
Suppose $\alpha, \gamma$ real so that $z = 1$ is a root of

$$z^3 - \frac{\alpha}{\gamma} z^2 + \frac{\alpha}{\gamma} z - 1 = 0$$

Here $\alpha, \gamma$ are real numbers subject to $\alpha^2 - \gamma^2 = 1$, i.e.
points on hyperbola $\gamma$.

Now I want to know when $\alpha - \gamma z = \alpha - \gamma$ is small.

It seems that $|\alpha - \gamma| < \frac{1}{\sqrt{2}}$ is the condition that

gives an $L^2$-solution. It clear that $\alpha - \gamma \to 0$ on
the hyperbola as $(\alpha, \gamma) \to \infty$ along the asymptotes $\alpha = \gamma$, but
not $\alpha = -\gamma$.

The other roots of the cubic are inverses since the
product of the roots is 1. Call the roots $1, \varepsilon, \varepsilon^{-1}$. Then

$$\frac{\alpha}{\gamma} = 1 + \varepsilon + \varepsilon^{-1}$$

If $(\alpha, \gamma)$ is far out in the first quadrant, then $0 < \varepsilon + \varepsilon^{-1} < 2$
and so $\varepsilon = e^{i\theta}$ where $0 < \theta < \pi/2$. As we come
along the hyperbola toward the origin, the

critical point for the roots is when $\varepsilon + \varepsilon^{-1} = 2$ or $\frac{\alpha}{\gamma} = 3$.

$$\frac{\alpha}{\gamma} = 3 \quad \alpha = 3\gamma$$

$$(3\gamma)^2 - \gamma^2 = 1 \quad 8\gamma^2 = 1$$

$$\gamma = \frac{1}{\sqrt{8}} = \frac{1}{2\sqrt{2}} \quad \alpha = \frac{3}{2\sqrt{2}} \quad \alpha - \gamma = \frac{1}{\sqrt{2}}$$

So this critical point for the roots corresponds exactly

\[ \alpha - \gamma = \frac{1}{\sqrt{2}}. \]
If $x > 0$, then $x > 1$ so if $y < 0$ then $x - y > 1$
so we don't get something in $l^2$. If $x > 0$, $y < 0$, then $x - y > 1$ so these
points are not good. Since in this case
\[
\frac{2}{y} = 1 + \varepsilon + \varepsilon^{-1} < 0 \quad \text{or} \quad \varepsilon + \varepsilon^{-1} < -1
\]
this case includes $\varepsilon$ on the unit circle, specifically
$\varepsilon$ near enough to $-1$.

So I see that for $x, y$ real, $x^2 - y^2 = 1$
$x > \frac{1}{\sqrt{2}}$, I do get examples where the impedance
exists, or at least there is a definite candidate
for the impedance.

The next project is to understand the Hilbert
space picture behind circuits so that I can make
the above example precise.

Let's begin with a finite resistance network, that
is a graph in which each edge has been assigned
a resistance $R_0$ with $0 < R_0 < \infty$. Assume the
graph is connected, pick a basepoint $*$ and attach a battery of 1-volt between $*$ and some other vertex $\ell$. Let $V$ be the function on the vertices giving the voltage relative to $*$. If $e$ is an edge of resistance $R_e$, then $V_{\ell_\sigma} - V_{\ell_0}$ is the voltage across $e$.

$$ \frac{1}{R_e} (V_{\ell_\sigma} - V_{\ell_0}) = \frac{1}{R_e} (\delta V)(e) $$

is the power dissipated by $e$. The total power dissipated is

$$ \sum_e \frac{1}{R_e} [\delta V(e)]^2 = \| \delta V \|^2 $$

If we use the $R_e$ to define an inner product on $C(X,R)$, for some reason the actual state seen is the minimum energy state, i.e. which minimizes $\| \delta V \|^2$ subject to the conditions $V_{\ell_0} = 1, V_{*e} = 0$. This implies that

$$(\delta V, \delta W) = 0$$

for all voltage functions $W$ vanishing at $\ell, *$. This is equivalent to $(\delta^* \delta V, W) = 0$ for all such $W$, i.e. $\delta^* \delta V = 0$ at all vertices except $\ell, *$, and this last condition should be equivalent to the KCL holding at all vertices $\neq \ell, *$. 
Generalize the preceding to the case where the resistances can be capacitors or inductors. If \((dV(t))\) is the voltage drop across \(\sigma\) which has an impedance \(Z_0\), then the current flow through \(\sigma\) is

\[ I = \frac{dV(t)}{Z_0} \]

hence the power dissipated is proportional to

\[ \text{Re} \left( \frac{dV(t)}{Z_0} \cdot \frac{dV(t)}{2Z_0} \right) = \text{Re} \left( \frac{1}{2Z_0} \right) |dV(t)|^2 \]

This is zero for reactances, so it is necessary to find another quadratic form.

