January 1, 1978:

Obstacle scattering with Dirichlet [redacted] boundary condition: Suppose ψ defined outside obstacle and

$$(\Delta + k^2)\psi = 0 \qquad \text{outside}$$

1) $$\psi = 0 \qquad \text{on } \partial X$$

$$\psi - \varphi \qquad \text{outgoing}$$

where φ is a global soln of $(\Delta+k^2)\varphi = 0$. Then we saw that for $\bar{r}$ outside the obstacle X

2) $$\psi(\underline{r}) = \varphi(\underline{r}) + \int_{\partial X} G(\underline{r},\underline{r}')\,\frac{\partial\psi}{\partial n}(r'^{+})\,dS_{r'}$$ [redacted]

However the right side makes sense inside the obstacle and can be used to define ψ there. We then have

$$\frac{\partial\psi}{\partial n}(r^+) - \frac{\partial\psi}{\partial n}(r^-) = \int\left\{\frac{\partial G}{\partial n_1}(r^+,r') - \frac{\partial G}{\partial n_1}(r^-,r')\right\}\frac{\partial\psi}{\partial n}(r'^+)\,dS_{r'}$$

$$= \frac{\partial\psi}{\partial n}(r^+),$$

so
$$\frac{\partial\psi}{\partial n}(r^-) = 0 \qquad \text{for } r \text{ on } \partial X.$$

But then both $\psi(\bar{r})$ and $\frac{\partial\psi}{\partial n}(r^-)$ are [redacted] 0 on ∂X (note ψ should be [redacted], continuous because of the nature of G.) Thus ψ has to be zero inside the obstacle. So we see that 1) is equivalent to the integral equation 2) over all space with the additional requirement that ψ vanish on ∂X.

What I want to understand is the nature of solutions of the integral equation 2). To begin, consider the homogeneous equation where $\varphi = 0$. A solution with $\psi = 0$ on ∂X or equivalently $\psi = 0$ inside is the same as a scattering eigenfunction, i.e. an outgoing solution of $(\Delta + k^2)u = 0$ in the exterior vanishing on ∂X. Such a non-trivial ψ exists when the exterior Dirichlet problem has a non-trivial soln.

A soln of 2) with $\varphi = 0$ which is non-zero on ∂X gives a non-trivial solution of the interior Neumann problem. Conversely given a non-trivial solution of the interior Neumann problem, combine it with the solution of the exterior Dirichlet problem with the same values on ∂X. Then ψ satisfies $(\Delta + k^2)\psi = 0$ off ∂X and on ∂X we have $\frac{\partial \psi}{\partial n}(r-) = 0$, so that $\frac{\partial \psi}{\partial n}(r+)$ is the jump in the normal derivative along ∂X. Thus it's clear

$$3) \qquad \psi(\underset{\sim}{r}) = \int G(\underset{\sim}{r}, \underset{\sim}{r}') \frac{\partial \psi}{\partial n}(r'^{+}) \, dS_{r'}$$

since the difference would be a global outgoing soln. of $(\Delta + k^2)u = 0$.

Thus we get non-trivial solutions of 3) when either the exterior Dirichlet problem has a non-trivial solution, or when it doesn't but the interior Neumann problem has a non-trivial solution.

Review yesterday's analysis when ∂X is the sphere $r = a$ and so the problem separates into uncoupled radial equations.

Let $j(r)$ be the solution of the radial equation

$$\left(\frac{1}{r}\frac{d}{dr} r \frac{d}{dr} - \frac{n^2}{r^2} + k^2\right) u = 0$$

which is nice at 0, and $h(r)$ the solution nice (i.e. outgoing) at ∞. Then $W(j,h)(r) \neq 0$ because we assume for all space there is no non-trivial outgoing soln.

So consider a non-zero solution of 3):

4) $$\psi(r) = G(r,a)\,\psi'(a^+)$$

Then $$G(r,a)\,\psi'(a^+) = G(r,a)\,G'(a^+,a)\,\psi'(a^+),$$ so as $\psi'(a^+) \neq 0$ we get

$$G'(a^+,a) = 1 \quad \text{or equivalently}$$

$$G'(a^-,a) = \frac{j'(a)\,h(a)}{W(a)} = 0$$

Thus either $h(a) = 0$ and the exterior Dirichlet problem is non-unique, or else $h(a) \neq 0$ and $j'(a) = 0$ or the interior Neumann problem is non-unique. If $h(a) = 0$, then

$$G(r,a) = \frac{j(r)\,h(a)}{W(a)} = 0 \quad \text{for} \quad r < a$$

so $\psi = 0$ inside the obstacle. If $j'(a) = 0$ and $h(a) \neq 0$, then the non-trivial solution of 4)

$$G(r,a)$$

is the non-trivial Neumann soln $j(r)$ in the interior pieced together with the appropriate outgoing solution. outside.

