Situation: For anisotropic scattering we obtained

\[ R(k) = \frac{\tilde{f}(2-2s)}{\tilde{f}(2s)} \quad s = \frac{1}{2} + \frac{k}{2} \]

\[ = \frac{\tilde{f}(1+2ik)}{\tilde{f}(1-2ik)} \]

and in \( \text{Im} k > 0 \) the singularities are a pole at \( k = \frac{1}{2} i \) and zeroes on the line (by Riemann Hyp) \( \text{Im} k = \frac{1}{4} \).

Now we sort of understand the pole as being due to a bound state. So the question is whether we might obtain something interesting by applying Marchenko-Faddeev to the scattering data \( R(k) \), bound state at \( k = \frac{1}{2} i \) with a suitable norming constant, to obtain a potential \( q(x) \) on a half-line \( x > a \). One problem with this is that in the representation for \( R(k) \):

\[ R(k) = \frac{B(k)}{A(k)} \]

\( B(k) = A(k) = \overline{A(k)} \)

\( A(k) \) is analytic in the UHP with zeroes at the bound states. Hence \( A(k) \) would vanish at \( k = \frac{1}{2} i \) and would have poles on \( \text{Im}(k) = -\frac{1}{4} \). Hence it seems that \( A(k) \) could not be \( \overline{\tilde{f}(1-2ik)} \).
Lippmann–Schwinger integral equations:

Suppose we have two one-parameter groups of operators \( U_0(t) = e^{iH_0 t} \), \( U(t) = e^{iHt} \) and we put \( H = H_0 + V \). Assume the trajectories \( e^{iH_0 t} u \) and \( e^{iHt} v \) are asymptotic as \( t \to \pm \infty \). I want them to be strongly asymptotic so that

\[
v = \lim_{t \to -\infty} U(-t)U_0(t)u
\]

\[
u = \lim_{t \to +\infty} U_0(-t)U(t)v
\]

For example if \( \|U(t)v - U_0(t)u\| \to 0 \) and \( U, U_0 \) are 1-par. unitary groups. One has

\[
\frac{d}{dt} U_0(-t)U(t)v = U_0(-t)(-iH_0)U(t)v + U_0(-t)(iH)U(t)v = U_0(-t)iV U(t)v
\]

so

\[
U_0(-t)U(t)v = u + \int_{-\infty}^{t} U_0(-t')iV U(t')v \, dt'
\]

\[
U(t)v = U_0(t)u + \int_{-\infty}^{t} U_0(t-t')iV U(t')v \, dt'
\]
Now take Fourier transform of the last equation.

\[ \hat{\mathcal{V}}(k) = \int_{-\infty}^{\infty} e^{-ikt} U(t) \nu \, dt \] and similarly for \( \hat{\mathcal{U}} \).

\[ \hat{\mathcal{V}}(k) = \hat{\mathcal{U}}(k) + \int_{-\infty}^{\infty} e^{-ikt} dt \int_{-\infty}^{t} e^{iH_0(t-t')} \, iVU(t')\nu \, dt' \]

\[ = \int_{-\infty}^{\infty} dt' \left( \int_{t'}^{\infty} dt e^{-ikt} e^{iH_0(t-t')} \, iVU(t')\nu \right) \]

\[ = \int_{-\infty}^{\infty} dt' e^{-ikt'} \frac{-1}{iH_0 - i(k\nu)} iVU(t')\nu \]

\[ \therefore \hat{\mathcal{V}}(k) = \hat{\mathcal{U}}(k) - \frac{i}{H_0 - i(k\nu)} \hat{\mathcal{V}}(k) \]

The explanation of the above: One wants to solve

\[ \frac{d}{dt} U(t)\nu = iH_0 U(t)\nu \]

with \( U(t)\nu \sim U_0(t)\nu \) as \( t \to -\infty \).

