Digression on Bessel functions – how to get the facts straight.

In polar coordinates $\Delta = \frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}$ so if $u(r,\theta) = \sum_{n\in\mathbb{Z}} u_n(r) e^{in\theta}$ is a solution of $(\Delta + k^2)u = 0$, then $u_n(r)$ satisfies the equation

$$\left(\frac{1}{r}\frac{d}{dr} r \frac{d}{dr} - \frac{n^2}{r^2} + k^2\right) u_n = 0 \qquad \text{or}$$

$$\left\{\left(r\frac{d}{dr}\right)^2 + (k^2 r^2 - n^2)\right\} u_n = 0$$

which is Bessel's DE with r replaced by kr. So take $k=1$ to get solutions of Bessel's DE.

$$(*) \qquad \left\{\left(r\frac{d}{dr}\right)^2 + (r^2 - n^2)\right\} u = 0.$$

$e^{iy} = e^{ir\sin\theta}$ satisfies $(\Delta + 1)u = 0$, so if we expand it in Fourier series

$$e^{ir\sin\theta} = \sum_{n\in\mathbb{Z}} J_n(r) e^{in\theta}$$

then $J_n(r)$ satisfies $(*)$. Put $t = e^{i\theta}$. Then

$$\sum_{n\in\mathbb{Z}} J_n(r) t^n = e^{r\frac{t-t^{-1}}{2}} = e^{\frac{r}{2}t} e^{-\frac{r}{2}t^{-1}}$$

$$= \sum_{m\geq 0} \frac{t^m}{m!}\left(\frac{r}{2}\right)^m \sum_{k\geq 0} \frac{t^{-k}}{k!}(-1)^k \left(\frac{r}{2}\right)^k$$

$$= \sum_{n\in\mathbb{Z}} t^n \sum_{\substack{m-k=n \\ m,k\geq 0}} \frac{(-1)^k}{m!\,k!}\left(\frac{r}{2}\right)^{m+k}$$

$$\therefore \quad J_n(r) = \sum_{\substack{m-k=n \\ m,k\geq 0}} \frac{(-1)^k}{m!\,k!}\left(\frac{r}{2}\right)^{m+k} \quad \blacksquare$$

$$= \begin{cases} \displaystyle\sum_{k\geq 0}' \frac{(-1)^k}{k!\,(k+n)!}\left(\frac{r}{2}\right)^{n+2k} & n \geq 0 \\[2ex] \displaystyle\sum_{m\geq 0}' \frac{(-1)^{m-n}}{m!\,(m-n)!}\left(\frac{r}{2}\right)^{-n+2m} = (-1)^{-n} J_{-n}(r) & n \leq 0. \end{cases}$$

We want to generalize the above so as to define J_n for arbitrary complex n. Starting point will be the formula

$$J_n(r) = \frac{1}{2\pi}\int_0^{2\pi} e^{ir\sin\theta}\, e^{-in\theta}\, d\theta$$

which we use to show that because $(\Delta+1)e^{ir\sin\theta} = 0$ it follows that $J_n(r)$ satisfies the n-th Bessel equation. Put

$$\varphi(r) = \int_C e^{ir\sin\theta}\, e^{-in\theta}\, d\theta$$

where C is a path in the complex plane going from a to b. We have

$$r^2\left(\frac{1}{r}\frac{d}{dr}\, r\frac{d}{dr} + 1\right)\varphi = \int_C \left(r^2\Delta - \frac{\partial^2}{\partial\theta^2}\right) e^{ir\sin\theta}\, e^{-in\theta}\, d\theta$$

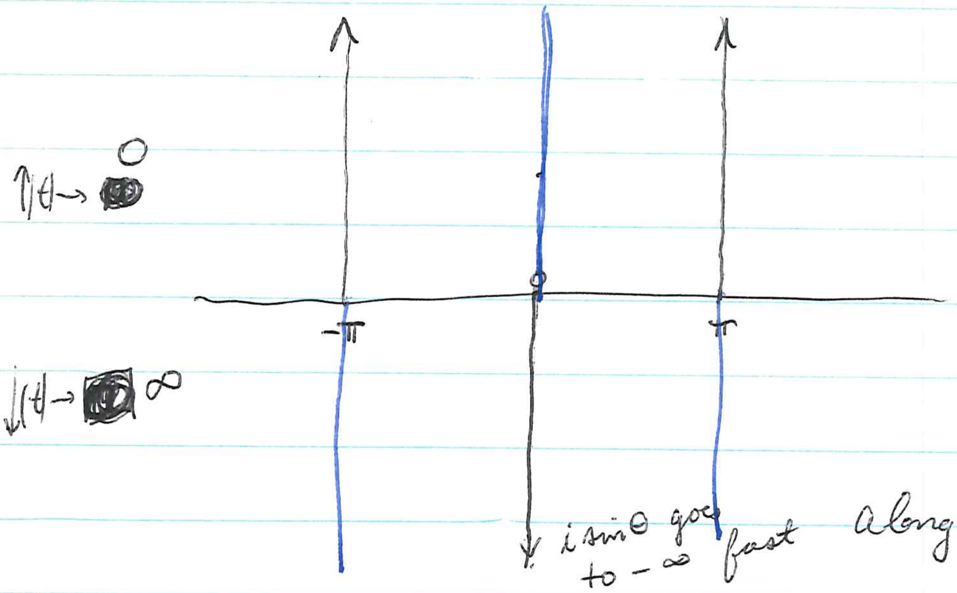
$$= \left[-\frac{\partial}{\partial\theta}\left(e^{ir\sin\theta}\right)e^{-in\theta} + e^{ir\sin\theta}\frac{\partial}{\partial\theta}\left(e^{-in\theta}\right)\right]_a^b$$

$$\underbrace{-\int_C e^{ir\sin\theta}\,\frac{\partial^2}{\partial\theta^2}\left(e^{-in\theta}\right) d\theta}_{n^2\varphi}$$

So we get a solution of Bessel's equation provided we choose the contour so that the endpoint term vanishes.

For example if $n \in \mathbb{Z}$, and $a=0$, $b=2\pi$ the endpoint term vanishes by periodicity. In general we can take a, b at ∞ and let the curve C go to ∞ along lines such that $i \sin\theta \to -\infty$ exponentially.

$$i \sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2}$$

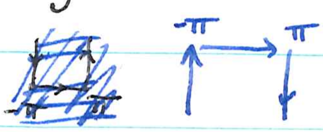


along these lines.

all wrong.

So we get a solutions of Bessel's DE. by coming ~~down~~ up $\operatorname{Re}\theta = -\pi$ to $-\pi$, going across to π, and ~~up~~ down to ∞.

$$J_n(r) = \frac{1}{2\pi} \int e^{ir\sin\theta} e^{-in\theta} \, d\theta$$



Notice this agrees with previous definition when $n \in \mathbb{Z}$, because the vertical terms cancel. We can also write this, putting $t = e^{i\theta}$:

$$J_n(r) = \frac{1}{2\pi i} \int e^{r(t-t^{-1})/2} \, t^{-n} \frac{dt}{t}$$

where the t-plane is cut along the negative real axis.

