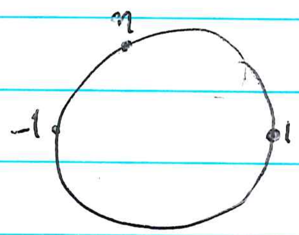
Consider a Dirac system on $0 \le x < \infty$ with $\varphi(0,\lambda) = \binom{1}{1}$. In order to get the spectral measure one can proceed as follows. Take the function

$$\frac{\psi_1(0,\lambda)}{\psi_2(0,\lambda)}$$

which has values in the disk for $\operatorname{Im}\lambda > 0$ and outside the disk for $\operatorname{Im}\lambda < 0$ and apply to it a fractional linear transformation taking 1 to ∞, [redacted] S^1 to $\mathbb{R}$ and the [redacted] upper half-plane. Then you get an analytic function on the upper half-plane with values in the upper half-plane which determines a measure.

What fractional linear transformations are involved?



Want $1 \mapsto \infty$. Let $\eta \mapsto 0$ so that

$$w = \frac{z-\eta}{z-1} \times \text{constant}$$

Then you want the w corresponding to -1 to be real, so

$$w = \frac{2}{1+\eta}\,\frac{z-\eta}{z-1} \cdot \text{real const } c$$

and the [redacted] sign of c is to be adjusted so that $\operatorname{Im} w > 0$ for $|z| < 1$.

Notice that for $\lambda \mapsto +i\infty$ we expect

$$\frac{\psi_1(0,\lambda)}{\psi_2(0,\lambda)} \to \infty$$

and hence the corresponding value of $\dfrac{2c}{1+\eta}\,\dfrac{\psi_1 - \eta\psi_2}{\psi_1 - \psi_2}$ is $\dfrac{2c}{1+\eta}$

In the periodic case the good analytic functions on the disk with positive imaginary part take purely imaginary values at $z=0$. Hence the indications are that one wants $\frac{2c}{1+\eta}$ to be purely imaginary. This forces $\eta = -1$.

Here's a simple problem you don't yet see the answer to. Let $d\nu$ be a probability measure on S^1 and $h_1, h_2, \dots$ the sequence of Schur parameters which give the associated orth. polys:

$$\begin{pmatrix} p_n \\ z^n p_n^* \end{pmatrix} = R(h_n) \begin{pmatrix} z & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} p_{n-1} \\ z^{n-1} p_{n-1}^* \end{pmatrix}$$

Let f be the endomorphism of the disk with these Schur parameters:

$$f(z) = R(\bar{h}_1) \begin{pmatrix} z & 0 \\ 0 & 1 \end{pmatrix} R(\bar{h}_2) \begin{pmatrix} z & 0 \\ 0 & 1 \end{pmatrix} \cdots\cdots\cdots \quad (o)$$

Then how are $f(z)$ and

$$g(z) = \int \frac{1+\zeta^{-1}z}{1-\zeta^{-1}z}\, d\nu(\zeta)$$

related? Conjecture that $g(z)$ is essentially $\frac{f(z)+1}{f(z)-1}$.

January 13, 1978

Suppose given $L = aT + b + T^{-1}a$ on $0 \le n < \infty$ and that

$$S(n,\lambda) = \begin{pmatrix} \varphi_{n+1} & \tilde{\varphi}_{n+1} \\ a_n\varphi_n & a_n\tilde{\varphi}_n \end{pmatrix}$$

is the solution matrix starting with $S(0,\lambda) = \text{identity}$. Suppose $d\mu(\lambda)$ is a measure on $\mathbb{R}$ which is a spectral measure for the system with the given φ-solution for $1 \le n \le \ell$, that is,

$$\int \varphi(n,\lambda)\varphi(n',\lambda)\,d\mu(\lambda) = \delta_{nn'}$$

for $1 \le n, n' \le \ell$. Put

$$m(\lambda) = \int \frac{d\mu(\hat{\lambda})}{\lambda - \hat{\lambda}} \qquad \psi(n,\lambda) = \int \frac{\varphi(n,\hat{\lambda})\,d\mu(\hat{\lambda})}{\lambda - \hat{\lambda}}$$

Note $\psi(n,\lambda)$ is defined for all $n \ge 0$ and $\psi(0,\lambda) = 0$. We have

$$[(\lambda - L)\psi_\lambda](n) = \int \varphi(n,\hat{\lambda})\,d\mu(\hat{\lambda}) = \delta_{n0} \qquad 1 \le n \le \ell.$$

