February 1, 1978

$$\frac{\partial^2 \tilde{u}}{\partial t^2} = \Delta \tilde{u} = y^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\tilde{u} \qquad \text{in the UHP.}$$

Assume

$$\tilde{u}(t,x,y) = \iint e^{i\omega t} e^{i\xi x} u(\omega,\xi,y) \frac{d\omega\, d\xi}{(2\pi)^2}$$

you get

$$-\omega^2 u = y^2\left(-\xi^2 + \frac{d^2}{dy^2}\right)u$$

(*) $$\left(y^2\frac{d^2}{dy^2} + \omega^2 - \xi^2 y^2\right)u = 0 \qquad \text{Put } u = y^{1/2} v$$

$$\Big[\underbrace{y^2\left(\frac{d}{dy} + \frac{1/2}{y}\right)^2}\, + \omega^2 - \xi^2 y^2\Big] v = 0$$

$$y^2\left(\frac{d^2}{dy^2} + \frac{1}{y}\frac{d}{dy} + \frac{1/4 - 1/2}{y^2}\right) = y^2\frac{d^2}{dy^2} + y\frac{d}{dy} - \frac{1}{4}$$

$$\left[\left(y\frac{d}{dy}\right)^2 + \omega^2 - \frac{1}{4} - \xi^2 y^2\right] v = 0$$

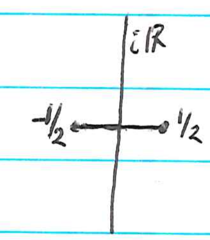
Recall $K_s(r)$ satisfies

$$-\left(r\frac{d}{dr}\right)^2 u + r^2 u = -s^2 u$$

$$\left[\left(r\frac{d}{dr}\right)^2 - s^2 - r^2\right] K_s(r) = 0$$

$$\left[\left(y\frac{d}{dy}\right)^2 - s^2 - \xi^2 y^2\right] K_s(\xi y) = 0$$

Note: $\omega \in \mathbb{R} \Leftrightarrow s \in$



$$-s^2 = \omega^2 - \frac{1}{4} \qquad s = \pm\sqrt{\frac{1}{4} - \omega^2}$$

So a solution of (*) decaying as $y \to +\infty$ is prop. to

$$u(\omega,\xi,y) = y^{1/2} K_{\sqrt{\frac{1}{4}-\omega^2}}(\xi y)$$

If ξ [redacted] $= 0$, then two independent solutions of (*) are

$$y^{\frac{1}{2} \pm \sqrt{\frac{1}{4} - \omega^2}}$$

With the idea of understanding scattering for the automorphic wave equation, we should examine the wave equation in the upper half-plane with the requirement of periodicity in x: $x \longmapsto x+1$.

First look at the Euclidean case, i.e. the wave equation in the cylinder $\mathbb{C}/\{z \sim z+1\}$.

$$\tilde{u}(t,x,y) = e^{i\omega t} e^{i\xi x} u(\omega, \xi, y)$$

Periodicity $\iff$ $e^{i\xi} = 1$ or $\xi \in 2\pi\mathbb{Z}$.

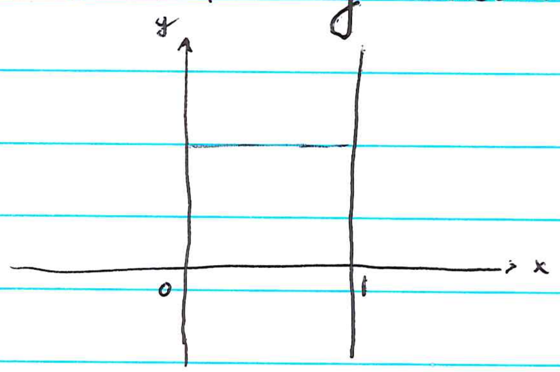
$$-\omega^2 u = \left(-\xi^2 + \frac{d^2}{dy^2}\right) u$$

$$\left[\frac{d^2}{dy^2} + (\omega^2 - \xi^2)\right] u = 0$$

which has the independent solutions $e^{\pm i\eta y}$ where

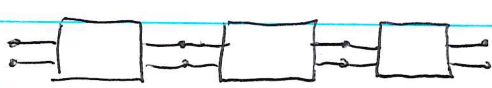
$$\eta^2 = \omega^2 - \xi^2$$

If you fix ξ, that is, you fix the x behavior of your solutions, then you have something like a transmission line in the y direction.



For each fixed t, y your states are the two-dimensional space spanned by $e^{i\xi x}$, $e^{-i\xi x}$. The propagation then takes place as in a transmission line in some sense to be made clear!