Put $u = \frac{rt}{2}$ or $t = \frac{2u}{r}$

$$J_n(r) = \frac{1}{2\pi i}\int e^{u-\frac{r^2}{4u}}\, u^{-n}\left(\frac{r}{2}\right)^n \frac{du}{u}$$

$$= \frac{1}{2\pi i}\int e^{u} \sum_{k=0}^{\infty} \frac{(-1)^k}{k!}\left(\frac{r}{2}\right)^{2k+n} u^{-n-k}\,\frac{du}{u}$$

$$\boxed{J_n(r) = \left(\frac{r}{2}\right)^n \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\,\Gamma(n+1+k)}\left(\frac{r}{2}\right)^{2k}}$$

because

$$\frac{1}{2\pi i}\int e^{u}\, u^{-n}\,\frac{du}{u} = \frac{1}{2\pi i}\left(e^{-i\pi n} - e^{i\pi n}\right)\int_0^{\infty} e^{-t}\, t^{-n}\,\frac{dt}{t}$$

$$= \frac{-\sin \pi n}{\pi}\,\Gamma(-n) = \frac{1}{\Gamma(1+n)}$$

Other solutions of Bessel's DE are the Hankel functions

$$H_n^1(r) = \frac{1}{\pi}\int e^{ir\sin\theta}\, e^{-in\theta}\, d\theta = \frac{1}{\pi i}\int e^{+r(t-t^{-1})/2}\, t^{-n}\,\frac{dt}{t}$$

π 0 $\qquad -1 \quad 0 \quad 1$

$$H_n^2(r) = \frac{1}{\pi}\int e^{ir\sin\theta}\, e^{-in\theta}\, d\theta = \frac{1}{\pi i}\int e^{r(t-t^{-1})/2}\, t^{-n}\,\frac{dt}{t}$$

$-\pi$ 0 $\qquad -1 \quad 0 \quad 1$

Clearly we have

$$J_n(r) = \frac{1}{2}\left(H_n^1(r) + H_n^2(r)\right)$$

Use steepest descent (or saddle point) method to determine the asymptotic behavior of Hankel functions as $r \to +\infty$.

Recall the principle of steepest descent. We have an integral

$$\int_C e^{rf(z)} g(z)\,dz$$

depending on a large parameter r. Assume we can deform the contour so as to make $\operatorname{Re} f(z) < 0$ away from a critical point, which say occurs at $z=0$, so that $f'(0)=0$; suppose $f''(0) \neq 0$. Assume $f(0)=0$.

Better: suppose we have an integral of the above type where the contour passes through a critical point for f; assume the point is $z=0$, and that $f(0)=0$, $f''(0)\neq 0$. Then near $z=0$ we can change variable and so arrange $f(z) = \alpha z^2$, $\alpha = \frac{f''(0)}{2}$. One moves the integration contour so that $e^{rf(z)}$ descends in steepest way from $z=0$. e.g. suppose $r\alpha < 0$, say $=-1$, so that $e^{rf(z)} = e^{-z^2} = e^{-x^2+y^2-2ixy}$. Then

$$\operatorname{Re} e^{-z^2} = e^{-x^2+y^2}$$

descends most steeply along $y=0$. Regarding anything that happens away from $z=0$ as negligible, the above integral which has major contribution

$$g(0)\int e^{r\alpha z^2}\,dz = \sqrt{\frac{\pi}{-r\alpha}}\,g(0)$$

Example 1: Stirling's formula

$$\Gamma(s+1) = \int_0^\infty e^{-t+s\log t}\,dt \qquad Re(s) > -1$$

$$f(t) = -t + s\log t \qquad f'(t) = -1 + \frac{s}{t} = 0 \quad \text{at } t = s$$

$$f''(t) = -\frac{s}{t^2} = -\frac{1}{s} \quad \text{at } t = s.$$

Saddle-point term is

$$e^{-s+s\log s}\int e^{-\frac{1}{s}\frac{z^2}{2}}\,dz = e^{-s+s\log s}\sqrt{2\pi s}\,.$$

Example 2: $\quad H_n^1(r) = \frac{1}{\pi}\int e^{ir\sin\theta}\, e^{-in\theta}\,d\theta$

$$f(\theta) = ir\sin\theta$$

$$f'(\theta) = ir\cos\theta = 0 \qquad \text{where} \quad \theta \in \frac{\pi}{2} + \mathbb{Z}$$

$$f''(\theta) = -ir\sin\theta = -ir \quad \text{at} \quad \theta = \frac{\pi}{2}$$

Saddle point term is

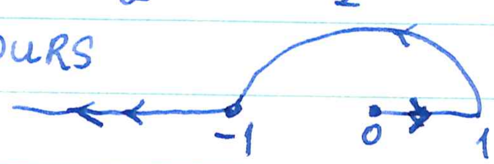
$$\frac{1}{\pi} e^{ir}\, e^{-in\frac{\pi}{2}} \int e^{-\frac{ir}{2}z^2}\,dz \qquad z = \theta - \frac{\pi}{2}$$

$$= \frac{1}{\pi} e^{ir}\, e^{-in\frac{\pi}{2}} \quad ?$$

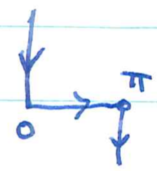
SIGNS ARE WRONG:

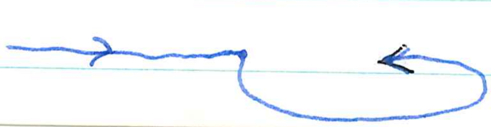
$$i\sin\theta = \frac{e^{i\theta}-e^{-i\theta}}{2} = \frac{t-t^{-1}}{2} \qquad e^{ir\sin\theta} = e^{r\left(\frac{t-t^{-1}}{2}\right)}$$

GOOD CONTOURS

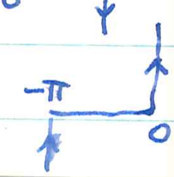


for H^1

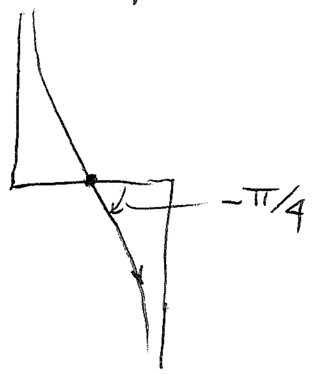




for H^2



For the saddle point term



$$\theta - \frac{\pi}{2} = z = e^{-i\pi/4} x$$

$$H_n^1(r) \sim \frac{1}{\pi} e^{ir} e^{-in\frac{\pi}{2}} \int e^{-i\frac{r}{2} z^2} dz$$

$$= \frac{1}{\pi} e^{ir - in\frac{\pi}{2} - i\frac{\pi}{4}} \sqrt{\frac{2\pi}{r}}$$

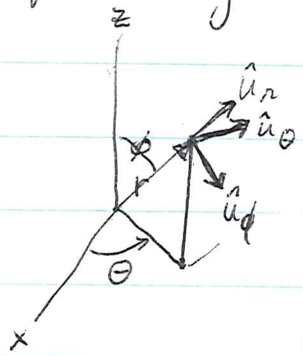
So

$$\boxed{H_n^1(r) \sim \sqrt{\frac{2}{\pi r}}\, e^{i(r - n\frac{\pi}{2} - \pi/4)}}$$

$$\boxed{H_n^2(r) \sim \sqrt{\frac{2}{\pi r}}\, e^{-i(r - n\frac{\pi}{2} - \pi/4)}}$$

$$\boxed{J_n(r) \sim \sqrt{\frac{2}{\pi r}} \cos\left(r - n\frac{\pi}{2} - \frac{\pi}{4}\right)}$$

Let us consider 3-dimensional scattering by a spherically symmetric potential $V(r)$.