Let $\tilde{\psi}$ be the solution of $(\lambda - L)u = 0$ agreeing with ψ in the range $1 \le n \le \ell+1$, whence

$$a_0\tilde{\psi}(0,\lambda) + b_0\psi(1,\lambda) + a_1\psi(2,\lambda) = \lambda\psi(1,\lambda)$$
$$1 + b_0\psi(1,\lambda) + a_1\psi(2,\lambda) = \lambda\psi(1,\lambda)$$

so that

$$a_0\tilde{\psi}(0,\lambda) = 1 \qquad \psi(1,\lambda) = m(\lambda).$$

Thus you see that

$$\tilde{\psi} = m\varphi + \tilde{\varphi}$$

and hence we have

$$\begin{pmatrix} \psi_{l+1} \\ a_l \psi_l \end{pmatrix} = S(l,\lambda) \begin{pmatrix} m \\ 1 \end{pmatrix}$$

I want to conclude that $\operatorname{Im}\left(\frac{\psi_{l+1}}{a_l \psi_l}\right) \leq 0$ for $\operatorname{Im}\lambda > 0$.

Use Bessel's inequality: $\psi(n,\lambda)$ is the coefficient of $\frac{1}{\lambda-\hat{\lambda}}$ in $L^2(d\mu(\hat{\lambda}))$ wrt the orthonormal system φ[redacted]$(n,\hat{\lambda})$ for $n=1,\dots,l$ hence

$$\sum_{n=1}^{l} |\psi(n,\lambda)|^2 \leq \int \frac{d\mu(\hat{\lambda})}{|\lambda-\hat{\lambda}|^2} = -\frac{\operatorname{Im}(m(\lambda))}{\operatorname{Im}\lambda}$$

$$\parallel$$

$$\frac{W(\tilde{\psi}_\lambda, \tilde{\psi}_{\bar{\lambda}})\Big]_0^l}{\lambda-\bar{\lambda}} = \frac{W(\tilde{\psi}_\lambda, \tilde{\psi}_{\bar{\lambda}})(l) - \begin{vmatrix} m(\lambda) & m(\bar{\lambda}) \\ 1 & 1 \end{vmatrix}}{2i \operatorname{Im}\lambda}$$

So we get
$$\frac{1}{2i}\,\frac{W(\tilde{\psi}_\lambda, \tilde{\psi}_{\bar{\lambda}})(l)}{\operatorname{Im}\lambda} \leq 0$$

which expresses that for $\operatorname{Im}\lambda$, $\operatorname{Im}\frac{\psi(l+1,\lambda)}{a_l \psi(l,\lambda)}$ [redacted] ≤ 0.

Thus we have proved:

<u>Prop:</u> If $d\mu$ is a spectral measure for L on [redacted] $1 \leq n \leq l$, then

$$m(\lambda) = \int \frac{d\mu(\hat{\lambda})}{\lambda-\hat{\lambda}}$$

is of the form

$$m(\lambda) = S(l,\lambda)^{-1}(m_l(\lambda))$$

where $m_\ell(\lambda)$ is the Stieltjes transform of some measure (or the constant ∞).

Let's see if this generalizes to Schroed. case. Let $d\mu(\lambda)$ be a spectral measure on $0 \le x \le \ell$, i.e.

$$\int \varphi(x,\lambda)\varphi(y,\lambda)\,d\mu(\lambda) = \delta(x-y)$$

for x, y in this interval or more precisely

$$\int |\hat{f}(\lambda)|^2\,d\mu(\lambda) = \int_0^\ell |f(x)|^2\,dx$$

for all $f \in L^2(0,\ell)$ where $\hat{f}(\lambda) = \int_0^\ell f(x)\varphi(x,\lambda)\,dx$

We assume that $\int \frac{d\mu(\lambda)}{1+\lambda^2} < \infty$ so

that the Stieltjes transform $m(\lambda) = \int \frac{d\mu(\bar{\lambda})}{\lambda - \bar{\lambda}}$ is defined, at least up to a real constant. Then $\frac{1}{\lambda_0 - \lambda}$ is square-integrable with respect to $d\mu$, so we can transform it to an element of $L^2(0,\ell)$:

$$\psi(x,\lambda_0) = \int \frac{\varphi(x,\lambda)\,d\mu(\lambda)}{\lambda_0 - \lambda}$$

The point here is that $f \mapsto \hat{f}(\lambda)$ [redacted] $= (f, \varphi_\lambda)$ is an isometric embedding of $L^2((0,\ell))$ into $L^2(d\mu)$, hence it has an adjoint $g \mapsto \tilde{g}$ [redacted] with

$$(\tilde{g}, f) = \int_0^\ell \tilde{g}(x) f(x)\,dx = (g, \hat{f}) = \int g(\lambda)\left[\int f(x)\varphi(x,\lambda)\,dx\right]d\mu$$

and [redacted] hence $\tilde{g}(x) = \int g(\lambda)\varphi(x,\lambda)\,d\mu$.