Example: Take an iterated 2-port with transfer matrix A:



$$\begin{pmatrix} iV_0 \\ I_0 \end{pmatrix} = A \begin{pmatrix} iV_1 \\ I_1 \end{pmatrix} = A^n \begin{pmatrix} iV_n \\ I_n \end{pmatrix}$$

and fix the frequency ω. There are 2 main possibilities:

1) $|\mathrm{tr} A| < 2$ hence the eigenvalues of A are $e^{i\theta}, e^{-i\theta}$. Then we have a state with

$$\begin{pmatrix} iV_n \\ I_n \end{pmatrix} = e^{-in\theta} \begin{pmatrix} iV_0 \\ I_0 \end{pmatrix}$$

and its conjugate. The time-dependent version is

$$\begin{pmatrix} i\,v(n,t) \\ i(n,t) \end{pmatrix} = e^{i\omega t} e^{-in\theta} \begin{pmatrix} iV_0 \\ I_0 \end{pmatrix}$$

$\omega t - x\theta = \text{const}$

$x = \frac{\omega}{\theta} t + \text{const}$

which is a wave travelling to the right with speed

$$\frac{\omega}{\theta}.$$

Notice that since the matrix A depends on ω, θ is a function of ω, hence wave speed is a function of frequency — this is the phenomenon of <u>dispersion</u>.

2) $|\mathrm{tr} A| > 2$. In this case signals of frequency ω attenuate, and the characteristic impedance of the half-line is defined.

Return to the case of the cylinder. If ξ is fixed, and ω is the frequency, then the possible modes with x dependence $e^{i\xi x}$ are

$$e^{i\omega t} e^{i\xi x} e^{\pm i\eta y}$$

where η satisfies the equation

$$\eta^2 = \omega^2 - \xi^2$$

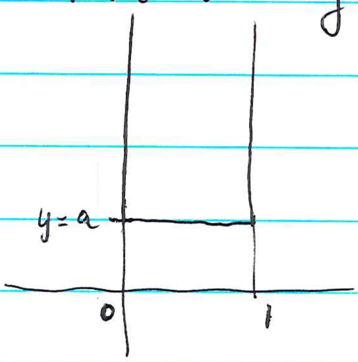
If $|\omega| < |\xi|$ one has attenuation, and if $|\omega| > |\xi|$ one has travelling waves with speed

$$\frac{\omega}{\sqrt{\omega^2 - \xi^2}} .$$

Return now the the upper half plane case: We have seen that solutions of $\frac{\partial^2 \tilde{u}}{\partial t^2} = \Delta \tilde{u}$ of frequency ω, x dependence $e^{i\xi x}$ are [redacted] proportional to

$$e^{i\omega t} e^{i\xi x} y^{1/2} K_{\sqrt{\frac{1}{4} - \omega^2}}(|\xi| y) \qquad \xi \neq 0$$

+ a similar function with K replaced by I. The other term blows up as $y \to +\infty$. Suppose we look for solutions of the wave equation periodic of period 1 in x which vanish on $y = a$. [redacted] Periodicity forces $\xi \in 2\pi\mathbb{Z}$, and vanishing on $y = a$ forces



$$K_{\sqrt{\frac{1}{4} - \omega^2}}(|\xi| a) = 0$$

which implies $\sqrt{\frac{1}{4} - \omega^2}$ is purely imaginary of the form $i \lambda_k([redacted]|\xi| a)$, whence

$$\omega^2 = \frac{1}{4} + \lambda_k(|\xi| a)^2 \qquad k = 0, 1, 2, \ldots$$

It seems therefore that the spectrum of Δ on this Fadeev half-cylinder is discrete. Maybe the [redacted] boundary condition of vanishing on $y = a$ is [redacted] too strong.

$$K_{\frac{1}{2}}(r) = \int_0^\infty e^{-\frac{r}{2}(t + t^{-1})} \frac{(t^{1/2} + t^{-1/2})}{2} \frac{dt}{t} \qquad u = t^{1/2} - t^{-1/2}$$

$$u^2 = t + t^{-1} - 2$$

$$e^r K_{\frac{1}{2}}(r) = \int_0^\infty e^{-\frac{r}{2}(t - 2 + t^{-1})} (t^{-1/2} + t^{-3/2}) \frac{dt}{2} \qquad du = \left(\frac{1}{2} t^{-1/2} + \frac{1}{2} t^{-3/2}\right) dt$$

$$e^{r} K_{\frac{1}{2}}(r) = \int_0^\infty e^{-\frac{r}{2}u^2}\,du = \sqrt{\frac{2\pi}{r}} \qquad \therefore \boxed{K_{\frac{1}{2}}(r) = \sqrt{2\pi}\,\frac{e^{-r}}{\sqrt{r}}}$$

Check: $\omega = 0$ $\quad e^{i\xi x} y^{1/2} K_{\frac{1}{2}}(\xi y) = \sqrt{2\pi}\,\frac{1}{\sqrt{|\xi|}}\, e^{i\xi x - \xi y}$ satisfies $\Delta u = 0$. Yes since $i\xi x - \xi y = i\xi z$.