$$\vec{\nabla} f = \frac{\partial f}{\partial r}\hat{u}_r + \frac{1}{r}\frac{\partial f}{\partial \phi}\hat{u}_\phi + \frac{1}{r\sin\phi}\frac{\partial f}{\partial \theta}\hat{u}_\theta$$

$$dV = r^2 \sin\varphi \, dr\, d\varphi\, d\theta$$

$$\Delta = \frac{1}{r^2\sin\varphi}\frac{\partial}{\partial r} r^2 \sin\varphi \frac{\partial}{\partial r} + \frac{1}{r^2\sin\varphi}\frac{\partial}{\partial\phi}\frac{1}{r} \cdot r^2 \sin\varphi \, \frac{1}{r}\frac{\partial}{\partial\varphi}$$

$$+ \frac{1}{r^2\sin\varphi}\frac{\partial}{\partial\theta}\frac{1}{r\sin\varphi} r^2\sin\varphi \frac{1}{r\sin\varphi}\frac{\partial}{\partial\theta}$$

$$\Delta = \frac{1}{r^2}\frac{\partial}{\partial r} r^2 \frac{\partial}{\partial r} + \frac{1}{r^2}\left\{\frac{1}{\sin\varphi}\frac{\partial}{\partial\varphi}\sin\varphi\frac{\partial}{\partial\varphi} + \frac{1}{\sin^2\varphi}\frac{\partial^2}{\partial\theta^2}\right\}$$

where the $\{\ \}$ term is the Laplacian on S^2. Eigenfunctions for the Laplacian on S^2 are called spherical harmonics. There is a natural basis $Y_{\ell,m}$ where m means that $e^{im\theta}$ appears as a factor times a suitable polynomial in $\cos\phi$. Here ℓ = degree, and the eigenvalue of the Laplacian is $-\ell(\ell+1)$.

You can remember the eigenvalue by using the fact that $r^2\Delta$ is homogeneous of degree 0 in r, so that any solution u of $\Delta u = 0$ near 0 is a series of homog. solutions $r^\ell u_\ell(\varphi,\theta)$. Then

$$0 = r^2\Delta(r^\ell u_\ell) = \ell(\ell+1) r^\ell u_\ell + r^\ell \{\text{Laplac. on } S^2\} u_\ell$$

For example $\Delta \frac{1}{r} = 0$, so translating we get

$$u(r,\varphi) = \frac{1}{\sqrt{1+r^2-2r\cos\varphi}} \quad \text{satisfies } \Delta u = 0,$$

Hence expanding as a power series in r

$$\frac{1}{\sqrt{1+r^2-2r\cos\varphi}} = \sum_{l=0}^{\infty} r^l P_l(\cos\varphi)$$

we get spherical harmonics $P_l(\cos\varphi)$. Since this is independent of θ we have

$$\left\{\frac{1}{\sin\varphi}\frac{d}{d\varphi}\sin\varphi\frac{d}{d\varphi} + l(l+1)\right\} P_l(\cos\varphi) = 0$$

or putting $z = \cos\varphi$, $\frac{d}{dz} = -\frac{1}{\sin\varphi}\frac{d}{d\varphi}$, gives the Legendre DE.

$$\left\{\frac{d}{dz}(1-z^2)\frac{d}{dz} + l(l+1)\right\} P_l(z) = 0$$

Note that

$$(1+r^2-2rz)^{-1/2} = \sum_{n=0}^{\infty} \frac{(-\frac{1}{2})(-\frac{3}{2})\cdots(-\frac{2n-1}{2})}{n!}(r^2-2rz)^n$$

$$r^n \sum_{m=0}^{n} \frac{n!}{m!(n-m)!}(-2z)^m r^{n-m}$$

$l = 2n-m$

$m = 2n-l$

$n-m = l-n$

$$= \sum_{0\le m\le n}{}' r^{2n-m} \frac{1\cdot 3\cdots(2n-1)}{2^n m!\,(n-m)!}(-1)^{n+m}(2z)^m$$

$$= \sum_{l}{}' r^l \left\{\sum_{l\ge n\ge \frac{l}{2}} \frac{1\cdot 3\cdots(2n-1)}{2^n(2n-l)!\,(l-n)!}(-1)^{l-n}(2z)^{2n-l}\right\}$$

so $$P_l(z) = \frac{1\cdot 3\cdots(2l-1)}{l!} z^l + \text{lower terms}$$

is a polynomial of degree l in z, the lth Legendre polynomial.

Let us now consider the reduced Schroedinger equation

$$(\Delta - V + k^2)\psi = 0$$

where $V = V(r)$ has finite range. Suppose ψ has the incoming part e^{ikz} which is a plane wave coming up the z axis. Then ψ should be independent of θ and we can expand it

$$\psi(r,\varphi) = \sum_{l=0}^{\infty} \psi_l(r)\, P_l(\cos\varphi)$$

The radial functions $\psi_l(r)$ satisfy

(*) $$\left\{\frac{1}{r^2}\frac{d}{dr}r^2\frac{d}{dr} - \frac{l(l+1)}{r^2} - V(r) + k^2\right\}\psi_l = 0.$$

and ψ_l has the incoming part $\psi_l^{(0)}$ where

$$e^{ikz} = e^{ikr\cos\varphi} = \sum_{l=0}^{\infty} \psi_l^{(0)}(r)\, P_l(\cos\phi).$$

It is convenient to change * so that it looks like a Schroedinger equation on $\mathbb{R}_{\geq 0}$:

$$r\frac{1}{r^2}\frac{d}{dr}r^2\frac{d}{dr}r^{-1} = \left(\frac{d}{dr}+\frac{1}{r}\right)\left(\frac{d}{dr}-\frac{1}{r}\right) = \frac{d^2}{dr^2}$$

(*) becomes:

(+) $$\left\{\frac{d^2}{dr^2} - \frac{l(l+1)}{r^2} - V(r) + k^2\right\}(r\psi_l) = 0$$

and hence for large r, ψ_l is asymptotic to a linear

Combination of $\frac{e^{ikr}}{r}$ and $\frac{e^{-ikr}}{r}$.

Assuming V isn't too singular at $r=0$, $r=0$ is a regular singular point for (+) and it's solutions near $r=0$ are asymptotic to linear combinations of r^{-l}, r^{l+1} so that

$$\psi_l \sim c_1 r^{-l-1} + c_2 r^{l} \qquad \text{as } r \to 0.$$

But physically ψ_l has to be bounded, so that the boundary condition at $r=0$ on ψ_l is that

$$\psi_l \sim c r^l.$$

Use this for ψ_l^{0} where $V=0$

$$r^{1/2}\frac{1}{r^2}\frac{d}{dr}r^2\frac{d}{dr}r^{-1/2} = \left(\frac{d}{dr}+\frac{3}{2r}\right)\left(\frac{d}{dr}-\frac{1}{2r}\right)$$

$$= \frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \frac{3}{4r^2} + \frac{1}{2r^2}$$

$$\therefore \quad \left\{\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \frac{1/4 + l(l+1)}{r^2} + k^2\right\}\left(r^{1/2}\psi_l^{(0)}\right) = 0$$

$$\left\{\left(r\frac{d}{dr}\right)^2 + k^2r^2 - \left(l+1/2\right)^2\right\} r^{1/2}\psi_l^{(0)}(r) = 0$$

This is Bessel's D.E. of order $l+\frac{1}{2}$ and we want the solution $\sim r^{l+\frac{1}{2}}$, so

$$\psi_l^{(0)}(r) = c_l (kr)^{-1/2} J_{l+\frac{1}{2}}(kr) \qquad c \text{ const.}$$

Let's determine c_l: $\quad e^{ir\cos\varphi} = \sum \psi_l^{(0)}(r)\, P_l(\cos\varphi)$

$$= \sum \frac{i^l r^l \cos^l\varphi}{l!}$$

$$\sum_l \frac{i^l r^l (\cos\varphi)^l}{l!} = \sum_l c_l \underbrace{r^{-1/2} J_{l+\frac{1}{2}}(r)}_{\text{terms } r^l,\, r^{l+1}, \ldots} \underbrace{P_l(\cos\varphi)}_{\text{terms } (\cos\varphi)^l,\, (\cos\varphi)^{l-1}, \ldots}$$