Let \( \frac{1}{Z_0} = Y_0 \) be the admittance matrix (assuming \( Y \) diagonal means that there is no coupling between inductors). Equip \( \mathbb{C}^N \) with the inner product so that it simplifies form on an orthonormal basis and \( \delta^* = 0 \).

The KCL become

\[ \forall YdV = 0 \text{ at vertices } \not= 0, \bar{g} \]

or

\[ (YdV, dV) = 0 \text{ for all } V \in \mathbb{C}^N \text{ vanishing at } 0, \bar{g}. \]

So it is clear now that we want to consider the form

\[ (YdV, dV) \]

on the space of \( V \in \mathbb{C}^N \) with \( V = 0 \) at \( \not= 0, \bar{g} \).

The essential for this form is the same as an equation of the KCL.

Note that this form depends on the hermitian part of \( Y \).

(\( Y \) in this context is the admittance matrix.)
Here's how to get at the existence of solutions.

Decompose $C^o$ into

$$C^o = C_1^o \oplus C_2^o$$

where the former consists of cochains $W$ vanishing at $x, y,$ and the latter consists of $W$ vanishing on the complementary set of vertices. Let $i : C_1^o \to C^o$ be the inclusion and let $V_0 \in C_2^o$ take the value $1$ at $x$ and $0$ at all other vertices. Then the voltages of interest are

$$V = V_0 + iW$$

The condition to be satisfied is that

$$i^*(\delta^* Y S V) = 0$$

$$i^*\delta^* Y S V_0 + i^*\delta^* Y S iW = 0$$

The operator $i^*\delta^* Y S i$ maps $C_1^o$ into itself. For the problem to have a unique solution it is necessary and sufficient that this operator be non-singular. Now observe that if $A = B + iC,$ then

$$(A, v) = (B, v) + i(C, v)$$

where

$$B > 0$$

Means $A$ non-singular. Now the hermitian part of $i^*\delta^* Y S i$ is

$$\frac{1}{2} (i^*\delta^* Y S i + i^*\delta^* Y^* S i) = i^*\delta^* \left( \frac{Y + Y^*}{2} \right) S i$$

Hence if $Re(Y) = \frac{Y + Y^*}{2} > 0$ this will be a positive operator (because $Si$ is injective by virtue of the graph being connected).
This proves the existence of $V$ and its uniqueness when $\Re(s) > 0$, e.g. $V = Cs$ for a capacitor provided $\Re(s) > 0$.

In the loss-less case, $Y$ is purely imaginary, so for uniqueness (which implies existence) one has to know that the hermitian form $(YSW, \delta W)$ on $C^o$ is non-degenerate.

It seems we can handle the loss-less case in the same way as a resistance network, except that there are negative resistances. The idea is to work with $iV, I, iY$

So I consider a network as being something like a resistance network where the resistance can be negative. The argument given for resistance networks shows that the KCL conditions on $V$ are equivalent to its being an extremal for the hermitian form $(YSV, \delta V)$.

Check: $(YSV + \varepsilon W, \delta V + \varepsilon W) = (YSV, \delta V) + \varepsilon (YSW, \delta V) + \varepsilon^2 (YSV, \delta W) + O(\varepsilon^2)$

Extremal: $(YSW, \delta V) + (YSV, \delta W) = 0$

because $Y = Y^*$

$2 \Re (YSV, \delta W)$

and since we can multiply $W$ by elements in $C$, we get

$(YSV, \delta W) = 0$ all $W \in C^o$

etc.
Next consider an infinite resistance network, and to simplify suppose all edges have resistance 1.

Pick a point \( q \). The problem is to find the physical voltage distribution which results when 1 volt is applied at \( q \). What this means is the limit of the voltage distribution obtained by grounding all vertices outside a compact set of nodes \( F \) and then letting \( F \) exhaust the graph.

Start with \( V_0 \) which is 1 at \( q \) and 0 elsewhere. Then you let \( W \) run over 0-cochains with compact support vanishing at \( q \) and form the closed subspace generated by these \( SW \) in the Hilbert space of square-integrable 1-cochains. The idea here is that I am trying to find a \( W \) minimizing \( \| S(V_0+W) \| \), that is, I am looking at the space \( \{ SW + SW \} \) and find the element of smallest norm here. This involves passing to the limit.