This adjoint has norm ≤ 1, so we get Bessel's inequality

$$\|\tilde g\|^2 \leq \|g\|^2$$

and in particular

$$\int |\psi(x,\lambda_0)|^2 dx \leqslant \int \frac{d\mu(\lambda)}{|\lambda-\lambda_0|^2} = -\frac{\operatorname{Im} m(\lambda_0)}{\operatorname{Im} \lambda_0}$$

Formally at least

$$(\lambda_0 - L)\,\psi_{\lambda_0}(x) = \int \varphi(x,\lambda)\,d\mu(\lambda) = \delta(x)$$

from which we see that the derivative of $\psi_{\lambda_0}(x)$ at $x=0$ must be $+1$. Since $\psi(0,\lambda_0) = m(\lambda_0)$ we therefore have

$\psi_{\lambda_0} = m(\lambda_0)$ [redacted] $\varphi_{\lambda_0} + \tilde\varphi_{\lambda_0}$.

Assuming this holds we have

$$-\frac{\operatorname{Im} m(\lambda_0)}{\operatorname{Im}(\lambda_0)} \geqslant \int_0^\ell |\psi(x,\lambda_0)|^2 dx = \frac{W(\psi_{\lambda_0}, \psi_{\bar\lambda_0})\Big]_0^\ell}{\lambda_0 - \bar\lambda_0}$$

$$= \frac{1}{2i}\frac{W(\psi_{\lambda_0},\psi_{\bar\lambda_0})(\ell)}{\operatorname{Im}\lambda_0} - \frac{1}{2i\operatorname{Im}\lambda_0}\begin{vmatrix} m(\lambda_0) & m(\bar\lambda_0) \\ 1 & 1 \end{vmatrix}$$

and hence

$$\frac{1}{2i}\frac{W(\psi_{\lambda_0},\psi_{\bar\lambda_0})(\ell)}{\operatorname{Im}\lambda_0} \leqslant 0$$

as in the J-matrix case.

Next I want to consider the [redacted] converse problem, namely you suppose $m(\lambda)$ is the Stieltjes transform of $d\mu$ and that $S(\ell,\lambda)\,m(\lambda)$ is a Stieltjes transform, and you want to prove $d\mu(\lambda)$ is a spectral measure on $0 \leq x \leq \ell$.

New notation: It seems silly to restrict to probability measures, [redacted] and we want formulas like

$$m_0(\lambda) = \frac{a_0^2}{\lambda - b_1 -}\ \frac{a_1^2}{\lambda - b_2 -} \cdots ,$$

[redacted] otherwise not all $m(\lambda)$ [redacted] will be admitted. [redacted] Moreover $a_0^2 = \int d\mu(\lambda)$ and we want $a_0 \varphi_1 = 1$ so that φ_1 is the first orthonormal poly. Hence we should work with $\begin{pmatrix} a_n u_{n+1} \\ u_n \end{pmatrix}$ as the data at position n. Then

$$\frac{a_n u_{n+1}}{u_n} + (b_n - \lambda) + \frac{a_{n-1}^2 u_{n-1}}{a_{n-1} u_n} = 0$$

can be written

$$\begin{pmatrix} a_n u_{n+1} \\ u_n \end{pmatrix} = \begin{pmatrix} (\lambda - b_n) u_n - a_{n-1} u_{n-1} \\ u_n \end{pmatrix} = \begin{pmatrix} 1 & \lambda - b_n \\ 0 & 1 \end{pmatrix} \begin{pmatrix} -a_{n-1} u_{n-1} \\ u_n \end{pmatrix}$$

$$= \begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -b_n \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -a_{n-1} \\ \frac{1}{a_{n-1}} & 0 \end{pmatrix} \begin{pmatrix} a_{n-1} u_n \\ u_{n-1} \end{pmatrix}$$

and $m_n(\lambda) = S(n, \lambda) \cdot m_0(\lambda)$ becomes

$$m_0(\lambda) = \frac{a_0^2}{\lambda - b_1 -} \cdots\cdots \frac{a_{n-1}^2}{\lambda - b_n - m_n(\lambda)}$$

[redacted]

Suppose that $m_1(\lambda)$ is a Stieltjes transform (by which I mean that $m_1(\lambda) = \int$ [redacted] $\left\{ \frac{1}{\lambda - \hat\lambda} + \frac{1}{1 + \hat\lambda^2} \right\} d\mu(\hat\lambda)$ + real const. and consequently

$$\lim_{\lambda \to +i\infty} \text{[redacted]}\ \frac{m_1(\lambda)}{\lambda} = 0.$$

