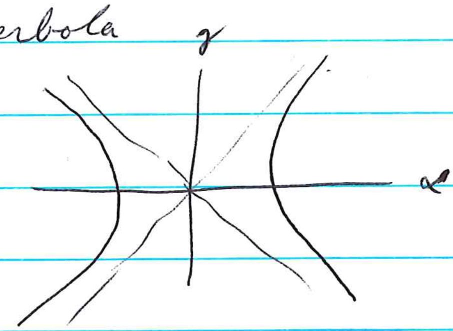
January 29, 1978.

Suppose $\alpha, \gamma$ real so that $z=1$ is a root of

$$z^3 - \frac{\alpha}{\gamma} z^2 + \frac{\alpha}{\gamma} z - 1 = 0$$

Here $\alpha, \gamma$ are real numbers subject to $\alpha^2 - \gamma^2 = 1$ i.e. points on hyperbola




Now I want to know when $\alpha - \gamma z = \alpha - \gamma$ is small. It seems that $|\alpha-\gamma| < \frac{1}{\sqrt{2}}$ [redacted] is the condition that gives an $\ell^2$-solution. It clear that $\alpha - \gamma \to 0$ on the hyperbola as $(\alpha,\gamma) \to \infty$ along the asymptotes $\alpha = \gamma$, but not $\alpha = -\gamma$.

The other roots of the cubic are inverses since the product of the roots is 1. Call the roots $1, \varepsilon, \varepsilon^{-1}$. Then

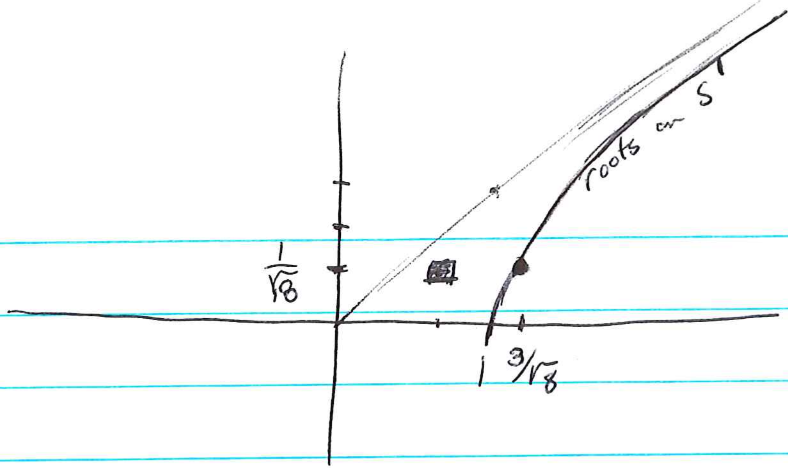
$$\frac{\alpha}{\gamma} = 1 + \varepsilon + \varepsilon^{-1}$$

If $(\alpha,\gamma)$ is far out in the first quadrant, then $0 < \varepsilon + \varepsilon^{-1} < 2$ and so $\varepsilon = e^{i\theta}$ where $0 < \theta < \pi/2$. As we come along the hyperbola toward the origin, the [redacted] critical point for the roots is when $\varepsilon + \varepsilon^{-1} = 2$ or $\frac{\alpha}{\gamma} = 3$.

$$\frac{\alpha}{\gamma} = 3 \qquad \alpha = 3\gamma \qquad (3\gamma)^2 - \gamma^2 = 1 \qquad 8\gamma^2 = 1$$

$$\gamma = \frac{1}{\sqrt{8}} = \frac{1}{2\sqrt{2}} \qquad \alpha = \frac{3}{2\sqrt{2}} \qquad \alpha - \gamma = \frac{1}{\sqrt{2}}$$

So this critical point for the roots corresponds [redacted] exactly to $\alpha - \gamma = \frac{1}{\sqrt{2}}$.




I can always suppose $\alpha > 0$ if I know it is real, without affecting much

If $\alpha > 0$, then $\alpha \geq 1$ so if $\gamma < 0$ then $\alpha - \gamma > 1$ so we don't get something in $\ell^2$.

If $\alpha > 0$, $\gamma < 0$, then $\alpha - \gamma > 1$ so these points are no good. Since in this case

$$\frac{\alpha}{\gamma} = 1 + \varepsilon + \varepsilon^{-1} < 0 \quad \text{or} \quad \varepsilon + \varepsilon^{-1} < -1$$

this case includes $\varepsilon$ on the unit circle, specifically $\varepsilon$ near enough to $-1$.

So I see that for $\alpha, \gamma$ real, $\alpha^2 - \gamma^2 = 1$ $\alpha > 3/\sqrt{8}$ I do get examples where the impedance exists, or at least there is a definite candidate for the impedance.

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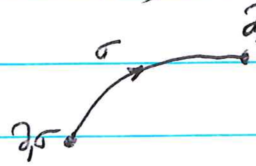
The next project is to understand the Hilbert space picture behind circuits, so that I can make the above example precise.

