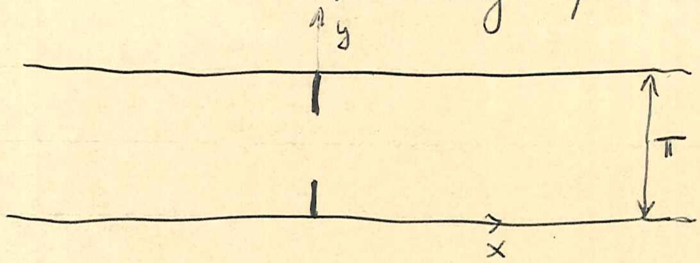
Return to the Schwinger problem




$$(\Delta + k^2)\psi = 0$$

$$\frac{\partial \psi}{\partial n} = 0 \quad \text{on metal}$$

First solve when no obstacle is present: Expand $\psi$ in a cosine series in $y$:

$$\psi(x,y) = \sum_{n=0}^{\infty}{}' \psi_n(x) \cos ny$$

$$\left(\frac{d^2}{dx^2} + k^2 - n^2\right) \psi_n(x) = 0$$

$$\psi_n(x) = \text{linear comb. of } e^{\pm i k_n x} \qquad k_n = \sqrt{k^2 - n^2}$$

For $\operatorname{Im} k > 0$ we choose $\operatorname{Im} k_n > 0$. Let's compute the Green's function by solving

$$(\Delta + k^2) u = f \qquad f = \sum_0^{\infty} f_n(x) \cos ny$$

$$u = \sum_0^{\infty} u_n(x) \cos ny \qquad f_n(x) = \int_0^{\pi} f(x,y) \cos ny \, dy \begin{cases} \frac{2}{\pi} & n > 0 \\ \frac{1}{\pi} & n = 0 \end{cases}$$

$$\left(\frac{d^2}{dx^2} + k_n^2\right) u_n = f_n \qquad \text{has } L^2 \text{ soln.}$$

$$u_n(x) = \int \frac{e^{ik_n|x-x'|}}{2ik_n} f_n(x') dx'$$

$$\therefore \quad u(x,y) = \iint \left\{ \frac{e^{ik|x-x'|}}{2ik} \frac{1}{\pi} + \sum_{n=1}^{\infty} \frac{e^{ik_n|x-x'|}}{2ik_n} \frac{2}{\pi} \cos ny \cos ny' \right\}$$

$$f(x',y') \, dx' dy'$$

Thus
$$\boxed{G_k(x,y,x',y') = \frac{e^{ik|x-x'|}}{2ik}\frac{1}{\pi} + \sum_{n=1}^{\infty} \frac{e^{ik_n|x-x'|}}{2ik_n}\frac{2}{\pi}\cos ny \cos ny'}$$
is the Green's function for the waveguide without obstacles.

Recall that [redacted] because of $x \mapsto -x$ symmetry we need only consider even or odd solutions $\psi$ in $x$, and the even case is trivial since it forces $\frac{\partial \psi}{\partial x} = 0$ at $x = 0$, which is the same as [redacted] a metal wall along $x = 0$. So we need only consider [redacted] $\psi$ which are odd in $x$.

Let $\varphi$ be a free solution which is odd in $x$, e.g.
$$\varphi(x,y) = \left(e^{-ik_n x} - e^{ik_n x}\right)\cos ny$$
and let $\psi$ be the [redacted] real solution differing from it by an outgoing solution:
$$\psi\text{[redacted]} = \varphi + \sum_{n=0}^{\infty} c_n e^{ik_n x}\cos ny \qquad x \geq 0$$

Because $\psi$ [redacted] is odd and continuous across the aperture we have
$$\psi(0^+,y) = 0 \quad \text{in aperture}$$
$$\varphi(0^{\text{[redacted]}},y) \equiv 0$$
so
$$c_n = \int_{obst}^{\text{[redacted]}} \cos ny'\, \psi(0^+,y')\,dy' \begin{cases} \frac{2}{\pi} & n>0 \\ \frac{1}{\pi} & n=0 \end{cases}$$

giving the following formula for $\psi$ in terms of $\psi(0^+,y)$

$$\psi(x,y) = \varphi(x,y) + \int_{\text{obst}} \left\{ e^{ikx}\frac{1}{\pi} + \sum_{1}^{\infty} e^{ik_n x}\frac{2}{\pi}\cos ny \cos ny' \right\} \psi(0^+,y')\,dy'$$