Somehow this means that cross-terms cancel.

$$\frac{i^l r^l (\cos\varphi)^l}{l!} = c_l\, r^{-1/2} \left(\frac{r}{2}\right)^{l+\frac{1}{2}} \frac{1}{\Gamma(l+\frac{3}{2})} \qquad (\cos\varphi)^l \, \frac{1\cdot 3 \cdots (2l-1)}{l!}$$

$$\Gamma(l+\tfrac{3}{2}) = (l+\tfrac{1}{2}) \cdots \tfrac{1}{2} \underbrace{\Gamma(\tfrac{1}{2})}_{\sqrt{\pi}} = \frac{1\cdot 3 \cdots (2l+1)}{2^{l+1}} \sqrt{\pi}$$

$$i^l = c_l \frac{1}{2^{l+\frac{1}{2}}} \frac{1}{\sqrt{\pi}/2^{l+1}} \quad \frac{1}{2l+1}$$

or

$$c_l = \quad \sqrt{\frac{\pi}{2}}\,(2l+1)\, i^l$$

So

$$\boxed{e^{ikr\cos\varphi} = \sum_{l=0}^{\infty} (2l+1) i^l \sqrt{\frac{\pi}{2kr}}\, J_{l+\frac{1}{2}}(kr)\, P_l(\cos\varphi)}$$

Recall large r behavior for Bessel function

$$J_{l+\frac{1}{2}}(r) \sim \sqrt{\frac{2}{\pi r}} \cos\left(r - (l+\tfrac{1}{2})\tfrac{\pi}{2} - \tfrac{\pi}{4}\right)$$

$$r - l\frac{\pi}{2} - \frac{\pi}{2}$$

$$\sim \sqrt{\frac{2}{\pi r}} \sin\left(r - l\frac{\pi}{2}\right)$$

So this leads one to introduce the Riccati-Bessel fn:

$$u_l(r) = \sqrt{\tfrac{1}{2}\pi r}\, J_{l+\frac{1}{2}}(r) \sim \sin\left(r - \tfrac{\pi}{2} l\right)$$

and to write the above as

$$\boxed{\begin{aligned} e^{ikr\cos\varphi} &= \sum_{\ell=0}^{\infty} (2\ell+1) i^{\ell} \frac{u_{\ell}(kr)}{kr} P_{\ell}(\cos\varphi) \\ &\sim \sum_{\ell=0}^{\infty} (2\ell+1) i^{\ell} \frac{\sin(kr - \frac{\pi}{2}\ell)}{kr} P_{\ell}(\cos\varphi) \\ [redacted] &= \sum_{\ell=0}^{\infty} (2\ell+1) \frac{e^{ikr} - (-1)^{\ell} e^{-ikr}}{2ikr} P_{\ell}(\cos\varphi) \end{aligned}}$$

We can expand ψ similarly

$$\psi = \sum_{\ell=0}^{\infty} (2\ell+1) i^{\ell} \frac{\tilde{\psi}_{\ell}(r)}{kr} P_{\ell}(\cos\varphi)$$

We want $\psi \sim e^{ikr\cos\varphi}$ provided $\mathrm{Im}(k) > 0$. This means that

$$\tilde{\psi}_{\ell}(r) \sim [redacted]\ u_{\ell}(kr) \qquad \mathrm{Im}\, k > 0$$

and $[redacted] \sim \dfrac{-e^{-ikr} e^{i\frac{\pi}{2}\ell}}{2i} \qquad \mathrm{Im}\, k > 0$

Thus for k [redacted] approaching the real axis we get

$$\tilde{\psi}_{\ell}(r) \sim \frac{1}{2} e^{i\frac{\pi}{2}(\ell+1)} \left(e^{-ikr} - e^{-i\pi\ell} S_{\ell}\, e^{ikr} \right)$$

where $S_{\ell}(k)$ is the scattering coefficient. Note that

$$\psi - e^{ikr\cos\varphi} \sim \sum_{\ell=0}^{\infty} (2\ell+1) \frac{S_{\ell} - 1}{2ik} \frac{e^{ikr}}{r} P_{\ell}(\cos\phi)$$

outgoing spherical waves.

so the scattering amplitude is the series

$$A(\varphi) = \sum_{l=0}^{\infty} (2l+1) \frac{S_l - 1}{2ik} P_l(\cos\phi).$$

Next thing to do is to work out this theory in the plane where

$$\Delta = \frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}.$$

and the spherical harmonics are replaced by $e^{in\theta}$, $n \in \mathbb{Z}$. Take incoming wave

$$e^{-ikx} = e^{-ikr\cos\theta} = \sum_{n\in\mathbb{Z}} \varphi_n(kr)\, e^{in\theta}$$

$$\varphi_n(r) = \int_0^{2\pi} e^{-ir\cos\theta - in\theta} \frac{d\theta}{2\pi}$$

We can determine the large r behavior of φ_n by the method of stationary phase.

$$\frac{d}{d\theta}\cos\theta = -\sin\theta = 0 \qquad \text{at } \theta = 0, \pi$$

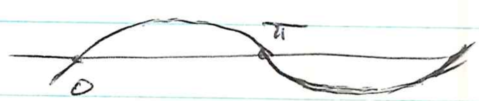
$$-i\cos\theta = -i + i\frac{\theta^2}{2} \qquad \text{near } \theta = 0$$

so steepest descent curve is  i.e. $\theta = e^{i\pi/4}x$

$$-i\cos\theta = i - i\frac{(\theta-\pi)^2}{2} \qquad \text{near } \theta = \pi$$

so steepest descent curve is  $\theta = \pi + e^{-i\pi/4}x$

Thus the contour gets deformed to and we have two contributions



Near $\theta = 0$ get

$$e^{-ir}\frac{1}{2\pi}\int e^{r\frac{i}{2}\theta^2}\,d\theta \qquad \theta = \sqrt{2}\,e^{i\pi/4}\frac{dx}{\sqrt{r}}$$

near π get

$$e^{ir}\frac{e^{-in\pi}}{2\pi}\int e^{-r\frac{i}{2}\theta^2}\,d\theta \qquad \theta = \sqrt{2}\,e^{-i\pi/4}\frac{dx}{\sqrt{r}}$$

so

$$\varphi_n(r) \sim \frac{e^{-ir}e^{i\pi/4}}{\sqrt{2\pi r}} + \frac{e^{ir}e^{-in\pi}e^{-i\pi/4}}{\sqrt{2\pi r}}$$

$$= \sqrt{\frac{2}{\pi r}}\,e^{-in\frac{\pi}{2}}\cos\left(r - \frac{n}{2}\pi - \frac{\pi}{4}\right)$$

Thus we have the asymptotic expansion

$$e^{-ikx} = e^{-ikr\cos\theta} \sim \sum_{n=0}^{\infty} \text{[redacted]}\ \frac{e^{i\frac{\pi}{4}}}{\sqrt{2\pi kr}}\left\{e^{-ikr} + e^{-i(n+\frac{1}{2})\pi}e^{ikr}\right\}e^{in\theta}$$

Next we look for a solution of the Schroedinger equation

$$(\Delta + k^2)\psi = V\psi \qquad V = V(r)$$

with the same incoming behavior as φ. Expand

$$\psi = \sum_{n\in\mathbb{Z}} \frac{e^{i\frac{\pi}{4}}}{\sqrt{2\pi kr}}\,\psi_n(r)\,e^{in\theta}$$

so that ψ_n satisfies the differential equation:

$$\left\{\frac{1}{r}\frac{d}{dr}r\frac{d}{dr} - \frac{n^2}{r^2} + k^2 - V\right\}\left(r^{-1/2}\psi_n\right) = 0.$$