Rewrite the DE

\[ \left( \frac{d}{dt} - iH_0 \right) U(t)\nu = iVU(t)\nu \]

and use the Green's function solution of this

\[ U(t)\nu = U_0(t)\nu + \int_{-\infty}^{\infty} G(t,t') \, iVU(t')\nu \, dt' \]

which is adapted to the boundary condition at \( t = -\infty \).
Thus
\[ G^-(t,t') = \begin{cases} 
0 & t < t' \\
U(t-t') & t > t'
\end{cases} \]

Then you take the Fourier transform of the integral equation using the fact that convolution goes into product and that
\[
\int_{-\infty}^{\infty} G^-(t) e^{-ikt} \, dt = \int_{-\infty}^{\infty} e^{-ikt+ik Ho t} \, dt = \frac{1}{i} \frac{1}{k-H_o+i\epsilon}
\]

where \( \epsilon \) is an infinitesimal positive quantity. One obtains the Lippmann–Schwinger equation

\[ \hat{V}(k) = \hat{u}(k) + \frac{1}{k-H_o-i\epsilon} \hat{V} \, \hat{v}(k) \]

Similarly if \( U(t) V^+ \sim U_0(t) u \) at \( t \to +\infty \) we get

\[ \hat{V}^+(k) = \hat{u}(k) + \frac{1}{k-H_o+i\epsilon} \hat{V} \, \hat{v}^+(k) \]
November 19, 1978

Review Zippmann-Schwinger: Let \( u(t) = e^{iH_0 t}u(0) \), \( v(t) = e^{iH_0 t}v(0) \) be asymptotic (strongly) as \( t \to +\infty \).

From
\[
\frac{d}{dt} (e^{-iH_0 t} v) = i(H-H_0) v = i V v
\]

we get
\[
e^{-iH_0 t} V(t) = e^{-iH_0 t} v(0) - \int_0^t e^{-iH_0 t'} i V v(t') dt'
\]

or
\[
v(t) = u(t) - \int_0^t e^{iH_0 (t-t')} i V v(t') dt'
\]

or
\[
v(t) = u(t) + \int_{-\infty}^t G^+(t-t') V v(t') dt'
\]

where
\[
G^+(t-t') = \begin{cases} 0 & t > t' \\ \frac{e^{iH_0 (t-t')}}{i} & t < t' \end{cases}
\]

Take Fi.
\[
\int_{-\infty}^\infty e^{-ikt} G^+(t) dt = \int_{-\infty}^\infty e^{-ikt} \frac{e^{iH_0 t}}{i} dt = \frac{1}{i} \frac{1}{k - H_0 + i\epsilon}
\]

\( \epsilon > 0 \)

Get LS equation
\[
\dot{V}(k) = A(k) + \frac{1}{k - H_0 + i\epsilon} V V(k)
\]
But the basic equation is
\[ v = u + G^+ V v \]
or \[ v = (I - G^+ V)^{-1} u, \] Thus \((I - G^+ V)^{-1}\) is essentially the Møller wave operator \(\Omega^+\):
\[ \Omega^+ = \lim_{t \to +\infty} U(t) U_0(t) \]
which associates to a free trajectory \(U(t) = U_0(t) u_0\) the perturbed trajectory \(v(t) = U(t) v_0\) asymptotic to it.

Since the scattering matrix is
\[ S = (\Omega^+)^{-1} \Omega^- \]
one gets the formula
\[ \det S(k) = \frac{\det (I - G^+ V)}{\det (I - G^- V)} \]

