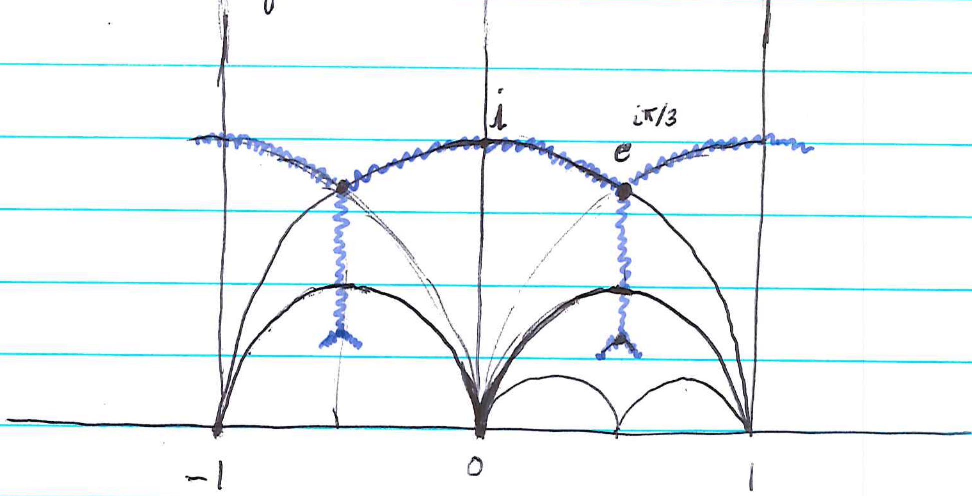
January 24, 1978

Problem: Describe an invariant wave equation on the $PSL_2(\mathbb{Z})$ tree. Picture of the tree:



The edge from i to $e^{i\pi/3}$ is a fundamental domain for the action of $\Gamma = PSL_2(\mathbb{Z})$. The stabilizer of i is the cyclic group of order 2 generated by $z \longmapsto -\frac{1}{z}$ and the stabilizer of $e^{i\pi/3}$ is the cyclic group of order 3 generated by

$$z \longmapsto z-1 \longmapsto -\frac{1}{z-1} = \frac{1}{1-z}$$

which rotates $120°$ at the point $e^{i\pi/3}$. The stabilizer of the edge is $\{1\}$, so we have

$$\Gamma \cong (\mathbb{Z}/2) * (\mathbb{Z}/3)$$

The problem is to set up an invariant wave equation on this tree. I am going to think of each edge in the tree as a transmission line and at each vertex we have junctions. Then a state of the system will be a function defined on each edge of the edge with values in a 2 diml complex vector space whose elements I think of as (iV, I) where V is voltage and I is current.

Thus given a point x in the interior of an edge I can speak of $(V(x), I(x))$. Orient each edge from the $\mathbb{Z}_2$ to the $\mathbb{Z}_3$ end.

On this space of states we should have an energy norm defined as follows. From $(V(x), I(x))$ we ought to be able to compute the energy stored in each edge and each junction and thus get a total energy. (Note: $V(x)$ is to be thought of as real, also $I(x)$.)

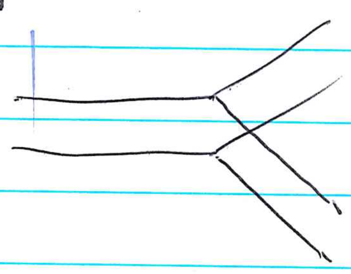
Time evolution should define a one-parameter unitary group, hence a self-adjoint operator on the Hilbert space of states. A fundamental question is whether one can choose a) transmission line b) i-junction c) $e^{i\pi/3}$-junction so that Γ acts on the Hilbert space of states preserving the unitary group, and such that the spectrum is <u>discrete</u>.

The basic problem is to show that at any frequency ω one half of the graph has a characteristic impedance.

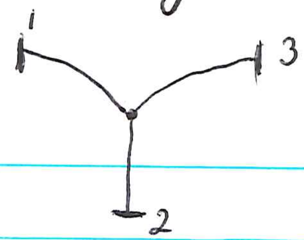
Suppose that our edge has the chain matrix $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ so that

$$iZ_1 = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \left(i Z_2 \right)$$

Suppose for the moment that the three port connection is the familiar one:



We maybe should be viewing.



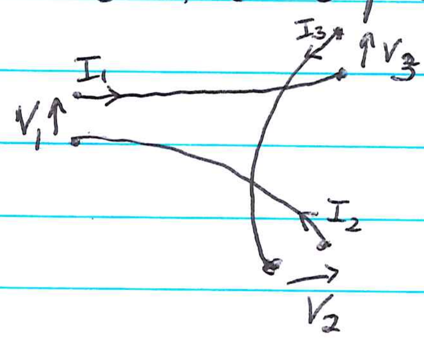
as a 3-port. Then there is a relation this 3-port gives between the impedences say

