Obstacle scattering with Dirichlet boundary condition:

Suppose \( \psi \) defined outside obstacle and

\[
(\Delta + k^2) \psi = 0 \quad \text{outside} \quad \Omega
\]

\( \psi = 0 \) on \( \partial \Omega \)

\( \psi - \psi^0 \) outgoing.

where \( \psi^0 \) is a global soln of \( (\Delta + k^2) \psi = 0 \). Then we saw that for \( \vec{r} \) outside the obstacle \( \chi \)

\[\begin{aligned}
\psi(\vec{r}) &= \psi^0(\vec{r}) + \int G(\vec{r}, \vec{r}') \frac{\partial \psi^0}{\partial n}(\vec{r}') \, dS' \\
\end{aligned}\]

However the right side makes sense inside the obstacle and can be used to define \( \psi \) there. We then have

\[
\frac{\partial \psi}{\partial n}(\vec{r}^+) - \frac{\partial \psi}{\partial n}(\vec{r}^-) = \int \left[ \frac{\partial G}{\partial n}(\vec{r}^+, \vec{r}') - \frac{\partial G}{\partial n}(\vec{r}^-, \vec{r}^-) \right] \frac{\partial \psi^0}{\partial n}(\vec{r}') \, dS'
\]

\[
= \frac{\partial \psi}{\partial n}(\vec{r}^+)
\]

so

\[
\frac{\partial \psi}{\partial n}(\vec{r}^-) = 0 \quad \text{for} \quad \vec{r} \text{ on } \partial \Omega.
\]

But then both \( \psi(\vec{r}) \) and \( \frac{\partial \psi}{\partial n}(\vec{r}) \) are 0 on \( \partial \Omega \) (note \( \psi \) is continuous because of the nature of \( G \)). Thus \( \psi \) has to be zero inside the obstacle. So we see that 1) is equivalent to the integral equation 2) over all space with the additional requirement that \( \psi \) vanish on \( \partial \Omega \).
What I want to understand is the nature of solutions of the integral equation 2). To begin, consider the homogeneous equation where \( \phi = 0 \). A solution with \( \phi = 0 \) on \( \partial X \) or equivalently \( \phi = 0 \) inside is the same as a scattering eigenfunction, i.e. an outgoing solution of \((\Delta + k^2)u = 0\) on the exterior vanishing on \( \partial X \). Such a non-trivial \( \phi \) exists when the exterior Dirichlet problem has a non-trivial solution.

A solution of 2) which is non-zero on \( \partial X \) gives a non-trivial solution of the interior Neumann problem. Conversely, given a non-trivial solution of the interior Neumann problem, combine it with the solution of the exterior Dirichlet problem with the same values on \( \partial X \). Then \( \phi \) satisfies \((\Delta + k^2)\phi = 0\) off \( \partial X \) and on \( \partial X \) we have \( \frac{\partial \chi}{\partial n} = 0 \), so that \( \frac{\partial \chi}{\partial n} (n^+) \) is the jump in the normal derivative along \( \partial X \). Thus it's clear

3) \[
\phi(x) = \int G(x, s) \frac{\partial \chi}{\partial n} (s^+) \, ds
\]
since the difference would be a global outgoing soln. of \((\Delta + k^2)u = 0\).

Thus we get non-trivial solutions of 3) when either the exterior Dirichlet problem has a non-trivial solution, or when it doesn't but the interior Neumann problem has a non-trivial solution.

Review yesterday's analysis when \( \partial X \) is the sphere \( r=a \) and so the problem separates into uncoupled
Let \( j(r) \) be the solution of the radial equation
\[
\left( \frac{1}{r} \frac{d}{dr} r \frac{d}{dr} - \frac{n^2}{r^2} + k^2 \right) u = 0
\]
which is nice at 0, and \( h(r) \) the solution nice (i.e. outgoing) at \( \infty \). Then \( W[j, h](r) \neq 0 \) because we assume for all space there is no non-trivial outgoing soln.

So consider a solution of 3):

4) \( \psi'(a) = \psi'(a^+) \)

Then \( G(a, a) \psi'(a^+) = G(a, a) G'(a^+, a) \psi'(a^+) \) so as \( \psi'(a^+) \neq 0 \) we get
\[
G'(a^+, a) = \frac{1}{1} \quad \text{or equivalently}
\]
\[
G'(a^-, a) = \frac{\psi'(a) h(a)}{W(a)} = 0
\]

Thus either \( h(a) = 0 \) and the exterior Dirichlet problem is non-unique, or else \( h(a) \neq 0 \) and \( \psi'(a) = 0 \) or the interior Neumann problem is non-unique. If \( h(a) = 0 \), then
\[
G(r, a) = \frac{\psi(r) h(a)}{W(a)} = 0 \quad \text{for} \quad r < a
\]
so \( \psi = 0 \) inside the obstacle. If \( \psi'(a) = 0 \) and \( h(a) \neq 0 \), then the non-trivial solution of 4)
\[
G(r, a)
\]
is the non-trivial Neumann soln. \( j(r) \) in the interior pieced together with the appropriate outgoing solution outside.
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Consider obstacle scattering: \( \psi = 0 \) on boundary. Suppose \( \omega \) is a function on \( \partial \Omega \) satisfying

\[
0 = \psi(x) + \int_{\partial \Omega} G(x, x') \psi(x') \, ds', \quad x \in \partial \Omega
\]