Let's begin with a finite resistance network, that is a graph in which each edge $\sigma$ has been assigned a resistance $R_\sigma$ with $0 < R_\sigma < \infty$. Assume the graph is connected, pick a basepoint $*$ and attach a battery of 1-volt between $*$ and some other vertex [redacted] $q$. Let $V$ be the function on the vertices giving the voltage relative to $*$. If $\sigma$ is an edge of resistance $R_\sigma$, then $\underbrace{V_{\partial_0\sigma} - V_{\partial_1\sigma}}_{(\delta V)(\sigma)}$ is the voltage




drop across $\sigma$, $\frac{1}{R_\sigma}(V_{\partial_0\sigma} - V_{\partial_1\sigma})$ is the current flowing through $\sigma$ and

$$\frac{1}{R_\sigma}(V_{\partial_0\sigma} - V_{\partial_1\sigma})^2 = \frac{1}{R_\sigma}(\delta V)(\sigma)^2$$

is the power dissipated by $\sigma$. The total power dissipated is

$$\sum_\sigma \frac{1}{R_\sigma}\left[(\delta V)(\sigma)\right]^2 = \|\delta V\|^2$$

if we use the $R_\sigma$ to define an inner product on $C^1(X,\mathbb{R})$. For some reason the actual state $V$ seen is the minimum energy state, i.e. which [redacted] minimizes $\|\delta V\|^2$ subject to the conditions $V_q = 1$, $V_* = 0$. This implies that

$$(\delta V, \delta W) = 0$$

for all voltage functions $W$ vanishing at $q, *$. [redacted] This is equivalent to $(\delta^*\delta V, W) = 0$ for all such $W$, i.e. $\delta^*\delta V = 0$ at all vertices except $(q,*)$ and this last condition should be equivalent to the KCL holding at all vertices $\neq q, *$.

Generalize the preceding to the case where the resistances can be capacitors or inductors. If $(\delta V)(\sigma)$ is the voltage drop across $\sigma$ which has an impedance $Z_\sigma$, then the current flow through $\sigma$ is

$$I = \frac{\delta V(\sigma)}{Z_\sigma}$$

hence the power dissipated is proportional to

$$Re\left(\overline{\delta V(\sigma)} \cdot \frac{\delta V(\sigma)}{Z_\sigma}\right) = Re\left(\frac{1}{Z_\sigma}\right) |\delta V(\sigma)|^2$$

This is zero for reactances, so it is necessary to find another quadratic form.

Let $Y_\sigma = \frac{1}{Z_\sigma}$ be the admittance matrix (assuming $Y$ diagonal means that there is no coupling between inductors). [redacted] Equip [redacted] $C_i, C^i$ with the standard inner product so that [redacted] simplices form an orthonormal basis and [redacted] $\delta^* = \partial$. The KCL become

$$\partial Y \delta V = 0 \quad \text{at vertices} \neq 0, q$$

or

$$(Y\delta V, \delta W) = 0 \quad \text{for all } W \in C^0 \text{ vanishing}$$

at $0, q$. ~~So it's clear now that we want to consider the hermitian form~~

$$\cancel{(Y\delta V, \delta V)}$$

~~on the space of $V \in C^0$ with $V=0$ at $x$, $V=1$ at $q$. An extremal for this form is the same as a solution of the KCL.~~

~~Note that this form depends on the hermitian part of $Y$ ?~~

Here's how to get at the existence of solutions.
Decompose $C^0$ into

$$C^0 = C_1^0 \oplus C_2^0$$

where the former consists of cochains $W$ vanishing at $*, q$ and the latter consists of $W$ vanishing [redacted] on the complementary set of vertices. Let $i: C_1^0 \hookrightarrow C^0$ be the inclusion, and let $V_0 \in C_2^0$ take the value 1 at $q$ and 0 at all other vertices. Then the voltages of interest are

$$V = V_0 + iW \qquad W \in C_1^0 .$$

The condition to be satisfied is that

$$i^*(\delta^* Y \delta V) = 0$$

$$i^* \delta^* Y \delta V_0 + i^* \delta^* Y \delta i W = 0$$

The operator $i^* \delta^* Y \delta i$ maps $C_1^0$ into itself. For the problem to have a unique solution it is necessary and sufficient that this operator be non-singular. Now observe that if [redacted] an operator $A$ is split into hermitian and anti-hermitian parts: $A = B + iC$, then

$$(Av, v) = \underbrace{(Bv, v)}_{\text{real}} + i \underbrace{(Cv, v)}_{\text{real}} .$$

Hence [redacted] $B > 0$ [redacted] $\Rightarrow$ $A$ non-singular. Now the hermitian part of $i^* \delta^* Y \delta i$ is

$$\frac{1}{2}\left(i^* \delta^* Y \delta i + i^* \delta^* Y^* \delta i\right) = i^* \delta^* \left(\frac{Y + Y^*}{2}\right) \delta i$$

Hence if $Re(Y) = \dfrac{Y + Y^*}{2} > 0$ this will be a positive operator (because $\delta i$ is injective, by virtue of the graph being connected).

This proves the existence of $V$ and its uniqueness when the $Y_\sigma$ have real parts $> 0$, e.g. $Y_\sigma = Cs$ for a capacitor, provided $\mathrm{Re}(s) > 0$.

In the loss-less case, $Y$ is purely imaginary, so for uniqueness (which implies existence) one has to know that the hermitian form $(Y\delta iW, \delta iW)$ on $C_1^0$ is non-degenerate.