Then you see that [redacted] $\frac{m_0(\lambda)}{\lambda} \to a_0^2$ which means

I think that $\int d\mu(\lambda) = a_0^2$

January 14, 1978:

Problem: Consider $Lu = -u'' + qu = \lambda u$ on $0 \leq x \leq l$, $\varphi(x,\lambda) = \varphi_1(x,\lambda)$. Let $\psi(x,\lambda)$ be a solution such that $\frac{\psi(l,\lambda)}{\psi'(l,\lambda)}$ is a Stieltjes transform. (For example, a real constant) (More precisely we want in the class N^- of functions $w(\lambda)$ analytic off the line with $\frac{\operatorname{Im} w(\lambda)}{\operatorname{Im} \lambda} \leq 0$ and $\overline{w(\lambda)} = w(\bar{\lambda})$.) Then I know that

$$\psi(0,\lambda) \in N^-$$

so there is a measure $d\mu$ associated to $\psi(0,\lambda)$. The problem is to show $d\mu$ is a spectral measure on $0 \leq x \leq l$. Assume $W(\varphi_\lambda, \psi_\lambda) = 1$.

Suppose $\frac{\psi(l,\lambda)}{\psi'(l,\lambda)}$ is a real constant to fix the ideas. The problem amounts to the completeness of the eigenfunctions. Be specific: One calculates $d\mu$ and finds it is the ~~measure~~ sum

$$\sum \delta(\lambda - \lambda_i) \frac{1}{\|\varphi_{\lambda_i}\|^2}$$

over the eigenvalues. Consequently

$$\int \varphi(x,\lambda)\varphi(y,\lambda)\,d\mu(\lambda) = \sum \frac{\varphi_{\lambda_i}(x)\,\varphi_{\lambda_i}(y)}{\|\varphi_{\lambda_i}\|^2}$$

and this will be $\delta(x,y)$ iff the $\frac{\varphi_{\lambda_i}}{\|\varphi_{\lambda_i}\|}$ form an orthonormal basis for $L^2(0,l)$.

It's getting obvious that one really has to understand completeness on the finite interval very well. Review various approaches to this question you know:

1) Integral equation method. Replace the operator L & bdry. conditions by Green's function, actually by L^{-1}, assuming 0 is not an eigenvalue. Then you have a completely continuous self-adjoint operator for which completeness is established by successively removing the highest eigenvalue.

2) Cauchy integral formula

$$\frac{1}{2\pi i}\oint_{|\lambda|=\infty} G(x,y,\lambda)\,d\lambda = \delta(x-y)$$

3) Possibly there is a wave equation method closely related to 2). (Victor's version of Poisson summation?)

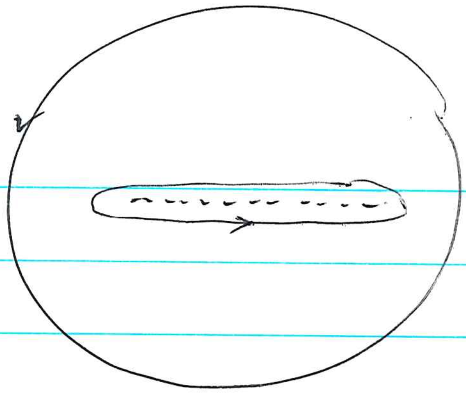
In connection with 2) here is a way to derive that $d\mu$ is a spectral measure when $d\mu$ is assoc. to $\phi(0,\lambda)$: Form Green's function as usual and establish the completeness relation

$$\delta(x,y) = \frac{1}{2\pi i}\oint_{|\lambda|=\infty} G(x,y,\lambda)\,d\lambda$$

More precisely:

$$f(x) = \frac{1}{2\pi i}\oint_{|\lambda|=\infty} d\lambda \left(\int_0^{\ell} G(x,y,\lambda)\, f(y)\,dy \right)$$

for any sufficiently nice f. Now deform this integral to a loop around the spectrum:



and you get

$$f(x) = -\frac{1}{2\pi i}\int_{-\infty}^{\infty} d\lambda \int_0^{\ell} [G^+(x,y,\lambda) - G^-(x,y,\lambda)] f(y)\, dy$$

