February 19, 1978

Consider again $\Delta u = \lambda u$ on the modular tree or rather a branch of it.

The calculation done yesterday amounts to looking for a solution depending on the distance from the left vertex, i.e.

$$u = \alpha^d$$

It is, so to speak, a radial solution. What one gets by using the retraction onto an apartment associated to a chamber. The condition that this be a solution is

$$1 + 2x^2 - 3x = \lambda x$$

and for it to be an $L^2$ solution means that

$$1^2 + |x|^2 + 2|x|^4 + 4|x|^6 + \cdots < \infty$$

i.e.

$$2|x|^2 < 1 \text{ or } |x| < \frac{1}{\sqrt{2}}.$$ 

So if we rewrite the above

$$\frac{1}{x} - 3 + 2x = \lambda$$

or

$$\frac{\sqrt{2}}{(\sqrt{2}x)} - 3 + \sqrt{2}(\sqrt{2}x) = \lambda \quad \frac{1}{2}\left(\sqrt{2}x + \frac{1}{\sqrt{2}x}\right) = \frac{3 + 1}{2\sqrt{2}}$$

we see the unit circle $|\sqrt{2}x| < 1$ gets mapped isomorphically onto the region in the $\lambda$ plane which is the complement of the slit $-1 \leq \frac{3 + \lambda}{2\sqrt{2}} \leq 1.$
For each \( \lambda \) outside the slit we can construct the corresponding Green's function with unit source at \( y \):

\[
G(x, y, \lambda) = \text{const.} \frac{d(x, y)}{\lambda}
\]

where the constant \( c \) is chosen so that

\[
(\Delta - \lambda)G(x, y, \lambda) = -\frac{\partial^2}{\partial y^2}, \quad \text{hence}
\]

\[
c(3\alpha - 3 - \lambda) = -1
\]

\[
c = \frac{-1}{3\alpha - (1 + 2\alpha)} = \frac{-1}{\alpha - \frac{1}{2}} = \frac{1}{\frac{1}{2} - \alpha}
\]

Check: Take \( \lambda = 0 \) where \( \alpha = \frac{1}{2} \) and \( c = \frac{1}{2 - \frac{1}{2}} = \frac{2}{3} \).

So

\[
G(x, y, \lambda) = \frac{1}{\frac{1}{2} - \alpha} d(x, y)
\]

This should be the kernel for the operator \((\Delta - \lambda)^{-1}\).

However, at the moment I don’t know if this kernel gives a bounded operator on \( L^2 \), although this seems likely.

Spherical functions: Fix a vertex \( 0 \) and consider functions on the vertices which are radial, i.e. depend only the distance from \( 0 \). Let \( f \) be a radial eigenfunction for \( \Delta \) with eigenvalue \( \lambda \). Then at \( 0 \) we have

\[
\Delta f(0) = 3 f(1) - 3 f(0) = 2 f(0)
\]

and at other values of \( r \) we have

\[
f(r-1) + 2 f(r+1) = 3 f(r) = \lambda f(r)
\]
One sees from these equations that there is a unique radial eigenfunction with given value \( f(0) \) and eigenvalue \( \lambda \). Moreover it has the form

\[
f(r) = c_1 r^{\lambda_1} + c_2 r^{\lambda_2}
\]

where \( \lambda_1, \lambda_2 \) are the roots of the characteristic equation

\[
\lambda^{-1} + 2 \lambda = (\lambda + 3)
\]

it should be

Now, clear that bounded eigenfunctions, in fact polynomial growth eigenfunctions, do not exist except for \( \lambda \) in the cut \([-3 - 2\sqrt{2}, -3 + 2\sqrt{2}]\). For if we had an eigenfunction \( u \neq 0 \), choose the origin to be a point where \( u \neq 0 \), then average over the compact group of automorphisms of the tree preserving \( 0 \) and you get a radial eigenfunction \( f(r) \neq 0 \) with polynomial growth. But there can't be any \(\lambda \) off the cut, but one would have to have \( f(r) = c_2 r^{\lambda_2} \) where \( \lambda_2 \) is the small root (in modulus < \( \frac{1}{2} \)). This would mean that

\[
(\lambda + 3) f(0) = \lambda_2 (\lambda + 3) c_2
\]

and also

\[
(\lambda + 3) f(0) = 3 f(1) = 3 c_2 \lambda_2
\]

so

\[
\lambda_2 = \lambda + 3 \quad \text{and} \quad \lambda_2 = -1
\]

which is impossible.