It seems we can handle the loss-less case in the same way as a resistance network, except that there are negative resistances. The idea is to work with $iV$, $I$, $iY$

So I consider a loss-less network as being something like a resistance network where the resistances can [redacted] be negative. The argument given for resistance networks shows that [redacted] the KCL conditions on $V$ are equivalent to its being an extremal for the hermitian form

$$(Y\delta V, \delta V).$$

Check: $$(Y\delta(V+\varepsilon W), \delta V+\varepsilon W) = (Y\delta V, \delta V) + \varepsilon(Y\delta W, \delta V) + \varepsilon(Y\delta V, \delta W) + O(\varepsilon^2)$$

Extremal: [redacted] $$\underbrace{(Y\delta W, \delta V) + (Y\delta V, \delta W)}_{2\,\mathrm{Re}(Y\delta V,\, \delta W)} = 0$$ because $Y = Y^*$

and since we can multiply [redacted]$W$ by elements in $\mathbb{C}$, we get

$$(Y\delta V, \delta W) = 0 \quad \text{all } W \in C_1^0$$

etc.

Next consider an infinite resistance network, and to simplify suppose all edges have resistance 1. [redacted] Pick a point $q$. The problem is to find the physical voltage distribution which results when 1 volts is applied at $q$. What this means is [redacted] the limit of the voltage distribution obtained by grounding all vertices outside a compact set of nodes $F$ and then letting $F$ exhaust the graph.

Start with $V_0$ which is 1 at $q$ and 0 elsewhere. Then you let $W$ run over 0-cochains with compact support vanishing at $q$ and form the closed subspace generated by these $\delta W$ in the Hilbert space of square-integrable 1-cochains. The idea here is that I am trying to find a $W$ minimizing $\|\delta(V_0+W)\|^2$, that is, I am looking at the [redacted] affine space $\{\delta V_0 + \delta W\}$ and find the element of smallest norm here. This involves passing to the limit.

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Slight reformulation: I have been looking for the voltage distribution $V$ such that $V$ is 1 at $q$ and $\delta^*\delta V = \Delta V = 0$ off $q$. Up to a constant this is the same as the solution of

$$\Delta V = \begin{cases} 0 & \neq q \\ 1 & = q \end{cases}$$

i.e. the Green's function for $\Delta$ with the source $q$. This function represents the [redacted] voltage distribution when 1 amp is fed into the network at the vertex $q$. The actual value of $V$ at [redacted] $q$ gives the impedance of the network at $q$.

Sign conventions: If $\sigma$ is an edge

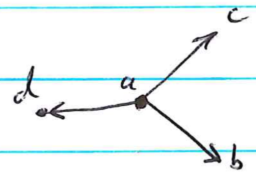

then $\quad V(a) = Z_\sigma I(\sigma) + V(b) \quad$ or

$$(\delta V)(\sigma) = V(\partial\sigma) = V(b) - V(a) = -Z_\sigma I(\sigma)$$

or $\quad I = -Y\delta V.\quad$ Thus if $Y = Id$, then

$$(-\delta^*\delta V)(q)$$

represents the net current flowing [redacted] into the node $q$. For example:




[redacted] $\quad (\delta^*\delta V)(a) = -(\delta V)(ac) - (\delta V)(ab) - (\delta V)(ad)$

$$= V(a) - V(c) - V(b) + V(a) - V(d) + V(a)$$

$$= 3V(a) - V(b) - V(c) - V(d)$$

So

$$\Delta V = -\delta^*\delta V$$

gives the sum of the $V$ values of the neighbors minus the number of neighbors times $V$ in the center. Want $\Delta V = -1$ at $q$

Stephen Adler's problem: Square lattice in the plane with vertices $\mathbb{Z}\times\mathbb{Z}$. Here we can use the Fourier transform which gives an isom. between $\ell^2(\mathbb{Z}\times\mathbb{Z})$ and $L^2(S^1\times S^1)$:

$$V(m,n) = \int_0^{2\pi}\int_0^{2\pi} \frac{d\theta_1}{2\pi}\frac{d\theta_2}{2\pi}\, \hat{V}(\theta_1,\theta_2)\, e^{i(m\theta_1 + n\theta_2)}$$

$$\hat{\Delta V} = (e^{i\theta_1} + e^{-i\theta_1} + e^{i\theta_2} + e^{-i\theta_2} - 4)\hat{V} = -1$$

so
$$\hat{V}(\theta_1, \theta_2) = \frac{[redacted]\ 1}{4 - (e^{i\theta_1} + e^{-i\theta_1} + e^{i\theta_2} + e^{-i\theta_2})}$$

$1 - \cos\theta = 2\sin^2\left(\frac{\theta}{2}\right)$

$$= \frac{1}{2} \cdot \frac{1}{2 - \cos\theta_1 - \cos\theta_2} = \frac{1}{4} \frac{1}{\sin^2\left(\frac{\theta_1}{2}\right) + \sin^2\left(\frac{\theta_2}{2}\right)}$$

The denominator [redacted] vanishes when $\theta_1 = \theta_2 = 0$, where it behaves like $\theta_1^2 + \theta_2^2$. Note that $\frac{1}{\theta_1^2 + \theta_2^2}$ is not square-integrable, not even integrable

$$\iint \frac{dx\,dy}{(x^2+y^2)^2} = \iint \frac{r\,dr\,d\theta}{r^4} \quad \text{like} \int \frac{dr}{r^3} \quad \text{bad at } r=0.$$

Hence this Green's function is not square integrable, even though it ought to be the only [redacted] reasonably behaved solution to $\Delta V = \begin{cases} -1 & \text{at } 0 \\ 0 & \text{elsewhere} \end{cases}$.