Slight reformulation: I have been looking for the voltage distribution \( V \) such that \( V \) is 1 at \( q \) and \( \delta \delta V = \Delta V = 0 \) off \( q \). Up to a constant this is the same as the solution of

\[
\Delta V = \begin{cases}
0 & \text{at } q \\
\delta & \text{elsewhere}
\end{cases}
\]

i.e. the Green's function for \( \Delta \) with the source \( q \). This function represents the voltage distribution when 1 amp is fed into the network at the vertex \( q \). The actual value of \( V \) at \( q \) gives the impedance of
Sign conventions: if $\sigma$ is an edge

Then $V(a) = z_{\sigma} I(\sigma) + V(b)$ or

$$(\delta V)(\sigma) = V(\partial \sigma) = V(b) - V(a) = -z_{\sigma} I(\sigma)$$

So $I = -\delta V$. Thus if $\gamma = Id$, then

$$(-\delta^* \delta V)(\gamma)$$

represents the net current flowing into the node $\gamma$. For example:

$$\delta^* \delta V(a) = -(\delta V(ac) - (\delta V(ab) + (\delta V(ad))$$

$$= V(a) - V(c) + V(b) + V(a) + V(d) + V(a)$$

$$= 3V(a) - V(b) - V(c) - V(d)$$

So $$\Delta V = -\delta^* \delta V$$

gives the sum of the $V$ values of the neighbors minus the number of neighbors times $V$ in the center. We get $\Delta V = -1$ at $\gamma$. 

Stephen Adler's problem: Square lattice in the plane with vertices $\mathbb{Z} \times \mathbb{Z}$. Here we can use the Fourier transform which gives an isomorphism between $l^2(\mathbb{Z} \times \mathbb{Z})$ and $L^2(S^1 \times S^1)$:

$$V(m,n) = \int_0^{2\pi} \int_0^{2\pi} e^{i(m_1 \theta_1 + n_2 \theta_2)} V(\theta_1, \theta_2)$$
\[ \hat{\Delta} V = (e^{i\theta_1} e^{-i\theta_1} + e^{i\theta_2} e^{-i\theta_2} - 4) V = -1 \]

so

\[ \hat{V}(\theta_1, \theta_2) = \frac{1}{4 - (e^{i\theta_1} e^{-i\theta_1} + e^{i\theta_2} e^{-i\theta_2})} \]

The denominator vanishes when \( \theta_1 = \theta_2 = 0 \), where it behaves like \( \theta_1^2 + \theta_2^2 \). Note that \( \frac{1}{\theta_1^2 + \theta_2^2} \) is not square-integrable, not even integrable.

\[ \iint \frac{d\theta_1 d\theta_2}{(\theta_1^2 + \theta_2^2)^2} = \iint \frac{r dr d\theta}{r^4} \]

Hence this Green’s function is not square integrable, even though it ought to be the only reasonably behaved solution to \( \hat{\Delta} V = \begin{cases} 1 & \text{at } 0 \\ 0 & \text{elsewhere} \end{cases} \).

To find the resistance of the network we take the V-value at zero:

\[ V(0,0) = \frac{1}{4} \int_0^{2\pi} d\theta_1 \int_0^{2\pi} d\theta_2 \frac{1}{1 - \frac{1}{4}(e^{i\theta_1} + e^{-i\theta_1} + e^{i\theta_2} + e^{-i\theta_2})} \]

or one expands \( \frac{1}{4} \frac{1}{1 - \frac{1}{4}(e^{i\theta_1} + e^{-i\theta_1} + e^{i\theta_2} + e^{-i\theta_2})} \)

in a Fourier series (say you use the geometric series) and find the constant term.

Actually since the function involved in the formula for \( V(0,0) \) isn’t integrable and is suspicious something is fishy... ??
Understand something about fundamental solutions.

In the plane \( \Delta = \frac{1}{r^2} \frac{\partial}{\partial r} r^2 \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} \); \( f(r) \) is a solution of \( \Delta f = 0 \) provided

\[
f'' + \frac{1}{r} f' = 0 \quad \text{ln}(f') + \text{ln} r = \text{const}
\]

\[
f' = \frac{c}{r} \quad \text{or} \quad f = c_1 \text{ln} r + c_2
\]

Green's formula

\[
\iint_R [(Au) v - u (Av)] dxdy = \oint_{\partial R} (\frac{\partial u}{\partial n} v - u \frac{\partial v}{\partial n}) \, ds
\]