Consider \( \Delta = y^2 \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) \) on the UHP. We have

\[
y^{-1/2} (\Delta + \frac{1}{4}) y^{1/2} = y^2 \frac{\partial^2}{\partial x^2} + y^2 \left( \frac{\partial}{\partial y} + \frac{1}{y} \right)^2 + \frac{1}{4} = y^2 \frac{\partial^2}{\partial x^2} + \frac{y^2}{2} \frac{\partial^2}{\partial y^2} + y^2 \frac{\partial}{\partial y} + \frac{1}{4}
\]
Thus 
\[ \Delta + \frac{1}{4} = y^2 \frac{\partial^2}{\partial x^2} + y^{1/2} \left( y \frac{\partial^2}{\partial y} \right)^2 y^{-1/2} \]

\[ \left( \Delta + \frac{1}{4} \right) u, v = \int \left( y^2 \frac{\partial^2 u}{\partial x^2} + y^{1/2} \left( y \frac{\partial^2}{\partial y} \right)^2 (y^{-1/2} u) \right) \frac{\partial u}{\partial y} \frac{\partial v}{\partial y} \]

\[ = \int \frac{\partial^2 u}{\partial x^2} u \, dx \, dy + \int \left( y \frac{\partial^2}{\partial y} \right)^2 (y^{-1/2} u) \cdot (y^{-1/2} v) \, dx \, dy \]

If \( u \) has compact support we can integrate by parts to get

\[ = -\int \left( \frac{\partial u}{\partial x} \right)^2 \, dx \, dy - \int \left[ \left( y \frac{\partial}{\partial y} \right)^2 (y^{-1/2} u) \right] \, dx \, dy \]

\[ \leq 0 \]

Consequently 
\[ \Delta + \frac{1}{4} \leq 0. \]

---

Formula for spherical functions in the UHP case:
Recall that in geodesic polar coordinates the Laplacian for the upper half plane is

\[ \Delta = \frac{1}{\sinh(s)} \frac{\partial}{\partial r} \left( \sinh(s) \frac{\partial}{\partial r} \right) + \frac{1}{\sinh^2(s)} \frac{\partial^2}{\partial \theta^2} \]

A radial eigenfunction satisfies the DE

\[ \frac{1}{\sinh(s)} \frac{d}{dr} \left( \sinh(r) \frac{du}{dr} \right) = \lambda u \]

This should be compared with

\[ \frac{1}{r} \frac{d}{dr} \left( r \frac{du}{dr} \right) = \lambda u \]
which is Bessel's DE of order 0 essentially: \[ -\lambda + k^2 r^2 u = 0. \]

The only solution regular at \( r = 0 \) is \( J_0(kr) \) up to scalar factors. The same should be true for the sinh \( r \) case so one sees there is a unique - up-to-scaler-factors radial eigenfunction.

To find it we can start with the eigenfunction \( y^* \) for \( \Delta \) and average it over the rotation group

\[
\left( \frac{\cos \theta \sin \theta}{-\sin \theta \cos \theta} \right)^* y = \frac{y}{|\cos \theta - 2\sin \theta|^2}
\]

hence we are interested in

\[
y^* \int_0^{\pi} |\cos \theta - 2\sin \theta|^{-2s} d\theta
\]

Restrict this to the \( y \) axis and recall \( y = e^r \) on this axis,

\[
y^* \int_0^{\pi} \left[ (\cos \theta)^2 + (\sin \theta)^2 \right]^{-2s} d\theta
\]

\[
= \text{const} \int_0^{\pi} \left[ (1 + \cos 2\theta) + \frac{y(1 - \cos 2\theta)}{2} \right]^{-2s} d\theta
\]

\[
= \text{const} \int_0^{2\pi} \left( \cosh r - \sinh r \cos \theta \right)^{-2s} d\theta
\]

which agrees with what's in Helgason's book except for the exponent \(-2s\).
February 15, 1978

Let the tree, \( K \) finite subtree. One has an exact sequence in real cohomology

\[
\begin{array}{cccc}
H^0(X) & \to & H^0(X-K) & \to & H^1(X, X-K) & \to & H^1(X) \\
\uparrow & & \uparrow & & \uparrow & & \uparrow \\
\mathbb{R} & \to & \text{Map}(\pi_0(x-K), \mathbb{R}) & \to & & & \\
\end{array}
\]

\( H^1(X, X-K) \) is constructed out of 1-cochains on \( X \) with support in the set of 1-simplices not in \( X-K \), i.e. which touch \( K \). Given such a cochain \( \alpha \), since \( X \) is contractible we get \( \alpha = \delta f \) where \( f \in C^0(X) \) is unique up to constants. Since \( \delta f = 0 \) vanishes on \( X-K \), \( f \) is locally constant on \( X-K \), hence constant on each component.

