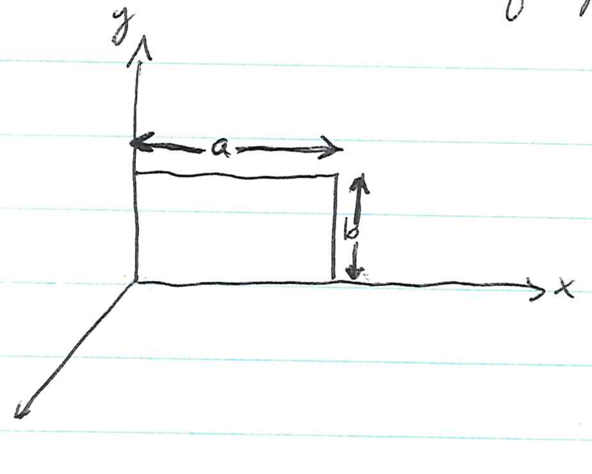
Maxwell's equations inside a wave-guide are

$$\nabla \cdot H = \nabla \cdot E = 0$$

$$\nabla \times E = -\frac{\partial H}{\partial t} \qquad \nabla \times H = \frac{\partial E}{\partial t}$$

assuming $\varepsilon = \mu = 1$. Assume we have a rectangular wave-guide with direction of propagation the z-axis.



Look for solutions with frequency dependence $e^{-i\omega t}$, with z dependence e^{iKz}, so that

$$H = e^{-i\omega t} e^{iKz} H(x,y) \qquad \text{etc.}$$

Then Maxwell's equations become

$$\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + iKE_z = 0 \qquad \frac{\partial H_x}{\partial x} + \frac{\partial H_y}{\partial y} + iKH_z = 0$$

$$\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z} = i\omega H_x = \frac{\partial E_z}{\partial y} - iKE_y \qquad \frac{\partial H_z}{\partial y} - iKH_y = -i\omega E_x$$

$$\frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x} = i\omega H_y = -\frac{\partial E_z}{\partial x} + iKE_x \qquad -\frac{\partial H_z}{\partial x} + iKH_x = -i\omega E_y$$

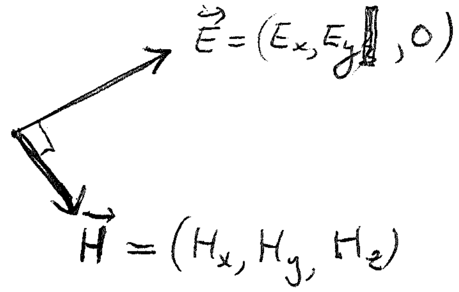
$$\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} = i\omega H_z \qquad \frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} = -i\omega E_z$$

Let us now consider TE waves (this means the electric field is transverse to the direction of propagation, i.e. $E_z = 0$).

If $E_z = 0$, then

$$H_x = -\frac{K}{\omega} E_y \qquad H_y = \frac{K}{\omega} E_x$$

so that E, H are perpendicular, and the x, y projection of H has length $= \frac{K}{\omega} \cdot$ length of E.



The equations become $\quad \dfrac{\partial E_x}{\partial x} + \dfrac{\partial E_y}{\partial y} = 0$

$$\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} = i\omega H_z \qquad -iKH_z = \underbrace{\frac{\partial}{\partial x}\left(-\frac{K}{\omega} E_y\right) + \frac{\partial}{\partial y}\left(\frac{K}{\omega} E_x\right)}_{\text{same}}$$

$$-i\omega E_x + iK\left(\frac{K}{\omega} E_x\right) = \frac{\partial}{\partial y}\left(\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y}\right)\frac{1}{i\omega}$$

or $$(\omega^2 - K^2) E_x = \frac{\partial^2 E_y}{\partial y \partial x} - \frac{\partial^2 E_x}{\partial y^2} = -\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) E_x$$

$$iK\left(-\frac{K}{\omega} E_y\right) + i\omega E_y = \frac{\partial H_z}{\partial x} = \frac{1}{i\omega}\frac{\partial}{\partial x}\left(\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y}\right)$$

$$(K^2 - \omega^2) E_y = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) E_y$$

$$\frac{\partial}{\partial x}\left(\frac{K}{\omega} E_x\right) - \frac{\partial}{\partial y}\left(-\frac{K}{\omega} E_y\right) = 0.$$

So the equations reduce to

$$\begin{cases} \dfrac{\partial E_x}{\partial x} + \dfrac{\partial E_y}{\partial y} = 0 \\ \left\{ \dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2} + (\omega^2 - K^2) \right\} \begin{pmatrix} E_x \\ E_y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \end{cases}$$

and then we can determine the other components by

$$E_z = 0$$

$$H_x = -\frac{K}{\omega} E_y \qquad H_y = +\frac{K}{\omega} E_x$$

$$H_z = \frac{1}{i\omega}\left(\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} \right)$$

But this is to miss the point, because the basic quantity to look at is H_z. In effect

$$\frac{\partial H_z}{\partial y} = +iK H_y - i\omega E_x = iK\left(+\frac{K}{\omega} E_x\right) - i\omega E_x$$

[redacted]

$$\boxed{i\omega \frac{\partial H_z}{\partial y} = \text{[redacted]} (\omega^2 - K^2) E_x}$$

$$\boxed{i\omega \frac{\partial H_z}{\partial x} = -(\omega^2 - K^2) E_y}$$

Better [redacted]

$$\boxed{\begin{aligned} \frac{i}{\omega^2 - K^2} \frac{\partial H_z}{\partial x} &= \frac{1}{K} H_x = -\frac{1}{\omega} E_y \\ \frac{i}{\omega^2 - K^2} \frac{\partial H_z}{\partial y} &= \frac{1}{K} H_y = \frac{1}{\omega} E_x \end{aligned}}$$

$$\left\{ \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + (\omega^2 - K^2) \right\} H_z = 0$$

Thus H_z appears as a potential for the TE mode. [redacted] It amounts to a Hertz vector with a single component in the z-direction.