To find the resistance of the network we take the V-value at zero:

$$V(0,0) = \frac{1}{4} \int_0^{2\pi}\int_0^{2\pi} \frac{d\theta_1}{2\pi} \frac{d\theta_2}{2\pi} \frac{1}{1 - \frac{1}{4}(e^{i\theta_1} + e^{-i\theta_1} + e^{i\theta_2} + e^{-i\theta_2})}$$

or one expands $\frac{1}{4} \frac{1}{1 - \frac{1}{4}(e^{i\theta_1} + e^{-i\theta_1} + e^{i\theta_2} + e^{-i\theta_2})}$ in a Fourier series (say you use the geometric series) and find the constant term.

Actually since the function involved in the formula for $V(0,0)$ isn't integrable and is positive [redacted] something is fishy. ??

January 30, 1978

Understand something about fundamental solutions.

In the plane $\Delta = \frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}$ ; $f(r)$ is a solution of $\Delta f = 0$ provided

$$f'' + \frac{1}{r} f' = 0 \qquad \ln(f') + \ln r = \text{const}$$

$$f' = \frac{c}{r} \qquad \text{or} \qquad f = c_1 \ln r + c_2$$

Green's formula

$$\iint_R \left[(\Delta u)v - u\Delta v\right] dx\,dy = \oint_{\partial R} \left(\frac{\partial u}{\partial n} v - u\frac{\partial v}{\partial n}\right) ds$$

n outward normal [redacted]

Take $v = \ln r$, $u \in C_0^\infty(\mathbb{R}^2)$ and let $R$ be the outside of the circle of radius $a$:

$$\iint_R (\Delta u)(\ln r)\, dx\,dy = \int_0^{2\pi} \left(-\frac{\partial u}{\partial r}\ln a + u\frac{1}{a}\right) a\, d\theta$$

Let $a \downarrow 0$ and use $a \ln a \to 0$ to get

$$\iint_{\mathbb{R}^2} (\Delta u)(\ln r)\, dx\,dy = 2\pi u(0,0)$$

Consequently because $\Delta$ is its own adjoint we have by definition in the theory of distributions that

$$\Delta\left(\frac{1}{2\pi}\ln r\right) = \delta(x)\delta(y)$$

[redacted] Interpretation: $\frac{1}{2\pi}\ln r$ is a locally integrable function [redacted] hence it defines a distribution, whose Laplacian is the $\delta$ distribution at $O$.

Notice that because $\ln(r) < r$ for out, $\frac{1}{2\pi}\ln r$

is a tempered distribution on $\mathbb{R}^2$ and hence it has a Fourier transform which is also a tempered distribution.

Suppose
$$u(x_1, x_2) = \int e^{ix\xi}\, \hat{u}(\xi_1, \xi_2)\, \frac{d\xi_1}{2\pi} \frac{d\xi_2}{2\pi}$$
Then
$$\Delta u = \int -(\xi_1^2 + \xi_2^2)\, \hat{u}(\xi_1, \xi_2)\, \frac{d\xi_1\, d\xi_2}{(2\pi)^2} = \delta$$
forces
$$-(\xi_1^2 + \xi_2^2)\, \hat{u} \text{ [redacted] } = 1$$

[redacted] This forces $\hat{u}$ to coincide with $-\frac{1}{\xi_1^2 + \xi_2^2}$ away from $0$. So we see the problem of division for distributions. Notice that any distribution $\varphi$ satisfying

(*) $$(\xi_1^2 + \xi_2^2)\,\varphi = 0$$

has support at $0$. I think that the only distributions with support at a point [redacted] are linear combinations of $\delta$ and its derivatives. Hence it seems to me that the only solns of (*) are [redacted] of the form

$$\left(a \frac{\partial}{\partial \xi_1} + b \frac{\partial}{\partial \xi_2} + c\right) \delta$$ could be polys.

<u>Consequently</u> <u>the</u> <u>only</u> <u>tempered</u> [redacted] <u>distribution</u> <u>solutions</u> <u>of</u> <u>Laplace's</u> <u>equation</u> <u>are</u> [redacted] ~~linear functions~~ + polys. ~~constants~~. Similarly the only [redacted] tempered distribution solutions [redacted] of $\Delta u = \delta$ are of the form $\frac{1}{2\pi} \ln r +$ [redacted] harmonic poly ~~$ax + by + c$~~, and among these one can single out $\frac{1}{2\pi} \ln r$ as having the smallest growth at $\infty$. (Actually this only pins $u$ down to $\frac{1}{2\pi} \ln r$ + constant.)

In the Adler problem, instead of $\hat{u}(\xi_1, \xi_2) = -\frac{1}{\xi_1^2 + \xi_2^2}$ off $0$ we have
$$\hat{u}(\theta_1, \theta_2) = -\frac{1}{4} \frac{1}{\sin^2\theta_1 + \sin^2\theta_2}$$

$\hat{u}(\theta_1, \theta_2)$ is a distribution on $S^1 \times S^1$ and it should correspond to a function on $\mathbb{Z} \times \mathbb{Z}$ with at most polynomial growth.