Take \( v = \text{ln} r \), \( u \in C^\infty_0 (R^2) \) and let \( R \) be the outside of the circle of radius \( r \):

\[
\iint_R (Au)(\text{ln} r) \, dxdy = \int_0^{2\pi} \left( \frac{\partial u}{\partial n} \text{ln} r + u \frac{\partial}{\partial r} \right) \, ad\theta
\]

Let \( a \to 0 \) and use \( a \text{ln} r \to 0 \) to get

\[
\iint_{R^2} (Au)(\text{ln} r) \, dxdy = 2\pi u(0,0)
\]

Consequently because \( \Delta \) is its own adjoint we have by definition in the theory of distributions that

\[
\Delta \left( \frac{\text{ln} r}{2\pi} \right) = \delta(x) \delta(y)
\]

Interpretation: \( \frac{1}{2\pi} \text{ln} r \) is a locally integrable function, hence it defines a distribution, whose Laplacian is the \( \delta \) distribution at \( 0 \).

Notice that because \( \text{ln}(r) \leq r \) for \( r \) far out, \( \frac{1}{2\pi} \text{ln} r \)
is a tempered distribution on $\mathbb{R}^2$ and hence it has a Fourier transform which is also a tempered distribution. 

Suppose 

$$ u(x_1, x_2) = \int e^{i\xi_1 x_1} \hat{u}(\xi_1, \xi_2) \frac{d\xi_1}{2\pi} \frac{d\xi_2}{2\pi} $$

Then 

$$ \Delta u = \int -\left( \xi_1^2 + \xi_2^2 \right) \hat{u}(\xi_1, \xi_2) \frac{d\xi_1}{2\pi} \frac{d\xi_2}{2\pi} = 0 $$

forces

$$ -\left( \xi_1^2 + \xi_2^2 \right) \hat{u}(\xi_1, \xi_2) = 1 $$

This forces $\hat{u}$ to coincide with $\frac{1}{\xi_1^2 + \xi_2^2}$ away from 0, so we see the problem of division for distributions. Notice that any distribution $\varphi$ satisfying

$$ (\xi_1^2 + \xi_2^2) \varphi = 0 $$

has support at 0. I think that the only distributions with support at a point are linear combinations of $\delta$ and its derivatives. Hence it seems to me that the only solutions of $(\ast)$ are of the form

$$ \left( a \frac{\partial}{\partial x_1} + b \frac{\partial}{\partial x_2} + c \right) \delta $$

could be polynomials. Consequently the only tempered distribution solutions of Laplace's equation are linear functions $+poly.$

Similarly the only tempered distribution solutions of $\Delta u = 0$ are of the form $\frac{1}{r} \sin m \theta + \frac{1}{r^2} \sin n \theta + \sin m \phi$ and among these one can single out $\frac{1}{r^2} \sin m \theta + \sin n \phi$ as having the smallest growth at $\infty$. (Actually this only pins $u$ down to $\frac{1}{r^2} \ln r + \text{constant}.$)

In the adler problem, instead of $\hat{u}(\xi_1, \xi_2) = -\frac{1}{\xi_1^2 + \xi_2^2}$ off 0 we have

$$ \hat{u}(\theta_1, \theta_2) = -\frac{1}{\sin^2 \frac{\theta_1}{2} + \sin^2 \frac{\theta_2}{2}} $$
\( u(\theta_1, \theta_2) \) is a distribution on \( S^1 \times S^1 \) and it should correspond to a function on \( \mathbb{Z} \times \mathbb{Z} \) with at most polynomial growth.

The first remark to make is the purely algebraic statement that the only solutions of the Laplace equation

\[ \Delta u = 0 \]

on \( \mathbb{Z} \times \mathbb{Z} \) having polynomial growth are functions. The proof is by Fourier transform: Such a \( u \) corresponds to a distribution \( \widehat{u} \) on \( S^1 \times S^1 \) killed by \( \sin^2 \theta_1 + \sin^2 \theta_2 \) and hence \( \widehat{u} \) has support \( \{0\} \), etc.

Because

\[ -\frac{1}{4} \sin^2 \theta_1 + \sin^2 \theta_2 \]

behaves like \( -\frac{1}{\theta_1^2 + \theta_2^2} \)

as \( \theta_1, \theta_2 \to 0 \), I expect the fundamental solution \( u(m, n) \) to be asymptotic to \( \frac{1}{2\pi} \log \sqrt{m^2 + n^2} \) for \( m, n \) large.

Another idea: Replace \( \Delta \) by \( \Delta - \mu^2 \) which has an inverse for \( \mu > 0 \) in \( L^2 \). Then let \( \mu \to 0 \) to get the desired fundamental solution.