General argument: suppose E, H satisfy the time-ind equations $\nabla \cdot E = \nabla \cdot H = 0$, $\nabla \times E = i\omega H$, $\nabla \times H = -i\omega E$, and suppose that $E_x = 0$. Then we have from $\nabla \cdot E = 0$

$$\frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} = 0$$

so there exists $\psi(x,y,z)$ a function, unique up to adding a function of x, such that

$$E_y = [redacted]\ \frac{\partial \psi}{\partial z} \qquad E_z = -[redacted]\ \frac{\partial \psi}{\partial y}$$

[redacted] Another way of writing this is

[redacted]

$$\nabla \times (\psi, 0, 0) = \left(0, \frac{\partial \psi}{\partial z}, -\frac{\partial \psi}{\partial y}\right) = E$$

Put $\Pi = (\psi, 0, 0)$; this is a Hertz vector. Next

$$i\omega H_x = \frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z} = -\frac{\partial^2 \psi}{\partial y^2} - \frac{\partial^2 \psi}{\partial z^2}$$

$$i\omega H_y = \cancel{\frac{\partial E_x}{\partial z}} - \frac{\partial E_z}{\partial x} = +\frac{\partial^2 \psi}{\partial x \partial y}$$

$$i\omega H_z = \frac{\partial E_y}{\partial x} - \cancel{\frac{\partial E_x}{\partial y}} = \frac{\partial^2 \psi}{\partial x \partial z}$$

and

$$
\begin{aligned}
& & -i\omega E_x &= \frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z} = \frac{1}{i\omega}\left\{\frac{\partial^3\psi}{\partial x\partial y\partial z} - \frac{\partial^3\psi}{\partial x\partial y\partial z}\right\} = 0 \\
-i\omega\frac{\partial\psi}{\partial z} &= & -i\omega E_y &= \frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x} = \frac{1}{i\omega}\left\{-\frac{\partial^2\psi}{\partial y^2\partial z} - \frac{\partial^2\psi}{\partial z^3} - \frac{\partial^2\psi}{\partial x^2\partial z}\right\} \\
i\omega\frac{\partial\psi}{\partial y} &= & -i\omega E_z &= \frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} = \frac{1}{i\omega}\left\{+\frac{\partial^2\psi}{\partial x^2\partial y} + \frac{\partial^2\psi}{\partial y^3} + \frac{\partial^2\psi}{\partial y\partial z^2}\right\}
\end{aligned}
$$

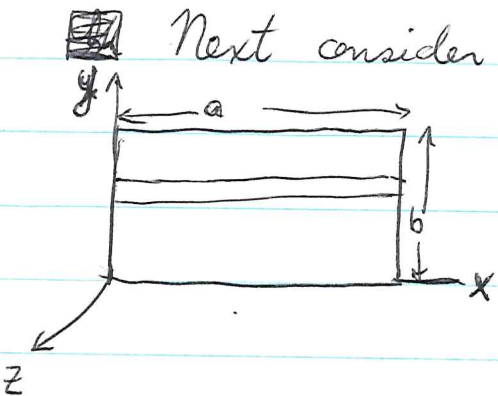
The last 2 equations show that $\Delta\psi + \omega^2\psi$ is a function of x alone; since we are free to modify ψ by a function of x, we can suppose ψ chosen so that

$$\Delta\psi + \omega^2\psi = 0.$$

So Maxwell's equations with $E_x = 0$ have been reduced to the function ψ satisfying the Helmholz equation, and

$$\begin{cases} E = \nabla\times(\psi, 0, 0) \\ i\omega H = \nabla\times\vec{E} \end{cases}$$

Next consider a rectangular wave-guide. An obstacle independent of x is placed in the guide. One assumes the exciting field has $E_x = 0$. Schwinger claims then that $E_x = 0$ in the guide, and hence the problem reduces to the function ψ. Since



$$i\omega H_x = -\frac{\partial^2\psi}{\partial y^2} - \frac{\partial^2\psi}{\partial z^2} = \frac{\partial^2\psi}{\partial x^2} + \omega^2\psi$$

if one separates out the x-direction H_x becomes a multiple of ψ. So assume $\psi(x,y,z) = \tilde{\psi}(y,z)\sin\frac{\pi x}{a}$

Then

$$E_x = 0$$

$$E_y = \frac{\partial \psi}{\partial z} = \frac{\partial \tilde{\psi}}{\partial z} \sin\left(\frac{\pi x}{a}\right)$$

$$E_z = -\frac{\partial \psi}{\partial y} = -\frac{\partial \tilde{\psi}}{\partial y} \sin\left(\frac{\pi x}{a}\right)$$

$$H_x = \frac{1}{i\omega}\left\{\omega^2 - \left(\frac{\pi}{a}\right)^2\right\} \tilde{\psi} \sin\left(\frac{\pi x}{a}\right)$$

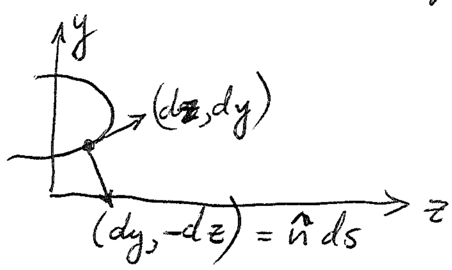
$$H_y = \frac{1}{i\omega}\frac{\pi}{a}\frac{\partial \tilde{\psi}}{\partial y} \cos\left(\frac{\pi x}{a}\right)$$

$$H_z = \frac{1}{i\omega}\frac{\pi}{a}\frac{\partial \tilde{\psi}}{\partial z} \cos\left(\frac{\pi x}{a}\right)$$

and

$$\boxed{\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\tilde{\psi} + \left\{\omega^2 - \left(\frac{\pi}{a}\right)^2\right\}\tilde{\psi} = 0}$$

The boundary conditions on $\tilde{\psi}$ must imply that E is $\perp$ to metallic surfaces.



We want $E_z dz + E_y dy = 0$ or $-\frac{\partial \tilde{\psi}}{\partial y} dz + \frac{\partial \tilde{\psi}}{\partial z} dy = 0.$

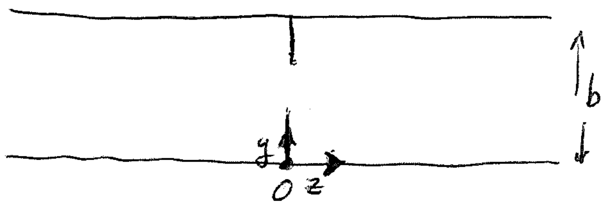
But

$$\frac{\partial \tilde{\psi}}{\partial n} ds = \frac{\partial \tilde{\psi}}{\partial z} dy - \frac{\partial \tilde{\psi}}{\partial y} dz$$

so that the boundary condition going with the above wave equation is the Neumann condition

$$\boxed{\frac{\partial \tilde{\psi}}{\partial n} = 0}$$

Consider a wave-guide with zero-thickness obstacle running in the x-direction. We have seen that the electromagnetic field can be described by a function $\psi(z,y)$ satisfying $(\Delta^2+k^2)\psi=0$ in the interior and $\frac{\partial\psi}{\partial n}=0$ on the boundary of the y,z crossection of the guide:



It is possible to handle this problem in a way analogous to the Sommerfeld problem, by taking F.T. in the y-direction. Notice that for $z\neq 0$, the condition $\frac{\partial\psi}{\partial y}(z,y)=0$ for $y=0,b$ permits integration by parts:

$$\int_0^b \frac{\partial^2\psi}{\partial y^2}\cos n\frac{\pi}{b}y\,dy = \left[\frac{\partial\psi}{\partial y}\cos\frac{n\pi}{b}y\right]_0^b + \frac{n\pi}{b}\int_0^b \frac{\partial\psi}{\partial y}\sin n\frac{\pi}{b}y\,dy$$

$$= \left[\frac{n\pi}{b}\psi\sin\frac{n\pi}{b}y\right]_0^b - \left(\frac{n\pi}{b}\right)^2\int_0^b \psi\cos\frac{n\pi}{b}y\,dy$$

and consequently if ψ is expanded in a cosine series

$$\psi(z,y) = a_0(z) + \sum_{n=1}^{\infty} a_n(z)\cos n\frac{\pi}{b}y$$

then for $z\neq 0$, we have

$$\left\{\frac{d^2}{dz^2} - \left(\frac{n\pi}{b}\right)^2 + k^2\right\} a_n(z) = 0$$

and hence

$$a_n(z) = \text{lin. comb. of } e^{\pm i k_n z} \qquad k_n = \sqrt{k^2 - (n\tfrac{\pi}{b})^2}$$

As I did before we want the solution ψ with incoming waves e^{-ikz} and so find it convenient to look at the difference u:

$$\psi(y,z) = e^{-ikz} + u(y,z)$$

Then $(\Delta + k^2)u = 0$, $\frac{\partial u}{\partial y} = 0$ for $y=0$ and $y=b$, and

$$\frac{\partial u}{\partial z} = ik \qquad \text{on the obstacle.}$$

For $\operatorname{Im} k > 0$, this Neumann problem should have a unique solution. By symmetry it should be odd:

$$u(y,z) = -u(y,-z) \qquad \therefore\ u(y,0) = 0 \quad y \in \text{aperture}$$

Since u should consist of outgoing waves, we have

$$u(y,z) = A_0 e^{ikz} + \sum_{n=1}^{\infty} A_n e^{ik_n z} \cos\left(n\frac{\pi}{b}y\right) \qquad z>0$$

where k_n is the square root of $k^2 - (n\frac{\pi}{b})^2$ in the UHP.
The quantity [redacted]

$$A_0 = \int_0^b u(y,0^+)\,dy/b = \int_{\text{obst}} u(y,0^+)\,dy/b$$

is the interesting one - it the reflection coefficient. Note that k is such that $k < \frac{\pi}{b}$ in practice, so that all the higher modes attenuate.

Since

$$A_n = \frac{2}{b} \int_{obst} \cos n\frac{\pi}{b}y \; u(y,0^+)\, dy$$

we have

$$u(y,z) = e^{ikz}\left(\frac{1}{b}\int_{obst} u(y',0^+)dy'\right) + \sum_{n=1}^{\infty} e^{ik_n z}\left(\frac{2}{b}\int_{obst} \cos n\frac{\pi}{b}y \cos\frac{n}{\pi}by' \; u(y',0^+)dy'\right)$$

Here is another way to understand this formula. Let G^* be the Green's function for $\Delta + k^2$ in the strip $0 \le y \le b$ with boundary conditions $\frac{\partial G^*}{\partial n} = 0$ for $y = 0, b$ and $G^* = 0$ for $z = 0$. Then Green's formula

$$\iint_{\boxed{///}} u(\Delta+k^2)G^* - G^*(\Delta+k^2)u = \int \left\{u\frac{\partial G^*}{\partial n} - G^*\frac{\partial u}{\partial n}\right\} ds$$

gives

$$u(y,z) = \int_{obst} u(y',0)\frac{\partial G^*}{\partial z'}(y,z,y',0)\; dy'$$

because $G^*\frac{\partial u}{\partial n} = 0$ on ∂, and $u = 0$ in the aperture, $\frac{\partial G^*}{\partial n} = 0$ on $=$. All this is exactly similar to the Sommerfeld problem.

Finally we get an integral equation for $u(y,0^+)$ on the obstacle by putting the last requirement that $\frac{\partial u}{\partial z} = ik$ on obstacle.

$$ik = \int_{obst} \frac{\partial^2 G^*}{\partial z \partial z'}(y,0,y',0)\; u(y',0^+)\,dy' \qquad \text{for } y \in \text{obst.}$$

or

$$ik = ik\left(\frac{1}{b}\int_{obst} u(y',0^+)dy'\right) + \sum_{n=1}^{\infty} ik_n\left(\frac{2}{b}\int_{obst} \cos n\frac{\pi}{b}y \cos n\frac{\pi}{b}y' \; u(y',0^+)dy'\right)$$

Equivalent circuit ideas: Suppose we modify ψ so that $\nabla \times (\psi, 0, 0) = ikE$, that is,

$$E_x = 0 \qquad ikE_y = \frac{\partial \psi}{\partial z} \qquad ikE_z = -\frac{\partial \psi}{\partial y}$$

and let us define the voltage and current at a transverse plane with coordinate z away from the obstacle to be

$$V(z) = \frac{1}{b}\int_0^b E_y(y,z)\,dy \qquad I(z) = \frac{1}{b}\int_0^b \psi(y,z)\,dy$$

Then

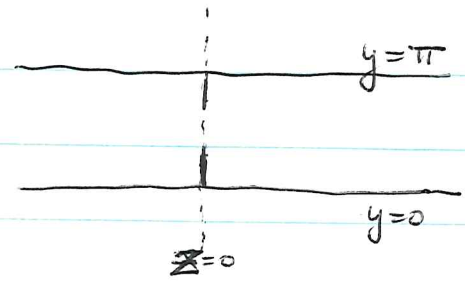
$$\boxed{\frac{\partial I}{\partial z} = ik\,V}$$

$$\frac{\partial V}{\partial z} = \frac{1}{b}\int_0^b \frac{1}{ik}\frac{\partial^2 \psi}{\partial z^2}\,dy = \frac{1}{b}\int_0^b \frac{1}{ik}\left(-\frac{\partial^2 \psi}{\partial y^2} - k^2\psi\right)dy$$

$$= -\frac{1}{bik}\left[\frac{\partial \psi}{\partial y}\right]_0^b - \frac{k^2}{ik}I$$

$$\boxed{\frac{\partial V}{\partial z} = ik\,I}$$

The above are transmission line equations, with wave number k, char. impedance = 1.