I problems here because \(\Omega^\pm\) are not operator functions of \(k\) the way \(1 - G^+ V\) is. See Dec. 17
Schwinger's variational business. One has an integral equation

\[ \mathcal{Y} = G K \]

where \( \mathcal{Y} \) is a given, \( G \) is a Green's function, and \( K \) is either a surface current or the field on an aperture. If

\[ \frac{1}{X} = (\mathcal{Y}, K) \]

then \( X \) is the admittance of an equivalent circuit. The idea is that one really wants to compute \( X \) by choosing an approximation to \( K \). If one uses the above formula for \( X \), then the error matters. The point is instead to use the expression

\[ X = \frac{(G K, K)}{(\mathcal{Y}, K)^2} \]

to compute \( X \) from an approximate \( K \), because the latter expression is stationary when \( K \) is correct. In effect

\[ \delta \frac{(G K, K)}{(\mathcal{Y}, K)^2} = \frac{(G \delta K, K) + (G K, \delta K)}{(\mathcal{Y}, K)^2} - \frac{2}{(\mathcal{Y}, K)^3} \frac{\delta (G K, K)}{(\mathcal{Y}, K)^2} \]

\[ = \frac{1}{(\mathcal{Y}, K)^2} \left\{ 0 \right\} \]

\[ \Rightarrow \quad G K = \frac{(G K, K)}{(\mathcal{Y}, K)} \mathcal{Y} \]

\[ \Rightarrow \quad G K \text{ is proportional to } \mathcal{Y} \]
More generally assume only that $G$ is hermitian.

Then

$$
\frac{\delta}{|\psi, K|^2} \frac{G(K, K)}{G(K, K)} = \left\{ \frac{\delta(GK, K)}{G(K, K)} - \frac{\delta(K, K)}{G(K, K)} \right\}
$$

$$
= \frac{G(K, K)}{|\psi, K|^2} \left\{ \frac{G(K, K)}{G(K, K)} - \frac{\delta(K, K)}{G(K, K)} \right\}
$$

$$
= \frac{G(K, K)}{|\psi, K|^2} 2 \left\{ \text{Re} \left( \frac{G(K, K)}{G(K, K)}, \delta(K, K) \right) - \text{Re} \left( \frac{\psi, K}{G(K, K)}, \delta(K, K) \right) \right\} = 0
$$

$$\iff \quad GK = \frac{G(K, K)}{\psi, K} \psi \iff \quad GK \text{ proportional to } \psi.
$$

---

Example of a variational process is needed.

---

Green's function for $\Delta + k^2$ in $\mathbb{R}^3$: In spherical coordinates the basic infinitesimals are $dr, r d\phi, r \sin \phi d\theta$, that is

$$
d s^2 = dr^2 + r^2 d\phi^2 + r^2 \sin^2 \phi d\theta^2
$$

and an orthonormal frame is

$$
\frac{\partial}{\partial r}, \frac{1}{r} \frac{\partial}{\partial \phi}, \frac{1}{r \sin \phi} \frac{\partial}{\partial \theta}
$$

The volume element is $dV = r^2 \sin \phi \, dr \, d\phi \, d\theta$ and the Laplacian is

$$
\frac{1}{r^2 \sin \phi} \frac{\partial}{\partial r} \left( r^2 \sin \phi \frac{\partial}{\partial r} \right) + \frac{1}{r^2 \sin \phi} \frac{\partial}{\partial \phi} \left( \frac{1}{r} \frac{\partial}{\partial \phi} \right) + \frac{1}{r^2 \sin \phi \sin \phi \, d\theta} \frac{\partial}{\partial \theta} \left( \frac{1}{r \sin \phi} \frac{\partial}{\partial \theta} \right)
$$
\[
\Delta = \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2}{\partial \phi^2}
\]
\[
= \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial}{\partial r} \right) + \frac{1}{r^2} (\text{Laplacian on } S^2)
\]

I guess it is known that \( r^l \psi^m(\rho, \theta) \) form a basis for the harmonic homogeneous polynomials of degree \( l \).