To get an integral equation for $\psi(0^+,y)$ on the obstacle use the boundary condition $\frac{\partial\psi}{\partial x}(0^+,y) = 0$ on obstacle:

$$0 = \frac{\partial\varphi}{\partial x}(0,y) + \int_{\text{obst.}} \left\{ ik\frac{1}{\pi} + \sum_{1}^{\infty} ik_n \frac{2}{\pi}\cos ny \cos ny' \right\} \psi(0^+,y')\,dy'$$

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I want to interpret the above integral eqn. using the Greens function. Let's first rectify a misunderstanding of yesterday:

Dirichlet problem: To solve $(\Delta+k^2)\psi = 0$ for $|r|>a$ with $\psi = 0$ on $|r|=a$ and $\psi-\varphi$ outgoing. Let $G(r,r')$ be the free space Green's function and apply Green's formulas to $u = \psi - \varphi$

$u(r) = \iiint_{|r'|>a}$ [redacted] $\left\{ (\Delta_{r'}+k^2)G(r,r')\cdot u(r') - G(r,r')(\Delta_{r'}+k^2)u(r') \right\} dr'$

$$= \iint_{|r'|=a} \left( -\frac{\partial G}{\partial n_2}(r,r')\,u(r') + G(r,r')\frac{\partial u}{\partial n}(r') \right) dS_{r'}$$

There are no boundary terms at $\infty$ because $u$ has the same behavior as $G$. Because $\varphi$ is also a solution of $(\Delta'+k^2)\varphi = 0$ inside the obstacle we get

$$0 = \iint_{|r'|=a} \left( -\frac{\partial G}{\partial n_2}(r,r')\,\varphi(r') + G(r,r')\frac{\partial \varphi}{\partial n}(r') \right) dS_{r'}$$

for $|r|>a$. Thus we get the equation

1) $$\boxed{\psi(r) = \varphi(r) + \iint_{|r'|=a} G(r,r')\frac{\partial\psi}{\partial n}(r')\, dS_{r'}}$$

by adding and using $\psi=0$ on $|r|=a$. The above seems to be the desired analogue of LS. It leads to the integral equation

2) $$\frac{\partial\psi}{\partial n}(r) = \frac{\partial\varphi}{\partial n}(r) + \iint_{|r'|=a} \frac{\partial G}{\partial n_1}(r,r')\frac{\partial\psi}{\partial n}(r')\, dS_{r'} \qquad |r|=a$$

for $\frac{\partial\psi}{\partial n}$ on $|r|=a$ [redacted]. The problem I ran into yesterday was why does a solution of 2) yield [redacted] in 1) a solution of the origin problem. It's clear that $(\Delta+k^2)\psi=0$, $\psi-\varphi$ is outgoing, but why is $\psi(r)=0$ for $|r|=a$? This problem remains. The real point is to look maybe instead at the integral equation.

3) $$\boxed{0 = \varphi(r) + \iint_{|r'|=a} G(r,r')\frac{\partial\psi}{\partial n}(r')\, dS_{r'} \qquad |r|=a}$$

Now if we solve this to get a $\frac{\partial\psi}{\partial n}$, then define $\psi$ via 1), the problem becomes to show that 2) is satisfied. The problem has shifted; the problem really is to show that solutions of 2), 3) are equivalent.

Reformulate: Suppose we solve

$$0 = \varphi(r) + \iint_{|r'|=a} G(r,r')\, w(r')\, dS_{r'} \qquad |r'|=a$$

for $w(r')$ on $|r'| = a$, and then define

$$\psi(r) = \varphi(r) + \iint_{|r'|=a} G(r,r')\, w(r')\, dS_{r'} \qquad \text{for } |r| \geq a$$

It follows that $(\Delta + k^2)\psi = 0$ for $|r| > a$ and that $\psi(r) = 0$ for $|r| = a$, and that $\psi - \varphi$ is outgoing. The only unclear point is whether $w = \frac{\partial \psi}{\partial n}$. However we know

$$\iint_{|r'|=a} G(r,r')\left(w(r') - \frac{\partial \psi}{\partial n}(r')\right) dS_{r'} = 0$$

so that if we, as part of solving for $w$, require uniqueness as well as existence, then we win.