To solve $\ast \begin{cases} (\Delta + k^2)\psi = 0 & \text{inside} \\ \dfrac{\partial \psi}{\partial n} = 0 & \text{on } \partial \end{cases}$

[redacted]

For a fixed $z \neq 0$ one can expand $\psi(y,z)$ in the eigenfunctions for $\frac{d^2}{dy^2}$ with boundary conditions $\frac{d}{dy} = 0$ at $y = 0, \pi$:

$$\psi(y,z) = \sum_{n=0}^{\infty} \hat{\psi}_n(z) \cos ny$$

and one sees that $\hat{\psi}_n$ satisfies the OE.

$$\left(\frac{d^2}{dz^2} + k^2 - n^2\right)\hat{\psi}_n = 0$$

so that the general solution of $\ast$ near z is

$$\psi(y,z) = \sum_{n=0} \left(\alpha_n e^{-ik_n z} + \beta_n e^{-ik_n z}\right) \cos ny$$

where $k_n = \sqrt{k^2 - n^2}$. [redacted]

Assume k [redacted] real, $0 < |k| < 1$ so that $k_n^2 = k^2 - n^2 < 0$ for $n \geq 1$, and let us choose $\operatorname{Im}(k_n) > 0$. Then for a physically relevant ψ one has

$$\psi(y,z) = \alpha_0^+ e^{ikz} + \beta_0^+ e^{-ikz} + \sum_{n=1}^{\infty} \alpha_n^+ e^{ik_n z} \cos ny \qquad z > 0$$

$$= \alpha_0^- e^{ikz} + \beta_0^- e^{-ikz} + \sum_{n=1}^{\infty} \beta_n^- e^{-ik_n z} \cos ny \qquad z < 0$$

The current and voltages are defined by

$$I^{\pm} = \frac{1}{\pi}\int_0^{\pi} \psi(y, 0^{\pm})\, dy = \alpha_0^{\pm} + \beta_0^{\pm}$$

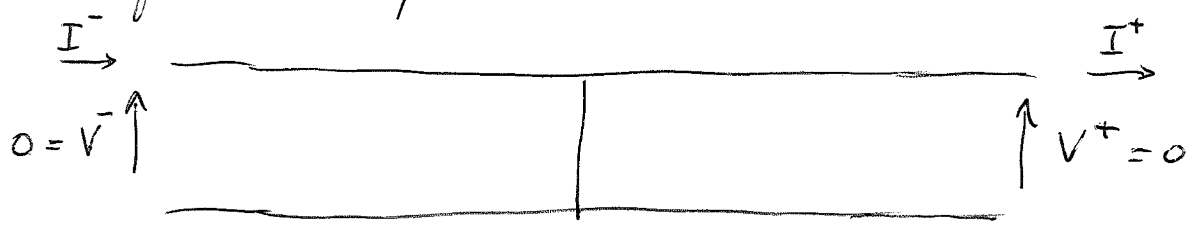
$$V^{\pm} = \frac{1}{\pi}\int_0^{\pi} \frac{1}{ik}\frac{\partial\psi}{\partial z}(y, 0^{\pm})\, dy = \alpha_0^{\pm} - \beta_0^{\pm}$$

Now we have to express the boundary conditions on $z=0$. This will tie together the voltage and current on one side with that on the other side.

Following Schwinger we consider separately even and odd ψ with respect to $z \longmapsto -z$. If ψ is even, then $\frac{\partial\psi}{\partial z} = 0$ for $z=0$, so it's the same as if there were no aperture. From

$$ik_n \alpha_n^{+} = \frac{2}{\pi}\int_{\text{aperture}} \frac{\partial\psi}{\partial z}(y, 0^{+}) \cos ny\, dy \qquad n \geq 1$$

one sees all $\alpha_n^{+} = \beta_n^{-} = 0$ for $n \geq 1$. From the above formula for V, I we get $\alpha_0^{+} = \beta_0^{+} = \alpha_0^{-} = \beta_0^{-}$. So for this ψ we have



$I^{+} = I^{-}$

Next consider odd ψ, whence

$$I^{+} = -I^{-} = \frac{1}{\pi}\int_{\text{obstacle}} \psi(y, 0^{+})\, dy \qquad V^{+} = V^{-}$$

$$\beta_0^{-} = -\alpha_0^{+}, \quad \alpha_0^{-} = -\beta_0^{+} \qquad \beta_n^{-} = -\alpha_n^{+}$$

Write $I = I^+ = -I^-$, $V = V^+ = V^-$. Then from

$$V = \alpha_0^+ - \beta_0^+ \qquad I = \alpha_0^+ + \beta_0^+$$

we get

$$\alpha_0^+ = \tfrac{1}{2}(V+I) \qquad \beta_0^+ = \tfrac{1}{2}(I-V)$$

and so

$$\begin{aligned}\psi(y,z) &= \tfrac{1}{2}(V+I)e^{ikz} + \tfrac{1}{2}(I-V)e^{-ikz} + \sum \cdots \\ &= I\cos(kz) + iV\sin(kz) + \sum_{n=1}^{\infty} \alpha_n^+ e^{ik_n z}\cos ny\end{aligned}$$

$$\begin{aligned}\alpha_n^+ &= \frac{2}{\pi}\int_0^\pi \cos(ny')\,\psi(y',0^+)\,dy' \\ &= \frac{2}{\pi}\int_{obst} \cos(ny')\,\psi(y',0^+)\,dy'\end{aligned}$$

So I get two possible integral equations: One uses $\psi(y,\overset{+}{0})$ on the obstacle and is obtained from

$$\psi(y,z) = I\cos(kz) + iV\sin(kz) + \sum_1^\infty e^{ik_n z}\frac{2}{\pi}\int_{obst}\cos ny\cos ny'\,\psi(y',0^+)\,dy'$$

by setting $\frac{\partial\psi}{\partial z} = 0$ on the obstacle:

$$\boxed{0 = kV + \sum_1^\infty k_n \frac{2}{\pi}\int_{obst.}\cos ny\cos ny'\,\psi(y',0^+)\,dy'}$$

The other is obtained from the other formula for α_n on the middle of the preceding page:

$$\psi(y,z) = I\cos kz + iV\sin kz + \sum_{1}^{\infty} \frac{e^{ik_n z}}{ik_n} \frac{2}{\pi}\int_{\text{apert.}} \cos ny \cos n y' \frac{\partial\psi}{\partial z}(y',0^+)\,dy'$$

and ~~setting~~ $\psi(y,0^+)=0$ on the aperture:

$$\boxed{0 = I + \sum_{n=1}^{\infty} \frac{1}{ik_n} \frac{2}{\pi}\int_{\text{apert.}} \cos ny \cos ny' \frac{\partial\psi}{\partial z}(y';0^+)\,dy'}$$