Try this for the Laplace equation in \( \mathbb{R}^2 \): Want a fundamental solution \( (\Delta - \mu^2) u = 0 \). If \( u = u(r) \) then we get

\[ \frac{1}{r} \frac{d}{dr} (r \frac{du}{dr}) - \mu^2 u = 0 \]

or

\[ (r \frac{d}{dr})^2 u - \mu^2 r^2 u = 0 \]

which is the imaginary Bessel's equation with \( \nu = 0 \). We
a solution with $I_0$, but not the $N_0$ solution which has a log in it. Notice that for $n \gg 0$, the equation
\[
\frac{d^2u}{dr^2} + \frac{1}{r} \frac{du}{dr} - \mu^2 u = 0
\]
should have solutions asymptotic to $e^{\pm \mu r}$. Now the $u$ we are after should decay as $r \to +\infty$, so we want the solution to be proportional to $K_0(\mu r)$.

Recall that the imaginary Bessel DE is
\[
-(r \frac{d}{dr})^2 u + r^2 u = -s^2 u
\]
and its solution decaying as \( r \to +\infty \) is
\[
K_s(r) = \int_0^\infty e^{-\frac{1}{2} (r + \frac{1}{r}) + s \frac{dt}{r}}
\]
Thus $K_s(\mu r)$ is the solution decaying as \( r \to +\infty \) of
\[
-(r \frac{d}{dr})^2 u + \mu^2 r^2 u = -s^2 u.
\]

It is easier to see in 3 dimensions where
\[
\Delta u = \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 \frac{\partial u}{\partial r}) + \frac{1}{r \sin \phi} \frac{\partial}{\partial \phi} (\sin \phi \frac{\partial u}{\partial \phi}) + \frac{1}{r^2 \sin^2 \phi} \frac{\partial^2 u}{\partial \theta^2}
\]
\[
ds^2 = dr^2 + r^2 d\phi^2 + (r \sin \phi)^2 d\theta^2
\]
\[
dV = r^2 \sin \phi \, dr \, d\phi \, d\theta
\]
\[
- \left( \frac{1}{r \sin \phi} \right) \frac{\partial^2}{\partial \phi \partial \phi} = \frac{1}{r^2 \sin \phi} \frac{\partial}{\partial \phi} \left( \frac{1}{r^2 \sin \phi} \right)
\]
\[
- \left( \frac{1}{r \sin \phi \, \partial \theta} \right)^2 = \frac{1}{r^2 \sin \phi \, \partial \theta} \frac{1}{r^2 \sin \phi}
\]
A radial solution of $\Delta u - \mu^2 u = 0$ satisfies
\[
\frac{1}{r^2} \frac{d}{dr} \left( r^2 \frac{du}{dr} \right) - \mu^2 u = 0
\]
\[
\frac{d^2 u}{dr^2} + \frac{2}{r} \frac{du}{dr} - \mu^2 u = 0
\]
\[
\frac{d^2}{dr^2} (ru) = \frac{d}{dr} \left( r \frac{du}{dr} + u \right) = r \frac{d^2 u}{dr^2} + 2 \frac{du}{dr} = \mu^2 (ru)
\]
hence
\[
u = c_1 e^{-\mu r} + c_2 e^{\mu r}.
\]
The solution decaying at $\infty$ is
\[
u = \frac{e^{-\mu r}}{r}
\]up to a constant.

\[
\iint_R (\nabla u \cdot \hat{n}) dV = 
\iiint_R \nabla u \cdot \hat{n} dS
\]
For $u = \frac{1}{r}$ and $R$ a ball of radius $a$ around zero, we have
\[
\iint_R \nabla u \cdot \hat{n} dS = \iiint_R -\frac{1}{a^2} dS = -\frac{1}{a^2} \frac{4\pi a^2}{3} = -\frac{4\pi}{3}
\]
Hence
\[
\Delta \left( \frac{1}{4\pi} \right) = -\delta. \quad \text{Thus} \quad u = -\frac{1}{4\pi} \frac{e^{\mu r}}{r}
\]is the fundamental solution $u$.

$$(\Delta - \mu^2) u = \delta$$

since $\frac{1}{r}$ is locally integrable: $dV = r^2 \sin \theta \, dr \, d\theta \, d\phi$, it defines a distribution; but because it is square-integrable around zero, $u = -\frac{1}{4\pi} \frac{e^{-\mu r}}{r} \in L^2(\mathbb{R}^3)$ for $\mu > 0$. As $\mu \to 0$ it goes to $-\frac{1}{4\pi}$ which is locally-square-integrable, but not globally so.