This simplifies using $\quad r^{1/2}\frac{1}{r}\frac{d}{dr}r\frac{d}{dr}r^{-1/2} = \left(\frac{d}{dr}+\frac{1}{2r}\right)\left(\frac{d}{dr}-\frac{1}{2r}\right)$

$$= \frac{d^2}{dr^2} - \frac{1}{4r^2} + \frac{1}{2r^2} = \frac{d^2}{dr^2} + \frac{1}{4r^2}$$

to a one-dimensional radial Schroed. equation:

$$(*) \qquad \left\{ \frac{d^2}{dr^2} - \frac{(n^2 - 1/4)}{r^2} - V(r) + k^2 \right\} \psi_n = 0$$

Assuming V nice at $r=0$, this DE has solutions behaving like r^λ near zero with

$$\lambda(\lambda - 1) = n^2 - \tfrac{1}{4} = (n+\tfrac{1}{2})(n-\tfrac{1}{2})$$

$$\text{or} \qquad \lambda = n + \tfrac{1}{2}, \; -n + \tfrac{1}{2}$$

Now $r^{-1/2}\psi_n(r)$ has to be bounded at $r=0$ so we get the boundary condition for $(*)$.

$$\psi_n(r) \sim \text{const} \cdot r^{|n| + \frac{1}{2}}$$

For $r \to +\infty$ we know $\psi_n(r) \sim e^{-ikr}$ if k is pushed into the UHP. There is a scattering coefficient $S_n(k)$ such that

$$\psi_n(r) \sim e^{-ikr} + e^{-i(n+\frac{1}{2})\pi} S_n(k) e^{ikr}$$

The total scattering is given by

$$\psi - \varphi \sim \sum_{n \in \mathbb{Z}} \frac{e^{\frac{i\pi}{4}}}{\sqrt{2\pi k r}} e^{-i(n+\frac{1}{2})\pi} \{ S_n(k) - 1 \} e^{in\theta} e^{ikr}$$

$$= A(\theta) \frac{e^{ikr}}{\sqrt{r}}$$

where $$A(\theta) = \sum_{n \in \mathbb{Z}} \frac{e^{-in\pi - \frac{i\pi}{4}}}{\sqrt{2\pi k}} \{ S_n(k) - 1 \} e^{in\theta}$$

is the scattering amplitude.

December 14, 1978

3-dimensional scattering for $(\Delta - V + k^2)\psi = 0$ with V of compact support. Expand ψ in spherical harmonics

$$\psi = \sum_{\substack{|m|\leq l \\ 0\leq l<\infty}} \tilde{\psi}_{lm}(r)\, Y_{lm}(\phi,\theta)$$

Since
$$\Delta = \frac{1}{r^2}\frac{\partial}{\partial r} r^2 \frac{\partial}{\partial r} + \frac{1}{r^2}\underbrace{\left(\frac{1}{\sin\varphi}\frac{\partial}{\partial\varphi}\sin\varphi\frac{\partial}{\partial\varphi} + \frac{1}{\sin^2\varphi}\frac{\partial^2}{\partial\theta^2}\right)}_{\Delta_{S^2}}$$

and $\Delta_{S^2} Y_{lm} = -l(l+1)$, one has for $r \gg 0$ that

$$\left\{\frac{1}{r^2}\frac{d}{dr} r^2 \frac{d}{dr} - \frac{l(l+1)}{r^2} + k^2\right\}\tilde{\psi}_{lm}(r) = 0$$

Since
$$r\,\frac{1}{r^2}\frac{d}{dr} r^2 \frac{d}{dr}\, r^{-1} = \left(\frac{d}{dr}+\frac{1}{r}\right)\left(\frac{d}{dr}-\frac{1}{r}\right) = \frac{d^2}{dr^2} \cancel{-\frac{1}{r^2}} \cancel{+\frac{1}{r^2}}$$
this can be written

$$\left\{\frac{d^2}{dr^2} - \frac{l(l+1)}{r^2} + k^2\right\}\left(r\tilde{\psi}_{lm}(r)\right) = 0$$

and hence for $r \gg 0$, one has

$$r\tilde{\psi}_{lm}(r) \sim c_1 e^{-ikr} + c_2 e^{ikr}$$

with constants c_1, c_2 depending on l, m. Thus ψ has the asymptotic behavior (at least formally)

$$\psi \sim \underbrace{\frac{e^{-ikr}}{r}\left(\sum c_1(lm) Y_{lm}\right)}_{\text{incoming part}} + \underbrace{\frac{e^{ikr}}{r}\left(\sum c_2(lm) Y_{lm}\right)}_{\text{outgoing part}}$$

The above calculation is formal. The point of the Radon transform is to make precise the whole business [redacted] of assigning [redacted] functions on S^2 describing the incoming and outgoing parts of a solution ψ.

When we do the scattering we start with a plane wave $\varphi = e^{-ikz}$ incoming along the positive z direction [redacted]. (It is also outgoing along the negative z-direction, One distinguishes these by thinking of k as being in the upper or lower half-planes.) We find ψ^+ with the same incoming part as φ. Think of φ as being standard; then ψ^+ has a standard incoming part and the scattering is given by [redacted] the deviation of its outgoing part from being standard.

Assume [redacted] $V = V(r)$ has spherical symmetry. Then since φ is θ-independant, so should be ψ.

$$\psi = \sum_{\ell=0}^{\infty} \tilde{\psi}_\ell(r)\, P_\ell(\cos\phi)$$

$$(*) \qquad \left\{ \frac{d^2}{dr^2} - \frac{\ell(\ell+1)}{r^2} - V(r) + k^2 \right\} r\tilde{\psi}_\ell(r) = 0$$

Assuming V nice at $r=0$, [redacted] a solution of the above behaves as $r \to 0$ like a linear comb. of r^λ where [redacted] $\lambda(\lambda-1) = \ell(\ell+1)$ so $\lambda = \ell+1$ or $\lambda = -\ell$. Now $\tilde{\psi}_\ell$ is to be bounded near 0 so the boundary condition for $(*)$ is

$$r\tilde{\varphi}_\ell(r) \sim r^{\ell+1}\cdot\text{const} \qquad \text{as} \quad r \to 0$$

So we solve the radial Schroedinger equation (*) [redacted] with the above boundary condition in order to get $\varphi_\ell(r) = r\tilde{\varphi}_\ell(r)$ [redacted] up to a constant factor. Then we compare its asymptotic behavior with that for $\varphi_\ell = r\tilde{\varphi}_\ell$ in order to get the scattering.

Notice the case $\ell = 0$ - so-called s-wave scattering (maybe s-wave means spherical wave). Then we get the usual Schroedinger equation on $r \geq 0$

$$\left\{\frac{d^2}{dr^2} - V(r) + k^2\right\}\varphi_0 = 0$$

with Dirichlet boundary condition $\varphi_0(0) = 0$.

Example of a potential wall at $r = a$. Write $\varphi(r)$ for $\varphi_0(r) = r\tilde{\varphi}_0(r)$. Free solutions are multiples of

$$\frac{\sin kr}{k} = \frac{e^{ikr} - e^{-ikr}}{2ik}$$

and perturbed solutions are multiples of

$$\frac{\sin k(r-a)}{k} = \frac{e^{ikr}e^{-ika} - e^{-ikr}e^{ika}}{2ik}$$

To go from φ to φ^+ one wants the same incoming part (which blows up as $r \to \infty$ when $\operatorname{Im} k > 0$). Thus

$$\varphi \mapsto \psi^+ \; : \; \frac{\sin kr}{k} \longmapsto \frac{e^{-2ika}e^{ikr} - e^{-ikr}}{2ik} = e^{-ika}\frac{\sin k(r-a)}{k}$$

Similarly

$$\varphi \mapsto \psi^- : \frac{\sin kr}{k} \longmapsto \frac{e^{ikr} - e^{2ika}e^{-ikr}}{2ik} = e^{ika}\frac{\sin k(r-a)}{k}$$

So the scattering matrix entry which transform ψ^+ to ψ^- is

$$S = e^{2ika}.$$

What I really want to get at is to understand how the scattering matrix which is a single number is related to Fredholm determinant of an integral operator.