Let's express this equation as an eigenvalue integral equation. Recall

$$ik_n = i\sqrt{k^2-n^2} = -\sqrt{n^2-k^2}$$

$$ikV = \frac{1}{\pi}\int \frac{\partial\psi}{\partial z}(y';0^+)\,dy'$$

So put $\mathcal{E}(y) = \frac{1}{ik}\frac{\partial\psi}{\partial z}(y;0^+)$ for the electric field in the aperture. Then one has

$$\frac{I}{i\,[redacted]} = \int_{\text{apert}} 2k\sum_{n=1}^{\infty} [redacted] \frac{\cos ny \cos ny'}{\sqrt{n^2-k^2}}\,\mathcal{E}(y')\,\frac{dy'}{\pi}$$

$$\|$$

$$\left(\frac{I}{iV}\right)V = [redacted]\left(-i\frac{I}{V}\right)\int_{\text{apert}} \mathcal{E}(y')\,\frac{dy'}{\pi}$$

Therefore $\left(-i\frac{I}{V}\right)$ appears [redacted] how? ?

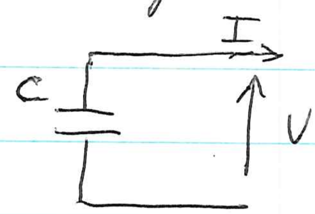
Put $\mathcal{E}(y) = \frac{1}{ik}\frac{\partial \psi}{\partial z}(y, 0^+)$ for the electric field in the aperture. Then the aperture integral equation is

$$\boxed{\frac{I}{ik} = 2\int_{\text{apert}} \sum_{n=1}^{\infty} \frac{\cos ny \cos ny'}{\sqrt{n^2-k^2}}\, \mathcal{E}(y')\, \frac{dy'}{\pi} \qquad y \in \text{apert}}$$

[redacted] Given the current I, one solves this for the electric field in the aperture $\mathcal{E}$, and the computes the voltage by

$$\boxed{V = \int_{\text{apert}} \mathcal{E}(y')\, \frac{dy'}{\pi}}$$

Now if you have a capacitor



one has [redacted]

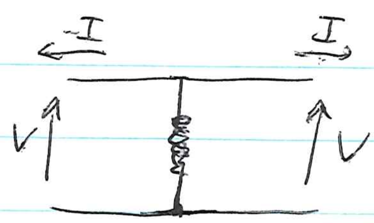
$$Q = CV$$

$$-I = \frac{dQ}{dt} = C\frac{dV}{dt} = C(-ik)V \qquad \text{if } V = e^{-ikt}V_0$$

so

$$\frac{I}{ikV} = C$$

Here we have



and the admittance is

[redacted] $$\frac{2I}{ikV}$$

hence the equivalent circuit is a shunt capacitance with this capacitance. ∎

Moreover using Schwinger's device of taking the inner product of both sides of the integral equation with $\mathcal{E}$ gives the capacitance as follows

$$\frac{I\int \mathcal{E}\,\frac{dy}{\pi}}{ik} = 2\iint \sum \frac{\cos ny\,\cos ny'}{\sqrt{n^2-k^2}}\,\mathcal{E}(y)\mathcal{E}(y')\,\frac{dy\,dy'}{\pi^2}$$

or

$$\boxed{C = \frac{2I}{ikV} = \frac{4\iint_{apert} \sum_1^\infty \frac{\cos ny\,\cos ny'}{\sqrt{n^2-k^2}}\,\mathcal{E}(y)\,\mathcal{E}(y')\,\frac{dy\,dy'}{\pi^2}}{\left(\int_{apert} \mathcal{E}(y')\,\frac{dy'}{\pi}\right)^2}}$$

Now the idea is that general nonsense shows that because the kernel $4\sum_1^\infty \frac{\cos ny\,\cos ny'}{\sqrt{n^2-k^2}}$ is positive definite, the actual aperture field $\mathcal{E}$ minimizes the right side and that hence C is the minimum value of the right side for all possible $\mathcal{E}$. So any guess for $\mathcal{E}$ will yield an upper bound for C.

~~The obstacle integral equation is~~

$$\cancel{ikV = 2\int_{obst} \sum_1^\infty \sqrt{n^2-k^2}\,\cos ny\,\cos ny'\,\frac{dy'}{\pi}}$$

The obstacle integral equation is

$$\boxed{ikV = 2\int_{obst} \sum_1^\infty \sqrt{n^2-k^2}\,\cos ny\,\cos ny'\;\psi(y')\,\frac{dy'}{\pi}}$$

where $\psi(y) = \psi(y, 0^+)$ is the current in the obstacle.

Solving this for ψ, we calculate the current

$$\boxed{I = \int_{\text{obst.}} \psi(y')\,\frac{dy'}{\pi}}$$

and then get the capacitance $C = \frac{2I}{ikV}$. Using Schwinger's device gives

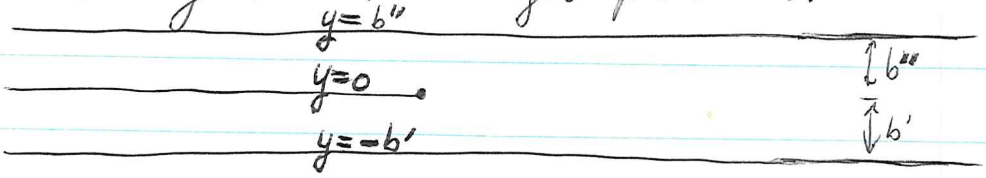
$$\frac{ikV}{2} I = \iint \sum \sqrt{}\; \cos ny \cos ny'\, \overset{\psi(y)}{\wedge}\, \psi(y')\,\frac{dy'\,dy}{\pi^2}$$

or

$$\boxed{\frac{1}{C} = \frac{ikV}{2I} = \frac{\iint_{\text{obst}} \sum_{n=1}^{\infty} \sqrt{n^2-k^2}\, \cos ny \cos ny'\, \psi(y)\psi(y')\,\frac{dy\,dy'}{\pi^2}}{\left(\int_{\text{obst}} \psi(y)\,\frac{dy}{\pi}\right)^2}}$$

By the variational principle, any choice of ψ will give a upper bound for $\frac{1}{C}$, hence a lower bound for C.