Here's a nice paradox: Calculate the non-eucl. distance between i and iy

$$\text{dist} = \int_1^y \frac{d\hat{y}}{\hat{y}} = \ln y$$

whence $t \mapsto e^t i$ is the geodesic heading vertically parameterized by arclength. Transfer this to the circle and you get

$$t \longmapsto \frac{e^t i - i}{e^t i + i} = \frac{e^t - 1}{e^t + 1} = \tanh\left(\frac{t}{2}\right)$$

In other words $x = \tanh\left(\frac{t}{2}\right)$ has non-euclidean distance t from 0. But calculate this [redacted] distance:

$$d = \int_0^x \frac{d\hat{x}}{1-\hat{x}^2} = \int_0^x \left(\frac{1}{1-\hat{x}} + \frac{1}{1+\hat{x}}\right)\frac{d\hat{x}}{2} = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)$$

or $\quad e^{2d} = \dfrac{1+x}{1-x} = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}(x) \quad$ so

$$x = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} e^{2d} = \frac{e^{2d}-1}{e^{2d}+1} = \tanh(d)$$

has distance d from 0. The explanation of the paradox is that the map from the [redacted] UHP to the disk

$$z \longmapsto \begin{pmatrix} 1 & -i \\ 1 & +i \end{pmatrix}(z) = w$$

has $\quad \dfrac{dw}{dz} = $ [redacted] $\dfrac{(z+i)1 - (z-i)1}{(z+i)^2} = \dfrac{2i}{(z+i)^2}$

$$= \frac{1}{2i} \quad \text{when } z = i$$

hence the formula for the metric in the circle should be

$$\|dz\| = \frac{2|dz|}{1-|z|^2} \quad \text{or} \quad ds^2 = \frac{4(dx^2+dy^2)}{(1-|z|^2)^2}$$

$$dV = \frac{4\,dx\,dy}{(1-|z|^2)^2}$$

Let ρ be the distance (non-euclidean) from the center of the disk so that

$$\rho = \int_0^r \frac{2dr}{1-r^2} = \ln\frac{1+r}{1-r} \quad \text{or} \quad r = \frac{e^\rho-1}{e^\rho+1} = \tanh\left(\frac{\rho}{2}\right)$$

Then (ρ,θ) represent geodesic polar coordinates, and we can express the metric, Laplacian in these coordinates.

$$ds^2 = \frac{4(dr^2+r^2d\theta^2)}{(1-r^2)^2} \qquad d\rho = \frac{2dr}{1-r^2}$$

$$\frac{2r}{1-r^2} = \frac{2\tanh\frac{\rho}{2}}{1-\tanh^2\frac{\rho}{2}} = \frac{2\sinh\frac{\rho}{2}\cosh\frac{\rho}{2}}{\cosh^2 - \sinh^2} = \sinh\rho$$

Thus

$$\boxed{ds^2 = d\rho^2 + \sinh^2\rho \cdot d\theta^2} \qquad dV = \sinh\rho\, d\rho\, d\theta$$

It follows that the circle of radius (non-eucl.) a has circumference $\int_0^{2\pi} \sinh a\, d\theta = 2\pi \sinh a$.

An orthonormal frame is

$$\frac{\partial}{\partial\rho} \qquad \frac{1}{\sinh\rho}\frac{\partial}{\partial\theta}$$

minus their adjoints are

$$\frac{1}{\sinh\rho}\frac{\partial}{\partial\rho}\sinh\rho \qquad \frac{1}{\sinh\rho}\frac{\partial}{\partial\theta}\frac{1}{\sinh\rho}\sinh\rho$$

hence

$$\boxed{\Delta = \frac{1}{\sinh\rho}\frac{\partial}{\partial\rho}\sinh\rho\frac{\partial}{\partial\rho} + \frac{1}{\sinh^2\rho}\frac{\partial^2}{\partial\theta^2}}$$

see Helgason p. 405 for analogy to $S^2 = SO(3)/SO(2)$

February 2, 1978

Put $\theta(\tau) = \sum_{n\in\mathbb{Z}} e^{i\pi n^2\tau} = \sum_{n\in\mathbb{Z}} q^{n^2/2}$ $\qquad q = e^{2\pi i\tau}$

so that $\hat{\zeta}(s) = \int_0^\infty \frac{\theta(it)-1}{2}\, t^{s/2}\,\frac{dt}{t}$ $\qquad \tau = it$