Do the same analysis for the Neumann problem: $(\Delta + k^2)\psi = 0$, $\frac{\partial \psi}{\partial n} = 0$ on $|r| = a$ and $\psi - \varphi$ outgoing. Put $u = \psi - \varphi$:

$$u(r) = \int_{|r'|=a} \left\{ -\frac{\partial G}{\partial n_2}(r,r')\, u(r') + G(r,r') \frac{\partial u}{\partial n}(r') \right\} dr'$$

$$0 = \int_{|r'|=a} \left\{ -\frac{\partial G}{\partial n_2}(r,r')\, \varphi(r') + G(r,r') \frac{\partial \varphi}{\partial n}(r') \right\} dr'$$

$$\therefore\ \psi(r) = \varphi(r) - \int_{|r'|=a} \frac{\partial G}{\partial n_2}(r,r')\, \psi(r')\, dr'$$

This last relation says nothing in the example on page 437 since $\varphi = 0$ on the obstacle.

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Understand Green's function formulas: Return to Dirichlet problem: $(\Delta + k^2)\psi = 0$ outside $X$, $\psi = 0$ on $\partial X$, $\psi - \varphi$ outgoing where $\varphi$ is a free solution. We showed that $\psi$ satisfies

$$(*) \qquad \psi(r) = \varphi(r) + \iint_{\partial X} G(r, r') \frac{\partial \psi}{\partial n}(r')\, dS_{r'}$$

It is good to think of $\psi$ as being defined in all space, like an electric field component. Thus $\psi$ is zero inside the obstacle and $(\Delta + k^2)\psi$ is a distribution concentrated on $\partial X$, like a surface current. The above formula can be interpreted as saying that the field due to the surface current cancels out the exciting field $\varphi$ inside the obstacle. Especially one should note that $(*)$ holds inside $X$.

So now it's clear how to solve the exterior problem in terms of the interior one. Given $\varphi$ you ?? This is not so clear!

There is a sense in which $\frac{\partial G}{\partial n}(r, r')$ behaves like a $\delta$-function. Specifically if $r, r' \in \partial X$, then

$$\frac{\partial G}{\partial n_1}(r^+, r') - \frac{\partial G}{\partial n_1}(r^-, r') = \delta(r - r')$$

To see this multiply the left by $f(r)$ and integrate. You get

$$\int_{\partial X} \frac{\partial G}{\partial n_1}(r,r')\, f(r)\, dS_r - \underbrace{\int G(r,r')\frac{\partial f}{\partial n}(r)\, dS_r}_{0}$$

$$= \underbrace{\iint (\Delta_r + k^2) G(r,r') f(r)\, dr}_{f(r')} - \underbrace{\iint G(r,r')(\Delta+k^2) f(r)\, dr}_{0}$$

which gives

$$\int_{\partial X} \left\{ \frac{\partial G}{\partial n_1}(r^+,r') - \frac{\partial G}{\partial n_1}(r^-,r') \right\} f(r)\, dS_r = f(r')$$

Thus

$$\int_{\partial X} \left\{ \frac{\partial G}{\partial n_1}(r^+,r') - \frac{\partial G}{\partial n_1}(r^-,r') \right\} f(r')\, dS_{r'} = f(r)$$

will also be true. Now apply this to $f = \frac{\partial \psi}{\partial n}$. You get

$$\frac{\partial \psi}{\partial n}(r) = \int_{\partial X} \frac{\partial G}{\partial n_1}(r^+,r') \frac{\partial \psi}{\partial n}(r')\, dS_{r'} - \int_{\partial X} \frac{\partial G}{\partial n_1}(r^-,r') \frac{\partial \psi}{\partial n}(r')\, dS_{r'}$$

The last term is $+\frac{\partial \varphi}{\partial n}(r)$ since in the interior.

$$\varphi(r) + \int_{\partial X} G(r,r') \frac{\partial \psi}{\partial n}(r')\, dS_{r'} = 0$$

Thus

$$\int_{\partial X} \frac{\partial G}{\partial n_1}(r^+,r') \frac{\partial \psi}{\partial n}(r')\, dS_{r'} = \frac{\partial \psi}{\partial n}(r) - \frac{\partial \varphi}{\partial n}(r)$$

But this is too confused. The point is that if I find $\omega$ on $\partial X$ such that

$$0 = \varphi(r) + \int G(r,r')\, \omega(r')\, dS_{r'} \qquad \text{in the interior}$$

[redacted] and define $\psi(r)$ to be this expression outside, then

$$w(r) = \int \left\{ \frac{\partial G}{\partial n_1}(r^+, r') - \frac{\partial G}{\partial n_1}(r^-, r') \right\} w(r')\, dS_{r'}$$

$$= \left\{ \frac{\partial \psi}{\partial n}(r) - \frac{\partial \varphi}{d n}(r) \right\} + \left\{ \frac{\partial \varphi}{\partial n}(r) \right\} = \frac{\partial \psi}{\partial n}(r)$$

as [redacted] desired.