$$Z_1 = F(Z_2, Z_3)$$

The characteristic impedance should be given by solution of

$$Z = F(Z, Z)$$

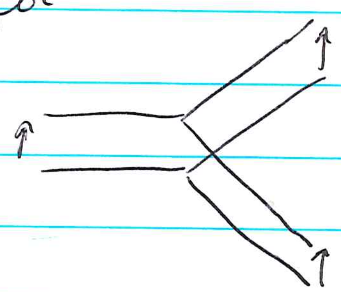
For example, take the 3-port



so that $I_1 = I_2 = I_3$, $V_1 + V_3 + V_2 = 0$. Then the impedance relation is

$$Z_1 = Z_2 + Z_3$$

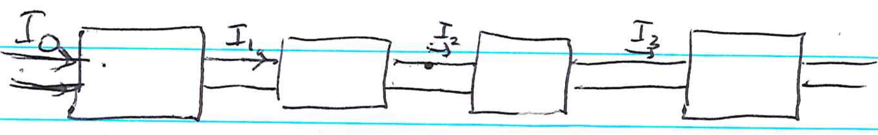
and so the characteristic impedance is either 0 or ∞. Similarly for



one has

$$\frac{1}{Z_1} = \frac{1}{Z_2} + \frac{1}{Z_3}$$

Characteristic impedance of a 2 port. Connect an infinite number of copies of the same two-port in cascade



Let $A=\begin{pmatrix}A&B\\C&D\end{pmatrix}$ be the chain matrix of the 2 port – defined so that

$$\begin{pmatrix} iV_1\\ I_1\end{pmatrix} = \begin{pmatrix}A&B\\C&D\end{pmatrix}\begin{pmatrix}iV_2\\ I_2\end{pmatrix}$$

Suppose the 2-port lossless operating at a real frequency so that $A\in SL_2(\mathbb{R})$. We have

$$\begin{pmatrix} iV_0\\ I_0\end{pmatrix} = A^{n}\begin{pmatrix} iV_n\\ I_n\end{pmatrix}$$

The good case occurs when A has real eigenvalues, i.e. $|\mathrm{tr}\,A| > 2$. If $\begin{pmatrix} iV_0\\ I_0\end{pmatrix}$ is an eigenvector for A with eigenvalue $|\lambda| > 1$, then

$$\begin{pmatrix} iV_n\\ I_n\end{pmatrix} = \lambda^{-n}\begin{pmatrix} iV_0\\ I_0\end{pmatrix}$$

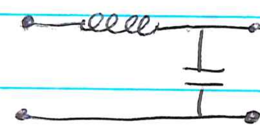
so that $\begin{pmatrix} iV_n\\ I_n\end{pmatrix} \longrightarrow 0$ as $n\to\infty$. Hence we get a physical "possibility". In some sense when I apply a voltage V_0 at the left end, the current I_0 flows in the network. Notice that the impedance $\frac{Z_0}{I_0}$ is purely imaginary because the eigenvalues are real, hence the eigenspaces are defined over $\mathbb{R}$. Hence we have a loss-less 1-port.

Let's think of an iterated 2-port [redacted] as an analogue of Hill's equation, i.e. Schroedinger's equation with a periodic potential. Suppose the transfer matrix $A(\lambda)$ has an eigenvalue on the unit circle. The other eigenvalue is also on the circle, necessarily conjugate to the first, hence $\operatorname{tr} A(\lambda) \in [-2,2]$ and conversely this condition implies that the eigenvalues are conjugate points on the circle.

When $\operatorname{tr} A(\lambda) \notin [-2,2]$ one eigenvalue is outside S^1, hence there is an exponentially decaying state, which means the characteristic ([redacted] better terminology: iterative impedance is defined). The set $\{\lambda \mid \operatorname{tr} A(\lambda) \in [-2,2]\}$ is a family of bands on the real line. ($\lambda \notin \mathbb{R} \Rightarrow \exists\ \ell^2$ eigenfunction, hence some eigenvalue of $A(\lambda)$ has to be outside S^1.)

I don't whether there exists a 2-port such that $\operatorname{tr} A(\lambda)$ is constant without $A(\lambda)$ being constant.

So it seems that iterative impedance is defined for a 2-port for all frequencies outside the pass-bands.

Example: For the 2-port [redacted]  one has

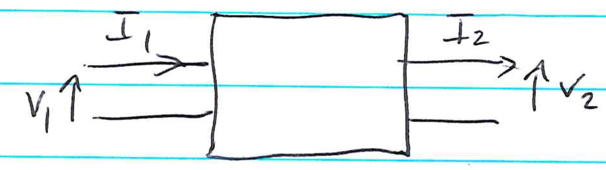
$$A = \begin{pmatrix} 1 & -L\omega \\ & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ C\omega & 1 \end{pmatrix} = \begin{pmatrix} 1 - LC\omega^2 & -L\omega \\ C\omega & 1 \end{pmatrix}$$

hence $\operatorname{tr}(A) = 2 - LC\omega^2$. The pass-bands are given by

$$-2 \le 2 - LC\omega^2 \le 2 \quad \text{or} \quad 0 \le LC\omega^2 \le 4 \quad \text{or}$$

$$|\omega| \le \frac{2}{\sqrt{LC}}$$

Recall that the reverse of a 2-port is obtained by interchanging labelling of the terminals. If the 2port is described by the transfer matrix A:



$$\begin{pmatrix} iV_1 \\ I_1 \end{pmatrix} = A \begin{pmatrix} iV_2 \\ I_2 \end{pmatrix}$$

then the reversed 2-port is described by the transfer matrix R_A given by

$$\begin{pmatrix} iV_2 \\ -I_2 \end{pmatrix} = R_A \begin{pmatrix} iV_1 \\ -I_1 \end{pmatrix} \qquad A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

i.e. $\qquad A^{-1} = D R_A D \qquad D = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

hence

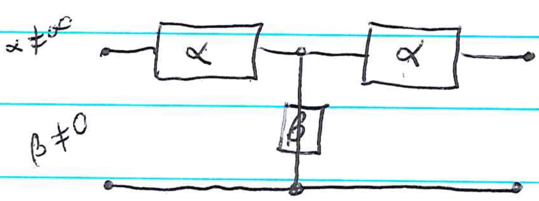
$$R_A = DA^{-1}D = D\begin{pmatrix} d & -b \\ -c & a \end{pmatrix} D = \begin{pmatrix} d & b \\ c & a \end{pmatrix}$$

and so R_A is obtained by interchanging diagonal entries. Reversible and symmetric 2-port are the same thing