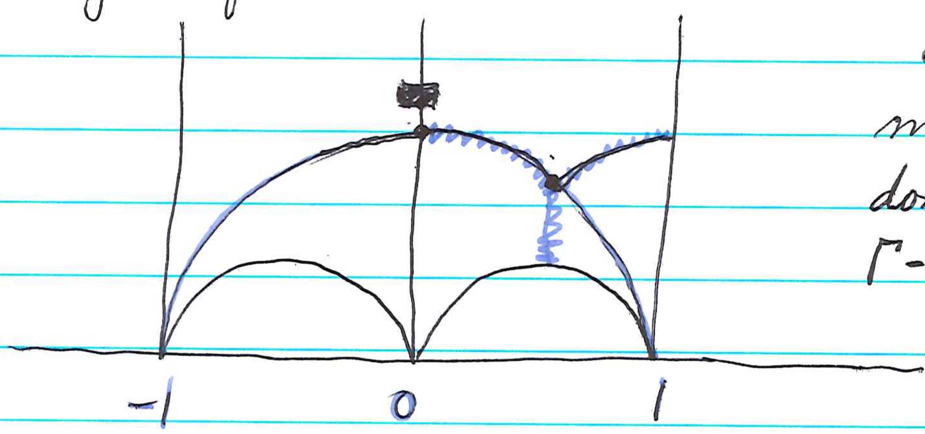
The functional equation for θ reads

$$\theta(it) = \frac{1}{t^{1/2}}\,\theta\left(i\frac{1}{t}\right) \quad \text{or} \quad \boxed{\begin{aligned}\theta\left(-\tfrac{1}{\tau}\right) &= \left(\tfrac{\tau}{i}\right)^{1/2}\theta(\tau)\\ \theta(\tau+2) &= \theta(\tau)\end{aligned}}$$

in this notation. Also we have ~~so~~ that θ is a modular form for the subgroup Γ' of $\Gamma = PSL_2(\mathbb{Z})$ generated by

$$T^2 : \tau \longmapsto \tau+2$$
$$S : \tau \longmapsto -\frac{1}{\tau}$$

According to Serre's book Γ' is of index 3 in Γ and its fundamental domain has 2 cusps to which correspond 2 types of Eisenstein series.

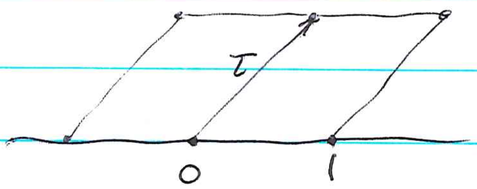


It seems that the Y might be a fundamental domain for Γ' acting on the Γ-tree.

Review the calculation of the fundamental domain for Γ.

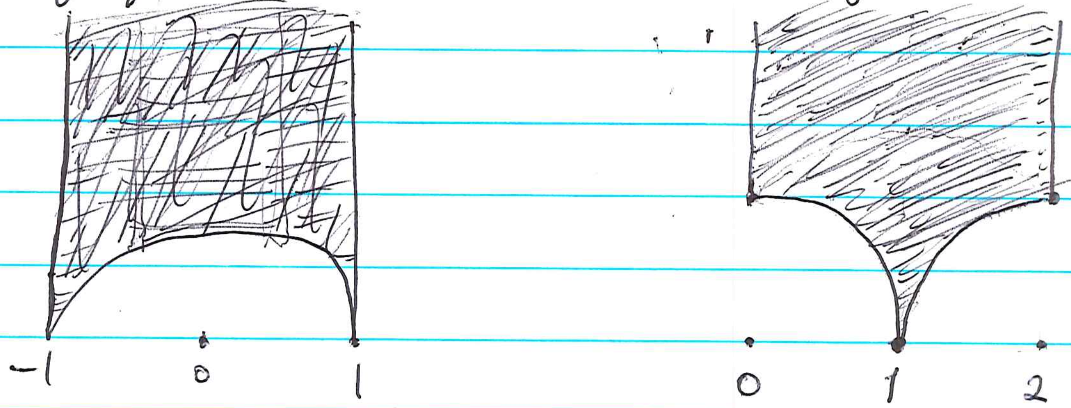
$$\operatorname{Im}\underset{\substack{\uparrow\\ SL_2(\mathbb{R})}}{\begin{pmatrix} a & b\\ c & d\end{pmatrix}}\tau = \operatorname{Im}\frac{(a\tau+b)(c\bar\tau+d)}{|c\tau+d|^2} = \operatorname{Im}\frac{ac|\tau|^2+bd+ad\tau+bc\bar\tau}{|c\tau+d|^2} = \frac{\operatorname{Im}\tau}{|c\tau+d|^2}$$

Now as $\begin{pmatrix}a&b\\c&d\end{pmatrix}$ runs over Γ, $c\tau+d$ runs over certain elements of the lattice spanned by $1, \tau$ (you get all elements α of the lattice $\ni \frac{1}{n}\alpha$ is not in the lattice), hence $|c\tau+d|$ is bounded >0. It follows $\{\mathrm{Im}(g\tau) \mid g\in\Gamma\}$ is bounded above, hence one can select in the Γ orbit of τ an element with maximum imaginary part.



One can move this by translation into the region $|\mathrm{Re}\,\tau| \leq \frac{1}{2}$, whence also $|\tau| \geq 1$ or else $-\frac{1}{\tau} = \begin{pmatrix}0&-1\\1&0\end{pmatrix}\tau$ would have larger imaginary part. Thus one sees that the usual $D=$ contains at least one point from every orbit under the group gen. by S, T.