The first remark to make is the purely algebraic statement that the only solutions of the Laplace equation

$$\Delta u = 0$$

on $\mathbb{Z} \times \mathbb{Z}$ having polynomial growth are [redacted] polynomial functions. [redacted] The proof is by Fourier transform: Such a $u$ corresponds to a distribution $\hat{u}$ on $S^1 \times S^1$ killed by $\sin^2 \frac{\theta_1}{2} + \sin^2 \frac{\theta_2}{2}$ and [redacted] hence $\hat{u}$ has support $\{0\}$, etc.

Because $-\frac{1}{4} \frac{1}{\sin^2 \frac{\theta_1}{2} + \sin^2 \frac{\theta_2}{2}}$ behaves like $-\frac{1}{\theta_1^2 + \theta_2^2}$ as $\theta_1, \theta_2 \to 0$ I expect the fundamental solution $u(m,n)$ to be asymptotic to $\frac{1}{2\pi} \log \sqrt{m^2+n^2}$ far out.

Another idea: Replace $\Delta$ by $\Delta - \mu^2$ which has an inverse for $\mu > 0$ [redacted] in $L^2$. Then let $\mu \downarrow 0$ to get the desired fundamental solution.

Try this for [redacted] Laplace's equation in $\mathbb{R}^2$: Want a fundamental solution $(\Delta - \mu^2)u = \delta$. If $u = u(r)$ then we get

[redacted]
$$\frac{1}{r}\frac{d}{dr}\left(r \frac{du}{dr}\right) - \mu^2 u = 0$$

or

$$\left(r\frac{d}{dr}\right)^2 u - \mu^2 r^2 u = 0$$

which is the imaginary Bessel's equation with $n = 0$. We [redacted] want, not $I_0$, but, a solution with the $N_0$ solution, which has a log r in it. Notice that for $r >> 0$ the equation

$$\frac{d^2u}{dr^2} + \frac{1}{r}\frac{du}{dr} - \mu^2 u = 0$$

should have solutions asymptotic to $e^{\pm\mu r}$. Now the $u$ we are after should decay as $r \to +\infty$, so we want the solution to be proportional to

$$K_0(\mu r).$$

Recall that the imaginary Bessel DE is

$$-\left(r\frac{d}{dr}\right)^2 u + r^2 u = -s^2 u$$

and its [redacted] solution decaying at $r \to +\infty$ is

$$K_s(r) = \int_0^\infty e^{-r/2(t+t^{-1})}\, t^s \frac{dt}{t}$$

Thus $K_s(\mu r)$ is the solution decaying at $r \to +\infty$ of

$$-\left(r\frac{d}{dr}\right)^2 u + \mu^2 r^2 u = -s^2 u.$$

---

It is easier to see in 3 dimensions where

$$\Delta u = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right) + \frac{1}{r^2\sin\phi}\frac{\partial}{\partial\phi}\left(\sin\phi\frac{\partial u}{\partial\phi}\right) + \frac{1}{r^2\sin^2\phi}\frac{\partial^2 u}{\partial\theta^2}$$


$$ds^2 = dr^2 + r^2 d\phi^2 + (r\sin\phi)^2 d\theta^2 \qquad dV = r^2\sin\phi\, dr\, d\phi\, d\theta$$

$$\frac{\partial}{\partial r} \qquad \frac{1}{r}\frac{\partial}{\partial\phi} \qquad \frac{1}{r\sin\phi}\frac{\partial}{\partial\theta}$$

$$-\left(\frac{1}{r}\frac{\partial}{\partial\phi}\right)^* = \frac{1}{r^2\sin\phi}\frac{\partial}{\partial\phi}\frac{1}{r}\, r^2\sin\phi$$

$$-\left(\frac{1}{r\sin\phi}\frac{\partial}{\partial\theta}\right)^* = \frac{1}{r^2\sin\phi}\frac{\partial}{\partial\theta}\frac{1}{r\sin\phi}\, r^2\sin\phi$$

A radial solution of $\Delta u - \mu^2 u = 0$ satisfies

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{du}{dr}\right) - \mu^2 u = 0$$

[redacted]

$$\frac{d^2u}{dr^2} + \frac{2}{r}\frac{du}{dr} - \mu^2 u = 0$$

$$\frac{d^2}{dr^2}(ru) = \frac{d}{dr}\left(r\frac{du}{dx} + u\right) = r\frac{d^2u}{dr^2} + 2\frac{du}{dx} = \mu^2(ru)$$

hence $ru = c_1 e^{-\mu x} + c_2 e^{\mu x}$. The solution decaying at $\infty$ is $u = \dfrac{e^{-\mu x}}{r}$ up to a constant.

$$\iiint_{R\,[redacted]} (\mathrm{div}\,\nabla u)\,[redacted]\,dV = \iint_{\partial R} \nabla u \cdot \hat{n}\, dS$$

For $u = \frac{1}{r}$ and $R$ a ball of radius $a$ around zero, we have

$$\iint_{\partial R} \nabla u \cdot \hat{n}\, dS = \iint_{\partial R} -\frac{1}{a^2}\, dS = -\frac{1}{a^2} 4\pi a^2 = -4\pi$$

Hence $\Delta\left(\frac{1}{4\pi r}\right) = -\delta$. Thus $u = -\frac{1}{4\pi}\frac{e^{-\mu r}}{r}$ is the fundamental solution [redacted]:

$$(\Delta - \mu^2)u = \delta$$

Since $\frac{1}{r}$ is locally-integrable: $dV = r^2 \sin\phi\, dr\, d\phi\, d\theta$, it defines a distribution; but because it is square-integrable around zero, $u = -\frac{1}{4\pi}$ [redacted] $\frac{e^{-\mu r}}{r} \in L^2(\mathbb{R}^3)$ for $\mu > 0$. As $\mu \searrow 0$ it goes to $-\frac{1}{4\pi r}$ which is locally-square-integrable, but not globally so.