Now provided ψ_λ is normalized so $W(\varphi_\lambda, \psi_\lambda) = 1$

$$G^{\pm}(x,y,\lambda) = \varphi(x_<,\lambda)\,\psi^{\pm}(x_>,\lambda)$$

and moreover on the spectrum $\varphi_\lambda, \psi_\lambda^{\pm}$ are proportional so that

$$\psi^{\pm}(x,\lambda) = m^{\pm}(\lambda)\,\varphi(x,\lambda)$$

Thus

$$G^{\pm}(x,y,\lambda) = m^{\pm}(\lambda)\,\varphi(x,\lambda)\,\varphi(y,\lambda)$$

and so we get

$$-\frac{1}{2\pi i}\left[G^+(x,y,\lambda) - G^-(x,y,\lambda)\right] d\lambda = -\frac{1}{2\pi i}\left[m^+(\lambda) - m^-(\lambda)\right] d\lambda\ \varphi_\lambda(x)\varphi_\lambda(y)$$

$$= d\mu(\lambda)\,\varphi(x,\lambda)\,\varphi(y,\lambda).$$

which gives the desired formula for the spectral measure.

Consider the problem of the completeness relation. According to Fadeev's article (which gives the Titchmarsh (?) argument) one uses the identity

$$(\lambda - L)^{-1} - \lambda^{-1} = (\lambda-L)^{-1}\{\lambda - (\lambda - L)\}\lambda^{-1}$$

$$= \frac{1}{\lambda}(\lambda-L)^{-1}L$$

hence

$$\frac{1}{2\pi i}\oint_{|\lambda|=\infty}(\lambda-L)^{-1} f\, d\lambda - \underbrace{\frac{1}{2\pi i}\oint_{|\lambda|=\infty}\lambda^{-1} f\, d\lambda}_{=f} = \frac{1}{2\pi i}\oint_{|\lambda|=\infty}(\lambda-L)^{-1}Lf\,\frac{d\lambda}{\lambda}$$

so the problem is to show the last integral is zero. Since $\frac{1}{2\pi i}\frac{d\lambda}{\lambda} = \frac{d\theta}{2\pi}$ it enough to show that

$$[(\lambda - L)^{-1} Lf](x) = \int_0^\ell G(x, y, \lambda)(Lf)(y)\,dy$$

goes to zero as $|\lambda| \to \infty$. We can suppose $f \in C_0^\infty((0, \ell))$ so we may as well look at $G(x, y, \lambda)$ for x, y fixed. The problem will be to show this tends to zero as $|\lambda| \to \infty$, or if it doesn't then at least its average of the circle does. This is not so clear:

Examples: 1) Take $\varphi(x, \lambda) = \cos\sqrt{\lambda}\,x$ on $0 \le x \le \pi$ with $u'(\pi) = 0$, so that we can take

$$\psi(x, \lambda) = \cos\sqrt{\lambda}(x - \pi)$$

Then

$$W(\varphi_\lambda, \psi_\lambda) = \begin{vmatrix} \cos\sqrt{\lambda}\,x & \cos\sqrt{\lambda}(\pi - x) \\ -(\sin\sqrt{\lambda}\,x)\sqrt{\lambda} & +\sqrt{\lambda}\sin\sqrt{\lambda}(\pi - x) \end{vmatrix}$$

$$= \sqrt{\lambda}\sin(\sqrt{\lambda}\,\pi)$$

so the eigenvalues are $\lambda = 0, 1, 2^2, 3^2, \cdots$.

$$G(x, y, \lambda) = \frac{\cos(\sqrt{\lambda}\,x_<)\cos(\sqrt{\lambda}(\pi - x_>))}{\sqrt{\lambda}\sin(\sqrt{\lambda}\,\pi)}$$

2) Same φ on $0 \le x < \infty$

$$\psi(x, \lambda) = e^{i\sqrt{\lambda}\,x} \qquad \sqrt{\lambda} \text{ always in UHP}$$

$$G(x, y, \lambda) = \frac{\cos(\sqrt{\lambda}\,x_<)\,e^{i\sqrt{\lambda}\,x_>}}{i\sqrt{\lambda}}$$

If $x \le y$, then

$$G(x,y,\lambda) = \frac{e^{i\sqrt{\lambda}(x+y)} + e^{i\sqrt{\lambda}(y-x)}}{2i\sqrt{\lambda}}$$

so

$$|G(x,y,\lambda)| \leqslant \frac{1}{2|\sqrt{\lambda}|}\left(e^{-(\mathrm{Im}\sqrt{\lambda})(x+y)} + e^{\text{[redacted]}-(\mathrm{Im}\sqrt{\lambda})(y-x)}\right)$$

$$\leqslant \frac{1}{\sqrt{\lambda}}$$

and it's clear that $G \to 0$ as $|\lambda| \to \infty$. (In the first example it isn't so clear because of the poles.)