Similarly if $\Gamma' = \langle S, T^2\rangle$, then the regions



contain at least one point from every Γ' orbit. But consider the homomorphism

$$\Gamma = PSL_2(\mathbb{Z}) \longrightarrow SL_2(\mathbb{Z}/2\mathbb{Z}) = GL_2(\mathbb{Z}/2\mathbb{Z}) \simeq \Sigma_3$$

Then Γ' gets mapped onto the subgroup gen. by $\begin{pmatrix}0&1\\1&0\end{pmatrix}$ which is a 2-cyclic, hence conjugate to $\begin{pmatrix}1&1\\&1\end{pmatrix}$. It follows that Γ' has index ≥ 3 and is conjugate to a subgroup of

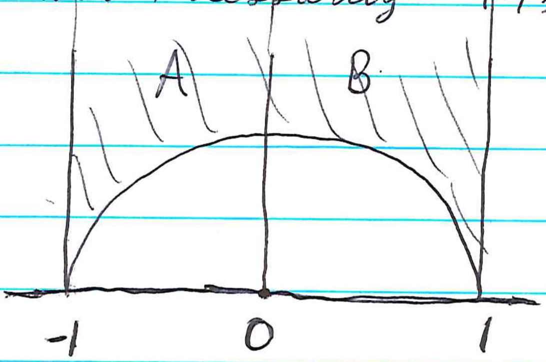
$$\Gamma_0(2) = \left\{ \begin{pmatrix}a&b\\c&d\end{pmatrix} \in \Gamma \;\middle|\; c \equiv 0 \bmod 2 \right\}$$

February 3, 1978:

$\Gamma = PSL_2(\mathbb{Z})$ $\quad$ Γ' subgp gen. by $S : \tau \mapsto -\frac{1}{\tau}$ $\quad$ $T^2 : \tau \mapsto \tau + 2$

$$\operatorname{Im}(\gamma\tau) = \frac{\operatorname{Im}\tau}{|c\tau+d|^2} \qquad \text{if } \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{R}).$$

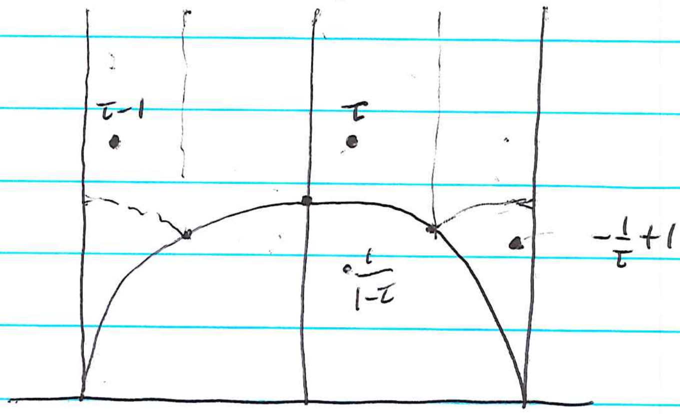
Let $\tau \in$ UHP have maximum imaginary part in its Γ' orbit. We can suppose $|\operatorname{Re}\tau| \leq +1$ by using T^{2n} and then necessarily $|\tau| \geq 1$. Hence the region $|\operatorname{Re}\tau| \leq 1$, $|\tau| \geq 1$ contains one point from each Γ' orbit.



By considering reduction mod 2

$$\Gamma \longrightarrow SL_2(\mathbb{Z}/2\mathbb{Z}) \cong \Sigma_3$$

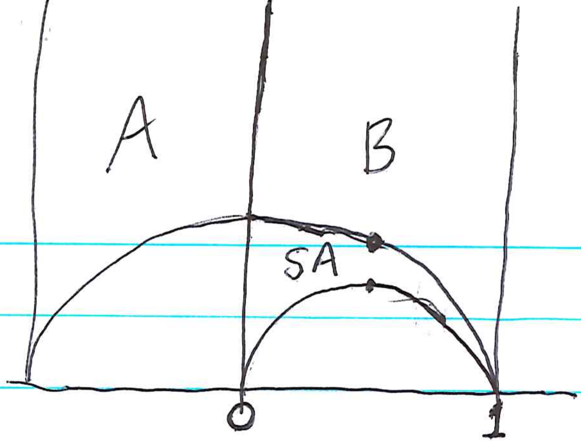
we know that Γ' has index ≥ 3. Take τ to be a generic element in the fundamental domain $D : \{|\operatorname{Re}\tau| \leq \frac{1}{2}, |\tau| \geq 1\}$ for Γ



Then there are exactly three points of the Γ orbit $\Gamma\tau$ in $A \cup B$, call them $\tau, \gamma_1\tau, \gamma_2\tau$. Then if $\gamma \in \Gamma$ one has $\exists\, \gamma' \in \Gamma'$ with $\gamma'\gamma\tau =$ one of these $\Rightarrow$ $\gamma \in \Gamma' \cup \Gamma'\gamma_1 \cup \Gamma'\gamma_2$, hence $(\Gamma : \Gamma') = 3$.