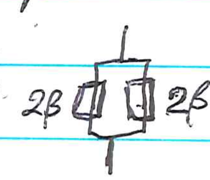
If B is any 2-port, $B \cdot R_B$ is reversible:

$$D(B \cdot R_B)^{-1} D = D(DB^{-1}D)^{-1} D \cdot DB^{-1}D = B \cdot R_B$$

and it is natural to inquire if every reversible 2-port can be decomposed in this form. Consider a T-network



where the boxes denote impedances. We can break β up into two get a 2-port cascades with its reverse.



The chain matrix is

$$\begin{pmatrix} i & \\ & 1 \end{pmatrix}\begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ \frac{1}{\beta} & 1 \end{pmatrix}\begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix}\begin{pmatrix} i & 0 \\ 0 & 1 \end{pmatrix}^{-1}$$

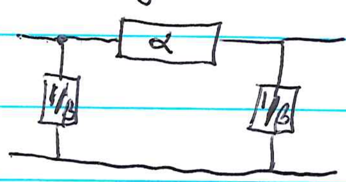
$$'' \quad \begin{pmatrix} 1+\frac{\alpha}{\beta} & \alpha \\ \frac{1}{\beta} & 1 \end{pmatrix}\begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix} \quad '' \quad = \begin{pmatrix} 1+\frac{\alpha}{\beta} & i\left(2\alpha+\frac{\alpha^2}{\beta}\right) \\ \frac{1}{i\beta} & \frac{\alpha}{\beta}+1 \end{pmatrix}$$

If we want this to be $\begin{pmatrix} a & b \\ c & a \end{pmatrix}$ we choose β so that

$$\frac{1}{i\beta} = c$$

and then α such that $1+\frac{\alpha}{\beta} = a$. Notice that β is not permitted to be zero, although $\beta = \infty$ is allowed, if one wants a transfer matrix. So if $c \neq 0$ we get unique α, β. If $c = 0$, then we have to take $\frac{1}{\beta} = 0$, hence a has to be 1.

Try a π-network



$\alpha \neq \infty$
$\beta \neq \infty$

$$\begin{pmatrix} -i & \\ & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ \beta & 1 \end{pmatrix}\begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ \beta & 1 \end{pmatrix}\begin{pmatrix} -i & 0 \\ 0 & 1 \end{pmatrix}$$

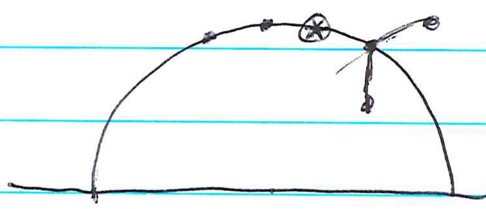
$$'' \quad \begin{pmatrix} 1 & \alpha \\ \beta & \beta\alpha+1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ \beta & 1 \end{pmatrix} \quad ''$$

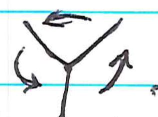
$$= \begin{pmatrix} 1+\alpha\beta & \alpha i \\ -i(2\beta+\beta^2\alpha) & \beta\alpha+1 \end{pmatrix}$$

hence if $b \neq 0$ we can put $i\alpha = b$ and then solve $1+\alpha\beta = a$ for β. If $b = 0$ we have to take $\alpha = 0$ hence a has to be 1.

So it seems that the only symmetric 2-port we might have trouble fitting into the form $B \cdot R_B$ is where $b=c=0$ and $a=-1$. However if $A = R_A$, then the same is true for $B^{-1} A R_{B^{-1}}$ so it's clear that we can handle $A = -I$ by this device.

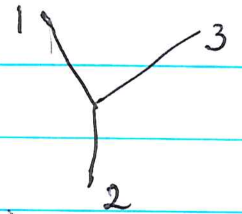
So recall the program. You want to construct an electrical system on the $PSL_2(\mathbb{Z})$ tree which is invariant under the group action. Pick a point* in the interior of the edge from i to $e^{i\pi/3}$, and suppose we choose a voltage-current basis for the system at ⊛. Since $\Gamma = PSL_2(\mathbb{Z})$ acts freely on the tree we then get a unique invariant way of choosing voltage-current at all the points $\gamma(⊛)$, $\gamma \in \Gamma$. Then I can think of the system as being built up out a [redacted] symmetrical 2 port and a 3-port



invariant under cyclic permutation .

The above calculations show that at least for a fixed real frequency one can split any symmetrical 2 port into a 2-port connected in cascade with its reverse. Hence I can attach arms to the [redacted] 3 port and so assume the symmetrical 2 port is the identity.

Let's assume the 3-port has an impedance matrix Z. Since it is supposed lossless, iZ is hermitian and even symmetric (hence real) if the 3-port is reciprocal. How is the cyclic symmetry to be expressed?