Substituting in the above shows that eigenvalues of the Laplacian on \( S^2 \) are \(-l(l+1)\). Consequently when the Schrödinger DE

\[
(-\Delta + V(r)) \psi = \lambda \psi
\]

is separated in spherical coord., for a component \( \psi \) of angular momentum \( l \) one gets

\[
\left\{ -\frac{1}{r^2} \frac{d^2}{dr^2} + \frac{l(l+1)}{r^2} + V(r) \right\} \psi = \lambda \psi.
\]

The Green's function centered at 0 will be a radial function \( u(r) \)

\[
0 = (\Delta + k^2) u = \left( \frac{1}{r^2} \frac{d}{dr} \left( r^2 \frac{d}{dr} \right) + k^2 \right) u
\]

\[
r \left( \frac{1}{r^2} \frac{d}{dr} \left( r^2 \frac{d}{dr} \right) \right) \frac{1}{r} = \left( \frac{d}{dr} + \frac{1}{r} \right) \left( \frac{d}{dr} - \frac{1}{r} \right) = \frac{d^2}{dr^2} - \frac{1}{r^2} + \frac{1}{r^2}
\]

\[
u = \frac{C}{r}
\]

ind. solutions. For \( In k > 0 \)
it must decay so $G$ is proportional to \( \frac{e^{ikr}}{r} \).

At the origin we want \((k^2 + \Delta)G = 0\). For $k = 0$ we know the appropriate $G$ is \(-\frac{1}{4\pi}r\):

\[
\begin{align*}
\iint_{B(0, \epsilon)} \Delta (-\frac{1}{4\pi}) \, dV &= \iint_{S(0, \epsilon)} (-\frac{1}{4\pi}) \cdot r \, dS = \iint_{S(0, \epsilon)} \frac{1}{4\pi r^2} \, dS = 1.
\end{align*}
\]

On the other hand the Fourier transform gives

\[
G_k(r) = \iiint e^{ik \cdot \xi} \frac{1}{k^2 - 1^2} \, \frac{d^3 \xi}{(2\pi)^3}
\]
On solves the Helmholtz equation \((\Delta + k^2) u = 0\) with \(u = 0\) on the plane, both \(u\) and \(\frac{\partial u}{\partial z}\) should be continuous across the aperture. The other boundary condition says that for \(z < 0\) the solution consists of an incoming plane wave and outgoing waves and that for \(z > 0\) it consists of outgoing waves.

It would be nice to understand precisely the meaning of incoming and outgoing. Let us consider a simpler example, the Sommerfeld (diffraction) problem: \(u\) by an infinite half-plane, which we take to be \(x \leq 0, y = 0\) in the \(x, y\) plane.
We want to solve \( \frac{\partial^2 \psi}{\partial t^2} = \Delta \psi \) with \( \Delta \psi = 0 \) on the half-line with
\[
\psi(x,y,t) = f(x+t) \quad t > 0
\]
if supported in \( \mathbb{R}_{x>0} \). Assume the solution exists. Then it will be a superposition of plane wave solutions:

\[
\psi(x,y,t) = \int e^{-ikt} u(x,y,k) \, dk/2\pi
\]
where \( u(x,y,k) \) satisfies \( (\Delta + k^2)u = 0 \), \( u = 0 \) on half-line, and
\[
u(x,y,k) = e^{-ikx} \text{ reflected wave } \tilde{\nu}(x,y,k)
\]
By causality,
\[
\int e^{-ikt} \tilde{\nu}(x,y,k) \, dk/2\pi = 0 \quad t < \sqrt{x^2 + y^2}
\]
(otherwise infinite function), hence \( \tilde{\nu}(x,y,k) \) should be "made up" of \( e^{ikr} \) with \( r > \sqrt{x^2 + y^2} \). Thus for \( \text{Im} k > 0 \) the function \( \tilde{\nu}(x,y,k) \) should be exponentially decaying as one goes to \( \infty \). Actually in this case if one considers \( \tilde{\nu} = u - e^{-ikx} \), it is a solution of Helmholtz with Dirichlet boundary condition \( -e^{-ikx} \) on the half-line and \( 0 \) at \( \infty \), so one expects the existence of \( \tilde{\nu} \) to come from elliptic theory. It would seem this works quite generally.