It might be easier to work with the set $SA \cup B$: If $R \in \Gamma$ is the rotation $120°$ around $e^{i\pi/3}$

$$R = \begin{pmatrix} 0 & +1 \\ -1 & +1 \end{pmatrix} \qquad R : \tau \mapsto \frac{+1}{-\tau+1} = \frac{1}{1-\tau}$$



Then $1, R, R^2$ are coset representatives for Γ' in Γ

$$d\left(\frac{az+b}{cz+d}\right) = \frac{(cz+d)\,a\,dz - (az+b)\,c\,dz}{(cz+d)^2} = \frac{dz}{(cz+d)^2}$$

Thus a section $f(z)\,dz^{\otimes(+k)}$ of $\Omega^{\otimes(+k)}$ is invariant under the action of $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ when

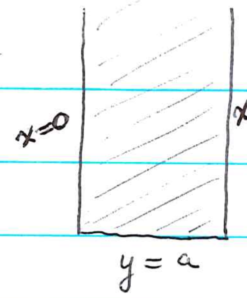
$$f\left(\frac{az+b}{cz+d}\right) d\left(\frac{az+b}{cz+d}\right)^{\otimes(+k)} = f\left(\frac{az+b}{cz+d}\right)(cz+d)^{2k}\,dz^{\otimes(-k)} = f(z)\,dz^{\otimes(+k)}$$

i.e. when
$$f\left(\frac{az+b}{cz+d}\right) = (cz+d)^{+2k} f(z)$$
(modular of weight $2k$). Recall the invariant volume form is $\frac{dx\,dy}{y^2}$ and that
$$\mathrm{Im}\left(\frac{az+b}{cz+d}\right) = \frac{\mathrm{Im}\, z}{|cz+d|^2}$$

Thus
$$|f(z)|^2 (\mathrm{Im}\, z)^{+2k} \frac{dx\,dy}{y^2} \longmapsto \left|f\left(\frac{az+b}{cz+d}\right)\right|^2 \mathrm{Im}\left(\frac{az+b}{cz+d}\right)^{+2k} \frac{dx\,dy}{y^2}$$
$$= |f(z)|^2 |cz+d|^{+4k} \left(\frac{\mathrm{Im}\, z}{|cz+d|^2}\right)^{+2k} \frac{dx\,dy}{y^2}$$
$$= |f(z)|^2 (\mathrm{Im}\, z)^{2k} \frac{dx\,dy}{y^2}$$

hence
$$\|f(z)\,dz^k\|^2 = \int_{D_\Gamma} |f(z)|^2 (\mathrm{Im}\, z)^{2k} \frac{dx\,dy}{y^2}$$
gives a good inner product (Petersson inner product on the space of forms of weight $2k$).

Return to $\Delta \tilde{u} = y^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\tilde{u} = \frac{\partial^2 \tilde{u}}{\partial t^2}$ in a cusp:



with periodicity conditions under $z \mapsto z+1$. Expand the solutions in terms of solutions of the form

$$e^{i\omega t}\, e^{i\xi x}\, u(y)$$

where

$$y^2\left(-\xi^2 u + \frac{d^2u}{dy^2}\right) = -\omega^2 u \qquad \text{or}$$

$$y^2\frac{d^2u}{dy^2} + (\omega^2 - \xi^2 y^2)\,u = 0.$$

If $u = y^{1/2} v$ this becomes

$$(*) \qquad \left[\left(y\frac{d}{dy}\right)^2 + \left(\omega^2 - \frac{1}{4} - \xi^2 y^2\right)\right] v = 0$$

which has the solution $K_s(|\xi| y)$ $\quad s^2 = \frac{1}{4} - \omega^2$ for $\xi \neq 0$.

The periodicity in x forces ξ [redacted] $= 2\pi n \quad n \in \mathbb{Z}$.
If $\xi \neq 0$, then the spectrum of the SL equation

$$\left[-\left(y\frac{d}{dy}\right)^2 + \xi^2 y^2\right] v = \text{[redacted]} \left(\omega^2 - \frac{1}{4}\right) v$$

on [redacted] $a < y < \infty$ is discrete. However if $\xi = 0$, then we get

$$-\left(y\frac{d}{dy}\right)^2 v = \left(\omega^2 - \frac{1}{4}\right) v$$

whose spectrum is continuous. Consequently the continuous part of the spectrum is all constant in x.

So the situation is similar to wave guides. The mode ξ can be transmitted down the guide[redacted], but the higher modes all get attenuated, hence they give bound states.

February 4, 1978.