If $I = \begin{pmatrix} I_1 \\ I_2 \\ I_3 \end{pmatrix}$ is a set of currents and $V = ZI$ is the resulting voltages, then the same should be true for FI, FV where F is the symmetry

$$F = \begin{pmatrix} & & 1 \\ 1 & & \\ & 1 & \end{pmatrix}$$

Hence we want $FZ = ZF$ and so Z has the form

$$iZ = \begin{pmatrix} a & b & c \\ c & a & b \\ b & c & a \end{pmatrix}$$

Be carefull:

$$FZ = \begin{pmatrix} & & 1 \\ 1 & & \\ & 1 & \end{pmatrix} \begin{pmatrix} z_{11} & z_{12} & z_{13} \\ z_{21} & z_{22} & z_{23} \\ z_{31} & z_{32} & z_{33} \end{pmatrix} = \begin{pmatrix} z_{31} & z_{32} & z_{33} \\ z_{11} & z_{12} & z_{13} \\ z_{21} & z_{22} & z_{23} \end{pmatrix}$$

$$ZF = \begin{pmatrix} z_{11} & & \\ & & \\ & & \end{pmatrix} \begin{pmatrix} & & 1 \\ 1 & & \\ & 1 & \end{pmatrix} = \begin{pmatrix} z_{12} & z_{13} & z_{11} \\ z_{22} & z_{23} & z_{21} \\ z_{32} & z_{33} & z_{31} \end{pmatrix}$$

$z_{11} = z_{22} = z_{33}$

$z_{12} = z_{23} = z_{31}$ OKAY

$z_{13} = z_{21} = z_{32}$

Since iZ is hermitean, a is real and $b = \bar{c}$. In the reciprocal case $b = c$ are real, and the system is also symmetric under the full symmetric group Σ_3.

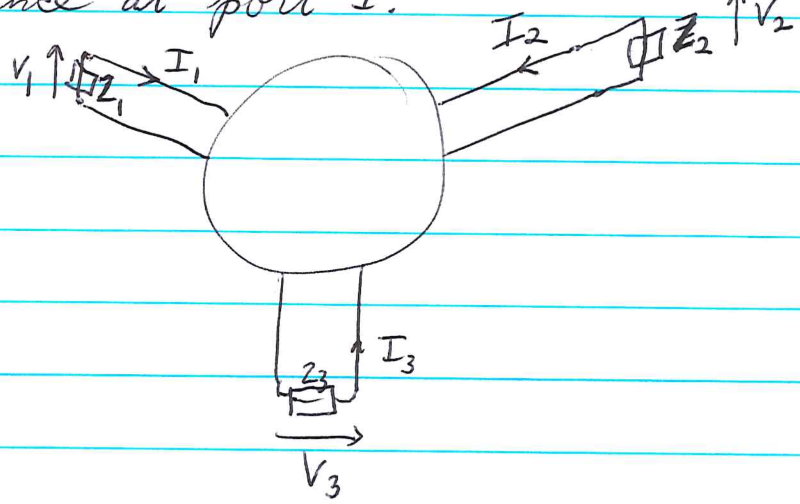
Now the problem is the formula giving the impedance at [redacted] port 1 when impedances Z_2, Z_3 are connected at the other ports.

Interesting Point: One might that it should be possible to connect up arbitrary impedances Z_2, Z_3 and get an impedance Z_1, but this would give you an alg. map

$$\mathbb{P}^1 \times \mathbb{P}^1 \longrightarrow \mathbb{P}^1$$

which seems unlikely. Somehow the positivity conditions have to enter.

Suppose that impedances Z_2, Z_3 are connected at ports 2,3 respectively and that Z_1 is the resulting impedance at port 1.



This means that [redacted] there exists a non-zero current distribution I_1, I_2, I_3 such that

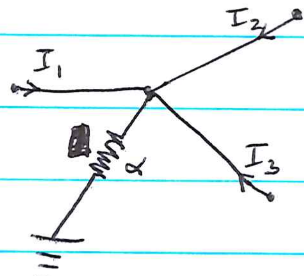
$$\begin{pmatrix} Z_1 I_1 \\ Z_2 I_2 \\ Z_3 I_3 \end{pmatrix} = Z \begin{pmatrix} I_1 \\ I_2 \\ I_3 \end{pmatrix}$$

which therefore gives the relation

$$\begin{vmatrix} a-Z_1 & b & c \\ c & a-Z_2 & b \\ b & c & a-Z_3 \end{vmatrix} = 0$$

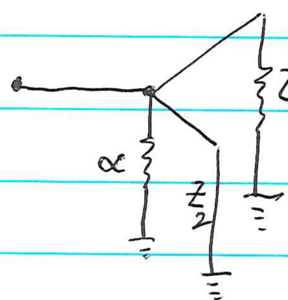
Notice that this gives Z_1 as a rational function of Z_2, Z_3.