Conversely, given \( f \in H^0(X-K) \subset C^0(X-K) \), we can extend \( f \) to a function \( \tilde{f} \) on the vertices of \( K \), say by \( 0 \) for example. Then \( \alpha = \delta \tilde{f} \) is a 1-cochain with support off \( X-K \) whose class depends only on \( f \). \( \tilde{f} \) is unique up to functions on the vertices of \( K \) and it is natural \( \delta \) in the \( L^2 \)-context to minimize \( \| \delta \tilde{f} \|^2 \). If \( \tilde{f} \) is the minimum then \( (\delta \tilde{f}, \delta g) = 0 \) for all \( g \in C^0(K) \) and so \( \delta^* \delta \tilde{f} = 0 \). Thus we are solving the Dirichlet problem, i.e. finding a function \( \tilde{f} \) on \( K \cup \partial K \), \( \partial K = \text{vertices joined to } K \) by edges, such that \( \Delta \tilde{f} = 0 \) in the interior. Physically we can think of \( f \) on \( \partial K \) as being applied external voltages, and then \( \tilde{f} \) is the resulting voltage internally.

Now we want to let \( K \) expand but keeping \( f \in H^0(X-K) \) fixed. In the limit we should get a harmonic function \( \tilde{f} \) on the vertices such that \( \tilde{f} = f \). But I have already analyzed the voltage...
distribution on a branch of the tree

which tends to zero as we go →, and I found the branch
to have a resistance of 2 ohms. So it's more or less
clear to find the limiting voltage distribution at a point
inside K. If I replace each branch issuing from K by
a resistance of 2 ohms in series with a voltage given
by the value of f on that branch.

In this manner it seems possible to associate to each
locally constant function on the space of ends of X a harmonic
function f on X having f as its boundary values.
In other words we can solve the Dirichlet problem for locally
constant boundary values.

Denote by $C^i_2(X) \subset C^i(X)$ the subspace of $l^2$-cochains.
Assuming $\Delta$ bounded away from zero on $C^0_2(X)$, I know
that $\delta: C^0_2(X) \rightarrow C^1_2(X)$ has a closed range so that
the cohomology

$$H^1_2(X) = \frac{C^1_2(X)}{\delta C^0_2(X)}$$

is a Hilbert space. Here's a simple proof that the
canonical map

$$H^1_c(X) \rightarrow H^1_2(X)$$

is injective. Let $\alpha \in C^1_c(X)$ be an $l^2$-coboundary $\alpha = \delta g$
with $g \in C^0_c(X)$. Then $g$ is constant far out, hence
as it is square integrable, it must be zero far out, hence
$g \in C^0(X)$, and so $\alpha$ represents 0.

Because $C^1_c(X)$ is dense in $C^1(X)$.
February 16, 1978

Yesterday I saw that given a locally constant function $f$ on $\partial X$, one could solve the Dirichlet problem: find a harmonic $\tilde{f}$ on $X$ such that $\tilde{f}$ has the boundary value $f$. This should imply that for each vertex $y$, if there is a Poisson measure $\mu_y$ on $\partial X$ associated to $y$, then, which represents $f \mapsto \tilde{f}(y)$.

By rotational symmetry around $y$, it's clear that since $\mu_y$ is a probability measure (the constant functions are harmonic) that for any $x$ the subset of ends $\{y : d(y,x) = d\}$ on the other side of $x$ from $y$ should have measure $\frac{1}{3} \cdot \frac{2}{2d(y,x)}$

We can prove this by showing that for any harmonic $u$, $u(y) =$ average of $u(x)$ as $x$ runs over the circle $C_d$ of radius $d$. Clear for $d = 1$. Next observe that for $d \geq 2$

$$\sum_{d(x,y) = d+1} u(x) + 2 \sum_{d(x,y) = d-1} u(x) = 3 \sum_{d(x,y) = d} u(x)$$

so that

$$\frac{1}{2} \sum_{d(x,y) = d+1} u(x) - \sum_{d(x,y) = d} u(x) + \sum_{d(x,y) = d-1} u(x) - \frac{1}{2} \sum_{d(x,y) = d} u(x) = 0$$

zero by induction
by induction if we know the averages over \( C_{d-1}, S_d \) are the same we can get the average of \( C_{d+1} \) to be the same, this for \( d \geq 2 \). Finally you should check for \( d = 1 \).

\[-3 \sum_{c_1} u + \sum_{c_2} u + 3 \sum_{c_3} u = 0 \quad \Rightarrow \quad \frac{\sum_{c_2} u}{c_2} = 6 u(0) \quad \text{OK.}\]

Finally let us fix an origin \( O \), whence we get a definite measure \( d\mu_0 \) on \( \partial X \), and then compute the other Poisson measures \( d\mu_y \) in terms of \( d\mu_0 \) and a function on \( \partial X \).