Note that G dies exponentially if [redacted] $\mathrm{Im}\sqrt{\lambda} \to +\infty$ unless $y = x$. This brings to [redacted] mind Hörmander's estimates.

Recall from page 649 a formal wave equation method for obtaining [redacted] the completeness relation. [redacted] ~~which you want to expand in eigenfunctio~~[redacted] You suppose you can solve the wave equation

$$i\frac{\partial u}{\partial t} = Lu$$

with arbitrary initial data, whence [redacted] e^{-iLt} [redacted] is defined. Given f let $u(t) = e^{-itL}f$ be the solution with initial data f. Take the Fourier transform

$$\hat{u}(\lambda) = \int_{-\infty}^{\infty} e^{i\lambda t} u(t)\,dt$$

and break it up into $\int_0^\infty + \int_{-\infty}^0$.

Then

$$\int_0^\infty e^{i\lambda t} e^{-iLt}\,dt = \int_0^\infty e^{i(\lambda - L)t}\,dt = i(\lambda - L)^{-1} = iG^+(\lambda)$$

since it is analytic for $\mathrm{Im}\,\lambda > 0$

$$\int_{-\infty}^{0} e^{i(\lambda-L)t}\,dt = \frac{1}{i}(\lambda-L)^{-1} = \frac{1}{i}G^{-}(\lambda).$$ By Fourier inversion

$$u(t) = \int e^{-i\lambda t}\,\hat{u}(\lambda)\,\frac{d\lambda}{2\pi} = \int e^{-i\lambda t}\,\frac{d\lambda}{2\pi i}\{G^{-}(\lambda)-G^{+}(\lambda)\}f$$

hence putting $t=0$ we get the completeness relation

$$\frac{1}{2\pi i}\int_{-\infty}^{\infty}\{G^{-}(\lambda)-G^{+}(\lambda)\}\,d\lambda = I.$$

January 16, 1978

deBranges approach to completeness: Let χ_ℓ be the char fn. of $[0,\ell]$. [redacted] The map

$$\lambda \longmapsto [redacted]\,\chi_\ell\varphi_{\bar\lambda} \in L^2(0,\ell)$$

defines the de B space $\mathcal{H}_\ell$.

Better: $\lambda \longmapsto \chi_\ell\varphi_{\bar\lambda}$ is an anti-holom. map with values in $L^2(0,\ell)$. Its image is "the" Hilbert space described by the kernel

$$K^\ell(\lambda,z) = (\chi_\ell\varphi_{\bar z},\chi_\ell\varphi_{\bar\lambda}) = (\varphi_\lambda,\varphi_{\bar z})_{[0,\ell]} = \frac{W(\varphi_\lambda,\varphi_{\bar z})(\ell)}{\lambda-\bar z}$$

which we denote by $\mathcal{H}_\ell$. $\mathcal{H}_\ell$ [redacted] is a Hilbert space of entire functions generated by the functions $\lambda\mapsto K^\ell(\lambda,z)$ for each z. [redacted] So we have an isometric embedding

$$\mathcal{H}_\ell \xhookrightarrow[i_\ell]{} L^2(0,\ell)$$

$$K^\ell_z \longmapsto \chi^\ell\varphi_{\bar z}$$

whose adjoint i_ℓ^* sends f to the entire function

$$(\imath_\ell^* f)(\lambda) = \hat{f}(\lambda) = (f, \varphi_{\bar\lambda})_{[0,\ell]} = \int_0^\ell f(x)\varphi(x,\lambda)\,dx$$

If $\ell < \ell'$ we have two embeddings:

$$\begin{tikzcd}
\mathcal{H}_{\ell'} \arrow[dr, hook, "\imath_{\ell'}"] & \\
& L^2(0,\infty) \\
\mathcal{H}_\ell \arrow[ur, hook, "\imath_\ell"'] &
\end{tikzcd}$$

Let's compute $\imath_{\ell'}^* \imath_\ell$ on a generator K_z^ℓ of $\mathcal{H}_\ell$.

$$\imath_\ell K_z^\ell = \chi_\ell \varphi_{\bar z}$$

$$\left(\imath_{\ell'}^* \imath_\ell K_z^\ell\right)(\lambda) = \left(\chi_\ell \varphi_{\bar z}, \chi_{\ell'}\varphi_{\bar\lambda}\right) = K_z^\ell(\lambda)$$

This shows that $K_z^\ell \in \mathcal{H}_{\ell'}$ for any z. Let g be an arbitrary element of $\mathcal{H}_\ell$. Then $g = \lim g_n$ where g_n is a finite linear combination of K_z^ℓ for various z, and where this limit is taken in the sense of the norm on $\mathcal{H}_\ell$. Since $\imath_{\ell'}^* \imath_\ell$ is a bounded operator, we have

$$\imath_{\ell'}^* \imath_\ell g = \lim \imath_{\ell'}^* \imath_\ell g_n$$

in the sense of the norm of $\mathcal{H}_{\ell'}$. But we've seen that $\imath_{\ell'}^* \imath_\ell g_n = g_n$, and we know that $g_n \to g$ in $\mathcal{H}_\ell \Rightarrow g_n \to g$ uniformly on compact sets. Thus $\imath_{\ell'}^* \imath_\ell g = g$.