Example: Consider the 3-port



which has the impedance matrix $\begin{pmatrix} \alpha & \alpha & \alpha \\ \alpha & \alpha & \alpha \\ \alpha & \alpha & \alpha \end{pmatrix}$

So now connect up impedances Z_2, Z_3



$$\frac{1}{Z_1} = \frac{1}{\alpha} + \frac{1}{Z_2} + \frac{1}{Z_3}$$

$$\begin{vmatrix} \alpha-Z_1 & \alpha & \alpha \\ \alpha & \alpha-Z_2 & \alpha \\ \alpha & \alpha & \alpha-Z_3 \end{vmatrix} = \alpha^3 \begin{vmatrix} 1-\frac{Z_1}{\alpha} & 1 & 1 \\ 1 & 1-\frac{Z_2}{\alpha} & 1 \\ 1 & 1 & 1-\frac{Z_3}{\alpha} \end{vmatrix} =$$

$$\alpha^3\left\{\left(1-\frac{z_1}{\alpha}\right)\left(1-\frac{z_2}{\alpha}\right)\left(1-\frac{z_3}{\alpha}\right)-\left(1-\frac{z_1}{\alpha}\right)+1-\left(1-\frac{z_3}{\alpha}\right)+1-\left(1-\frac{z_2}{\alpha}\right)\right\}$$

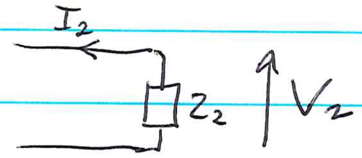
$$=\alpha^3\left\{\cancel{1}-\cancel{\frac{z_1+z_2+z_3}{\alpha}}+\frac{z_1z_2+z_2z_3+z_1z_3}{\alpha^2}-\frac{z_1z_2z_3}{\alpha^3}\right\}$$

$$=\alpha(z_1z_2+z_2z_3+z_1z_3)-z_1z_2z_3=0$$ This

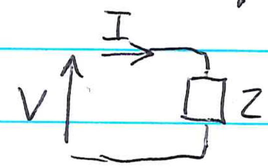
becomes $\frac{1}{z_1}+\frac{1}{z_2}+\frac{1}{z_3}=\frac{1}{\alpha}$,

so I have to get the signs straight.

The signs are as follows. To connect up Z_2 at port 2 means that $V_2=-Z_2I_2$



In effect I think of impedance as measuring ratio of voltage to current flowing in:



$V=ZI$

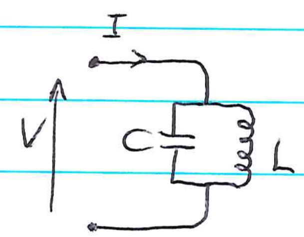
Hence we see that the impedance relation has to be written

$$\begin{vmatrix} a-z_1 & b & c \\ c & a+z_2 & b \\ b & c & a+z_3 \end{vmatrix}=0$$

January 26, 1978

Consider an invariant electrical system on the $PSL_2(\mathbb{Z}) = \Gamma$ tree. Assume ω is a frequency such that there is an ℓ^2 solution and the space K_ω of these is 1-dimensional. Consider reflection at the point i of the tree; this has to be ± 1 on K_ω. If $+1$, then $I=0$ at i; if -1, then $V=0$ at i. Similarly $120°$ rotation around $e^{i\pi/3}$ has to act on K_ω by a cube root of unity. This implies that (V,I) changes by 6th roots of unity as we move around the tree via elements of Γ. So we can't have an ℓ^2-solution.

I still don't understand very well what should be meant by the impedance of an infinite tree-like circuit.

Example: 

$\cancel{Z = \frac{1}{\frac{1}{Cs} + Ls} = \frac{Cs}{1+LCs^2}}$

$$Z = \frac{1}{\frac{1}{1/Cs} + \frac{1}{Ls}} = \frac{1}{Cs + \frac{1}{Ls}} = \frac{Ls}{LCs^2+1} = \frac{iL\omega}{1-LC\omega^2}$$

This blows up at $\omega = \frac{1}{\sqrt{LC}}$, the resonant frequency.

Try to understand networks. Suppose X is a connected oriented graph. A <u>current-flow</u> on X is a 1-cycle, i.e. to each edge you have a current assigned which is a complex number whose real part is relevant, and the Kirchoff Current laws state you have a 1-cycle. Voltage is a 1-coboundary, that is, to each edge you have a potential difference, and it is possible to find a potential function defined at all vertices explaining these differences.

In each edge the current and voltage are related (This ignores mutual inductance.) as follows:

$$\frac{dV}{dt} = \frac{1}{C} I \qquad \text{for a capacitance}$$

$$(*) \qquad V = RI \qquad \text{for a resistance}$$

$$V = L\frac{dI}{dt} \qquad \text{for an inductance}$$

Hence a state of the system consists of a function $t \longmapsto (V(t), I(t))$, $V(t)$ a 1-coboundary, $I(t)$ a 1-cycle, such that on each edge the above equations hold.

One would like to find independent variables for all these equations, i.e. describes the states as a manifold with a flow. Treat it like an over-determined system of DE's.