So the problem already occurs at the level of scattering on the line. Recall: We get two rank 2 bundles over the λ plane given by

$$\overset{\circ}{V}_\lambda = \operatorname{Ker}(\Delta + \lambda) \qquad V_\lambda = \operatorname{Ker}(\Delta - V + \lambda)$$

For $\lambda \notin \mathbb{R}_{\geq 0} \cup \{\text{bound eigenvalues}\}$ we get an isomorphism between these two vector spaces by solving

$$\varphi \longmapsto \psi = (1 - (\Delta+\lambda)^{-1}V)^{-1}\varphi$$

What is of interest to me is the determinant

$$\det\left(1 - (\Delta+\lambda)^{-1}V\right) = \det\left((\Delta+\lambda)^{-1}(\Delta+\lambda-V)\right)$$

which is an infinite-dimensional determinant, which turns out to be computable using Wronskians.

The key situation to be understood: Take scattering on the line with V of compact support. For $\lambda \notin \mathbb{R}_{\geq 0}$ we know that

$$\det\left(1-(\Delta+\lambda)^{-1}V\right) = \det\left((\Delta+\lambda)^{-1}(\Delta+\lambda-V)\right)$$

is given by a quotient of Wronskians:

$$\frac{W(\phi,\psi)}{W(\phi^0,\psi^0)} = \frac{W\left(A(k)e^{-ikx}+B(k)e^{ikx},\ e^{ikx}\right)}{W\left(e^{-ikx},e^{ikx}\right)} = A(k)$$

where k is the square root of λ in the UHP. We know that the scattering matrix is given by

$$S = \begin{pmatrix} \frac{B}{A} & \frac{1}{A} \\ \frac{1}{A} & -\frac{\bar{B}}{A} \end{pmatrix} \qquad \text{so} \qquad \det(S) = -\frac{\bar{A}}{A}$$

(Sign due to $\det S = -1$ if no interaction). Thus we find

$$(1) \qquad -\det(S) = \frac{\det\left(1-(\Delta+\lambda-i\varepsilon)^{-1}V\right)}{\det\left(1-(\Delta+\lambda+i\varepsilon)^{-1}V\right)}$$

Melrose remarked to me that this should be a priori clear, because $1-(\Delta+\lambda\pm i\varepsilon)^{-1}V$ is the Møller wave operator $\Omega^{\pm}$ and

$$(2) \qquad S = (\Omega^{-})^{-1}\Omega^{+}$$

Now what I want to do is to make Melrose's remark precise. The problem is that (2) is an operator formula whereas (1) is a formula of functions of k.

December 15, 1978

Review the problem: We consider scattering on $\mathbb{R}$ by a compact support potential V. If

$$e^{-ikx} \longleftrightarrow A(k)e^{-ikx} + B(k)e^{ikx}$$

then the scattering matrix is

$$S = \begin{pmatrix} \frac{1}{A} & -\frac{\bar{B}}{A} \\ \frac{B}{A} & \frac{1}{A} \end{pmatrix}$$

(normalized so that $S = I$ when $V = 0$), and

(1) $$\det\left(S(k)\right) = \frac{\overline{A(k)}}{A(k)}$$

But we also know that

(2) $$\det\left(1 - G_k^+ V\right) = A(k)$$

and that $\left(1 - G_k^+ V\right) : \operatorname{Ker}\left(\Delta + k^2 - V\right) \longrightarrow \operatorname{Ker}\left(\Delta + k^2\right)$ represents the Møller wave operator $(\Omega^+)^{-1}$. Since

(3) $$S = (\Omega^-)^{-1}(\Omega^+)$$

Melrose claims (1) follows from (2) and (3) by applying det.

The trouble here is that (1) is a finite dimensional determinant, whereas (2) is infinite-dimensional.

Explanation of De Witt, Phys. Rev. 103, 1565 (1956).
Fix k and compute the operator

$$(*) \qquad (1-G_k^- V)(1-G_k^+ V)^{-1}$$

on the k' eigenspace of Δ

$$(1-G_k^+ V)(\psi_{k'}) = \psi_{k'} - G_k^+ (\Delta + k'^2)\psi_{k'} \quad ?$$

~~Somehow~~ he sees that this operator is the identity for $k' \neq k$ and S_k for $k' = k$. He concludes therefore that the determinant of (*) is $\det(S_k)$. Perhaps the point is that $k+i\varepsilon$ is not real so that the resolvents exist, hence

$$(1-G_k^- V)(1-G_k^+ V)^{-1} = \left(\Delta + (k-i\varepsilon)^2\right)^{-1}\left(\Delta - V + (k-i\varepsilon)^2\right)\cdot$$

$$\left(\Delta - V + (k+i\varepsilon)^2\right)^{-1}\left(\Delta + (k+i\varepsilon)^2\right)$$

should cancel out to the identity ?

December 16, 1978:

On the line G_k^+ is the operator with kernel

$$G_k^+(x,x') = \frac{e^{-ik|x-x'|}}{2ik}$$

For V of compact support $I - G_k^+ V$ is a well-defined operator on distributions such that

$$(\Delta + k^2)(I - G_k^+ V) = (\Delta - V + k^2)$$

so it gives us a map for all k

$$(I - G_k^+ V) : \operatorname{Ker}(\Delta - V + k^2) \longrightarrow \operatorname{Ker}(\Delta + k^2)$$

I should have said that for $\operatorname{Im} k > 0$, G_k^+ restricted to L^2 coincides with the resolvent $(\Delta + k^2)^{-1}$.

Another point: We get rank 2 vector bundles over the k-plane with the fibres

$$\operatorname{Ker}(\Delta + k^2) \quad , \quad \operatorname{Ker}(\Delta - V + k^2)$$

Assuming no bound states we have isomorphisms between these bundles

$$\begin{array}{ll} I - G_k^+ V & \text{for } \operatorname{Im} k > 0 \\ I - G_k^- V & \text{for } \operatorname{Im} k < 0 \end{array}$$

Consider carefully the discrete case. Think of $\mathcal{H}$ as $L^2(S^1)$ with $\mathcal{U}_0 = $ mult. by z and of $\mathcal{U}$ as a unitary perturbation where the perturbation moves only [redacted] z^n for $|n| \leq N$. Thus $U z^n = z^{n+1}$ for $|n| >> 0$. Then the wave operators

$$\Omega^+ = \lim_{n \to \infty} U^n U_0^{-n} \qquad \Omega^- = \lim_{n \to -\infty} U^n U_0^{-n}$$

are defined as well as $S = (\Omega^-)^{-1}\Omega^+$. (There should be no trouble with Ω^+, Ω^- having [redacted] as image the orthogonal complement of the bound states)

Are there analogues of Lippmann-Schwinger in this situation? Since

$$U\Omega^\pm = \Omega^\pm U_0$$

it is clear that $\Omega^\pm$ intertwine on the level of the spectral resolution. Specifically [redacted] let

$$\varphi_\zeta = \sum_{n \in \mathbb{Z}} \zeta^{-n} z^n$$

denote the (formal) eigenvector for U_0 with eigenvalue ζ. (In other words I am looking at $\mathrm{Ker}\,(U_0 - \zeta)$ on infinite sequences). There are then eigenvectors for U

$$\psi_\zeta = \sum_{n \in \mathbb{Z}} \psi_\zeta(n) z^n$$

where $\psi_\zeta(n) =$ [redacted] $\begin{cases} c_1 \zeta^{-n} & n << 0 \\ c_2 \zeta^{-n} & n >> 0 \end{cases}$

and $\Omega^\pm \varphi_\zeta$ corresponds to $c_1 = 1$ (resp. $c_2 = 1$). The scattering operator $S(\zeta)$ gives us the ratio [redacted] c_2/c_1.