Try the same device with Adler's problem.

\[
\Delta(\theta_1, \theta_2) = \frac{1}{(e^{i \theta_1} e^{-i \theta_2} + e^{-i \theta_1} e^{i \theta_2} - 4 - \mu^2) - \frac{1}{4}}
\]
\[ u(0,0) = \int_0^{2\pi} \int_0^{2\pi} \frac{d\theta_1 d\theta_2}{2\pi \cdot 2\pi} \chi(\theta_1, \theta_2) \]

Try to evaluate this. If \( a > 1 \)

\[ \frac{1}{2\pi} \int_0^{2\pi} \frac{d\theta}{a - \cos \theta} = \frac{1}{2\pi i} \int_{\gamma} \frac{dz}{a - \frac{z^2 + 1}{2}} = \frac{1}{2\pi i} \int (-1) \frac{dz}{z^2 - 2a^2 + 1} \]

Roots \( z^2 - 2a^2 + 1 = 0 \)

\( z = a \pm \sqrt{a^2 - 1} = a - \sqrt{a^2 - 1} \)

inside \( S' \)

\[ \int_{S'} \frac{dz}{z^2 - 2a^2 + 1} = 4 - 2 \cdot \frac{1}{-2\sqrt{a^2 - 1}} = \frac{1}{\sqrt{a^2 - 1}} \]

Thus

\[ \frac{1}{2\pi} \int_0^{2\pi} \frac{d\theta}{a - \cos \theta} = \frac{1}{\sqrt{a^2 - 1}} \]

\( |a| > 1 \)

outside \( \gamma \)

We want

\[ \int_0^{2\pi} \frac{d\theta_1}{2\pi} \int_0^{2\pi} \frac{d\theta_2}{2\pi} \frac{1}{a - \cos \theta_1 - \cos \theta_2} = \int_0^{2\pi} \frac{d\theta}{2\pi} \frac{1}{\sqrt{(a - \cos \theta)^2 - 1}} \]

\( a > 2 \)

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\[ \int_0^{2\pi} \frac{1}{(a-1-\cos \theta)(a+1-\cos \theta)} \frac{d\theta}{2\pi} = \int_0^{2\pi} \frac{1}{\sqrt{(a-1-\frac{z^2 + 1}{2})(a+1-\frac{z^2 + 1}{2})}} \frac{dz}{2\pi i} \]

As \( a - 1 > 1 \), the roots of \( z^2 - 2(a-1)z + 1 \) are \( \lambda, \frac{1}{\lambda} \) with \( \lambda > 1 \) and \( \lambda + 1 \) as \( a \neq 2 \). Roots of \( z^2 - 2(a+1)z + 1 \) are \( \mu, \frac{1}{\mu} \) with \( \mu > 1 \). So put the plane:

The radical is defined to be \( + \sqrt{(a-2)(a+2)} \) at \( z = 1 \).

What happens as \( a \neq 2 \) as \( a + 1 \).
Problem: \[
\int_{-\pi}^{\pi} \frac{d\theta}{2\pi} \frac{1}{(a-\cos \theta)^2 - 1} \text{ approaches } +\infty \text{ as } a \to 2
\]

This is because the integrand is monotone increasing in \( a \) and positive. When \( a = 2 \) we have

\[
\frac{1}{(a-1-\cos \theta)(a+1-\cos \theta)} \to \frac{1}{\sqrt{2 \sin^2 \theta / 2}} = \frac{1}{|\sin \theta / 2| \sqrt{2(3-\cos \theta)}},
\]

which is not integrable.

So it appears that if

\[
U_{\mu}(\theta_1, \theta_2) = -\frac{1}{2} \frac{1}{a - \cos \theta_1 - \cos \theta_2}, \quad a = 2(1 + \frac{\mu^2}{4})
\]

then

\[
\lim_{\mu \to 0} U_{\mu}(0, 0) = -\infty.
\]

In fact

\[
U_{\mu}(0, 0) = \int_{\mathbb{R}} \frac{d\theta_1}{2\pi} \frac{d\theta_2}{2\pi} \left( \frac{1}{a - \cos \theta_1 - \cos \theta_2} \right)
\]

and

\[
\frac{1}{a - \cos \theta_1 - \cos \theta_2}
\]

is a positive function increasing as \( a \to 2 \) to a non-integrable function.

On the other hand I can argue that there has to be a fundamental solution for \( \mu = 0 \), by using theorems on the division of distributions by polynomials.