Since we have a system of DE's with constant coefficients one looks for exponential solutions, (or even exponential-polynomial solutions in cases of high multiplicity. If

$$\begin{pmatrix} V(t) \\ I(t) \end{pmatrix} = \begin{pmatrix} V e^{i\omega t} \\ I e^{i\omega t} \end{pmatrix}$$

then the equations (*) get translated into impedance equations linking the voltage and current "phasors" associated to each edge. Thus one looks for solutions of the equations

$$V \text{ coboundary, i.e. } \int_C V = 0 \quad C \text{ cycle}$$

$$I \text{ cycle, i.e. } dI = 0.$$

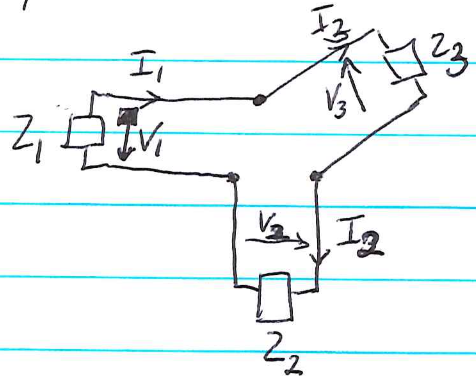
$$V = Z \cdot I$$

where Z is the impedance matrix. If one treats the cycles as independent variables, then the condition is that $\int_C ZI = 0$ for all cycles C. Thus we have the same number of equations as unknowns, and so only for certain ω will there exist solutions. The really good case is whether the multiplicity is 1, i.e. there is at most one solution + some non-degeneracy condition.

January 27, 1978

Can one derive the Lee-Yang theorem electrically? The idea might be to consider the c_{ij} as giving 2-ports running along the edges, and at each vertex you connect a variable impedance given by one of the z_i. The vanishing of the Lee-Yang poly thus tells when the z_i's are such as to allow natural oscillations of the system.

Example: Consider the series connection 3-port



The equations are:

Impedance equations for branches

$$V_i = Z_i I_i \qquad i = 1, 2, 3$$

KCL $\quad I_1 = I_3 = I_2$

KVL $\quad V_1 + V_2 + V_3 = 0$

Eliminating the I_i and the V_i we get the impedance relation:

$$Z_1 + Z_2 + Z_3 = 0$$

If this holds, $\exists$ a solution (V_i, I_i) unique up to a scalar. But I want to understand this from the viewpt of $Z_i \in P^1$. So we want to write the impedance equations in the form

$$A_i V_i = B_i I_i \qquad i = 1, 2, 3$$

where not both A_i and B_i vanish for each i. If A_1, A_2, A_3 are all $\neq 0$ the impedance relation is

$$\frac{B_1}{A_1} + \frac{B_2}{A_2} + \frac{B_3}{A_3} = 0 .$$

What happens when some A_i is zero? If A_2 is 0, i.e. $Z_2=\infty$, then since $B_2 \neq 0$ we have $I_2=0$, hence all currents are zero, so the equations become

$$\begin{cases} V_1+V_2+V_3=0 \\ A_1V_1=0 \\ A_3V_3=0 \end{cases}$$

If $A_3 \neq 0$, i.e. $Z_3 \neq \infty$, then we must have $A_1 = 0$ for there to exist a non-zero soln, which is then essentially unique. But if $A_3=0$, then A_1 can be anything and there will exist non-zero solutions. If $A_1 \neq 0$, one has $V_1=0$, V_2 arbitrary. So you have solutions with $(V_1,I_1)=0$, hence not defining a Z_1.

In general you have a space

$C' \oplus C_1 = W_1 \times \cdots \times W_p$ $\qquad$ p = number of edges

where W_i is the 2-diml space belonging to an edge whose elements are (V_i, I_i) pairs. Giving the impedances (or more precisely A_i, B_i for each edge) gives a subspace

$$\Gamma_Z \subset C' \times C_1$$

Maybe it would be better to cut down to the subspace K of $C' \times C_1$ defined by the Kirchoff conditions:

$$0 \to K \to C' \times C_1 \longrightarrow H' \times \tilde{C}_0 \to 0$$

so $\qquad K \xrightarrow{\sim} \tilde{C}^0 \times H_1$. But then

$$\Gamma_Z = L_1 \times \cdots \times L_p \subset W_1 \times \cdots \times W_p$$

so our configurations are $\Gamma_Z \cap K$. Now let's look at at the edge i belonging to a port. In the generic case $\Gamma_2 \cap K$ is one-dimensional and its image in W_i is L_i.

~~The important thing here is K. It doesn't change if we remove an edge from the graph without disconnecting it~~

If all $L_i \subset W_i$ are given $i = 2, \dots, n$, then we we can form

$$K \cap W_1 \times L_2 \times \dots \times L_p$$

$p = c_1$ no of edges
$1 - \beta_1 = c_0 - p$

Codim(K) is $\beta_1 + c_0 - 1 = p$, so that in good cases this intersection is 1-dimensional and its projection into W_1 is a line L_1 giving us an impedance Z_1.

July 28, 1978:

Recall (p.695) that on a transmission line with $L=C=1$ the quantity $a = \frac{1}{2}(V+I)$ behaves like a wave travelling to the right, that is, it is of the form $f(x-t)$; also $b = \frac{1}{2}(V-I)$ behaves like a wave travelling to the left. If one uses (a,b) instead of (V,I) to describe ports, then the behavior of an n-port is described by a scattering matrix S:

$$b = Sa$$

One has $V = a+b$, $I = a-b$ so that if an impedance matrix Z exists one has

$$a+b = Z(a-b) \qquad (Z+1)b = (Z-1)a$$

so

$$\boxed{S = \frac{Z-1}{Z+1}}$$

Given an n-port with scattering matrix S, let's derive the relation between the impedances at the ports. Thus we want to have $b_i = \rho_i a_i$ at the i-th port, as well as, $b = Sa$ hence eliminating, we get the condition

$$\left| \begin{pmatrix} \rho_1 & & \\ & \ddots & \\ & & \rho_n \end{pmatrix} - S \right| = 0$$

