X)\cdot(\Gamma - X\times pt - q\, pt\times X)$$

$$= \underbrace{\Gamma\cdot\Gamma}_{q(2-2g)} - 2\,\underbrace{\Gamma\cdot(X\times pt + q\, pt\times X)}_{q+q} + q + q$$

$$= -2qg$$

So we will get

$$|\Delta\cdot\Gamma - 1 - q|^2 \leq 2g\, 2g\, q$$

or

$$|\Delta\cdot\Gamma - 1 - q| \leq 2g\sqrt{q}$$ as desired.

May 6, 1979

Look at Ising model again. The Hamiltonian is already diagonalized by the base states which are spin assignments $s : I \to \{\pm 1\}$. The energy of s is

$$E_s = -B\sum s_i - \sum J_{ij}\, s_i s_j$$

where B is the external magnetic field. ($B>0$ means s_i wants to be $+1$). The partition fn. is

$$Z(\beta, B) = \sum_s e^{-\beta E_s}$$

and

$$U = -\frac{\partial}{\partial\beta}\log Z = \frac{\sum E_s e^{-\beta E_s}}{Z}$$

I can think of B as a variable like the "volume". Increasing B by dB changes E_s by $(-\sum s_i)\,dB$ which changes the energy by

$$-\frac{\sum (\sum s_i)e^{-\beta E_s}}{\sum e^{-\beta E_s}}\, dB$$

Thus the magnetization

$$M(\beta, B) = \frac{\sum(\sum s_i)e^{-\beta E_s}}{Z} = \frac{1}{\beta}\frac{\partial}{\partial B}\log Z$$

is the "pressure" belonging to the "volume" B. Then it would be nice to show that the basic thermodynamic relation

$$\boxed{dU + M\,dB = T\,dS}$$

holds. Here $T = 1/\beta$ and S is the entropy defined by

$$S = -\sum_s \frac{e^{-\beta E_s}}{Z} \log \frac{e^{-\beta E_s}}{Z}$$

Note that if the free energy F is defined by

$$e^{-\beta F} = Z \qquad \text{or} \qquad \beta F = -\log Z,$$

then

$$S = +\log Z \underbrace{\sum \frac{e^{-\beta E_s}}{Z}}_{1} + \beta \underbrace{\sum E_s \frac{e^{-\beta E_s}}{Z}}_{U}$$

so that

$$S = -\beta F + \beta U \qquad \text{or} \qquad \boxed{F = U - TS}$$

So start with

$$-d \log Z = \sum (d\beta\, E_s + \beta\, dE_s) e^{-\beta E_s}/Z$$

$$d(\beta F) = d\beta \cdot U + \beta(-M dB)$$

[redacted]

$$\parallel$$

$$d(\beta U - S) = d\beta \cdot U + \beta\, dU - dS$$

So

$$\beta dU - dS = \beta(-M dB) \qquad \text{or}$$

$$dU + M dB = \frac{1}{\beta} dS = T dS \qquad \text{as desired}$$

The above thing works for a finite number of sites. What one wants is to take a suitable limit. It seems that, at least for lattices, the limit of

$$\frac{1}{N} \log Z_N \qquad \text{exists.}$$

Example: Take a periodic linear model. The sites are $0, 1, 2, \ldots, N$ with $0 = N$, and the energy is

$$-E_s = +B\sum_{i=1}^{N} s_i + J\sum_{i=1}^{N} s_{i-1} s_i$$

You work out the partition function as a matrix product

$$Z_N = \sum_{\substack{s_1,\ldots,s_N \\ s_0 = s_N}} \left(e^{\beta J}\right)^{s_0 s_1}\left(e^{\beta B}\right)^{s_1}\left(e^{\beta J}\right)^{s_1 s_2}\left(e^{\beta B}\right)^{s_2} \cdots\cdots \left(e^{\beta J}\right)^{s_{N-1} s_N}\left(e^{\beta B}\right)^{s_N}$$

$$= \operatorname{tr}\left[\begin{pmatrix} e^{\beta J} & e^{-\beta J} \\ e^{-\beta J} & e^{\beta J} \end{pmatrix}\begin{pmatrix} e^{\beta B} & 0 \\ 0 & e^{-\beta B} \end{pmatrix}\right]^N = \lambda_1^N + \lambda_2^N$$

where the λ_1, λ_2 are the eigenvalues. Then it's clear that

$$\frac{1}{N}\log Z_N \longrightarrow \log \lambda_1$$

where λ_1 is the larger in absolute value of the two eigenvalues.