Study the Schwinger problem:



$$\Delta\psi + k^2\psi = 0$$

$$\frac{\partial\psi}{\partial n} = 0 \quad \text{on } \partial$$

$$\psi(x,y) \sim \tfrac{1}{2}e^{ikx} \qquad x \longrightarrow -\infty \qquad y>0$$

$$\sim -\tfrac{1}{2}e^{ikx} \qquad \text{"} \qquad y<0$$

Put

$$\psi(x,y) = -\tfrac{1}{2}e^{ikx} + u_1(x,y) \qquad 0\le y\le b''$$

$$\psi(x,y) = +\tfrac{1}{2}e^{ikx} + u_2(x,y) \qquad b'\le y\le 0$$

where u_1, u_2 are supposed to have nice Fourier transforms in x when $\operatorname{Im} k > 0$. Our boundary conditions become

$$\frac{\partial u_1}{\partial y} = 0 \qquad y = b''$$

$$\frac{\partial u_2}{\partial y} = 0 \qquad y = -b'$$

and on $y=0$ we get

$$\frac{\partial u_1}{\partial y} = \frac{\partial u_2}{\partial y} = 0 \qquad \text{for} \quad x<0$$

$$\frac{\partial u_1}{\partial y} = \frac{\partial u_2}{\partial y} \qquad u_1 - u_2 = e^{ikx} \qquad \text{for} \quad x>0$$

One of the problems is to understand how to interpret $\frac{\partial\psi}{\partial n} = 0$ on ∂ at the point $(x,y) = (0,0)$. This becomes apparent when one considers $\frac{\partial\psi}{\partial y}$ which vanishes on the lines and also decays as $|x| \to \infty$. It would be zero except for the different normal directions at $(0,0)$ which must enter into its boundary values.

For the moment just make some guesses. We will suppose that $\frac{\partial u_1}{\partial y} - \frac{\partial u_2}{\partial y}$ on $y=0$ which vanishes both for $x<0$, $x>0$ also vanishes at 0 in a suitable sense. Better: think of having problems in each of the strips $y>0$, $y<0$ to be fitted together along the boundary, and assume that the transition point $x=0, y=0$ is a benign singularity.

$$u_1 = \int e^{-i\xi x}\, \hat{u}_1(\xi, y)\, d\xi/2\pi$$

$$\left(\frac{d^2}{dy^2} + k^2 - \xi^2\right) \hat{u}_1(\xi, y) = 0$$

Put $\eta^2 = k^2 - \xi^2$
$\operatorname{Im} \eta > 0$ for ξ real.

$$\frac{\partial \hat{u}_1}{\partial y}(\xi, b'') = 0$$

Then

$$\hat{u}_1(\xi, y) = \hat{u}_1(\xi, 0)\,\frac{\cos \eta(b''-y)}{\cos \eta b''} \qquad 0 \le y \le b''$$

similarly

$$\hat{u}_2(\xi, y) = \hat{u}_2(\xi, 0)\,\frac{\cos \eta(y+b')}{\cos \eta b'} \qquad -b' \le y \le 0.$$

$$\frac{\partial \hat{u}_1}{\partial y}(\xi, 0) = \hat{u}_1(\xi, 0)\,\eta \tan \eta b''$$

$$\shortparallel$$

$$\frac{\partial \hat{u}_2}{\partial y}(\xi, 0) = -\hat{u}_2(\xi, 0)\,\eta \tan \eta b'$$

Moreover because $\frac{\partial u_1}{\partial y}(x,0)$ is supported in $x \geq 0$, we have that

$$\hat{u}_1(\xi, 0)\,\eta \tan \eta b'' = -\hat{u}_2(\xi, 0)\,\eta \tan \eta b'$$

is analytic in the UHP, probably in H^+.

From the fact that

$$u_1 - u_2 - e^{ikx} = 0 \qquad x > 0,\ y = 0$$

we get

$$\hat{u}_1(\xi,0) - \hat{u}_2(\xi,0) - \underbrace{\int_0^\infty e^{i\xi x + ikx}\,dx} = \int_{-\infty}^{\infty} e^{i\xi x}(u_1-u_2)(x,0)\,dx \in H^-$$

$\int_0^\infty e^{(k+i\xi)x}\,dx = -\frac{1}{ik+i\xi}$

$$\boxed{\begin{array}{l} \hat{u}_1(\xi,0) - \hat{u}_2(\xi,0) - \dfrac{i}{\xi+k} \in H^- \\[2ex] \hat{u}_1(\xi,0)\,\eta\tan\eta b'' = -\hat{u}_2(\xi,0)\,\eta\tan\eta b' \in H^+ \end{array}}$$

Special case: Let $b' = b''$ go to ∞. Then $\hat{u}_1 + \hat{u}_2 = 0$ so we get

$$\eta\hat{u}_1 \in H^+$$

$$2\hat{u}_1 - \frac{i}{\xi+k} \in H^-$$

One splits: $\eta = K^-/K^+$ with $K^\pm$ analytic $\neq 0$ in UHP (resp. LHP).

Then one wants

$$K^-\hat{u}_1 \text{ anal. UHP}$$

$$2K^-\hat{u}_1 - \frac{iK^-}{\xi+k} \text{ anal. UHP}$$

and so we write $-\dfrac{iK^-(-k)}{\xi+k} - \dfrac{i(K^-(\xi)-K^-(-k))}{\xi+k}$

and require $2K^-\hat{u}_1$ to cancel the first ~~last~~ term.

?

December 2, 1978:

The equations or conditions are

$$\begin{cases} \hat{u}_1(\xi,0)\,\eta\tan\eta b'' = -\hat{u}_2(\xi,0)\,\eta\tan\eta b' \in H^+ \\ \hat{u}_1(\xi,0) - \hat{u}_2(\xi,0) - \dfrac{i}{\xi+k} \in H^- \end{cases}$$

Put $v(\xi) = \hat{u}_1(\xi,0)\,\eta\tan\eta b''$. Then $v(\xi) \in H^+$ is to satisfy

$$v(\xi)\left[\frac{1}{\eta\tan\eta b''} + \frac{1}{\eta\tan\eta b'}\right] - \frac{i}{\xi+k} \in H^-$$

$$\frac{\cos\eta b''}{\eta\sin\eta b''} + \frac{\cos\eta b'}{\eta\sin\eta b'}$$

or

$$K(\xi)\,v(\xi) - \frac{i}{\xi+k} \in H^-$$

where

$$K(\xi) = \frac{\sin\eta b}{\eta\sin\eta b'\,\sin\eta b''} \qquad \eta = \sqrt{k^2-\xi^2}$$