Try the same device with Adler's problem.

$$\hat{u}(\theta_1, \theta_2) = \frac{1}{(e^{i\theta_1} + e^{-i\theta_1} + e^{i\theta_2} + e^{-i\theta_2}) - 4 - \mu^2} = -\frac{1}{4}\frac{1}{\left(1 + \frac{\mu^2}{4}\right) - \frac{1}{4}(\qquad)}$$

$$u(0,0) = \int_0^{2\pi}\!\!\int_0^{2\pi} \frac{d\theta_1\, d\theta_2}{2\pi\cdot 2\pi}\, \hat{u}(\theta_1,\theta_2)$$

Try to evaluate this. If $a > 1$

$$\frac{1}{2\pi}\int_0^{2\pi}\frac{d\theta}{a-\cos\theta} = \frac{1}{2\pi i}\oint_{|z|=1} \frac{1}{a-\frac{z+z^{-1}}{2}}\,\frac{dz}{z} = \frac{1}{2\pi i}\int \frac{dz}{(-\frac{1}{2})[z^2-2az+1]}$$

roots $z^2-2az+1=0$

$z = a \pm\sqrt{a^2-1} = a-\sqrt{a^2-1}$ inside $S^1$ $\quad = -2\cdot\frac{1}{-2\sqrt{a^2-1}} = \frac{1}{\sqrt{a^2-1}}$

$f'(z) = 2(z-a) = -2\sqrt{a^2-1}$

Thus $\boxed{\frac{1}{2\pi}\int_0^{2\pi}\frac{d\theta}{a-\cos\theta} = \frac{1}{\sqrt{a^2-1}} \quad |a|>1 \text{ branch outside cut}}$

We want

$$\int_0^{2\pi}\frac{d\theta_1}{2\pi}\int_0^{2\pi}\frac{d\theta_2}{2\pi}\,\frac{1}{a-\cos\theta_1-\cos\theta_2} = \int_0^{2\pi}\frac{d\theta}{2\pi}\,\frac{1}{\sqrt{(a-\cos\theta)^2-1}}$$

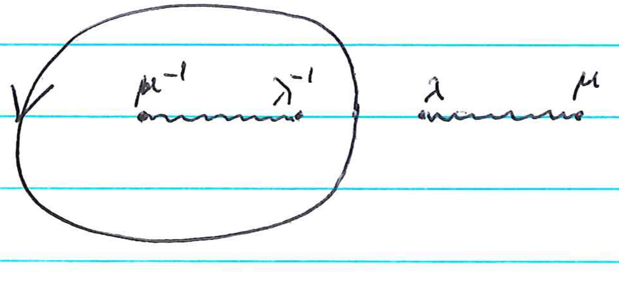
$2(1+\frac{\mu^2}{4})$
$\|$
$a>2$

January 31, 1978

$$\int_0^{2\pi}\frac{1}{\sqrt{(a-1-\cos\theta)(a+1-\cos\theta)}}\,\frac{d\theta}{2\pi} = \oint \frac{1}{\sqrt{\left(a-1-\frac{z+z^{-1}}{2}\right)\left(a+1-\frac{z+z^{-1}}{2}\right)}}\,\frac{dz}{2\pi i z}$$

$$= 2\oint \frac{1}{\sqrt{(z^2-2(a-1)z+1)(z^2-2(a+1)z+1)}}\,\frac{dz}{2\pi i}$$

As $a-1>1$, the roots of $z^2-2(a-1)z+1$ are $\lambda, \frac{1}{\lambda}$ with $\lambda>1$ and $\lambda\downarrow 1$ as $a\downarrow 2$. Roots of $z^2-2(a+1)z+1$ are $\mu, 1/\mu$ with $\mu>\lambda$. So cut the plane:




The radical is defined to be $+\sqrt{(4-2a)(-2a)} \overset{+}{=} 2\sqrt{a(a-2)}$ at $z=1$.

What happens as $a\downarrow 2$ as $\lambda\downarrow 1$.

$\exists$ Problem: $\displaystyle\int_{-\pi}^{\pi} \frac{d\theta}{2\pi} \frac{1}{\sqrt{(a-\cos\theta)^2-1}}$ approaches $+\infty$ as $a \downarrow 2$

This is because the integrand is monotone increasing in $a$ and positive. When $a = 2$ one has

$$\frac{1}{\sqrt{(a-1-\cos\theta)(a+1-\cos\theta)}} \longrightarrow \frac{1}{\sqrt{2\sin^2\frac{\theta}{2}\,(3-\cos\theta)}} = \frac{1}{\left|\sin\frac{\theta}{2}\right|\sqrt{2(3-\cos\theta)}}$$

which is not integrable.