$$\lambda = \frac{e^{\beta(J+B)} + e^{\beta(J-B)} \pm \sqrt{\left(e^{\beta(J+B)} \pm e^{\beta(J-B)}\right)^2 - 4\left(e^{2\beta J} - e^{-2\beta J}\right)}}{2}$$

$$= e^{\beta J}\cosh(\beta B) \pm \frac{1}{2}\sqrt{e^{2\beta J}\left(e^{\beta B} - e^{-\beta B}\right)^2 + 4e^{-2\beta J}}$$

$$\lambda = e^{\beta J}\left(\cosh(\beta B) \pm \sqrt{\sinh(\beta B)^2 + e^{-4\beta J}}\right)$$

Notice that the matrix involved has entries >0, hence has a largest positive eigenvalue, and the other has to be positive also.

Let's work up the dictionary [redacted] between Ising models and lattice gases. The idea here is that we can interpret $s_i = +1$ as meaning the ith site is occupied and $s_i = -1$ as unoccupied. To remove a molecule to the reservoir requires a certain energy μ, so $\mu = 2B$ in the dictionary. We want to change variables:

$$x_i = \frac{s_i + 1}{2}$$

$$-E(s) = B \sum{}' s_i + \sum_{i<j}{}' J_{ij} s_i s_j$$

$$= B\sum{}'(2x_i - 1) + \sum_{i<j}{}' J_{ij} \underbrace{(2x_i-1)(2x_j-1)}_{2[x_i \text{[redacted]} (x_j-1) + (x_i-1)x_j] + 1}$$

So the point is that except for an additive constant

$$-E(s) = 2B\sum{}' x_i \;\text{[redacted]}\; 2\sum_{i<j}{}' J_{ij} \underbrace{\left[x_i(1-x_j) + x_j(1-x_i)\right]}_{\begin{array}{ll} 1 & x_i \neq x_j \\ 0 & x_i = x_j \end{array}}$$

so

$$Z = \sum_s{}' e^{-\beta E(s)} = \sum_{J \subset I} \left(e^{2\beta B}\right)^{\text{card} J} \prod_{\substack{i \in J \\ j \in I - J}} e^{-2\beta J_{ij}}$$

is a Lee-Yang polynomial in $z = e^{(2B)\beta} = e^{\beta\mu}$. Notice that since z depends on $\beta\mu$, the effect of increasing β (lowering temperature) is [redacted] effectively to increase the interaction strengths J_{ij}. In fact the relevant quantity now is

[redacted]

$$\frac{1}{\beta}\frac{\partial}{\partial\mu}\log Z = \text{[redacted]}\; z\frac{\partial}{\partial z}\log Z = \sum (\text{card } J)\,\frac{e^{-\beta E_J}}{Z}$$

which gives the average number of molecules in the gas corresponding to a given value of z, (or actually the number of unfilled sites).

The partition function $Z = \sum_{n=0}^{N} z^n \sum_{|J|=n} c_J$ is a monic polynomial of degree N in z where N = number of sites. By the Lee-Yang theorem all its roots lie on $|z| = 1$. So if

$$Z_N = \prod_{i=1}^{N} (z - \lambda_i)$$

$$z\frac{\partial}{\partial z}\log Z_N = \sum_{i=1}^{N} \frac{z}{z-\lambda_i} = \sum_{i=1}^{N} \frac{1}{1-\lambda_i z^{-1}}$$

Thus

$$z\frac{\partial}{\partial z}\frac{1}{N}\log Z_N = \frac{1}{N}\sum_{i=1}^{N}\frac{1}{1-\lambda_i z^{-1}} = \int_{S^1} \frac{1}{1-\lambda z^{-1}}\, d\sigma_N(\lambda).$$

By compactness of S^1 we can assume by passing to a subsequence that the measures $d\sigma_N$ converge. In fact assume that the limit does exist

$$z\frac{\partial}{\partial z}\frac{1}{N}\log Z_N \longrightarrow \rho.$$

ρ is the density of unfilled sites. Then we have

$$\rho(z) = \int_{S^1} \frac{1}{1-\lambda z^{-1}}\, d\sigma(\lambda)$$

Now there are really two cases: If the support of $d\sigma$ does not include $\lambda = 1$, then $\rho(z)$ will be an analytic function for $z > 0$.