Thus we see that $\mathcal{H}_\ell \subset \mathcal{H}_{\ell'}$ and the inclusion is $\imath_{\ell'}^* \imath_\ell$. It follows that the inclusion is of norm ≤ 1. Assume that we can show it is an isometry (This depends only on the solution matrices $S(\ell,\lambda)$, $S(\ell',\lambda)$). Then it follows that the image of i_ℓ is contained in the image of $i_{\ell'}$, hence

$$i_{\ell'}(i^*_{\ell'} i_\ell) = i_\ell$$

which means that

$$\begin{tikzcd} \mathcal{H}_{\ell'} \arrow[dr, "i_{\ell'}"] & \\ \cup & L^2(0,\infty) \\ \mathcal{H}_\ell \arrow[ur, "i_\ell"'] & \end{tikzcd}$$

commutes.

Now we can establish completeness: $i_\ell \mathcal{H}_\ell = L^2(0,\ell)$ as follows. Let $f \in L^2(0,\ell)$ be $\perp$ to $i_\ell \mathcal{H}_\ell$. Then it's $\perp$ to $i_{\ell'}\mathcal{H}_{\ell'}$ for all $\ell' \leq \ell$. But $i_{\ell'}\mathcal{H}_{\ell'}$ contains $i_{\ell'} K_0^{\ell'} = \chi^{\ell'} \varphi_0 = \chi^{\ell'}$, and these characteristic functions span all step functions, hence span $L^2(0,\infty)$. $\therefore f = 0$.

Let's look at scattering for Schroedinger's equation on $\mathbb{R}$:

$$Lu = -\frac{d^2u}{dx^2} + qu = \lambda u$$

where q has compact support. Put $k^2 = \lambda$ and suppose there are no bound states. Introduce solutions

$$u^+_{>0}(x,k) = e^{ikx} \qquad x \gg 0$$

$$u^+_{<0}(x,k) = e^{ikx} \qquad x \ll 0$$

$$u^-_{>0}(x,k) = u^+_{>0}(x,-k) \qquad \text{and similarly } u^-_{<0}.$$

As long as $k \neq 0$ $u^+_{>0}, u^-_{>0}$ are ind. solutions, and the same holds for $u^+_{<0}, u^-_{<0}$. Let $A(k)$, $B(k)$ be $\ni$

$$u^+_{>0} = A(k)\, u^+_{<0} + B(k)\, u^-_{<0}$$

Then conjugation interchanges $\pm$ for k real so

$$u^-_{>0} = \overline{B(k)}\, u^+_{<0} + \overline{A(k)}\, u^-_{<0}$$

Also $k \longmapsto -k$ interchanges $\pm$ so we see that

$$\overline{B(k)} = B(-k) \qquad \overline{A(k)} = A(-k).$$

Write this

$$\begin{pmatrix} u^+_{>0} & u^-_{>0} \end{pmatrix} = \begin{pmatrix} u^+_{<0} & u^-_{<0} \end{pmatrix} \begin{pmatrix} A & \overline{B} \\ B & \overline{A} \end{pmatrix}$$

and take Wronskian. One gets the matrix is unimodular i.e.

$$|A|^2 - |B|^2 = 1.$$

Hence

$$\begin{pmatrix} u^+_{<0} & u^-_{<0} \end{pmatrix} = \begin{pmatrix} u^+_{>0} & u^-_{>0} \end{pmatrix} \begin{pmatrix} \overline{A} & -\overline{B} \\ -B & A \end{pmatrix}$$

Consider

$$\frac{1}{A(k)} u^+_{>0} = u^+_{<0} + \frac{B(k)}{A(k)} u^-_{<0}$$

i.e.

$$\underset{x \gg 0}{\frac{1}{A(k)} e^{ikx}} \longleftrightarrow \underset{x \ll 0}{e^{ikx} + \frac{B(k)}{A(k)} e^{-ikx}}$$