Example: $S = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$ Then $\begin{vmatrix} \rho_1 & 0 & -1 \\ -1 & \rho_2 & 0 \\ 0 & -1 & \rho_3 \end{vmatrix} = \rho_1\rho_2\rho_3 - 1$

"called the Y-circulator"

More generally suppose S is a permutation matrix belonging to an n-cycle, then

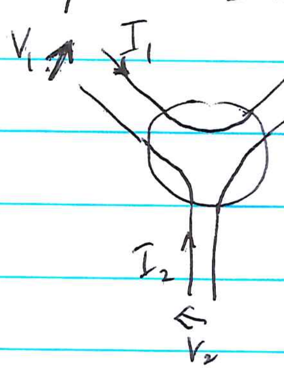
$$|\zeta - S| = \zeta_1 \cdots \zeta_n - 1 .$$

Consider 3 ports with cyclic symmetry. Put $\theta = e^{2i\pi/3}$ and note that S has the eigenvectors

$$\begin{pmatrix}1\\1\\1\end{pmatrix} \quad \begin{pmatrix}1\\ \theta\\ \theta^2\end{pmatrix} \quad \begin{pmatrix}1\\ \theta^2\\ \theta\end{pmatrix}$$

hence it is determined by what it does [redacted] to each of these.

Example: <u>Series 3-port</u>: $\quad V_1 + V_2 + V_3 = 0 \qquad I_1 = I_2 = I_3$



Suppose we take $\quad a = \begin{pmatrix}\theta^{0j}\\ \theta^{1j}\\ \theta^{2j}\end{pmatrix}$

i.e.

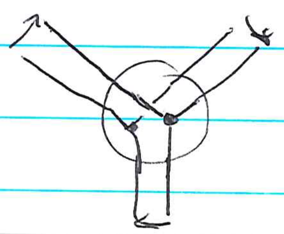
$$V_i + I_i = \theta^{ij}$$

Using that all I_i are equal [redacted] and adding up we get

$$0 + 3I = \begin{cases} 0 & j = 1,2 \\ 3 & j = 0 \end{cases}$$

If the first two cases $I = 0$ so $V_i = e^{ij}$, and so $b_i = V_i - I_i = V_i + I_i = a_i$ for these eigenspaces. In the last case $I = 1$ so $V_i = 0$ so $b_i = V_i - I_i = -a_i$. Hence the eigenvalues of S are $(-1, 1, 1)$.

<u>Parallel 3-port</u>: $\quad V_1 = V_2 = V_3, \quad I_1 + I_2 + I_3 = 0.$



[redacted] Here the eigenvalues are $(1, -1, -1)$

A 3-port which full Σ_3 symmetry has the form

$$\begin{pmatrix} \alpha & \beta & \beta \\ \beta & \alpha & \beta \\ \beta & \beta & \alpha \end{pmatrix}.$$

The eigenvalue for $\begin{pmatrix}1\\1\\1\end{pmatrix}$ is $\alpha+2\beta$

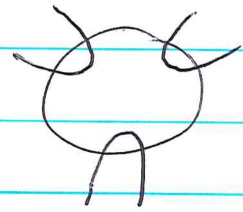
The other 2 eigenvalues are $\alpha-\beta$

Hence the scattering matrices for the above examples are:

series: $\alpha+2\beta=-1$, $\alpha-\beta=1$ [redacted] $\beta=-\frac{2}{3}$ $\alpha=\frac{1}{3}$

parallel: $\alpha+2\beta=1$, $\alpha-\beta=-1$ $\alpha=-\frac{1}{3}$ $\beta=\frac{2}{3}$

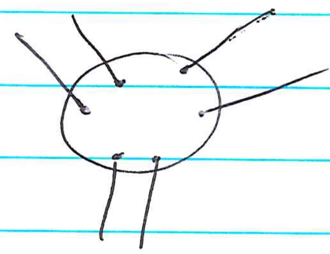
Shorted 3-port:



$V_1=V_2=V_3=0$

$$S=\begin{pmatrix} -1 & & \\ & -1 & \\ & & -1 \end{pmatrix}$$

Open 3-port:



$I_1=I_2=I_3=0$

$$S=\begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix}$$

The general [redacted] loss-less 3-port [redacted] with $\mathbb{Z}/3$ symmetry has [redacted] S matrix in the form

$$S=\begin{pmatrix} \alpha & \beta & \gamma \\ \gamma & \alpha & \beta \\ \beta & \gamma & \alpha \end{pmatrix}$$ and S is unitary

If the port is reciprocal, S has to be symmetric so $\beta=\gamma$ and we have full $\mathbb{Z}/3$ symmetry

Idea: Let's allow change of variables of the form

$$\begin{pmatrix} \tilde{V}_i \\ \tilde{I}_i \end{pmatrix} = \underbrace{\begin{pmatrix} a & b \\ c & d \end{pmatrix}}_{A} \begin{pmatrix} V_i \\ I_i \end{pmatrix}$$

where A is the transfer matrix of an arbitrary 2-port. In other words attach this 2 port to each of the ports of the 3-port.

$$V = ZI$$

$$(aV + bI) = \tilde{V} = \tilde{Z}\tilde{I} = \tilde{Z}(cV + dI)$$

$$(a - \tilde{Z}c)V = (\tilde{Z}d - b)I$$

$$Z = \frac{d\tilde{Z} - b}{-c\tilde{Z} + a} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \tilde{Z}$$