Then put \( \psi(x) = \) right hand side for any \( \Omega \). Clearly \( \psi \) satisfies \( (\Delta + k^2)\psi \) off \( \partial \Omega \), \( \psi = 0 \) on \( \partial \Omega \), \( \psi - \psi \) is outgoing. If the interior Dirichlet problem has no non-trivial solution, then \( \psi = 0 \) inside, so using the jump in \( G \) we conclude

\[
\omega(x) = \frac{\partial \psi}{\partial n}(x^+) - \frac{\partial \psi}{\partial n}(x^-) \quad \text{on} \quad \partial \Omega.
\]

Suppose we can separate the problem into radial equations. Then we want to solve

\[
\psi(a) + G(a, a) \omega = 0
\]

where

\[
G(a, a) = \frac{j(a) h(a)}{W(a)}.
\]

We can do this when \( j(a), h(a) \neq 0 \) \( W(a) \) in which case the interior Dirichlet problem is uniquely solvable.

Return to Schwinger problem:
Suppose we consider periodic conditions vertically, with the obstacle given by boundary conditions on \( x = 0 \). Solutions without the obstacle are given by linear combinations of

\[ e^{i k_n x} e^{i n y} \quad k_n^2 = k^2 - n^2 \]

and the Green's function is

\[ G(x,y,x',y') = \sum_{n \in \mathbb{Z}} \frac{e^{i k_n |x-x'|}}{2 i k_n} e^{i n (y-y')} \frac{1}{2\pi} \]

The boundary conditions will be, that on a certain subset of \( x = 0 \) called the obstacle, we have \( y = 0 \), and on the complementary subset called the aperture we have \( \frac{\partial y}{\partial x} = 0 \). Furthermore, we only look for a solution in \( x \geq 0 \).

The Green's function determines the outgoing waves.

Let's try to determine the reflection coefficient. Thus, we look for a \( y \) of the form

\[ y(x,y) = e^{-i k x} + \sum_{n \in \mathbb{Z}} c_n e^{i k_n x} e^{i n y} \]

satisfying the boundary conditions. Then

\[ c_n = \frac{1}{2\pi} \int_{\text{aperture}} y(0,y) e^{-i n y} \, dy \quad n \neq 0 \]

\[ c_0 = \frac{1}{2\pi} \int_{\text{aperture}} y(0,y) \, dy \quad n = 0 \]
Also
\[ \frac{\partial \psi}{\partial x}(x, y) = -i ke^{-ikx} + \sum_{n \in \mathbb{Z}} ik_n c_n e^{ik_n x} e^{-iny} \]

so
\[ \frac{1}{2\pi} \int \frac{\partial \psi}{\partial x}(0, y) e^{-iny} dy = \left\{ \begin{array}{ll} -ik + ik c_0 & n = 0 \\ ik_n c_n & n \neq 0 \end{array} \right. \]

Then we get two integral equations as follows. From the second formula for \( c_n \):

\[ \psi(x, y) = e^{-ikx} + e^{ikx} \left\{ 1 + \frac{1}{2\pi} \int \frac{1}{ik} \frac{\partial \psi}{\partial x}(0, y) dy \right\} \]

\[ + \sum_{n \neq 0} e^{ik_n x} e^{-iny} \frac{1}{2\pi} \int \frac{1}{ik_n} e^{-iny} \frac{\partial \psi}{\partial x}(0, y) dy \]

\[ \psi(x, y) = e^{-ikx} + e^{ikx} + \int \sum_{n \in \mathbb{Z}} \frac{e^{ik_n x} e^{-iny}}{ik_n} \frac{\partial \psi}{\partial x}(0, y) dy \]

Setting \( x = 0 \) you get an integral equation for \( \frac{\partial \psi}{\partial x}(0, y') \) on the obstacle:

\[ 0 = 2 + \int \sum_{n \in \mathbb{Z}} \frac{e^{iny} \frac{\partial \psi}{\partial x}(0, y') dy'}{ik_n} \]

which seems to be the same as

\[ 0 = \varphi(r) + \int G(r, r') \frac{\partial \psi}{\partial x}(r') dS_r \quad \text{on obst.} \]
The other equation comes from the other formula for $c_n$:

$$\frac{\partial \psi}{\partial x}(x, y) = -ik e^{-ikx} + \frac{\sum_{n} ik_n e^{ik_n x} e^{i(n-y')y}}{2\pi} \psi(0, y') \frac{dy'}{2\pi}$$

leading to

$$0 = -2ik + \frac{\sum_{n} ik_n e^{i(n-y')y}}{2\pi} \psi(0, y') \frac{dy'}{2\pi}$$