Let u be an eigenfunction for Δ in the UHP which is Γ-invariant. Expand u as a Fourier series

$$u = \sum_{n\in\mathbb{Z}}{}' e^{2\pi i n x} u_n(y)$$

Then u_n satisfies

$$y^2\left(-(2\pi n)^2 + \frac{d^2}{dy^2}\right) u_n(y) = \lambda u_n(y)$$

Moreover, if u is related to the spectral decomposition of u, [redacted] [redacted] i.e. $\lambda \in$ Spectrum of Δ, then I feel it ought to be the case that u_n decays as y[redacted]$\to +\infty$, or [redacted] least it doesn't blow-up. For $n \neq 0$ this means that

$$u_n(y) = \text{const}\, y^{1/2} K_{\sqrt{\lambda+\frac{1}{4}}}(|\xi| y) \sim \text{const}\; e^{-|\xi| y}$$

where $\xi = 2\pi n$, because the other solutions is $\sim \text{const}\cdot e^{|\xi| y}$ which grows much too fast. For $n=0$ however we have

$$y^2 \frac{d^2}{dy^2} u_0 = \lambda u_0$$

[redacted] so u_0 is a linear combination of the solutions y^s, y^{1-s} where

$$s(s-1) = \lambda.$$

Suppose $\quad u_0(y) = \alpha y^s + \beta y^{1-s} = \alpha y^{\frac{1}{2}+i\omega} + \beta y^{\frac{1}{2}-i\omega}$

Then I have

(*) $\quad u \sim \alpha y^{\frac{1}{2}+i\omega} + \beta y^{\frac{1}{2}-i\omega} \qquad$ as $\quad y \longrightarrow +\infty$

where ω is related to the eigenvalue λ by

$$\left(\tfrac{1}{2}+i\omega\right)\left(\tfrac{1}{2}-i\omega\right) = -\lambda \qquad\qquad \tfrac{1}{4}+\omega^2 = -\lambda$$

In order to make a good connection with the wave equation, we take the wave equation to be

$$\frac{\partial^2 \psi}{\partial t^2} = \left(\Delta + \tfrac{1}{4}\right)\psi$$

so that $\quad \psi = e^{-i\omega t} u \quad$ will be a solution when

$$-\omega^2 u = \Delta u + \tfrac{1}{4} u \qquad \text{or} \qquad \Delta u = -\left(\omega^2 + \tfrac{1}{4}\right) u$$

Correspond to (*) above we get

$$\psi \sim \text{[redacted]} \quad \alpha y^{\frac{1}{2}}\, e^{i\omega(\log y - t)} + \beta y^{\frac{1}{2}}\, e^{i\omega(-\log y - t)}$$

The first term represents an outgoing and the second an incoming wave. These are plane waves moving vertically with unit speed relative to the metric on the UHP. The scattering operator is

$$S(\omega) = \frac{\alpha}{\beta}$$

This should be independent of whatever eigenfunction u is chosen, because two eigenfunctions have to differ by a bound state which has negligible size as $y \to \infty$ (its like $e^{-|\xi| y}$).

Now we construct u by an Eisenstein series. Start with y^s which is already an eigenfunction for Δ with eigenvalue $s(s-1)$. It is also is invariant under the subgroup Γ_0 of Γ generated by $T: z \mapsto z+1$, hence provided the following converges it will be Γ-inv.

$$e(z) = \sum_{A \in \Gamma/\Gamma_0}' A \cdot y^s$$

natural action on functions $(A \cdot f)(z) = f(A^{-1}z)$

$$(A \cdot y)(z) = \operatorname{Im}(A^{-1}z) = \operatorname{Im}\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1}(z) = \operatorname{Im}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}(z) = \frac{\operatorname{Im} z}{|-cz+a|^2}$$

On the other hand we have

$$\begin{array}{ccc} \Gamma/\Gamma_0 & \longrightarrow & \mathbb{P}_1\mathbb{Q} \\ \begin{pmatrix} a & b \\ c & d \end{pmatrix} & \longmapsto & \begin{pmatrix} a & b \\ c & d \end{pmatrix}(\infty) = \frac{a}{c} \end{array}$$

Thus

$$e(z) = \sum_{\frac{a}{c} \in \mathbb{Q} \cup \infty} \frac{y^s}{|cz-a|^{2s}}$$

where rational numbers are represented in lowest terms + positive denominator. Now the usual form of the absolute value Eisenstein series is

$$E(z,s) = \sum_{m,n}' \frac{1}{|mz+n|^{2s}}$$

$$= \sum_{(a,c)=1} \frac{1}{|cz-a|^{2s}} \sum_{d=1}^{\infty} \frac{1}{d^{2s}}$$

$$= 2\zeta(2s) \sum_{\frac{a}{c} \in \mathbb{Q} \cup \infty} \frac{1}{|cz-a|^{2s}}$$

This Eisenstein series converges for $\operatorname{Re}(s) > 1$, so it's clear from convergence theory for Dirichlet series that the series defining $e(z)$ is absolutely convergent for $Re(s) > 1$.