On one hand we see $\Omega^\pm$ as operators on the space of Laurent polys. $\mathbb{C}[z, z^{-1}]$.

<u>Example:</u> Let $U = U_0 \theta$ where $\theta z^n = \begin{cases} z^n & n \neq 0 \\ \alpha & n = 0 \end{cases}$

Then

$$U z^n = \begin{cases} z^{n+1} & n < 0 \\ \alpha z & n = 0 \\ z^{n+1} & n > 0 . \end{cases}$$

$$\Omega^+ z^n = \lim_{m \to \infty} U^m z^{-m+n}$$

can eval. at $-m+n \leq 0$

$$= \begin{cases} z^n & n \leq 0 \\ \alpha z^n & n > 0 \end{cases}$$

$$\Omega^- z^n = \lim_{m \to \infty} U^{-m} z^{m+n}$$

can eval $m+n \geq 1$

$$= \begin{cases} z^n & n \geq 1 \\ \alpha^{-1} z^n & n \leq 0 \end{cases}$$

So that

$$S = (\Omega^-)^{-1} (\Omega^+) = \text{mult. by } \alpha.$$

This example shows that the determinant of $\Omega^\pm$ as an infinite-dimensional operator has no meaning.

Next let us find time-independent versions of $\Omega^\pm$. First approach will be to take an eigenfunction φ for U_0: $\quad U_0 \varphi = \zeta \varphi \quad$ where $\zeta \notin S^1$ and to construct an eigenfunction ψ for U

such that $\psi - \varphi$ is ℓ^2. Then

$$u\psi = \zeta\psi$$

$$\theta\psi = \zeta u_0^{-1}\psi$$

$$\underbrace{(1-\theta)\psi}_{\substack{\text{compact}\\ \text{support}}} \therefore \text{in } \ell^2 = (1-\zeta u_0^{-1})\psi = \underbrace{(1-\zeta u_0^{-1})}_{\substack{\text{invertible}\\ \text{on } \ell^2}}(\psi-\varphi)$$

so we get the equation:

$$\boxed{\psi = \varphi + (1-\zeta u_0^{-1})^{-1}(1-\theta)\psi}$$

Try a time-dependent version: Let $\psi(t) = u^t\psi(0)$.

Then

$u_0^{-t}\psi(t)$ satisfies

$$u_0^{-t-1}\psi(t+1) = u_0^{-t-1}u_0\theta\,\psi(t) = u_0^{-t}\theta\,\psi(t)$$

or

$$u_0^{-t-1}\psi(t+1) - u_0^{-t}\psi(t) = u_0^{-t}(\theta-1)\psi(t)$$

Sum from $-\infty$ to t

$$u_0^{-t}\psi(t) - \underbrace{\lim_{t\to-\infty} u_0^{-t}\psi(t)}_{\varphi(0)} = \sum_{t'<t} u_0^{-t'}(\theta-1)\psi(t')$$

So

$$\psi(t) = \varphi(t) + \sum_{t'<t} u_0^{t-t'}(\theta-1)\psi(t')$$

Now take the Fourier transform

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$$\hat{\psi}(z) = \sum_{n\in\mathbb{Z}} z^{-n}\psi(n) \qquad \text{so that}$$

$$U\hat{\psi} = \sum z^{-n}\psi(n+1) = z\hat{\psi}.$$

One gets

$$\hat{\psi} = \hat{\varphi} + \sum_{t}\sum_{t'<t} z^{-t+t'} U_0^{t-t'}(\theta-1) z^{-t'}\psi(t')$$

$$= \hat{\varphi} + \Big(\sum_{n>0} z^{-n}U_0^n\Big)(\theta-1)\hat{\psi}$$

Now $\sum_{n>0} z^{-n}U_0^n$ is analytic outside S^1

$$\shortparallel$$

$$(z^{-1}U_0)(1-z^{-1}U_0)^{-1} = (zU_0^{-1}-1)^{-1} = -(1-zU_0^{-1})$$

Thus we get the Lippmann-Schwinger equation

$$\hat{\psi} = \hat{\varphi} + (1-zU_0^{-1})^{-1}(1-\theta)\hat{\psi}$$

with the following interpretation: $\hat{\psi}, \hat{\varphi}$ are functions on S^1 (so we change the variable to z), $\psi = \Omega^+\varphi$ and this is indicated by the fact that in the above formula z is approached from outside S^1.

<u>Summary</u>:

We consider $\mathcal{H} = \ell^2$ [redacted] with $U_0 =$ unit shift operator to the right. Via Fourier transform

$$\{a_n\} \longmapsto \sum_{n\in\mathbb{Z}}{}' a_n z^n$$

$$\int f(z)\, z^{-n} \frac{dz}{2\pi i z} \longleftarrow f(z)$$

we get an isomorphism of $\mathcal{H}$ with $L^2(S^1)$ such that U_0 becomes multiplication by z. Notice that the compact support elements of ℓ^2 correspond to Laurent polynomials.

Now let $U = U_0(I+V)$ be a unitary perturbation of U_0 where V is a transformation with finite support (meaning it [redacted] its matrix relative to the obvious basis of ℓ^2 is finite). Then we can define wave operators

$$\Omega^{\pm} = \lim_{t\to\infty} U^t U_0^{-t}$$

and a scattering operator $S = (\Omega^-)^{-1}\Omega^+$. The time-dependent interpretation [redacted] of these operators goes as follows.

Consider a trajectory $\varphi(t) = U_0^t\varphi(0)$ for the shift operator with $\varphi(0)$ of compact support to simplify. Then for $t \ll 0$ the support of $\varphi(t)$ doesn't meet the support of the perturbation V so that $U\varphi(t) = \varphi(t+1)$ [redacted] for $t \ll 0$. Thus there is a trajectory $\psi(t) = U^t\psi(0)$ for U with $\varphi(t) = \psi(t)$ for $t \ll 0$, namely

$$\psi(0) = U^t\varphi(t) = U^t U_0^{-t}\varphi(0) = \Omega^+\varphi(0).$$

Similarly $\Omega^- \varphi(0)$ describes the U-trajectory coinciding with $\varphi(t)$ for $t \gg 0$.

Now we want to work out the time-independent version of these operators. The basic idea here is to take the Fourier transform of $\varphi(t)$, $\psi(t)$ or what amounts to the same thing the spectral decompositions of $\varphi(0)$ wrt U_0 (resp. $\psi(0)$ wrt U): Put

$$\hat{\varphi}(\zeta) = \sum_{t \in \mathbb{Z}} \varphi(-t)\zeta^t$$

so that $$U_0 \hat{\varphi}(\zeta) = \sum \varphi(-t+1)\zeta^{t-1+1} = \zeta\hat{\varphi}(\zeta),$$ and hence $\hat{\varphi}(\zeta)$ is a multiple of the sequence $\{\zeta^{-n}\} \in \ell^2$. Then

$$\varphi(t) = \int \hat{\varphi}(\zeta)\zeta^t \frac{d\zeta}{2\pi i \zeta}$$

so that

$$\varphi(0) = \int \hat{\varphi}(\zeta)\frac{d\zeta}{2\pi i \zeta}$$

is the decomposition of $\varphi(0)$ into eigenfunctions for U_0.