So it appears that if

$$\hat{u}_\mu(\theta_1,\theta_2) = -\frac{1}{2}\,\frac{1}{2\left(1+\frac{\mu^2}{4}\right)-\cos\theta_1-\cos\theta_2} \qquad a = 2\left(1+\frac{\mu^2}{4}\right)$$

then

$$\lim_{\mu\downarrow 0} u_\mu(0,0) = -\infty$$

In fact

$$u_\mu(0,0) = \iint \frac{d\theta_1}{2\pi}\frac{d\theta_2}{2\pi}\left(-\frac{1}{2}\right)\frac{1}{a-\cos\theta_1-\cos\theta_2}$$

and $\dfrac{1}{a-\cos\theta_1-\cos\theta_2}$ is a positive function increasing as $a \downarrow 2$ to a non-integrable function.

On the other hand I can argue that there has to be a fundamental solution for $\mu = 0$, by using theorems on the division of distributions by polynomials.

<u>Example</u>: Take $\mathbb{R}$: $\Delta = +\dfrac{d^2}{dx^2}$. If we want

$$(\Delta-\mu^2)\,G_y = \delta(x-y) \qquad \left(\frac{d^2}{dx^2}-\mu^2\right)f = 0\,,\quad e^{\pm\mu x}$$

the formula is

$$G(x,y) = \frac{e^{\mu x_<}\,e^{-\mu x_>}}{\begin{vmatrix} 1 & 1 \\ \mu & -\mu \end{vmatrix}} = \frac{e^{-\mu|x-y|}}{-2\mu}$$

So the fundamental solution for $\mu > 0$ is

$$\frac{e^{-\mu|x|}}{-2\mu}$$

Calculate its F.T.

$$\int e^{-ix\xi}\frac{e^{-\mu|x|}}{-2\mu}dx = \int_0^\infty e^{-x(i\xi+\mu)}\frac{dx}{-2\mu} + \int_{-\infty}^0 e^{x(-i\xi+\mu)}\frac{dx}{-2\mu}$$

$$= -\frac{1}{2\mu}\left\{\frac{1}{i\xi+\mu} + \frac{1}{-i\xi+\mu}\right\} = -\frac{1}{\xi^2+\mu^2}$$

Now if you let $\mu \downarrow 0$, then the fundamental solution doesn't converge to a fundamental solution for $\Delta$.

However if you [redacted] add a suitable solution of $\Delta u = 0$ you get

$$\frac{e^{-\mu|x|} - e^{\mu x}}{-2\mu} = \begin{cases} 0 & x \leq 0 \\ \dfrac{e^{-\mu x} - e^{\mu x}}{2\mu} & x \geq 0 \end{cases}$$

$$\lim_{\mu\to 0}\frac{e^{-\mu|x|} - e^{\mu x}}{-2\mu} = \begin{cases} 0 & x \leq 0 \\ x & x \geq 0 \end{cases}$$

which is a fundamental solution for [redacted] $\Delta$.

Example: $\mathbb{Z}$. Homogeneous solutions of

$$(\Delta - \mu^2)u = 0$$

are $e^{in\theta}$ where $e^{i\theta} + e^{-i\theta} - 2 - \mu^2 = 0$ or

$$\cos\theta = \lambda \qquad \lambda = 1 + \frac{\mu^2}{2}$$

Let $z = \lambda - \sqrt{\lambda^2 - 1}$ be the root with $0 < z < 1$. Then we have $\qquad \phi(n) = z^{-n} \qquad \psi(n) = z^{+n}$

and
$$W(\phi,\psi) = \begin{vmatrix} 1 & 1 \\ z^{-1} & z \end{vmatrix} = z - z^{-1}$$

so the fundamental solution for $\Delta - \mu^2$ is

$$u(n) = \frac{z^{+|n|}}{z - z^{-1}}$$

Check: $\frac{1}{z-z^{-1}}\left\{ \frac{z^1+z^1-2-\mu^2}{2z-(z+z^{-1})} \right\} = 1$

As $\mu \downarrow 0$, $\lambda \downarrow 1$ and $z \uparrow 1$ and this fundamental solution blows up. On the other hand the fund. solution

$$\frac{z^{|n|} - \frac{1}{2}(z^n + z^{-n})}{z - z^{-1}} = \begin{cases} \frac{1}{2}\,\frac{z^n - z^{-n}}{z - z^{-1}} & n \geq 0 \\ \frac{1}{2}\,\frac{z^{-n} - z^n}{z - z^{-1}} & n \leq 0 \end{cases}$$

$$\xrightarrow{\text{as } z \uparrow 1} \begin{cases} \frac{1}{2} n & n \geq 0 \\ -\frac{1}{2} n & n \leq 0 \end{cases}$$

and so we get the fundamental solution $\frac{1}{2}|n|$ for $\Delta$.

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It appears therefore that the [redacted] "voltage distribution which results by pushing 1 ampere into a node of an infinite network" is a subtle thing which gets explained adequately [redacted] with distributions, at least for periodic networks where this is a good deal.

Notice that for the Adler problem I expect the fundamental solution to grow like $\frac{1}{2\pi}\log r$, hence the voltage is $-\infty$ at $\infty$? Actually even for $\Delta$ on $\mathbb{R}^2$, where we expect the fundamental solution to be $\frac{1}{2\pi}\log r$, we can alter this by [redacted] adding a constant. So now I don't understand how one can even speak of the resistance of this network?

The thing to do it seems is to put in the boundary condition that $u - \frac{1}{2\pi}\log(r) \to 0$.