In fact, for potential scattering to get the solution \( u(x,k) \) with a certain incoming behavior \( e^{-ikx} \) one works with \( \tilde{\nu}(x,k) = u(x,k) - e^{-ikx} \) which satisfies \( (\Delta + k^2)\tilde{\nu} = \Delta u \) and hence is found by
solving the equation (Lippmann–Schwinger).

\[ u(x, k) - e^{-ikx} = \int G_k(x, x') g(x') u(x', k) dx'. \]

So now let's solve the Dirichlet problem or Helmholtz equation by introducing parabolic coordinates.

\[(x + iy) = \frac{1}{2} (\xi + i\eta)^2 \]

\[ x = \frac{1}{2} (\xi^2 - \eta^2) \quad y = \frac{1}{2} i \eta \]

\[ dx = \frac{1}{2} (d\xi - \eta d\eta) \quad dy = \frac{1}{2} i (d\xi + \eta d\eta) \]

\[ ds^2 = dx^2 + dy^2 = (\xi^2 + \eta^2) (d\xi^2 + d\eta^2) \]

An orthonormal frame is

\[ \frac{1}{\sqrt{\xi^2 + \eta^2}} \frac{\partial}{\partial \xi} \quad \frac{1}{\sqrt{\xi^2 + \eta^2}} \frac{\partial}{\partial \eta} \]

and

\[ dx dy = (\xi^2 + \eta^2) d\xi d\eta \]

So

\[ \Delta = \frac{1}{\xi^2 + \eta^2} \frac{\partial}{\partial \xi} \left( \frac{1}{\sqrt{\xi^2 + \eta^2}} \frac{\partial}{\partial \xi} \right) + \text{same for } \eta \]

\[ = \frac{1}{\xi^2 + \eta^2} \left( \frac{\partial^2}{\partial \xi^2} + \frac{\partial^2}{\partial \eta^2} \right) \]

so the Helmholtz equation becomes

\[ \left\{ \left( \frac{\partial^2}{\partial \xi^2} + \frac{\partial^2}{\partial \eta^2} \right) + k^2 (\xi^2 + \eta^2) \right\} u = 0 \]

which separates. We work in the domain \( \xi > 0; \) the boundary curve \( \xi = 0, \eta \in \mathbb{R} \) corresponds to the half-line \( x < 0, \eta = 0. \)

Let us first find \( u = e^{-ikx} + \tilde{u} \) so that we want \( \tilde{u}(x, 0) = -e^{-ikx} \) for \( x < 0. \) In parabolic
coordinates this becomes

\[ \tilde{u}(0, \eta) = -e^{i k \eta^2 / 2} \]

Try \( \tilde{u}(\xi, \eta) = f(\xi) g(\eta) \) whence \( g(\eta) = e^{i k \eta^2 / 2} \)

up to a constant. Then

\[
\left( \frac{\partial^2}{\partial \eta^2} + k^2 \eta^2 \right) e^{i k \eta^2 / 2} = e^{i k \eta^2 / 2} \left\{ \left( \frac{\partial}{\partial \eta} + ik \eta \right)^2 + k^2 \eta^2 \right\} \]

\[
= e^{i k \eta^2 / 2} \left\{ \frac{\partial^2}{\partial \eta^2} + 2i k \frac{\partial}{\partial \eta} - k^2 \eta^2 + ik + k \eta \right\}
\]

\[
= i k \eta g
\]

Hence \( f \) must satisfy

\[
\left( \frac{\partial^2}{\partial \xi^2} + k^2 \xi^2 + i k \right) f = 0
\]

\[
= \left( \frac{\partial^2}{\partial \xi^2} + k^2 \xi^2 + i k \right) e^{-i k \xi^2 / 2} \left( e^{i k \xi^2 / 2} f(\xi) \right)
\]

\[
= e^{-i k \xi^2 / 2} \left( \frac{\partial^2}{\partial \xi^2} - 2i k \frac{\partial}{\partial \xi} \right) \left( e^{i k \xi^2 / 2} f(\xi) \right)
\]

so

\[
\frac{d}{d \xi} e^{i k \xi^2 / 2} f(\xi) = c e^{i k \xi^2 / 2}
\]

\[
f(\xi) = c e^{-i k \xi^2 / 2} \int_{-\infty}^{\xi} e^{i k \xi^2 / 2} d\xi' + c e^{-i k \xi^2 / 2}
\]