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Suppose we consider scattering by [redacted] a circle in $\mathbb{R}^2$.

$$\psi = \sum_{n \in \mathbb{Z}} \psi_n(r)\, e^{in\theta}$$

$$\left( \frac{1}{r} \frac{d}{dr} r \frac{d}{dr} - \frac{n^2}{r^2} + k^2 \right) \psi_n(r) = 0$$

For the interior problem $\psi_n$ has to be well-behaved at $r = 0$, so $\psi_n(r) = \text{const}\; J_n(kr)$.

Find the Green's function by solving $(\Delta + k^2)\psi = f$.

$$f = \sum f_n(r) e^{in\theta} \qquad f_n(r) = \int_0^{2\pi} e^{-in\theta} f(r,\theta)\, d\theta / 2\pi$$

$$\left( \frac{1}{r} \frac{d}{dr} r \frac{d}{dr} - \frac{n^2}{r^2} + k^2 \right) \psi_n = f_n$$

has solution $$\psi_n(r) = \int_0^\infty G_n(r, r') f_n(r')\, dr'$$

where $G_n$ is the Green's function for the Bessel operator:

$$G_n(r, r') = \frac{J_n(kr_<)\, K_n(-ikr_>)}{W_n(k)}$$

Thus $$\psi = \sum e^{in\theta} \int_0^\infty G_n(r,r') \int_0^{2\pi} e^{-in\theta'} f(r',\theta')\, d\theta'/2\pi \; dr'$$

so
$$G(r,\theta;r',\theta') = \sum_{n\in\mathbb{Z}} G_n(r,r')\, e^{in(\theta-\theta')}/2\pi$$

Next compute

$$\int_{\theta'=0}^{2\pi} G(r,\theta;a,\theta')\, \frac{\partial \psi}{\partial r}(a,\theta')\, d\theta'$$

$$= \int_0^{2\pi} \sum_n G_n(r,a)\, e^{in(\theta-\theta')} \sum_m \frac{d\psi_m}{dr}(a)\, e^{im\theta'}\, d\theta'/2\pi$$

$$= \sum_n G_n(r,a)\, \frac{d\psi_n}{dr}(a)\, e^{in\theta}$$

Therefore the Lippmann-Schwinger equation becomes

$$\varphi_n(r) + G_n(r,a)\frac{d\psi_n}{dr}(a) = \begin{cases} 0 & r<a \\ \psi_n(r) & r>a \end{cases}$$

Yesterday I considered scattering by the circle $r = a$ in the plane, and showed the equation

$$\left.\begin{matrix}\psi(\underline{r}) \\ 0\end{matrix}\right\} = \varphi(\underline{r}) + \int_{\partial X} G(\underline{r}, \underline{r}') \frac{\partial \psi}{\partial n}(r')\, dr' \qquad \begin{matrix}\underline{r} \text{ outside} \\ \underline{r} \text{ inside}\end{matrix}$$

decomposes into uncoupled equations

$$\left.\begin{matrix}\psi_n(r) \\ 0\end{matrix}\right\} = \varphi_n(r) + G_n(r, a) \frac{d\psi_n}{dr}(a) \qquad \begin{matrix} r > a \\ r < a .\end{matrix}$$

But let's take seriously the philosophy that $\psi$ is defined in the whole plane. Suppose

$$\psi_n(r) = \varphi_n(r) + G_n(r, a)\, \omega_n \qquad r > a$$

where $\psi_n = 0$ for $r < a$.
Then

$$\psi_n'(a^+) = \varphi_n'(a) + G_n'(a^+, a)\, \omega_n$$

$$0 = \psi_n'(a^-) = \varphi_n'(a) + G_n'(a^-, a)\, \omega_n$$

so subtracting and using $\left[ G_n'(r, a) \right]_{a^-}^{a^+} = 1$, we get

$$\psi_n'(a^+) = \omega_n .$$

Conversely suppose $\psi_n(r)$ is a function defined for $0 < r < \infty$ such that

$$\psi_n(r) = \varphi_n(r) + G_n(r, a)\, \psi_n'(a^+)$$

where $\varphi_n$ is a regular solution of Bessel's DE: $\varphi_n = \text{const} \cdot J_n(kr)$