**Example:** Take \( \mathcal{R} \). \( \Delta = \frac{d^2}{dx^2} \). If we want

\[
(\Delta - \mu^2) G = \delta(x-y)
\]

\[
(\frac{d^2}{dx^2} - \mu^2) f = 0 \quad e^{\pm \mu x}
\]

the formula is

\[
G(x, y) = \frac{e^{\mu x} e^{-\mu y}}{\left| \begin{array}{cc} 1 & -\mu \\ \mu & -\mu \end{array} \right|} = \frac{e^{-\mu |x-y|}}{-2\mu}
\]
So the fundamental solution for $\mu > 0$ is
\[
\frac{e^{-\mu x}}{-2\mu}
\]
Calculate its F.T.
\[
\int e^{-ix} \frac{e^{-\mu|x|}}{-2\mu} \, dx = \int_0^\infty e^{-x(i\frac{\pi}{2} + \mu)} \frac{dx}{-2\mu} + \int_{-\infty}^0 e^{x(i\frac{\pi}{2} + \mu)} \frac{dx}{-2\mu}
\]
\[
= -\frac{1}{2\mu} \left\{ \frac{1}{i\frac{\pi}{2} + \mu} + \frac{1}{-i\frac{\pi}{2} + \mu} \right\} = \frac{-1}{\frac{\pi^2}{4} + \mu^2}
\]
Now if you let $\mu \to 0$, then the fundamental solution doesn't converge to a fundamental solution for $\Delta$.
However if you add a suitable solution of $\Delta u = 0$ you get
\[
\frac{e^{-\mu|x|} - e^{\mu x}}{-2\mu} = \begin{cases} 
0 & x \leq 0 \\
\frac{e^{\mu x} - e^{-\mu x}}{2\mu} & x > 0 
\end{cases}
\]
\[
\lim_{\mu \to 0} \frac{e^{-\mu|x|} - e^{\mu x}}{-2\mu} = \begin{cases} 
0 & x \leq 0 \\
x & x > 0 
\end{cases}
\]
which is a fundamental solution for $\Delta$.

Example: $\mathbb{Z}$, Homogeneous solutions of
\[
(\Delta - \mu^2) u = 0
\]
are $e^{i\theta}$ where $e^{i\theta} + e^{-i\theta} - 2 - \mu^2 = 0$ or
\[
\cos \theta = \lambda, \quad \lambda = 1 + \frac{\mu^2}{2}
\]
Let $z = \lambda - \sqrt{\lambda^2 - 1}$ be the root with $0 < z < 1$. Then
we have \( \phi(n) = z^{-n} \quad \psi(n) = z^n \)

and

\[
W(\phi, \psi) = \begin{vmatrix} 1 & 1 \\ z^{-1} & z \end{vmatrix} = z - z^{-1}
\]

so the fundamental solution for \( \Delta - \mu^2 \) is

\[
u(n) = \frac{z^{-1} \ln|z|}{z - z^{-1}}
\]

As \( \mu \to 0 \), \( z \to 1 \) and \( z \to 1 \) and this fundamental solution blows up. On the other hand the fund. solution

\[
z^{-1} - \frac{1}{2} (z^n + z^{-n})
\]

\[
\frac{z^{-n} - z^n}{z - z^{-1}}
\]

\[
\begin{cases}
\frac{1}{2} \frac{z^n - z^{-n}}{z - z^{-1}} & n > 0 \\
\frac{1}{2} \frac{z^{-n} - z^n}{z - z^{-1}} & n < 0
\end{cases}
\]

\[
\text{as } z \to 1
\]

\[
\begin{cases}
\frac{1}{2} n & n > 0 \\
-\frac{1}{2} n & n < 0
\end{cases}
\]

and so we get the fundamental solution \( \frac{1}{2} \ln|z| \)

for \( \Delta \).

It appears therefore that the "voltage distribution which results by pushing 1 ampere into a node of an infinite network" is a subtle thing which gets explained adequately with distributions, at least for periodic networks where this is a good dual.

Notice that for the Adler problem I expect the fundamental solution to grow like \( \frac{1}{2\pi} \log r \), hence the
voltage is $-\infty$ at $\infty$? Actually even for $\Delta \log \mathbf{r}$, we can alter this by adding a constant. So now I don’t understand how one can even speak of the resistance of this network?

The thing to do it seems is put in the boundary condition that $u = \frac{1}{2\pi} \log \mathbf{r} \to 0$.