Now we want \( f \) to vanish as \( \xi \to \infty \), so \( c_2 = 0 \)
and so

\[
\tilde{u}(\xi, \eta) = c_1 \left( e^{-i k \xi^2 / 2} \int_{-\infty}^{\xi} e^{i k \xi^2 / 2} d\xi' \right) e^{i k \eta^2 / 2}
\]
Put \( \text{Erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} \, dt \) and recall \( \int_0^\infty e^{-x^2} \, dx = \frac{\sqrt{\pi}}{2} \).

Here if \( s = k/\omega \) as usual

\[
\int_0^\infty e^{-i k \xi^2} \, \xi^2 \, d\xi = \int_0^\infty e^{-s \xi^2} \, d\xi = \frac{1}{\sqrt{s}} \int_0^\infty e^{-s \xi^2} \, d\xi
\]

\[
= \frac{\sqrt{\pi}}{2\sqrt{s}} \text{ Erf}(\sqrt{s} \xi)
\]

So \( \tilde{u}(\xi, \eta) = -e^{-i k \xi^2/2} \text{ Erf}(\sqrt{s} \xi) \)

Now \( x = \frac{\xi^2 - \eta^2}{2}, \quad y = \xi \eta \)

\( 2x \xi^2 = \xi^4 - \eta^2, \quad \xi^4 - 2x \xi^2 - \eta^2 = 0 \)

\( \xi^2 = x \pm \sqrt{x^2 + \eta^2}, \quad \xi = \sqrt{x \pm \sqrt{x^2 + \eta^2}} \)

So it appears that the desired solution is

\[
u(x, y, k) = e^{-ikx} \left\{ \frac{1}{\sqrt{s} \sqrt{x + \sqrt{x^2 + \eta^2}}} \text{ Erf}(\sqrt{s} \sqrt{x + \sqrt{x^2 + \eta^2}}) \right\}
\]

\[
= e^{-ikx} \frac{2}{\sqrt{\pi}} \int_0^\infty e^{-u^2} \, du
\]

where \( s = k/\omega \). But you want to move \( k \) to

the real axis from the UHP and so you obtain a Fresnel

integral instead of Erf.
Try Wiener-Hopf approach to the Sommerfeld problem.
To solve $(\Delta + k^2) \tilde{u} = 0$ in the plane $-\infty \leq x \leq 0$ with
$\tilde{u}(x,0) = e^{-ikx}$ for $x < 0$. Here $Im k > 0$ and $\tilde{u}$ is
to be zero at $\infty$. There are two approaches
depending on whether we concentrate on $\tilde{u}(x,0)$ for $x > 0$
or the discontinuity in $\frac{\partial \tilde{u}(x,0)}{\partial x}$ for $x < 0$.

Denote by $G_k(\vec{r};\vec{r}')$ the Green's function for
the plane. One has $G_k(\vec{r};\vec{r}') = G_k(1\vec{r};\vec{r}')$ where
$G_k(\vec{r})$ satisfies

$$
\left( \frac{d}{dr} \frac{d}{dr} + k^2 \right) u = 0 \quad k > 0
$$

or

$$
\left( \frac{d^2}{dr^2} + k^2 r^2 \right) u = 0
$$

which is essentially Bessel's DE of order 0, except for
$k^2$ being substituted for $\nu$. If we need this we can
find it later.