Put

$$R(k) = \frac{1}{A(k)} \qquad \text{reflection coeff}$$

$$T(k) = \frac{B(k)}{A(k)} \qquad \text{transmission coeff.}$$

Reason for terminology: Let $\alpha(k) \in C_0^\infty(\mathbb{R})$ and form

$$u(x,t) = \int e^{-ikt}\Big(u^+_{<0} + \underbrace{\frac{B}{A}}_{=R} u^-_{<0}\Big)\alpha(k)\,dk = \int e^{-ikt}\Big(\underbrace{\frac{1}{A}}_{=T} u^+_{>0}\Big)\alpha(k)\,dk$$

which is a solution of the wave equation

$$-\frac{\partial^2 u}{\partial t^2} = Lu$$

We have

$$u = \hat{\alpha}(x-t) + \widehat{R\alpha}(-x-t) \quad \longleftrightarrow \quad \widehat{T\alpha}(x-t)$$
$$x \ll 0 \qquad\qquad\qquad\qquad x \gg 0$$

For large negative t, the $\widehat{R\alpha}$ term is negligible, as well as the $\widehat{T\alpha}$ term, hence we have a wave $\hat{\alpha}(x-t)$ travelling toward the right toward the obstacle represented by the potential. As $t \longrightarrow +\infty$, the $\hat{\alpha}$ term is negligible as we see a reflected wave represented by the $\widehat{R\alpha}(-x-t)$ term travelling to the left, and a transmitted ~~wave~~ wave $\widehat{T\alpha}(x-t)$ travelling to the right. Notice that

$$|R|^2 + |T|^2 = \frac{|B|^2}{|A|^2} + \frac{1}{|A|^2} = 1$$

which expresses energy conservation.

To see what happens to a wave coming from the left we want to use $\int e^{-ikt} e^{-ikx} \beta(k)\, dk = \hat{\beta}(-x-t)$ for $x \gg 0$, so we want an equation relating $u^-_{<0}, u^-_{>0}, u^+_{>0}$ or

$$u^-_{<0} = -\bar{B}\, u^+_{>0} + A\, u^-_{>0}$$

$$\frac{1}{A} u^-_{<0} = u^-_{>0} - \frac{\bar{B}}{A} u^+_{>0}$$

$$\left(\tfrac{1}{A}\beta\right)^{\wedge}(-x-t) \longleftrightarrow \hat{\beta}(-x-t) + \left(\widehat{-\tfrac{\bar{B}}{A}\beta}\right)(x-t)$$
$$x \ll 0 \qquad\qquad\qquad\qquad x \ll 0$$

So it's more or less clear that we want to use the solutions ~~[illegible]~~ $u^+_{<0}$, $u^-_{>0}$ to represent incoming

solutions and $u^+_{>0}$, $u^-_{<0}$ to represent outgoing solution.
[redacted] The scattering matrix should relate these bases:

$$u^+_{<0} = \frac{1}{A} u^+_{>0} - \frac{B}{A} u^-_{<0}$$

$$u^-_{>0} = \frac{\bar{B}}{A} u^+_{>0} + \frac{1}{A} u^-_{<0}$$

So S is essentially $\begin{pmatrix} \frac{1}{A} & -\frac{B}{A} \\ \frac{\bar{B}}{A} & \frac{1}{A} \end{pmatrix}$

Note this is a unitary matrix, but not necessarily of determinant 1. Infact $\det = \frac{|A|^2}{A^2} = \frac{\bar{A}}{A}$.

Analyticity properties: If one supposes only that q decays as $|x| \to \infty$, then one has to define $u^{\pm}_{>0}$, $u^{\pm}_{<0}$ using the asymptotic behavior. For $\operatorname{Im} k > 0$, the asymptotic behavior determines only the "small" solution. So $u^+_{>0}$ and $u^-_{<0}$ are analytic in the upper half-plane, hence so is their Wronskian. Compute this Wronskian for k [redacted] real

$$W(u^+_{>0}, u^-_{<0}) = \text{[redacted]}$$

$$= W\left(u^+_{>0}, -\bar{B} u^+_{>0} + A u^-_{>0}\right)$$

$$= A(k)\, W(u^+_{>0}, u^-_{>0}) \qquad \begin{vmatrix} 1 & 1 \\ ik & -ik \end{vmatrix}$$

$$= A(k)(-2ik)$$

Thus $A(k)$ has an analytic extension to the upper half-plane. Notice also that a zero of $A(k)$ in the upper half-plane represents a bound state because it means a solution like $u^-_{<0} = e^{-ikx}$ for $x \ll 0$ and are $\sim$ to $u^+_{>0} = e^{ikx}$ for $x \gg 0$.

These zeroes [redacted] of A correspond to negative λ, hence as $\lambda = k^2$, they occur on the [redacted] imaginary axis.