This is a Wiener-Hopf type equation which one solves by factoring K

$$K(\xi) = K^+(\xi)/K^-(\xi)$$

where $K^\pm$ is analytic and non-vanishing in the upper (lower) half-plane. Then we want

$$K^+ v - \frac{iK^-}{\xi+k} \in H^-$$

$$= K^+ v - \frac{iK^-(-k)}{\xi+k} - \frac{i}{\xi+k}\left(K^-(\xi) - K^-(-k)\right)$$

and so we get

$$K^+ v - \frac{iK^-(-k)}{\xi + k} = 0$$

or

$$v = + \frac{iK^-(-k)}{(\xi + k)K^+}$$

Thus

$$\hat{u}_1(\xi, 0) = + \frac{iK^-(-k)}{(\xi + k)K^+(\xi)} \frac{1}{\eta \tan \eta b''}$$

and so

$$u_1(x,y) = \int e^{-i\xi x} \frac{iK^-(-k)}{(\xi+k)K^+(\xi)} \frac{1}{\eta \tan \eta b''} \frac{\cos \eta (b''-y)}{\cos \eta b''} \frac{d\xi}{2\pi}$$

Now to compute the reflection coefficient we want the asymptotic behavior as $x \to -\infty$. For $x<0$ $e^{-i\xi x}$ decays in the UHP, so we push the integration contour upward. The first residue picked up is at $\xi = k$ where the pole is simple

$$\frac{d}{d\xi} \eta \sin(\eta b'')\Big|_{\xi=k} = \frac{\eta\left(\eta b'' - \frac{\eta^3 b''^3}{3!} + \right)}{\xi - k}\Bigg|_{\xi=k} = \frac{(k^2-\xi^2)b''}{\xi-k}\Bigg|_{\xi=k} = -2kb''.$$

Thus

$$u_1(x,y) \sim e^{-ikx} \frac{iK^-(-k)}{2k\, K^+(k)} \frac{1}{-2kb''} \frac{1}{2\pi} \cdot 2\pi i$$

$$= \frac{K^-(-k)}{4k^2 K^+(k) b''} e^{-ikx} \qquad \begin{array}{l} x \to -\infty \\ y \geq 0 \end{array}$$

Similarly

$$u_2(x,y) \sim -\frac{K^-(-k)}{4k^2 K^+(k) b'} e^{-ikx} \qquad \begin{array}{l} x \to -\infty \\ y \leq 0. \end{array}$$

It appears now that the assumption that

$$\psi(x,y) \sim \begin{cases} -\frac{1}{2}e^{ikx} & y>0 \\ +\frac{1}{2}e^{ikx} & y<0 \end{cases} \qquad \text{as } x \to \cancel{\pm}\infty$$

should have no outgoing wave to the [redacted] right is wrong. Notice that u_1, u_2 do not depend on the $\frac{1}{2}$'s, only that $u_1 - u_2 = e^{ikx}$. Calculate the asymptotics of u_1, u_2 as $x \to \infty$.

$$u_1(x,y) = \int e^{-i\xi x} \frac{iK^-(-k)}{(\xi+k)K^+(\xi)} \frac{K^-(\xi)}{K^-(\xi)} \frac{\cos\eta(b''-y)}{\eta\sin\eta b''} \frac{d\xi}{2\pi}$$

$$= \int e^{-i\xi x} \frac{iK^-(-k)}{(\xi+k)K^-(\xi)} \frac{\eta \sin\eta b' \cancel{\sin\eta b''}}{\sin\eta b} \frac{\cos\eta(b''-y)}{\cancel{\eta\sin\eta b''}} \frac{d\xi}{2\pi}$$

For $x>0$, $e^{-i\xi x}$ decays in LHP and the first pole is at $\xi=-k$, a simple pole. Here $\eta=0$, $\frac{\sin\eta b'}{\sin\eta b} = \frac{b'}{b}$. Thus

$$u_1(x,y) \sim (-2\pi i)\, e^{ikx} \frac{iK^-(-k)}{K^-(-k)} \frac{b'}{b} \frac{1}{2\pi}$$

$$\boxed{u_1(x,y) \sim \frac{b'}{b} e^{ikx} \qquad x \to +\infty}$$

Similarly

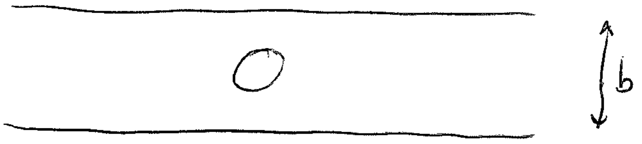
$$\boxed{u_2(x,y) \sim -\frac{b''}{b} e^{ikx} \qquad x \to +\infty}$$

Hence the solution without a right-outgoing or incoming wave is

$$\psi(x,y) = -\frac{b'}{b}e^{ikx} + u_1(x,y) \qquad 0 \le y \le b''$$

$$= \frac{b''}{b}e^{ikx} + u_2(x,y) \qquad -b' \le y \le 0$$

Consider in a waveguide a metal obstacle X



To solve $(\Delta+k^2)\psi = 0$ in the guide with $\psi=0$ on boundary and $\psi \sim \varphi$ far out, where φ is an incoming wave, in practice

$$\varphi = e^{-ik_1x} \sin\left(\frac{\pi}{b}y\right) \qquad k_1^2 = k^2 - \left(\frac{\pi}{b}\right)^2$$

Then one sets

$$\psi = \varphi + u$$

where u is to be L^2. Then we have the Dirichlet problem:

$$(\Delta+k^2)\,u = 0 \qquad u = 0 \quad \text{on walls } y=0,b$$
$$u = -\varphi \quad \text{on } \partial X$$

which we solve to find ψ.

What I want to do is to transform this Dirichlet problem into a problem on ∂X. Introduce the Green's function for the guide without the obstacle - $G(r,r')$ so that

$$(\Delta+k^2)\,G(r,r') = \delta(r-r') \qquad r = x\hat{i}+y\hat{j}$$

Green's formula gives

$$u(r') = \iint_{\text{guide}-X} \{u(\Delta+k^2)G - G(\Delta+k^2)u\}dV = -\int_{\partial X}\left(u\frac{\partial G}{\partial n} - G\frac{\partial u}{\partial n}\right)ds$$

where the minus comes as $\frac{\partial}{\partial n}$ is taken pointing outward.