First method is to use the Green's function for the
half-plane to determine $u$ off the
line in terms of $\tilde{u}$ on the line. This Green's function is
obtained by reflection

$$
G^*(x,y;x',y') = G(x,y;x',y') - G(x+y+y',x'-y')
$$

Using Green's formula

$$
\iint (u \Delta v - v \Delta u) = \int (u \frac{\partial v}{\partial n} - v \frac{\partial u}{\partial n})
$$

one gets

$$
\tilde{u}(x,y) = \int_{-\infty}^{\infty} \tilde{u}(x,0) \overline{G^*(x,0;x',y')} dx \quad \text{for } y > 0
$$
The same formula holds for $y' < 0$ (flux is a sum: $d\mathbf{s} = dx$). The condition to be satisfied is that $\frac{\partial \tilde{u}}{\partial y}$ is continuous on $y = 0, x > 0$.

Apply Green's formula to

$$\tilde{u}(x', y') = \iint (\tilde{u} \Delta G - (\Delta \tilde{u}) G) = \int (\tilde{u} \frac{\partial G}{\partial n} - \frac{\partial \tilde{u}}{\partial n} G)$$

$$\tilde{u}(x', y') = -\int_{-\infty}^{0} \left\{ \frac{\partial \tilde{u}}{\partial y}(x_0^+, x', y') - \frac{\partial \tilde{u}}{\partial y}(x_0^-, x', y') \right\} G(x_0, x', y') \, dx$$

If we know $\tilde{u}$ on $R$, then we can compute its value off the real axis using the formula at the bottom of the preceding page. By symmetry we necessarily have $\tilde{u}(x', y') = \tilde{u}(x', -y')$ and hence the only way for $\frac{\partial \tilde{u}}{\partial y}$ to be continuous across $x > 0, y = 0$ is for it to vanish. Thus we get the integral equation

$$\int_{-\infty}^{\infty} \tilde{u}(x_0) \frac{\partial}{\partial y'} \left\{ \frac{\partial G^*}{\partial y}(x_0, x', y') \right\} \, dx = 0 \text{ for } x' > 0$$

which, if we put $K(x, x') = \frac{\partial G^*}{\partial y}(x_0, x', y')$, becomes
a Wiener–Hopf equation

\[ \int_{-\infty}^{\infty} e^{-ikx} K(x, x') dx = \int_{-\infty}^{\infty} \tilde{u}(x, 0) K(x, x') dx \quad x > 0. \]

We can calculate \( K(x, x') \) as follows:

\[ G(x, y; x', y') = G\left( \sqrt{(x-x')^2 + (y-y')^2} \right) \]

\[ G^*(x, y; x', y') = G\left( \sqrt{(x-x')^2 + (y+y')^2} \right) - G\left( \sqrt{(x-x')^2 + (y-y')^2} \right) \]

\[ \frac{\partial G^*}{\partial y} = \frac{G'(\sqrt{(x-x')^2 + (y-y')^2})}{\sqrt{(x-x')^2 + (y-y')^2}} (y-y') - \frac{G'(\sqrt{(x-x')^2 + (y+y')^2})}{\sqrt{(x-x')^2 + (y+y')^2}} (y+y') \]

\[ \frac{\partial G^*}{\partial y}(x, 0, x', y') = -2y' \frac{G'(\sqrt{(x-x')^2 + y'^2})}{\sqrt{(x-x')^2 + y'^2}} \]

Since this vanishes at \( y' = 0 \) we get

\[ K(x, x') = \frac{\partial G^*}{\partial y}(x, 0, x', y') = -2 \frac{G'(1x-x')}{1x-x'} \]

Since \( G(r) \) behaves like \( \frac{1}{2\pi} \log(r) \) as \( r \to 0 \), this kernel has singularity \( -\frac{1}{\pi} \frac{1}{r^2} \), which indicates that we don't really know it. So there should be a Fourier transform approach which makes things clearer.

Solve the Helmholtz equation in the UHP using F.T.

in \( x \):

\[ u(x, y) = \int e^{-izx} \hat{u}(z, y) d\zeta / 2\pi \]

Then \( (\Delta + k^2) u = 0 \) yields \( \left( -\frac{\partial^2}{\partial y^2} + k^2 \right) \hat{u} = 0 \).