Geometry of the UHP: $\Im z > 0$. First determine the Riemann metric which is the $SL_2 \mathbb{R}$ invariant metric

with $ds^2 = |dz|^2 = dx^2 + dy^2$ at $i$. Let $(a \ b) \in SL_2 \mathbb{R}$, and

$$z = \frac{a+i+b}{ci+d} = \frac{(ai+b)(-ci+d)}{c^2+d^2} = \frac{((a+b)+i)}{c^2+d^2}$$

$$y = \frac{1}{|ci+d|^2}$$

Suppose we have a tangent vector $i + \varepsilon$ at $i$. Its image is

$$a(i+\varepsilon)+b = \frac{a+b+i\varepsilon}{ci+d}$$

$$c(i+\varepsilon)+d = \frac{c(i+\varepsilon)+d}{ci+d}$$

Hence the Jacobian mapping the tangent space at $i$ to the tangent space at $ai+b = \frac{1}{ci+d}$ is

$$\frac{ai+b}{ci+d} = i \iff c = -b, \ d = a \iff \begin{pmatrix} a & b \\ -b & a \end{pmatrix} = \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix}$$

In this case $\frac{1}{(ci+d)^2} = \frac{1}{(\cos \theta - i \sin \theta)^2} = e^{2i\theta}$ preserves the metric $|dz|^2$ as it should. The Riemann
metric is defined so that
\[ \| a + b + \frac{c}{c + d} \| = \| a + e \| = |e| \]
\[ \| a + b + \eta \| = |\eta| |c + d| = \frac{|\eta|}{y} \]
In other words, using Cartan's notation $dz$ for a displacement from $z$ to $z + dz$, the length (non-Euclidean) of this displacement is
\[ \| dz \| = \frac{|dz|}{y} \quad \text{i.e.} \quad ds^2 = \frac{dx^2 + dy^2}{y^2} \]
where $|dz| = \sqrt{dx^2 + dy^2}$ is the Euclidean length. The non-Euclidean volume is therefore
\[ dV = \frac{dx dy}{y^2} \]

Pass to the circle. \[ |z| < 1, \quad (a, b) = (\frac{a}{b}, \frac{b}{a}) \]
\[ z = \begin{pmatrix} a & b \\ c & d \end{pmatrix} (0) = \frac{1}{d} \]
\[ a c + b = a c + b - c e + d = \frac{bd + e}{d^2} = \frac{b}{d} + \frac{e}{d^2} \]
So\[ \| a c + b + \eta \| = |\eta| d^2 = |\eta| \cdot |d| \]
But\[ |z|^2 = \frac{|b|^2}{|a|^2} = \frac{|b|^2}{|a|^2} \Rightarrow 1 - |z|^2 = 1 - \frac{|b|^2}{|a|^2} = \frac{1}{|a|^2} = \frac{1}{|d|^2}, \text{ so} \]
\[ \| z + \eta \| = \frac{|\eta|}{1 - |z|^2} \quad \text{or in Cartan's form:} \]
\[ \| dz \| = \frac{|dz|}{1 - |z|^2} \quad \text{dV} = \frac{dx dy}{(1 - |z|^2)^2} \]

Compute $\Delta$ for UHP. We have
\[ \| \frac{\partial}{\partial x} \| = \frac{1}{y} |\frac{\partial}{\partial x}| = \frac{1}{y} \quad \| \frac{\partial}{\partial y} \| = \frac{1}{y} \]
hence an orthonormal pair of vector fields is
$$y \frac{\partial}{\partial x}, \ y \frac{\partial}{\partial y}$$
Their adjoints with respect to $dv = \frac{1}{y^2} \, dx \, dy$ are minus
$$y^2 \frac{\partial}{\partial x} \frac{1}{y} \quad y^2 \frac{\partial}{\partial y} \frac{1}{y}$$
so
$$\Delta = \left( y^2 \frac{\partial}{\partial x} \frac{1}{y} \frac{\partial}{\partial x} \frac{1}{y} \right) + \left( y^2 \frac{\partial}{\partial y} \frac{1}{y} \frac{\partial}{\partial y} \frac{1}{y} \right)$$
$$\Delta = y^2 \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right)$$
A similar calculation shows that an orthonormal frame in the case of the circle is
$$\frac{1}{(1-|z|^2)} \frac{\partial}{\partial x}, \ \frac{1}{(1-|z|^2)} \frac{\partial}{\partial y}$$
and hence
$$\Delta = \frac{1}{(1-|z|^2)} \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right)$$
by a similar calculation.

It should follow that a fundamental solution for Laplace's equation: $\Delta u = \delta(0)$ is the same in the Euclidean case: $u = \frac{1}{2\pi} \ln r$, solutions of $(\Delta - \mu^2)u = \delta$ should however be more interesting.