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Geometry of the UHP: $\operatorname{Im} z > 0$. First determine the Riemann metric which is the $SL_2\mathbb{R}$ invariant metric with $ds^2 = |dz|^2 = dx^2 + dy^2$ at $i$. Let $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2\mathbb{R}$, and

$$z = \frac{ai+b}{ci+d} = \frac{(ai+b)(-ci+d)}{c^2+d^2} = \frac{(ac+bd)+i}{c^2+d^2} \qquad \therefore y = \frac{1}{|ci+d|^2}$$

Suppose we have a [redacted] tangent vector $i+\varepsilon$ at $i$. Its image is

$$\frac{a(i+\varepsilon)+b}{c(i+\varepsilon)+d} = \frac{ai+b+a\varepsilon}{ci+d+c\varepsilon} \; \frac{ci+d-c\varepsilon}{ci+d-c\varepsilon}$$

$$= \frac{ai+b}{ci+d} + \varepsilon \frac{(ai+b)(-c)+a(ci+d)}{(ci+d)^2}$$

$$= \frac{ai+b}{ci+d} + \varepsilon \frac{1}{(ci+d)^2}$$

Hence the Jacobian mapping the tangent space at $i$ to the tangent space at $\frac{ai+b}{ci+d}$ is $\frac{1}{(ci+d)^2}$

$$\frac{ai+b}{ci+d} = i \iff c=-b,\ d=a \iff \text{[redacted]} \begin{pmatrix} a & b \\ -b & a \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

In this case $\frac{1}{(ci+d)^2} = \frac{1}{(\cos\theta - i\sin\theta)^2} = e^{2i\theta}$

preserves the metric $|dz|^2$ as it should. The Riemann metric is defined so that

$$\left\| \frac{ai+b}{ci+d} + \frac{\varepsilon}{(ci+d)^2} \right\| = \| i+\varepsilon \| = |\varepsilon|$$

$$\left\| \frac{ai+b}{ci+d} + \eta \right\| = |\eta|\,|ci+d|^2 = \frac{|\eta|}{y}$$

In other words using Cartan's notation $dz$ for a displacement from $z$ to $z+dz$, the length (non-Euclidean) of this displacement is

$$\|dz\| = \frac{|dz|}{y} \qquad \text{i.e. } ds^2 = \frac{dx^2+dy^2}{y^2}$$

where $|dz| = \sqrt{dx^2+dy^2}$ is the Euclidean length. The non-Euclidean volume is therefore

$$dV = \frac{dx\,dy}{y^2}$$

Pass to the circle: $|z|<1$. $\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & b \\ \bar{b} & \bar{a} \end{pmatrix}$

$|a|^2-|b|^2=1$

$$z = \begin{pmatrix} a & b \\ c & d \end{pmatrix}(0) = \frac{b}{d}$$

$$\frac{a\varepsilon+b}{c\varepsilon+d} = \frac{a\varepsilon+b}{c\varepsilon+d}\,\frac{-c\varepsilon+d}{-c\varepsilon+d} = \frac{bd+\varepsilon}{d^2} = \frac{b}{d} + \frac{\varepsilon}{d^2}$$

So

$$\left\| \frac{b}{d} + \eta \right\| = |\eta d^2| = |\eta|\cdot|d|^2$$

But $|z|^2 = \dfrac{|b|^2}{|d|^2} = \dfrac{|b|^2}{|a|^2} \implies 1-|z|^2 = 1-\dfrac{|b|^2}{|a|^2} = \dfrac{1}{|a|^2} = \dfrac{1}{|d|^2}$, so

$$\|z+\eta\| = \frac{|\eta|}{1-|z|^2} \quad \text{or in Cartan's form:}$$

those are off by factors of 2, see p. 766

$$\boxed{\|dz\| = \frac{|dz|}{1-|z|^2}} \qquad dV = \frac{dx\,dy}{(1-|z|^2)^2}$$

Compute $\Delta$ for UHP. We have

$$\left\|\frac{\partial}{\partial x}\right\| = \frac{1}{y}\left|\frac{\partial}{\partial x}\right| = \frac{1}{y} \qquad \left\|\frac{\partial}{\partial y}\right\| = \frac{1}{y}$$

hence an orthonormal pair of vector fields is

$$y\frac{\partial}{\partial x} \quad , \quad y\frac{\partial}{\partial y}$$

Their adjoints wrt $dV = \frac{1}{y^2}\,dx\,dy$ are minus

$$y^2\frac{\partial}{\partial x}\,y\,\frac{1}{y^2} \qquad\qquad y^2\frac{\partial}{\partial y}\,y\,\frac{1}{y^2}$$

so

$$\Delta = \left(y^2\frac{\partial}{\partial x}\,y\,\frac{1}{y^2}\right)\left(y\frac{\partial}{\partial x}\right) + \left(y^2\frac{\partial}{\partial y}\,y\,\frac{1}{y^2}\right)\left(y\frac{\partial}{\partial y}\right)$$

$$\boxed{\Delta = y^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)}$$

A similar calculation shows that an orthonormal frame in the case of the circle is

$$(1-|z|^2)\frac{\partial}{\partial x} \qquad\qquad (1-|z|^2)\frac{\partial}{\partial y}$$

and hence

$$\Delta = (1-|z|^2)\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)$$

by a similar calculation.

It should follow that a fundamental solution for Laplace's equation: $\Delta u = \delta(0)$ is the same in the Euclidean case: $u = \frac{1}{2\pi}\ln r$. Solutions of $(\Delta - \mu^2)u = \delta$ should however be more interesting.