Let \( J \) be an end and \( 0 \) an origin. It makes sense to talk about vertices equally distant from \( J \).

\[
\begin{align*}
\text{In want we can define a function } \\
h(x, y) &= \lim_{z \to J} d(x, z) - d(y, z)
\end{align*}
\]

Then you get the following picture for vertices equally distant from \( J \).
Moreover we get a harmonic function:

$$\eta(y) = \frac{1}{2} \int \frac{d\gamma}{\gamma(x,0)} = \lim_{z \to y} \frac{2}{2 \pi i} \frac{d\gamma(x,\bar{z})}{d(x,\bar{z})}$$

The choice of origin suffices to normalize \( \eta_0(x) \) to be 1 at \( x = 0 \).

**Classical version:** The harmonic function \( y = \text{Im}(z) \) in the UHP when considered on the disk \( |w| < 1 \) via the transformation

$$w = \frac{2-i}{z+i} = \left( \frac{1-i}{1+i} \right) (z) \quad z = \left( \frac{1+i}{-1+i} \right) \left( \frac{w-i}{w+i} \right)$$

becomes

$$\text{Im} \left( i \frac{1+w}{1-w} \right) = \text{Re} \left( \frac{1+w}{1-w} \right) = \frac{1-|w|^2}{|1-w|^2}$$

This blows up at \( w = 1 \) but vanishes at other points of \( |w| = 1 \). Rotated so that the singularity occurs at \( z \) it becomes

$$\frac{1-|w|^2}{|z-w|^2}$$

The Poisson measure on \( |w| = 1 \) belonging to \( w = 0 \) in

$$\frac{d\gamma}{2\pi i a} = \frac{d\omega}{2\pi i w} \quad \omega - \frac{b}{a}$$

becomes

$$\omega = \frac{1}{2\pi i (aw+b)(bw+a)} \frac{d\omega}{d\Theta} = \frac{1}{2\pi i (ae^{-i\Theta} + b)(ae^{i\Theta} + a)}$$

$$\frac{1}{a^2 - |b|^2} = \frac{1}{2\pi} \frac{d\Theta}{|ae^{i\Theta} + b|^2} = \frac{d\Theta}{2\pi} \frac{(1-|b|^2)}{|e^{i\Theta} + \frac{b}{a}|^2}$$
Hence the Poisson measure belonging to the point \( w = w_0 = -\frac{b}{a} \) is

\[
\frac{1 - |w|^2}{|w_0|^2 - 2i} \int_{\mathbb{C}} \frac{d\Theta}{|e^{i\Theta} - w|^2}
\]

so it maybe it's clear that the Poisson kernel for the tree \( X \) with origin \( O \) is

\[
\frac{1}{2 \rho_1(\rho_0)} \, d\mu_0(\rho).
\]

There seems to be an interesting inner product on \( H_c^1(X) \), namely the one obtained from the embedding

\[
H_c^1(X) \hookrightarrow H_2^1(X)
\]

Note that because \( C_c^1 \) is dense in \( C_c^1(X) \), it follows the the above embedding is dense. The inner product can be described as follows. Given \( f \) locally constant on \( \partial X \), let \( \tilde{f} \) be its harmonic extension. Then \( \|f\|_2^2 = \|\tilde{f}\|_2^2 \). Put another way, you replace an \( L^2 \)-cochain with compact support by its harmonic equivalent and you take the norm.

Return to \( \Delta \) on UHP. Using the Dirichlet problem we can identify smooth functions on \( S^1 \) with harmonic functions in the disk. Mod-ifying by constants this should be an irreducible representation of \( \text{PSL}_2(\mathbb{R}) \). Hence there should be a unique invariant inner product up to scalars.
Because we've seen that \( \Delta \leq -\frac{1}{4} \) on \( L^2(\text{UHP}) \) the irreducible representation of functions on \( S^1 \) described above does not occur as an eigenspace of \( \Delta \). In other words if \( L^2(\text{UHP}) \) is written as an integral of irreducible representations then only \( \lambda = s(s-1) \) occurs for \( \lambda \leq -\frac{1}{4} \), i.e. \( s \in \frac{1}{2} + i\mathbb{R} \). It seems that the space of harmonic functions on the disk with Dirichlet norm and constant is the irreducible representation belonging to \( s = 1 \), whereas to \( s = 0 \) belongs the trivial representation.