It remains to determine the asymptotic behavior of $e(z)$ as $y \to +\infty$. The hope is that

$$\sum_{\frac{a}{c} \in \mathbb{Q}}' \frac{1}{|cz-a|^{2s}} \sim \frac{\text{const}}{y^{2s}} \cdot y \qquad \text{as } y \to +\infty$$

for then we will have $e(z) \sim y^s + \text{const.}\, y^{1-s}$ and hence the const. will be the scattering matrix (or its inverse).

$$\sum_{m,n}' \frac{1}{|mz+n|^{2s}} \pi^{-s}\Gamma(s) = \sum_{m,n}' \frac{1}{|mz+n|^{2s}} \int_0^\infty e^{-\pi t} t^s \frac{dt}{t}$$

$$= \int_0^\infty \sum{}' e^{-\pi|mz+n|^2 t}\, t^s \frac{dt}{t}$$

$$\sum_{\mathbb{Z}\times\mathbb{Z}} e^{-\pi|mz+n|^2 t} = \sum e^{-\pi(m^2y^2 + (mx+n)^2)t}$$

To simplify suppose that $x = 0$. Then this is $\Theta(y^2 t)\Theta(t)$.

so

$$E(iy,s)\Gamma(s)\pi^{-s} = \int_0 \left[\Theta(y^2t)\Theta(t) - 1\right] t^s \frac{dt}{t}$$

$$= \underbrace{\int_0^\infty [\Theta(t)-1]\, t^s \frac{dt}{t}}_{2\zeta(2s)\,\pi^{-s}\Gamma(s)} + \int_0 \left(\Theta(y^2 t) - 1\right)\Theta(t)\, t^s \frac{dt}{t}$$

$$+ \underbrace{\int_0^\infty (\Theta(t)-1)\,\Theta\!\left(\frac{t}{y^2}\right) \frac{t^s}{y^{2s}} \frac{dt}{t}}$$

$$\frac{1}{y^{2s-1}} \int_0^\infty (\Theta(t)-1) \underbrace{\frac{t^{1/2}}{y}\,\Theta\!\left(\frac{t}{y^2}\right)}_{\to 1 \text{ fast as } y \to +\infty} t^{s-\frac{1}{2}} \frac{dt}{t}$$

So

$$\pi^{-s}\Gamma(s)\, E(iy,s) \sim 2\zeta(2s)\pi^{-s}\Gamma(s) + \frac{1}{y^{2s-1}}\, 2\zeta(2s-1)\, \pi^{-s+\frac{1}{2}}\Gamma(s-\tfrac{1}{2})$$

$$e(iy,s) = \frac{y^s E(iy,s)}{2\zeta(2s)} = \frac{y^s E(iy,s)\,\pi^{-s}\Gamma(s)}{2\zeta(2s)\,\pi^{-s}\Gamma(s)}$$

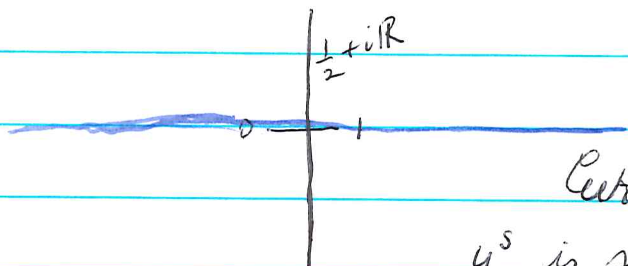
$$\sim y^s + y^{1-s}\left\{\frac{\cancel{2}\,\zeta(2s-1)\,\pi^{-s+\frac{1}{2}}\Gamma(s-\frac{1}{2})}{\cancel{2}\,\zeta(2s)\,\pi^{-s}\Gamma(s)}\right\}$$

$$e(iy,s) \sim y^s + y^{1-s}\left\{\frac{\zeta(2s-1)\,\pi^{1/2}\,\Gamma(s-\frac{1}{2})}{\zeta(2s)\,\Gamma(s)}\right\}$$

A good formula is

$$* \qquad \tfrac{1}{2}\pi^{-s}\Gamma(s)\, E(iy,s) y^s \sim \hat{\zeta}(2s)\, y^s + \hat{\zeta}(2(1-s))\, y^{1-s}$$

On the left is an eigenfunction for Δ invariant under Γ. The heartbreak, according to Harold, is that $\zeta(2s)$ instead of $\zeta(s)$ appears, because if one had an s, then $\hat{\zeta}(s)=0$ $\Rightarrow$ also $\hat{\zeta}(1-s)=0$ and so the eigenfunction will be square-integrable, so the eigenvalue $s(s-1)$ would be real which means s $\in \mathbb{R} \cup (\frac{1}{2}+i\mathbb{R})$ ~~is on the cross~~, and so you would get the Riemann hypothesis.



Curious consequence of $(*)$: Note that y^s is square integrable for $Re(s)<\frac{1}{2}$, hence if $\zeta(2(1-s))=0$ we would get an eigenvalue, so s would on $Re(s)=\frac{1}{2}$. Thus $Re(s)<\frac{1}{2} \Rightarrow \hat{\zeta}(2(1-s)) \neq 0$ hence $\hat{\zeta}(s)\neq 0$ for $Re(s)>1$. even $Re(s)=1$ see p. 788

Scattering operator is $\dfrac{\zeta(2-2s)}{\zeta(2s)}$ which for $Re(s)=\frac{1}{2}$ uses ζ on the line $Re(s)=1$.