(Put $x = z$

$$\frac{1}{2}\left(\frac{1}{1-\lambda x^{-1}} + \frac{1}{1-\bar{\lambda} x^{-1}}\right) = \frac{x}{2}\left(\frac{1}{x-\lambda} + \frac{1}{x-\bar{\lambda}}\right)$$

Approaches 1 [redacted] as $x \to \infty$ and 0 as $x \to 0$. It would be nice if it were [redacted] monotone in x, but this seems unlikely, e.g. if $\lambda = 1$.)

So when $\rho(z)$ is analytic on $\mathbb{R}_{\geq 0}$ then there is no phase transition. On the other hand if the zeroes of Z_N accumulate in such a way that $\lambda = 1$ is in the support of [redacted] $d\sigma$, then $\rho(z)$ will have a jump at $z = 1$, and we have a phase transition.

When working with a grand canonical ensemble the basic variables are

inverse temperature β

fugacity $z = e^{\beta\mu}$

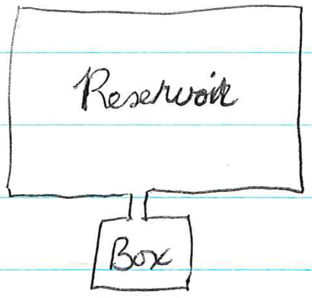
density ρ [redacted]

instead of β, V, P.

[redacted]

May 7, 1979

Classical grand canonical ensemble. Let's suppose



one has a huge number N of particles in the reservoir + box. One forms the partition function and integrates out the momentum variables. This gives the partition fn. the factor

$$\int e^{-\beta\frac{p^2}{2m}}\,\frac{dp}{2\pi} = \sqrt{\frac{2m}{\beta}}\,\frac{\sqrt{\pi}}{2\pi} = \sqrt{\frac{m}{2\pi\beta}}$$

raised to the $3N$ power, and we will concentrate on the rest which is an integral over positions:

$$\int e^{-\beta V(x)}\, d^{3N}x$$

Here $V(x) = V(x_1, \dots, x_N)$ is a symmetric function. Now decompose $3N$-dimensional space into parts according to how many particles are in the box. Thus 3-space is broken into 2 pieces, $3N$-space is broken into 2^N pieces and the number of these 2^N with exactly n particles in the box is $\binom{N}{n}$. So if we use symmetry we get

$$\int e^{-\beta V} d^{3N}x = \sum_{n=0}^{N} \binom{N}{n} \int\limits_{\substack{x_1,\dots,x_n \text{ inside box} \\ x_{n+1},\dots,x_N \text{ outside}}} e^{-\beta V}\, d^{3N}x$$

Assume that particles inside and outside don't interact, and that the potential energy of a particle outside is zero. If Vol is the volume of the reservoir, then

$$\int_{\substack{x_1\cdots x_n \text{ inside}\\ x_{n+1},\cdots,x_N \text{ outside}}} e^{-\beta V}\, d^{3N}x = \left(\int_{\substack{x_1,\dots,x_n\\ \text{inside}}} e^{-\beta V(x_1,\dots,x_n)}\, d^{3n}x\right)(\text{Vol})^{N-n}$$

Now $\binom{N}{n} = \dfrac{N(N-1)\cdots(N-n+1)}{n!} = \dfrac{(1-\frac{1}{N})\cdots(1-\frac{n-1}{N})}{n!}N^n$

So

$$\binom{N}{n}(\text{Vol})^{-n} \cong \frac{1}{n!}\left(\frac{N}{\text{Vol}}\right)^n$$

Thus if we let the volume of the reservoir and N increase so that the density $\frac{N}{\text{Vol}} = \rho$ is fixed, then our partition function is

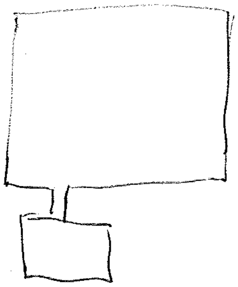
$$\sim \left(\frac{m}{2\pi}\right)^{3/2 N} \beta^{-3/2 N} (\text{Vol})^N \sum_{n\geq 0} \frac{\rho^n}{n!} Z_n$$

where

$$Z_n = \int_{\substack{x_1,\dots,x_n\\ \text{in box}}} e^{-\beta(\text{Pot. En})}\, d^{3n}x$$