February 17, 1978

**Dirichlet norm:** On a Riemann surface let \( \omega \) be a 1-form. Then we get a 1-form \( \overline{\omega} \) by conjugating and a 2-form \( \omega \wedge \overline{\omega} \) which can be integrated to get a number. For example if \( \omega = f \, dz \) is of type \((1,0)\) then \( \overline{\omega} = \overline{f} \, d\overline{z} \) and

\[
\omega \wedge \overline{\omega} = |f|^2 \, dz \wedge d\overline{z} = |f|^2 \left[ dxdy + i \, dx \wedge dy \right]
\]

\[
= |f|^2 (-2i) \, dx \, dy
\]

so \( \frac{i}{2} \omega \wedge \overline{\omega} \) is an intrinsic norm for forms of type \((1,0)\).

Take \( \omega = \frac{\partial f}{\partial z} \). Then

\[
\left\| \frac{\partial f}{\partial z} \right\|^2 = \frac{1}{2} \left( \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} \right)^2 = \frac{1}{4} \left( |\frac{\partial f}{\partial x}|^2 + |\frac{\partial f}{\partial y}|^2 + i \frac{\partial f}{\partial x} \frac{\partial f}{\partial y} - i \frac{\partial f}{\partial y} \frac{\partial f}{\partial x} \right)
\]

\[
\left\| \frac{\partial f}{\partial \overline{z}} \right\|^2 = \frac{1}{4}
\]
Consequently for a function \( f \) on the Riemann surface the Dirichlet integral
\[
\iint \left( \left| \frac{\partial f}{\partial x} \right|^2 + \left| \frac{\partial f}{\partial y} \right|^2 \right) \, dx \, dy
\]
is invariant for biholomorphic maps, because it can be expressed in terms of
\[
\int \bar{f} \, df + \int \overline{df} \, f
\]
for now I consider all smooth functions on \( S' \) as a space on which \( G = \text{PSL}_2(\mathbb{R}) \) acts. Then to each \( f \) on \( S' \) we can associate the unique harmonic extension \( \tilde{f} \) of \( f \) to the disk and take its Dirichlet norm. In this way one gets an inner product which is \( G \)-invariant on the space of smooth functions on the circle modulo constants.

Notice that on \( S' \) functions and densities (i.e., 1-forms as the orientation is preserved) are dual. Hence the inner product maybe gives a method of associating to \( f \), a density \( dg \) of measure 0. It seems what I should look for is an operator on the space of smooth functions mod constants (perhaps the Hilbert transform) such that the norm I am after is
\[
\| f \|^2 = \iint_{S'} f \, d\bar{f}.
\]

Suppose \( u \) harmonic in the UHP with finite Dirichlet integral and that \( u \) is real-valued.
\[
\iint \left( \left( \frac{\partial u}{\partial x} \right)^2 + \left( \frac{\partial u}{\partial y} \right)^2 \right) \, dx \, dy = \iint \left( \frac{\partial u}{\partial x} \frac{\partial u}{\partial y} + \frac{1}{y} \right) \, dx \, dy = \iint u' \, du \, dx \, dy
\]
\[
\int_{x=-\infty}^{\infty} u \frac{\partial u}{\partial x} \, dy - u \frac{\partial u}{\partial y} \, dx = -\int_{-\infty}^{\infty} u^2 \frac{\partial u}{\partial y} \, dx
\]

Now if \( v \) is a conjugate harmonic function to \( u \), i.e., \( u+iv \) is analytic, then Cauchy-Riemann equations give

\[
\frac{\partial u}{\partial y} = \frac{\partial v}{\partial x}
\]

so we get

\[
\int \left\{ \left( \frac{\partial u}{\partial x} \right)^2 + \left( \frac{\partial u}{\partial y} \right)^2 \right\} \, dx \, dy = \int_{-\infty}^{\infty} u \, dv
\]

where \( v \) is the conjugate harmonic function to \( u \).

Let's understand the operator \( T: u \to v \) on \( S^1 \).

If \( f \) is analytic, with \( f = u + iv \), \( u, v \) real

\[
f(z) = \sum_{n=0}^{\infty} a_n z^n \quad \bar{f}(z) = \sum_{n=0}^{\infty} \bar{a}_n z^{-n} \quad \text{on} \ S^1
\]

then

\[
u = \frac{f + \bar{f}}{2} = \sum_{n=0}^{\infty} \frac{a_n}{2} z^n + \sum_{n=0}^{\infty} \frac{\bar{a}_n}{2} z^{-n}
\]

\[
v = \frac{f - \bar{f}}{2i} = \sum_{n=0}^{\infty} \frac{a_n}{2} z^n + \sum_{n=0}^{\infty} i \frac{\bar{a}_n}{2} z^{-n}
\]

Hence in general given

\[
u(z) = \sum_{n \in \mathbb{Z}} c_n z^n
\]

then

\[
T(u) = \sum_{n=0}^{\infty} \frac{1}{2} c_n z^n + \sum_{n<0} i c_n z^n
\]
which is well-defined modulo constants. Also it makes sense even when \( u \) is not real. Clearly \( T \) is a unitary operator in \( L^2(S^1, \frac{d\theta}{2\pi})/c \) with
\[
T^2 = -1.
\]

Note that because \( T^2 = -1 \), \( T \) will be unitary with respect to any inner product for which its \( +i \) and \(-i\) eigenspaces are orthogonal.