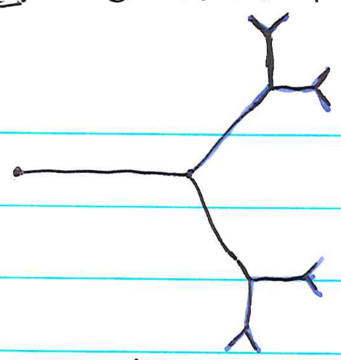
or

$$\boxed{\tilde{Z} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot Z = \frac{aZ + b}{cZ + d}}$$

Hence all that one has done is to move the eigenvalues of Z around by the transformation $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

The [redacted] same sort of thing will happen with scattering matrices. We can move the eigenvalues around by a transformation $\begin{pmatrix} a & b \\ \bar{b} & \bar{a} \end{pmatrix}$ in $SU(1,1)$. In the case of a 3-port with cyclic symmetry we can see the equivalences. [redacted] In order to be equivalent to a circulator, you need three distinct eigenvalues for the scattering matrix. You put the one belonging to $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ at 1, the others at θ, θ^2; according to the orientation [redacted] you get the two types of circulators.

Problem: Consider the $PSL_2(\mathbb{Z})$-tree with circulators at the nodes.

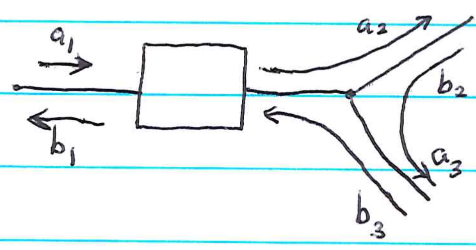


Take a matrix A in $SL_2(\mathbb{R})$ for your 2-port and put it on each of the branches. Can you determine those A for which the tree has an impedance?

Example: Take the $\mathbb{Z}$-tree:

In this case I have seen that [redacted] the impedance exists when A has an eigenvalue > 1, i.e. when $|\mathrm{tr}\, A| > 2$.

Assuming the existence of an impedance for [redacted] the tree we compute it as a fixpoint.



Let the relations given by the 2-port be:

$$\begin{pmatrix} a_1 \\ b_1 \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} a_2 \\ b_3 \end{pmatrix}$$

where $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ is the transfer matrix; for real frequencies it belongs to $SU(1,1)$.

Assuming the impedance ($\frac{a}{b}$ [redacted] version) of the tree is z, we have a non-zero state a_i, b_i in the above diagram with

[redacted] $z b_i =$ [redacted] a_i

so $a_2 = z b_2 = z a_3 = z^2 b_3$ and we get the equation

$$z = \frac{\alpha z^2 + \beta}{\gamma z^2 + \delta}$$

which is a cubic equation:

(*) $\boxed{\gamma z^3 + \delta z = \alpha}$ [redacted] $\boxed{z^2 + \beta}$

Suppose $\gamma = \bar{\beta}$, $\delta = \bar{\alpha}$

$$\bar{\beta} z^3 + \bar{\alpha} z = \alpha z^2 + \beta$$

[redacted] Dividing by z^3 and conjugating we get

$$\beta + \alpha (z^*)^2 = \bar{\alpha}(z^*) + \bar{\beta}(z^*)^3$$

This shows the roots of (*) are invariant under $z \mapsto z^*$ hence at least one lies on S^1, assuming $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in SU(1,1)$

The next thing to do is to take a root of (*) and to see if a state can be constructed on the tree. Here we want to know that as we propagate the solution we get something decaying fast enough. In particular we want to know that for the a_i, b_i determined above [redacted] one has

$$|b_1| > |b_2|, |b_3|$$

(I might need $\sqrt{2}$ in here to get an ℓ^2-state).

We can formulate the equation (*) as coming from

an eigenvalue problem: You want $a_i = z b_i$ for all i.
To calculate the matrix:

$$b_1 = \gamma a_2 + \delta b_3 \qquad a_2 = \frac{1}{\gamma}(b_1 - \delta b_3)$$

$$a_1 = \alpha a_2 + \beta b_3 = \frac{\alpha}{\gamma}(b_1 - \delta b_3) + \beta b_3 = \frac{\alpha}{\gamma} b_1 + \frac{\beta\gamma - \alpha\delta}{\gamma} b_3$$

$$a_3 = b_2$$

$$\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} \frac{\alpha}{\gamma} & 0 & \frac{\beta\gamma-\alpha\delta}{\gamma} \\ \frac{1}{\gamma} & 0 & -\frac{\delta}{\gamma} \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$$

$$\begin{pmatrix} \frac{\alpha}{\gamma} - z & 0 & \frac{\beta\gamma-\alpha\delta}{\gamma} \\ \frac{1}{\gamma} & -z & -\frac{\delta}{\gamma} \\ 0 & 1 & -z \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = 0$$

Suppose we assume $\alpha\delta - \beta\gamma = 1$. Then we find

$$(\alpha - \gamma z) b_1 = b_3$$

hence

$$\frac{b_3}{b_1} = \alpha - \gamma z$$

and we want this to be < 1, maybe even $< \frac{1}{\sqrt{2}}$ in absolute value.

On physical grounds, if an impedance z exists, we expect it to be [redacted] on S^1

Note that

$$\bar{\beta} z^3 + \bar{\alpha} z = \alpha z^2 + \beta$$

has the root $z = 1$ when α, β are real.