Then one sees that

$$\psi_n'(a^+) - \psi_n'(a^-) = \psi_n'(a^+)$$

so that $\psi_n'(a^-) = 0$. Thus $\psi_n(r)$ for $r < a$ is a regular solution of Bessel's D.E. satisfying $\psi_n'(a^-) = 0$, so either $\psi_n(r) = 0$ for $r \leq a$ or we have non-uniqueness for the interior Neumann problem.

So it's clear we should write the basic integral equation as

$$\psi(\underline{r}) = \varphi(\underline{r}) + \int_{\partial X} G(\underline{r}, \underline{r}') \frac{\partial \psi}{\partial n}(\underline{r}'+)\, dS_{r'}$$

because then the jump in the normal derivative of G forces

$$\frac{\partial \psi}{\partial n}(r^-) = 0 \qquad r \text{ on } \partial X$$

which implies $\psi = 0$ if the interior Neumann problem has unique solutions.

<u>Question</u>: Does the operator $f \longmapsto f(r) - G_n(r,a) f'(a^+)$ have a determinant? The operator $f \mapsto G_n(r,a) f'(a^+)$ has rank 1, so we get its trace by computing the eigenvalue $\lambda$:

$$\lambda \cdot G_n(r,a) = G_n(r,a)\; G_n'(a^+, a)$$

$$\lambda = G_n'(a^+, a)$$

Recall $\quad G_n(r,r') = \dfrac{u(r_<)\, v(r_>)}{W(u,v)(r')} \qquad u(r) = J_n(kr)$

$\qquad v(r) = K_n(-ikr)$

so
$$G_n'(a^+; a) = \frac{u(a)\,v'(a)}{W} = \frac{u(a)\,v'(a)}{u(a)\,v'(a) - u'(a)\,v(a)}$$

and the determinant I am after is

$$1 - \lambda = 1 - \frac{uv'}{uv' - u'v}(a) = \frac{u'v}{u'v - uv'}(a)$$

Thus
$$\boxed{\det_n = \frac{-k\, J_n'(ka)\, K_n(-ika)}{W\big(J_n(kr),\, K_n(-ikr)\big)(a)}}$$

To work out the denominator recall

$$K_s(r) = \Gamma(-s)\, I_s(r) + \Gamma(s)\, I_{-s}(r) \qquad \text{where}$$

$$I_s(r) = \left(\frac{r}{2}\right)^s \sum_{n=0}^{\infty} \frac{1}{n!}\left(\frac{r}{2}\right)^{2n} \frac{1}{(s+1)\cdots(s+n)}$$

is defined for $s \notin \mathbb{Z}_{<0}$.

Hence
$$W(K_s, I_s) = \Gamma(s)\, W(I_{-s}, I_s)$$

Now we know from Abel's formula that

$$W(I_{-s}, I_{+s}) = \text{const} \cdot \frac{1}{r}$$

and to evaluate the constant let $r \to 0$

$$r\, W(I_{-s}, I_{+s}) \sim r \begin{vmatrix} \left(\frac{r}{2}\right)^{-s} & \left(\frac{r}{2}\right)^{s} \\ -s\left(\frac{r}{2}\right)^{-s-1}\frac{1}{2} & s\left(\frac{r}{2}\right)^{s-1}\frac{1}{2} \end{vmatrix} = s$$

Thus
$$W(K_s, I_s) = \frac{s\,\Gamma(s)}{r} = \frac{\Gamma(s+1)}{r}$$

which means that the denominator of the $\det_n$ never vanishes.

Consequently the determinant of the LS operator for Dirichlet scattering vanishes if either $J_n'(ka) = 0$, which means the interior Neumann problem has a non-trivial, or if $K_n(-ika) = 0$, which means one has a scattered eigenstate.

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Modifications for the Neumann problem:

$$(\Delta + k^2)\psi = 0 \qquad \text{outside } \partial X$$

$$\frac{\partial \psi}{\partial n} = 0 \qquad \text{on } \partial X$$

$\psi - \varphi$ outgoing.