Real line version. Let \( u(x) \) be a smooth function of rapid descent on \( \mathbb{R} \) and \( \hat{u}(\xi) \) its Fourier transform:
\[
u(x) = \int_{\mathbb{R}} e^{ix\xi} \hat{u}(\xi) \frac{d\xi}{2\pi}
\]
Define its Hilbert transform by
\[
Tu(x) = \int_{\mathbb{R}} e^{ix\xi} \hat{u}(\xi) \frac{d\xi}{2\pi} - \frac{1}{i} \int_{\mathbb{R}} e^{ix\xi} \hat{u}(\xi) \frac{d\xi}{2\pi}
\]
Note that
\[
u(x) + iTu(x) = 2\int_{0}^{\infty} e^{ix\xi} \hat{u}(\xi) \frac{d\xi}{2\pi}
\]
is analytic for \( \text{Re}(\xi) > 0 \) and it decays exponentially as \( \text{Im}(\xi) \to \infty \). \( T \) is a singular integral operator of order 0.

Formulas for the Hilbert transform: First we want the formulas giving \( f = u + iTu \). On the circle
\[
u = \sum_{n \in \mathbb{Z}} c_n e^{in\theta} \quad c_{-n} = \overline{c}_n \quad \text{if } u \text{ real}
\]
\[
f = e^{i\theta} \sum_{n \geq 1} 2c_n e^{in\theta} = \int_{0}^{2\pi} u(e^{i\theta}) \left( 1 + 2 \sum_{n \geq 1} e^{-in\theta} \right) e^{i\theta} \frac{d\theta}{2\pi} = \int_{0}^{2\pi} u(e^{i\theta}) \frac{1 + e^{-i2\theta}}{1 - e^{-i2\theta}} \frac{d\theta}{2\pi}
\]
On the line
\[ f(z) = \int_0^{\infty} e^{i\theta} \left( \int_{-\infty}^{\infty} e^{-ix^2} u(x) \, dx \right) \frac{d\theta}{2\pi} \]
\[ = \int_{-\infty}^{\infty} \hat{u}(x) \left\{ \int_{-\infty}^{\infty} e^{i(x-z)^2} \frac{dx}{2\pi} \right\} \, dx \quad \text{for } \text{Im} \, z > 0 \]
\[ = \int_{-\infty}^{\infty} \hat{u}(x) \left\{ \frac{i}{\pi} \frac{e^{-1/4}}{z-x^2} \right\} \, dx = \frac{i}{\pi} \int_{-\infty}^{\infty} \frac{u(x) \, dx}{z-x^2} \]

Check: assuming \( u \) real
\[ \text{Re} \, f = \frac{1}{\pi} \int_{-\infty}^{\infty} u(x) \, dx \left( \frac{\text{Re} \left( \frac{1}{z-x^2} \right)}{|z-x^2|^2} \right) = \frac{1}{\pi} \int_{-\infty}^{\infty} u(x) \frac{\frac{z-x}{(z-x)^2+y^2}}{(z-x)^2+y^2} \, dx \]
\[ \text{Im} \, f = \frac{1}{\pi} \int_{-\infty}^{\infty} u(x) \frac{\frac{x-z}{(z-x)^2+y^2}}{(z-x)^2+y^2} \, dx \]

Thus
\[ Tu(x) = \lim_{y \to 0^+} \frac{1}{\pi} \int_{-\infty}^{\infty} u(x) \frac{x-z}{(z-x)^2+y^2} \, dx \]
\[ = \frac{1}{\pi} P \int_{-\infty}^{\infty} \frac{u(x) \, dx}{x-z} \]

where \( P \) denotes the Cauchy principal value defined as follows: No problem with the definition if \( u(x) = 0 \) for then \( u(x) \) is divisible by \( x-x \). For constant functions you define the \( P \) value to be zero. The good definition for differentiable \( u \) is
\[ \lim_{\varepsilon \to 0} \int_{-\infty}^{\infty} + \int_{\varepsilon}^{\infty} \]