Since $u = -\varphi$ on ∂X, we get the integral equation:

$$-\varphi(r') = \int_{r\in\partial X} \frac{\partial G}{\partial n}(r,r')\,\varphi(r) + G(r,r')\frac{\partial u}{\partial n}(r)\, ds$$

Solving for $\frac{\partial u}{\partial n}$ on ∂X, we get u and ψ. Another equation for $\frac{\partial u}{\partial n}$ on ∂X is

$$\frac{\partial u}{\partial n'}(r') = \int_{r\in\partial X} \left\{ \frac{\partial^2 G}{\partial n'\partial n}(r,r')\,\varphi(r) + \frac{\partial G}{\partial n'} G(r,r')\frac{\partial u}{\partial n}(r) \right\} ds .$$

So in the above way we can conveniently describe the operator $\varphi \mapsto \psi$. However I want to be able to make sense of the determinant of this operator. So perhaps the thing to do is to view the obstacle somehow as a limit of potentials.

For the n-th time I want to understand LS equation. We consider the DE

$$\frac{\partial u}{\partial t} = -iHu \qquad\qquad H = H_0 + V$$

whose solution is

$$u(t) = e^{-iHt}u_0 \,.$$

For $t>0$ we can use the Laplace transform to represent $u(t)$ in terms of the resolvent $(k-H)^{-1}$ with $\mathrm{Im}(k) \gg 0$:

$$u(t) = -\frac{1}{2\pi i}\int_{-\infty+i\varepsilon}^{\infty+i\varepsilon} e^{-ikt}(k-H)^{-1}\, dt \cdot u_0 \qquad t>0$$

Similarly

$$u(t) = \frac{1}{2\pi i}\int_{-\infty-i\varepsilon}^{\infty-i\varepsilon} e^{-ikt}(k-H)^{-1}\, dt \cdot u_0 \qquad t<0$$

Note that $(k-H)^{-1}u_0$ may have an analytic continuation in some sense across the real axis. If one can push ε negative, then one gets exponential decay for $u(t)$ as $t \to +\infty$.

With scattering one wants a "free trajectory" $e^{-iH_0t}\Omega u_0$ asymptotic to $u(t)$. We've seen this leads to the equation

$$u(t) = \underbrace{e^{-iH_0t}\Omega u_0}_{v(t)} - \int_t^\infty e^{-iH_0(t-t')}\left(\frac{1}{i}V\right)u(t')\,dt'$$

Take the F.T. of this

$$\int_{-\infty}^{0} e^{ikt} \; e^{-iH_0 t} \, dt = \frac{1}{i}(k-H_0)^{-1} \qquad \operatorname{Im} k<0$$

$$u(t) = \int e^{-ikt} \, \hat{u}(k) \, dk/2\pi$$

One obtains:

$$\boxed{\hat{u}(k) = \hat{v}(k) + (k-i0-H_0)^{-1} V \hat{u}(k)}$$

Interesting question: Do $\hat{u}(k)$, $\hat{v}(k)$ have a meaning in the LHP? For the examples I have dealt with $\hat{v}(k)$ is an incoming wave like e^{-ikx}; it is not L^2, but rather $\hat{u}-\hat{v}$ is L^2. Thus the above equation is solved in the form:

$$\hat{u}-\hat{v} = (k-H_0)^{-1} V(\hat{u}-\hat{v}) + \underbrace{(k-H_0)^{-1} V \hat{v}}_{\text{this lies in } L^2 \text{ because } V \text{ has compact support.}}$$

Note that this is in the form

$$(I-K)(\hat{u}-\hat{v}) = (k-H_0)^{-1} V \hat{v}$$

where $K = (k-H_0)^{-1} V$. In order to solve it one has to avoid the solutions of the homogeneous equation:

$$(I-K)\varphi = 0 \qquad \varphi \in L^2$$

or $\quad (k-H_0-V)\varphi = 0 \quad \Rightarrow \quad \varphi$ bound state.

Still trying to understand the Lippmann–Schwinger business. Consider potential scattering by a V of compact support on [redacted] $\mathbb{R}$. [redacted] Solutions of $(\Delta+k^2)u=0$ are linear combinations of e^{ikx}, e^{-ikx}. Let $\varphi(k)$ be a solution of $(\Delta+k^2)u$. One can ask for a solution $\psi(k)$ of $(\Delta+k^2)u = Vu$ which is asymptotic to $\varphi(k)$ as $|x| \longrightarrow \infty$. This has a definite meaning for $\operatorname{Im}(k) \neq 0$, because we can look for $\psi = \varphi + u$ with $u \in L^2$, and then we have to solve

$$(\Delta+k^2)u = (\Delta+k^2)\psi = [redacted]\, V\psi = V\varphi + Vu$$

or

$$\left\{I - (\Delta+k^2)^{-1}V\right\}u = V\varphi$$

The last equation is an integral equation with the kernel

$$K = (\Delta+k^2)^{-1}V = \frac{e^{-ik|x-x'|}}{2ik}V(x')$$

and so the Fredholm theory [redacted] tells us that as long

$$(\Delta+k^2)v = Vv$$

has no L^2 solutions, that is, $-k^2$ is not a bound state energy, then u hence $\psi = \varphi + u$ exists.

[redacted] For example suppose k is in the UHP, then if we take $\varphi(x,k) = e^{-ikx}$ we get the solution

$$T(k)e^{-ikx} \xleftrightarrow{\psi(x,k)} e^{-ikx} + R(k)e^{ikx}$$

and if we take e^{ikx} we get the solution

$$e^{ikx} + \left(-\bar{R}\frac{T}{\bar{T}}\right)e^{-ikx} \longleftrightarrow T(k)e^{ikx}$$

So it seems that the k-plane minus the real axis we have two rank 2 vector bundles which are canonically isomorphic (assuming no bound states). Both vector bundles extend across the real axis, but the isomorphism between them doesn't. There is a discontinuity measured by the scattering matrix.

The next point is the isomorphism over the UHP analytically continues to part of the LHP because the kernel K has an analytic continuation. Note $(\Delta+k^2)^{-1}$ doesn't have an obvious meaning. In fact the Fredholm theory shows that the isomorphism should be meromorphic.

For examples, we $\varphi(x,k) = e^{-ikx}$ in the UHP gives rise to $\psi(x,k)$:

$$T(k)e^{-ikx} \longleftrightarrow e^{-ikx} + R(k)e^{ikx}$$

which is meromorphic in k. Recall that if

$$\phi: \quad e^{-ikx} \longleftrightarrow A(k)e^{-ikx} + B(k)e^{ikx}$$

then

$$T(k) = \frac{1}{A(k)} \qquad R(k) = \frac{B(k)}{A(k)} .$$

More

$$\det(I-K) = \det\left((\Delta+k^2)^{-1}(\Delta-V+k^2)\right) = \frac{W(\phi, e^{ikx})_{x\gg 0}}{W(e^{-ikx}, e^{ikx})}$$

$$= A(k)$$