Green's formula gives

$$\psi(\underline{r}) = \int_\infty \left\{ \frac{\partial G}{\partial n_2}(\underline{r}, \underline{r}')\,\psi(\underline{r}') - G(\underline{r}, \underline{r}')\,\frac{\partial \psi}{\partial n}(\underline{r}') \right\} dS_{\underline{r}'}$$

$$- \int_{\partial X} \left\{ \qquad \text{"} \qquad \right\}$$

Replace $\psi$ by $\varphi$ in first integral using $\psi - \varphi$ outgoing; then first integral evaluates to $\varphi(\underline{r})$. So

1) $$\psi(\underline{r}) = \varphi(\underline{r}) - \int_{\partial X} \frac{\partial G}{\partial n_2}(\underline{r}, \underline{r}')\,\psi(\underline{r}'^{+})\, dS_{\underline{r}'} \qquad \underline{r} \text{ outside } \partial X$$

On $\partial X$ we have

$$\psi(\underline{r}^{+}) = \int_{\partial X} \left\{ \frac{\partial G}{\partial n_2}(\underline{r}^{-}, \underline{r}') - \frac{\partial G}{\partial n_2}(\underline{r}^{+}, \underline{r}') \right\} \psi(\underline{r}'^{+})\, dS_{\underline{r}'}$$

Comparing we get $$\varphi(\underline{r}) = \int_{\partial X} \frac{\partial G}{\partial n_2}(\underline{r}^{-}, \underline{r}')\,\psi(\underline{r}'^{+})\, dS_{\underline{r}'} \qquad \underline{r} \text{ on } \partial X$$

so if the interior Dirichlet problem has a unique solution, then we get

$$2) \quad 0 = \varphi(\underline{r}) - \int_{\partial X} \frac{\partial G}{\partial n_2}(\underline{r}, \underline{r}') \psi(\underline{r}'^{+}) \, dS_{\underline{r}'} \qquad \underline{r} \text{ inside } \partial X.$$

Separate the integral equation given by 1) and 2) into polar coords to get

$$\varphi_n(r) - \frac{\partial G_n}{\partial r'}(r,a)\, \psi_n(a^+) = \begin{cases} \psi_n(r) & r > a \\ 0 & r < a \end{cases}$$

Suppose $\varphi_n$ is a given free solution regular at $r=0$, $h_n$ is the free solution decaying at $\infty$ so that

$$G_n = \frac{\varphi_n(r_<)\, h_n(r_>)}{W(\varphi_n, h_n)(r')}$$

Let $w_n$ satisfy

$$\varphi_n(a) - \underbrace{\frac{\partial G_n}{\partial r'}(a^-, a)}_{\dfrac{\varphi_n(a)\, h_n'(a)}{W_n(a)}} w_n = 0$$

$$\text{so} \quad w_n = \frac{W_n(a)}{h_n'(a)}$$

Now define $\psi_n(r)$ for $r > a$ by

$$\psi_n(r) = \varphi_n(r) - \frac{\partial G_n}{\partial r'}(r,a)\, w_n$$

$$= \varphi_n(r) - \frac{\varphi_n'(a)\, h_n(r)}{W_n(a)} w_n = \varphi_n(r) - \frac{\varphi_n'(a)}{h_n'(a)} h_n(r)$$

and it is obvious that $\varphi_n'(a) = 0$

$$\varphi_n(a^+) = \frac{W_n(a)}{h_n'(a)} = \omega_n .$$

Finally compute the determinant of the operator

$$f(r) \longmapsto f(r) + \frac{\partial G_n}{\partial r'}(r,a)\, f'(a^+)$$

$$\cancel{\lambda = \frac{\partial^2 G_n}{\partial r\,\partial r'}(a^+,a) = \frac{\varphi_n'(a)\, h_n'(a)}{W_n(a)}}$$

$$\lambda = \frac{\partial G_n}{\partial r'}(a^+,a) = \frac{\varphi_n'(a)\, h_n(a)}{W_n(a)}$$

$$\det{}_n = 1 + \lambda = \frac{\varphi_n h_n' - \varphi_n' h_n + \varphi_n' h_n}{W_n}(a)$$

$$\boxed{\det{}_n = \frac{\varphi_n(a)\, h_n'(a)}{W_n(a)}}$$

Thus the $\det_n$ vanishes whenever there are non-trivial solutions of the interior Dirichlet problem or exterior Neumann problem with outgoing boundary conditions.