Be careful: \( f(z) \) is defined only for \( \text{Im} \, z > 0 \) but will have an analytic continuation if \( u \) is analytic. The integral
\[
\frac{1}{\pi} \int \frac{u(z)dz}{z-\lambda}
\]

will not represent this analytic continuations. Let us take \( f \) to be what is given by the integral. Then we have \( \bar{f}(\lambda) = -f(\lambda) \) so that on approaching the \( x \)-axis we have

\[
f^+(x) = u(x) + iTu(x)
\]

\[
f^-(x) = -u(x) + iTu(x)
\]

Also we have

\[
\frac{\partial}{\partial x} (z) = f'(z) = \frac{1}{\pi i} \int \frac{u(x)dx}{(z-x)^2}
\]

and

\[
\frac{\partial f^+}{\partial x} = \frac{\partial u}{\partial x} + i \frac{\partial Tu}{\partial x}
\]

\[
\frac{\partial f^-}{\partial x} = -\frac{\partial u}{\partial x} + i \frac{\partial Tu}{\partial x}
\]

Since

\[
\int_{-\infty}^{\infty} \frac{\partial u}{\partial x} \, dx = \int_{-\infty}^{\infty} \frac{\partial (\frac{u^2}{z})}{\partial x} \, dx = 0
\]

it follows that we can evaluate from either side:

\[
\int_{-\infty}^{\infty} u \frac{\partial}{\partial x} (Tu) \, dx = \int_{-\infty}^{\infty} u \, dx \frac{\partial f^+}{\partial x} \bigg|_i
\]

\[
= \lim_{\gamma \to 0} \int_{-\infty}^{\infty} u(x) \, dx \int_{-\infty}^{\infty} \frac{u(\lambda) \, d\lambda}{(x+iy-\lambda)^2} \left( -\frac{1}{\pi} \right)
\]

\[
= -\frac{1}{\pi} \int_{-\infty}^{\infty} u(x)u(\lambda) \, dx \, d\lambda
\]

\[
- \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{u(x)u(\lambda)}{(x-\lambda)^2}
\]
Circle approach. Different approaches to the Dirichlet norm: On a Riemann surface define \( * \) on the real cotangent bundle to be the transpose of multiplication by \( \frac{1}{i} \) on the tangent bundle. Thus
\[
*dx = d\Re \frac{1}{z} z = dy
\]
\[
*dy = d\Im \frac{1}{z} z = -dx
\]
Extend \( * \) conjugate linearly to complex 1-forms. Then
\[
du = \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy
\]
\[
*du = \frac{\partial u}{\partial x} dy - \frac{\partial u}{\partial y} dx
\]
\[
du * du = \left( \left| \frac{\partial u}{\partial x} \right|^2 + \left| \frac{\partial u}{\partial y} \right|^2 \right) dx dy
\]
and so \( \iint du * du \) is the Dirichlet norm. Let's use this in polar coordinates:
\[
*dr = r d\theta
\]
\[
*rd\theta = -dr
\]
\[
du = \frac{\partial u}{\partial r} dr + \frac{1}{r} \frac{\partial u}{\partial \theta} r d\theta
\]
\[
*du = \frac{\partial u}{\partial r} r d\theta + \frac{1}{r} \frac{\partial u}{\partial \theta} (-dr)
\]
\[
\iint du * du = \iint \left( \left| \frac{\partial u}{\partial r} \right|^2 + \frac{1}{r^2} \left| \frac{\partial u}{\partial \theta} \right|^2 \right) r dr d\theta
\]
Suppose we're given a harmonic function (say real-valued)
\[
u = \sum_{n \neq 0} \lambda^n (e_n e^{i\theta} + \overline{e_n} e^{-i\theta}) + \text{const}
\]
The conjugate harmonic function is

\[ v = \sum_{n > 0} n^n (-i c_n e^{in\theta} + i \overline{c}_n e^{-in\theta}) + \text{real} \]

I expect the Dirichlet norm of \( u \) to be

\[ \iint \nabla u \cdot \nabla u \, dV = \iint (\nabla (u \nabla u) - u \Delta u) \, dV \]

\[ = \oint u \nabla u \cdot n \, ds \quad \text{n outward normal} \]

\[ = \int_0^{2\pi} u \frac{\partial u}{\partial r} \, d\theta \quad \text{Cauchy-Riemann:} \]

\[ \frac{\partial u}{\partial \theta} = \frac{1}{i} \frac{\partial v}{\partial r} \]

\[ = \int_0^{2\pi} u \frac{\partial v}{\partial \theta} \, d\theta \]

\[ \frac{\partial v}{\partial \theta} = \sum_{n > 0} n^n \left( n c_n e^{in\theta} + n \overline{c}_n e^{-in\theta} \right) \]