May 8, 1979

Review box + reservoir. Let V = Volume of reservoir



N = total number of particles. The partition function for the whole system is

$$Z = \sum_{\text{states}} e^{-\beta(\text{Energy})} = \left(\frac{m}{2\pi\beta}\right)^{3/2 N} \sum_{\substack{\text{positions} \\ x_1,\dots,x_N}} e^{-\beta(\text{Pot. En.})}$$

$$= \left(\frac{m}{2\pi\beta}\right)^{\frac{3}{2}N} \sum_{\substack{n = \text{number} \\ \text{of particles} \\ \text{in box}}} \binom{N}{n} \underbrace{\left(\sum_{\substack{x_1,\dots,x_n \\ \text{in box}}} e^{-\beta(\text{Pot.En})}\right)}_{Z_n \,=\, \text{function of } \beta \text{ and parameters for box}} V^{N-n}$$

Thus

$$Z = \left(\frac{m}{2\pi\beta}\right)^{\frac{3}{2}N} \sum_n \binom{N}{n} V^{N-n} Z_n$$

Now one of the things we are interested in is the average number $\langle n \rangle$ of particles in the box. If

$$Z = \sum_s e^{-\beta E_s}$$

then

$$\langle n \rangle = \sum_s n(s) \frac{e^{-\beta E_s}}{Z} .$$

The above calculation makes it clear that

$$\sum_{n(s)=n} e^{-\beta E_s} = \left(\frac{m}{2\pi\beta}\right)^{3/2 N} \binom{N}{n} V^{N-n} Z_n .$$

So

$$\langle n \rangle = \frac{\sum_n n \binom{N}{n} V^{N-n} Z_n}{\sum_n \binom{N}{n} V^{N-n} Z_n}$$

So let us now assume the "max no. of atoms which can be crammed into the box" is much smaller than N, and use that

$$\binom{N}{n} = \frac{1}{n!}\left(1-\frac{1}{N}\right)\cdots\left(1-\frac{n-1}{N}\right)N^n \sim \frac{N^n}{n!}$$

Then

$$\langle n \rangle \doteq \frac{\sum_n n \frac{1}{n!}\left(\frac{N}{V}\right)^n Z_n}{\sum \frac{1}{n!}\left(\frac{N}{V}\right)^n Z_n}$$

Thus if we put $\rho =$ density $\frac{N}{V}$ and

$$Z^{(g)} = \sum_{n\geq 0} \frac{\rho^n}{n!} Z_n \qquad \text{grand partition function for box}$$

we find

$$\boxed{\langle n \rangle = \rho \frac{\partial}{\partial \rho} \log Z^{(g)}} \; .$$

Moreover

$$Z = \underbrace{\left(\left(\frac{m}{2\pi\beta}\right)^{3/2} V\right)^N}_{\substack{\text{partition function}\\ \text{for ideal gas}\\ \text{in the reservoir}}} Z^{(g)}$$

Notice that $\rho = \frac{N}{V}$ is temperature independent, so that it can be represented $e^{\beta\mu}$ only with a μ depending on temperature. So one sees the rationale in using a different variable in the grand partition function called "fugacity" which is then adjusted to yield the desired value for $\langle n \rangle$.

Actually I have left out the [redacted] kinetic part in the Z_n.

$$\sum \frac{z^n}{n!} \underbrace{\left(\frac{m}{2\pi\beta}\right)^{3/2\, n} Z_n^{Pot.}}_{Z_n^{full\ energy}} = \sum \frac{\rho^n}{n!} Z_n^{Pot}$$

$$z\left(\frac{m}{2\pi\beta}\right)^{3/2} = \rho \qquad \text{Still} \quad z \neq e^{\beta\mu}$$

Some examples: Suppose the particles in the box are independent. Then $Z_n = (Z_1)^n$ so

$$Z_{gr} = \sum \frac{z^n Z_1^n}{n!} = e^{zZ_1}$$

Typical example of Z_1 is

$$Z_1 = \sum e^{-\beta\varepsilon_a}$$

where ε_a is a family of energy levels (e.g. $\varepsilon_a = n^2$ const, $n = 1, 2, \cdots$). Then the above "distinguishable particles" grand partition function is to be contrasted with

Bose Einstein $\qquad \prod \frac{1}{1 - ze^{-\beta\varepsilon_a}}$

Fermi-Dirac $\qquad \prod \left(1 + ze^{-\beta\varepsilon_a}\right)$

So what one gets in the three cases is a product over a of the factors

$$e^{ze^{-\beta\varepsilon_a}}, \quad \frac{1}{1 - ze^{-\beta\varepsilon_a}}, \quad 1 + ze^{-\beta\varepsilon_a}$$

May 10, 1979

Far out idea: Form a model of a liquid using [redacted] K-theory. Thinks of a liquid as being made up of [redacted] particles which can form clumps. A clump of n-particles, (molecule might be a better term), has a higher group of symmetry than just Σ_n. So one has a mixture of groups G_n as in K-theory. Now the problem is to write down the partition function for such a liquid.