Similarly

$$\hat{\psi}(\zeta) = \sum \psi(-t)\zeta^t$$

is an eigenfunction for U. Now what I have to do is to find how to compute $\hat{\psi}$ from $\hat{\varphi}$ where $\psi = \Omega^+\varphi$. Method:

$$U_0^{-t-1}\psi(t+1) - U_0^{-t}\psi(t) = U_0^{-t-1}\underbrace{U}_{U_0(I+V)}\psi(t) - U_0^{-t}\psi(t)$$

$$= U_0^{-t} V \psi(t)$$

so
$$U_0^{-t}\psi(t) - \underbrace{\lim_{t\to-\infty} U_0^{-t}\psi(t)}_{\varphi(0)} = \sum_{t'<t} U_0^{-t'} V \psi(t')$$

or
$$\boxed{\psi(t) = \varphi(t) + \sum_{t'<t} U_0^{t-t'} V \psi(t')}$$

Now take F.T. (multiply by ζ^{-t} and add)

$$\hat{\psi}(\zeta) = \hat{\varphi}(\zeta) + \sum_{n>0} \zeta^{-n} U_0^n \, V \hat{\psi}(\zeta).$$

Since $\sum_{n>0} \zeta^{-n} U_0^n$ converges to $\zeta^{-1}U_0(1-\zeta^{-1}U_0)^{-1} = -(1-\zeta U_0^{-1})^{-1}$ for $|\zeta|>1$, this becomes

$$\hat{\psi}^+(\zeta) = \hat{\varphi}(\zeta) - G_\zeta^+ V \hat{\psi}^+(\zeta)$$

where $G_\zeta^+ = (1-\zeta U_0^{-1})^{-1}$ ζ approached from outside S^1.

(Perhaps it would be nicer to use $\varphi(t) = U^{-t}\varphi(0)$ and the standard F.T. $\hat{\varphi}(\zeta) = \sum \varphi(t)\zeta^t$. The net effect would be to have G_ζ^+ obtained by an approach to ζ from inside S^1. This is nicer because of the dictionary: $e^{ik} = z$.)   USE THIS CONVENTION

Direct approach goes as follows: Choose ζ off

S^1 so that $1-\zeta U_0^{-1}$ is invertible on ℓ^2.

Then we want to solve $U\psi = \zeta\psi$ with $\psi-\varphi \in \ell^2$ and φ given satisfying $U_0\varphi = \zeta\varphi$. Then

$$U_0(1+V)\psi = \zeta\psi$$

$$(1+V)\psi = \zeta U_0^{-1}\psi$$

$$\underset{\in \ell^2}{-V\psi} = (1-\zeta U_0^{-1})\psi = \underbrace{(1-\zeta U_0^{-1})}_{\text{invertible on } \ell^2}\underbrace{(\psi-\varphi)}_{\text{in } \ell^2}$$

so we get the equation

$$\psi = \varphi - (1-\zeta U_0^{-1})^{-1} V\psi$$

which we can analytically continue to S^1 from either side.

<u>Question</u>: What is the kernel for $G^+_\zeta = (1-\zeta U_0^{-1})^{-1}$ for $|\zeta|<1$ and can it be analytically continued to the rest of the ζ-plane?

$$\left\{(1-\zeta U_0^{-1})^{-1} f\right\}(n) = f(n) + \zeta f(n+1) + \zeta^2 f(n+2) + \cdots$$

$$= \sum_{\ell \geq n} \zeta^{-(n-\ell)} f(\ell)$$

$$= \left(\begin{Bmatrix} \zeta^{-n} & n \leq 0 \\ 0 & n > 0 \end{Bmatrix} * f \right)(n)$$

Thus G^+_ζ is convolution with $\begin{Bmatrix} \zeta^{-n} & n \leq 0 \\ 0 & 0 \end{Bmatrix}$

or

$$G^+_\zeta(n,n') = \begin{cases} \zeta^{n'-n} & n' \geq n \\ 0 & n' < n \end{cases}$$

Similarly $\quad G_\zeta^-(n,n') = \begin{cases} 0 & n \geq n' \\ -\zeta^{n'-n} & n < n' \end{cases}$

because
$$(G_\zeta^- f)(n) = -\left\{(\zeta^{-1}u_0)(1-\zeta^{-1}u_0)^{-1}\right\}f$$
$$= -\zeta^{-1}f(n-1) - \zeta^{-2}f(n-2) - \cdots\cdots$$

Anyway it's clear that as kernels G^+, G^- have analytic continuations to the whole ζ plane minus $0, \infty$. Note that G_ζ^+ is analytic at $\zeta = 0$ and G_ζ^- is analytic at $\zeta = \infty$.

<u>Question</u>: What can one say about
$$\det(1 + G_\zeta^+ V)\ ?$$

Consider the example on p. 389, where $Vz^n = \begin{cases} 0 & n \neq 0 \\ (a-1) & n = 0 \end{cases}$
Then V has rank 1, so also will $G_\zeta^\pm V$ and so
$$\det(1 + G_\zeta^\pm V) = 1 + \operatorname{tr}(G_\zeta^\pm V)$$
$$= 1 + G_\zeta^\pm(0,0)(a-1)$$
$$= \begin{cases} a & \text{for } G^+ \\ 1 & \text{for } G^- \end{cases}$$

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$$G_\zeta^+ = (1-\zeta u_0^{-1})^{-1} \qquad \text{for } |\zeta|<1$$

$$\begin{aligned} 1+G_\zeta^+V &= 1+(1-\zeta u_0^{-1})^{-1}V \\ &= 1+(u_0-\zeta)^{-1}u_0V \\ &= (u_0-\zeta)^{-1}(u_0-\zeta+u_0V) \end{aligned}$$

$$\boxed{1+G_\zeta^+V = (u_0-\zeta)^{-1}(u-\zeta) \qquad \text{for } |\zeta|<1}$$

In the above formula both sides are to be interpreted as operators on ℓ^2, which is when $(u_0-\zeta)^{-1}$ makes sense. In the general case when we analytically continue, so that G_ζ^+ is simply a matrix, all we can say is that

$$(u_0-\zeta)(1+G_\zeta^+V) = (u-\zeta)$$

Periodic potential V on $\mathbb{R}$: We can form

$$(\Delta+k^2)^{-1}(\Delta+k^2-V) = 1-G_k^+V$$

as before, but the operator G_k^+V doesn't have a trace anymore. But perhaps $1-G_k^+V$ has an Atiyah mod Γ-style determinant.

$$(G_k^+Vf)(x) = \int \underbrace{\frac{e^{-ik|x-x'|}}{2ik}}_{K(x,x')} V(x')f(x')\,dx'$$

Clearly $K(x+\gamma, x'+\gamma) = K(x,x')$ for $\gamma \in \Gamma =$ the group of periods. So we can define

$$\mathrm{tr}_\Gamma(K) = \int_{\mathbb{R}/\Gamma} K(x,x)\,dx$$

But we also want to consider the other coefficients in the Fredholm expansion.

$$\begin{vmatrix} K(x_1,x_1) & K(x_1,x_2) \\ K(x_2,x_1) & K(x_2,x_2) \end{vmatrix}$$

$$= \begin{vmatrix} 1 & e^{ik|x_1-x_2|} \\ e^{ik|x_1-x_2|} & 1 \end{vmatrix} \frac{V(x_1)\,V(x_2)}{(2ik)^2}$$

This is not a function on $\mathbb{R}/\Gamma \times \mathbb{R}/\Gamma$, so it doesn't seem to work. However we could work with the periodic Green's function which is

$$\sum_{n\in\mathbb{Z}} G_k^+(x, x'+n\ell) = \frac{1}{2ik} \sum_{n\in\mathbb{Z}} e^{ik|x-x'-n\ell|}$$

where ℓ is a fundamental period.