Hence

\[ \int_0^{2\pi} u \frac{\partial v}{\partial \theta} \, d\theta = 4\pi \sum_{n > 0} n |c_n|^2 = \text{Dirichlet norm of } u. \]

Also

\[ \frac{\partial u}{\partial r} = \sum_{n > 0} n^n \left( n c_n e^{in\theta} + n \overline{c}_n e^{-in\theta} \right) \]

\[ \int_0^l dr \int_0^{2\pi} \left( \frac{\partial u}{\partial r} \right)^2 \, d\theta \]

\[ = \int_0^l dr \cdot 2\pi \cdot n^2 n^{2n-2} \left( |c_n|^2 + |\overline{c}_n|^2 \right) \]

\[ = 4\pi n^2 |c_n|^2 \int_0^l r^{2n-1} \, dr = 2\pi n |c_n|^2 \]

\[ \frac{\partial u}{\partial \theta} = \sum_{n > 0} n^n \left( nc_n e^{in\theta} - in \overline{c}_n e^{-in\theta} \right) \]

\[ \int_0^l dr \int_0^{2\pi} \left( \frac{\partial u}{\partial \theta} \right)^2 \, d\theta \]

\[ = \int_0^l dr \sum_{n > 0} \frac{n^{2n} 2n^2 |c_n|^2}{2\pi} = 2\pi n |c_n|^2 \]

So we get again \[ 4\pi \sum_{n > 0} n |c_n|^2 \] for \( \iint \text{div}(u) \, dV \).
February 18, 1978:

If \( u = \sum c_n e^{i\theta} \) is a function on \( S^1 \), then its unique harmonic extension to the disk is \( u = \sum c_n r^n e^{i\theta} \).

\[
\frac{\partial u}{\partial r} = \sum_n n^2 c_n r^{n-1} e^{i\theta} \quad \frac{\partial u}{\partial \theta} = \sum_n c_n r^n e^{i\theta}
\]

\[
\int_0^{2\pi} \int_0^1 \left[ \left( \frac{\partial u}{\partial r} \right)^2 + \frac{1}{r^2} \left( \frac{\partial u}{\partial \theta} \right)^2 \right] d\theta = 2\pi \int_0^1 \left\{ \sum_n n^2 r^{n-2} |c_n|^2 + \sum_n |c_n|^2 r^{2n-2} \right\}
\]

\[
= 2\pi \sum_{n \in \mathbb{Z}} |n| |c_n|^2
\]

Write this in terms of a kernel on \( S^1 \).

\[
\int d\theta_1 d\theta_2 \sum_n c_n e^{i\theta_1} \sum_n \overline{c}_n e^{-i\theta_2} \sum_n |n| e^{i(n\theta_2 - \theta_1)} \frac{1}{2\pi}
\]

so the kernel is the distribution

\[
\frac{1}{2\pi} \sum_n |n| e^{i(n\theta_2 - \theta_1)}
\]

Now

\[
\sum_{n>0} n e^{i\theta} = \frac{d}{dz} \sum_{n=0}^\infty z^n = \frac{d}{dz} \frac{1}{1-z} = \frac{z}{(1-z)^2}
\]

\[
= \frac{e^{i\theta}}{(1-e^{i\theta})^2} = \frac{1}{(e^{i\theta} - e^{-i\theta})^2} = \frac{1}{-4 \sin^2 \frac{\theta}{2}}
\]

for \( \text{Im} \theta > 0 \). Hence it seems that

\[
\sum_n |n| e^{i\theta} = \frac{1}{-2 \sin^2 \frac{\theta}{2}} \quad \text{for } \theta \neq 0
\]

At \( \theta = 0 \) it is a distribution of some sort. Notice the same peculiar negative which means the zero part is very positive.
Remark: There is a nice description of the ends of the modular tree. Suppose you define Dedekind cut (incorrectly) as a partition of the rational numbers: $R = A + Q - A$ such that every member of $A$ is less than every member of $Q - A$. Then each rational number $r$ defines two of these cuts depending on whether $r \in A$ or $r \in Q - A$, and an irrational number determines exactly one such cut. Such a cut can be identified with an end of the modular tree. Note that at $\infty$ belong the two cuts $A = \varnothing$ and $A = Q$ which correspond to the 2 ways of getting to $\infty$.

Notice also that if an end is fixed then the other ends form a linearly ordered set. Moreover if the 2 ends belonging to $\infty$ are removed, then $\mathbb{Z} = \langle T \rangle$ acts nicely on the rest of the ends.