First thing to try is to suppose one has a discrete set X of possible positions for the molecules. Thus a state of the liquid will be a positive divisor on X. This is like a lattice gas except that the number of particles occupying the same position is any finite $n \geq 0$. We have to give the energy of a n-molecule ε_n. Then [redacted] given a state $(n_x, x \in X)$, the energy of this state is

$$\sum_{x\in X} \varepsilon_{n_x} .$$

The grand partition function is

$$\sum_{\substack{\text{[redacted]} \\ n : X \to \mathbb{N}}} z^{|n|} e^{-\beta \sum_x \varepsilon_{n_x}} = \text{[redacted]} \sum_{\{(n_x \geq 0)\}} \prod_x z^{n_x} e^{-\beta \varepsilon_{n_x}}$$

$$= \prod_{x\in X} \left(\sum_{n_x \geq 0} z^{n_x} e^{-\beta \varepsilon_{n_x}} \right)$$

Let's review the loop space on a compact Lie group K, say connected, such as $K = SU_2$. The nice model for ΩK consists of Laurent loops, i.e. algebraic maps $S^1 \longrightarrow K$ preserving basepoint. For $K = U_n$ these are given by Laurent polynomial matrices. Let $\Omega^L K$ denote Laurent loops.

Next we want a principal bundle for $\Omega^L K$ with base K, denoted $\Gamma^L K$. It consists of paths $h : \mathbb{R} \longrightarrow K$ with $h(0) = id$ which are in the form

$$h(t) = \alpha(e^{2\pi i t}) \exp(tX)$$

where $\alpha \in \Omega^L(K)$ and $X \in \text{Lie alg}(K)$. One has then a map

$$\begin{aligned} \Gamma^L K &\longrightarrow K \\ h &\longmapsto h(1) \qquad (= \exp(X) \text{ for the above } h.) \end{aligned}$$

Recall that the lemma that makes things work here is that

$$\exp(X) = \exp(Y) \implies \exp(tX)\exp(-tY) = \theta(e^{2\pi i t})$$
$$\text{with } \theta \in \Omega^L(K).$$

This allows one to see that left multiplication by $\Omega^L K$ on $\Gamma^L K$ is free with quotient K. So we get a principal bundle

$$\Omega^L K \longrightarrow \Gamma^L K \longrightarrow K$$

Next let $\mathcal{K}$ = all alg. maps $S^1 \longrightarrow K$, so that

$$\mathcal{K} = K \ltimes \Omega^L K$$

One makes $\mathcal{K}$ act on $\Gamma^L K$ by

$$(\alpha \cdot h)(t) = \alpha(e^{2\pi i t})\, h(t)\, \alpha(1)^{-1}$$

One has
$$\mathcal{K}\backslash\Gamma^L K = K\backslash K$$

where K acts on itself via conjugation.

Finally I recall that $\Gamma^L K$ is a Tits building in a suitable sense, but this is complicated. In fact there is a whole theory generalizing the Bott-Borel-Weil business for flag manifolds. The point is that the $\mathcal{K}$ orbits on $\Gamma^L K$ are of the form

$$\mathcal{K}/K_i$$

where K_i is the centralizer of ~~[illegible]~~ an element of K.

Take $h(t) = e^{tX}$ and find its stabilizer in $\mathcal{K}$:

$$\alpha(e^{2\pi i t})\, e^{tX} \alpha(1)^{-1} = e^{tX}$$

If you set $t = 1$, you find $\alpha(1) \in K^{(e^X)}$, and that

$$\alpha(e^{2\pi i t}) = e^{tX} \alpha(1)\, e^{-tX}$$

As this formula works for any element of $K^{(e^x)}$ it follows the stabilizer $\mathcal{K}^h \simeq K^{h(1)}$.

Somewhat easier is to consider ΓK = all paths $h: I = [0,1] \longrightarrow K$ starting at id, with the group K^I acting on its.