January 2, 1978:

Consider obstacle scattering: $\psi = 0$ on boundary. Suppose ω is a function on ∂X satisfying

$$1) \qquad 0 = \varphi(r) + \int_{\partial X} G(r, r')\, \omega(r')\, dS_{r'} \qquad r \in \partial X$$

Then put $\psi(r)$ = right hand side for any r. Clearly ψ satisfies $(\Delta + k^2)\psi$ off ∂X, $\psi = 0$ on ∂X, $\psi - \varphi$ is outgoing. If the interior Dirichlet problem has [redacted] no non-trivial solution, then $\psi = 0$ inside, so using the jump in G we conclude

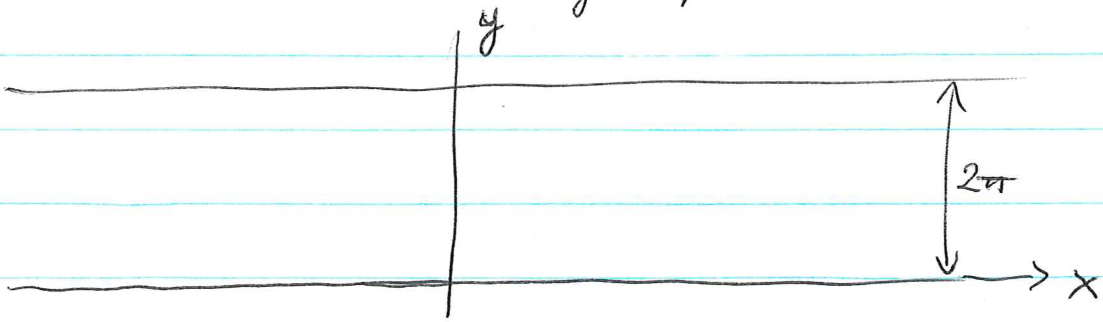
$$\omega(r) = \frac{\partial \psi}{\partial n}(r^+) \qquad \text{on } \partial X.$$

Suppose we can separate the problem into radial equations. Then we want to solve

$$\varphi(a) + G(a,a)\, \omega \; [redacted] = 0$$

where $G(a,a) = \dfrac{j(a)\, h(a)}{W(a)}$. We can do this when $j(a), h(a) \neq 0$ in which case the interior Dirichlet problem is uniquely solvable.

Return to Schwinger problem:



Suppose we consider periodic conditions vertically, with the obstacle given by boundary conditions on $x=0$. Solutions without the obstacle are given by linear combinations of

[redacted]

$$e^{\pm i k_n x} e^{iny} \qquad k_n^2 = k^2 - n^2$$

and the Green's function is

$$G(x,y,x',y') = \sum_{n\in\mathbb{Z}}{}' \frac{e^{ik_n|x-x'|}}{2ik_n} e^{in(y-y')} \frac{1}{2\pi}$$

The boundary conditions will be, that on a certain subset of $x=0$ called the obstacle, we have $\psi = 0$, and on the complementary subset called the aperture we have $\frac{\partial\psi}{\partial x} = 0$. Furthermore we only look for a solution in $x \geq 0$.

The Green's function determines the outgoing waves. Let's try to determine the reflection coefficient. Thus we look for a ψ of the form

$$\psi(x,y) = e^{-ikx} + \sum_{n\in\mathbb{Z}} c_n e^{ik_n x} e^{iny}$$

satisfying the boundary conditions. Then

$$\left.\begin{matrix} c_n \\ 1+c_0 \end{matrix}\right\} = \frac{1}{2\pi}\int_{\text{apert}} \psi(0,y)\, e^{-iny}\, dy \qquad \begin{matrix} n\neq 0 \\ n=0 \end{matrix}$$

Also

$$\frac{\partial \psi}{\partial x}(x,y) = -ik e^{-ikx} + \sum_{n\in\mathbb{Z}} ik_n c_n e^{ik_n x} e^{iny}$$

so

$$\frac{1}{2\pi}\int_{\text{obst}} \frac{\partial \psi}{\partial x}(0,y)\, e^{-iny}\, dy = \begin{cases} -ik + ikc_0 & n=0 \\ ik_n c_n & n\neq 0 \end{cases}$$

Then we get two integral equations as follows. From the second formula for c_n:

$$\psi(x,y) = e^{-ikx} + e^{ikx}\left\{1 + \frac{1}{2\pi}\int \frac{1}{ik}\frac{\partial \psi}{\partial x}(0,y)\,dy\right\}$$

$$+ \sum_{n\neq 0} e^{ik_n x} e^{iny} \frac{1}{2\pi}\int \frac{1}{ik_n} e^{-iny'} \frac{\partial \psi}{\partial x}(0,y')\,dy'$$

$$\psi(x,y) = e^{-ikx} + e^{ikx} + \int_{\text{obst}} \sum_{n\in\mathbb{Z}} \frac{e^{ik_n x}}{ik_n} e^{in(y-y')} \frac{\partial \psi}{\partial x}(0,y')\frac{dy'}{2\pi}$$

Setting $x=0$ you get an integral equation for $\frac{\partial \psi}{\partial x}(0,y')$ on the obstacle:

$$\boxed{0 = 2 + \int_{\text{obst}} \sum_{n\in\mathbb{Z}} \frac{e^{in(y-y')}}{ik_n} \frac{\partial \psi}{\partial x}(0,y')\frac{dy'}{2\pi}}$$

which seems to be the same as

$$0 = \varphi(r) + \int_{\text{obst}} G(r,r')\frac{\partial \psi}{\partial x}(r'+)\,dS_{r'} \qquad \text{on obst.}$$

The other equation comes from the other formula for c_n:

$$\frac{\partial \psi}{\partial x}(x,y) = -ike^{-ikx} - ike^{ikx} + \int_{apert} \sum_n ik_n e^{ik_n x} e^{in(y-y')} \psi(0,y') \frac{dy'}{2\pi}$$

leading to

$$\boxed{0 = -2ik + \int_{apert} \sum_n ik_n e^{in(y-y')} \psi(0,y') \frac{dy'}{2\pi}}$$