We want the solution decaying as \( y \to \infty \), so
\[ \hat{u}(\xi, y) = \hat{u}(\xi, 0) e^{-\left(\frac{\xi^2 - k^2}{y}\right)} \]

The point is that because \( \Im k > 0, \ k^2 \in \mathbb{R}_{>0} \), so \( \frac{\xi^2 - k^2}{y} \neq 0 \) for \( \xi \) real, and so there is a unique branch for \( \left(\frac{\xi^2 - k^2}{y}\right)^{1/2} \) asymptotic to \( \frac{1}{y} (1 - \frac{k^2}{\xi^2})^{1/2} \) for \( y \) large. So the solution of Helmholtz is given by

\[ u(x, y) = \int e^{-ix} e^{-\sqrt{\xi^2 - k^2} y} \hat{u}(\xi, 0) d\xi / 2\pi \]

Now we are interested in

\[ \frac{\partial u}{\partial y}(x, 0) = -\int e^{-ix} \sqrt{\xi^2 - k^2} \hat{u}(\xi, 0) d\xi / 2\pi \]

\[ = \int K(x-x') u(x', 0) dx \]

where

\[ K(x) \] has the F.T. \( -\sqrt{\frac{k^2 - \xi^2}{y}} \)

Rough check: Compute F.T. of \( \mathsf{sgn}(\xi) \)

\[ \int e^{-ix} e^{-\xi|\xi|} \mathsf{sgn}(\xi) d\xi / 2\pi \]

\[ = \int e^{-ix - \xi x} d\xi / 2\pi - \int e^{-ix + \xi x} d\xi / 2\pi \]

\[ = \frac{1}{2\pi} \frac{1}{x + i} - \frac{1}{2\pi} \frac{1}{x - i} = \frac{1}{\pi} \frac{-ix}{x^2 + e^2} \]

\[ = \frac{1}{\pi} \mathsf{P} \frac{1}{x} \]
Differentiating given
\[ \frac{1}{\sqrt{\pi}} P \left( -\frac{x}{2} \right) = \int -i/\sqrt{\pi} e^{-i x^2/2 \pi} \, dx \]
or
\[ \int e^{-i x^2} \sqrt{\pi} \, dx \ll \frac{i}{\pi} P \left( -\frac{x}{2} \right) \text{ which agrees with our earlier analysis of } K(x). \]

So let's return to the integral equation of Wiener-Hopf type which we want to solve.
\[ \int K(x-x') u(x',0) \, dx' = 0 \quad \text{for } x > 0 \]

where \( u(x,0) = -e^{-i\pi/k} x \) for \( x < 0 \). Let's put \( v(x) = u(x,0) \) for \( x > 0 \) and \( v = 0 \) for \( x < 0 \). Then
\[ V(\xi) = \int e^{i\xi x} v(x) \, dx \]
is analytic in the UHP, whereas
\[ \int_{-\infty}^{0} e^{-i\pi/k} e^{i\xi x} \, dx = \frac{i}{\xi - k} \]
is analytic in the LHP. We want to determine \( \hat{v} \) so that
\[ \hat{K}(\xi) \left\{ \frac{i}{\xi - k} + \hat{v}(\xi) \right\} \text{ analytic in LHP.} \]
\[ \frac{1}{\sqrt{\pi^2 k^2}} \]
Note that \( \hat{K} \) is analytic near the real axis, so its
Now \( \sqrt{\frac{2}{3} - k^2} = \sqrt{\frac{2}{3} + k} \sqrt{\frac{2}{3} - k} \), where the latter is analytic and invertible in the LHP so our condition is

\[
\sqrt{\frac{2}{3} + k} \frac{d}{d\zeta} + \frac{d}{d\zeta} \mathring{V}(\zeta) \quad \text{analytic LHP}
\]

If first term were already analytic in UHP we could take \( \mathring{V}(\zeta) \) to make the sum = 0. Instead write

\[
\frac{i \sqrt{2k}}{\frac{2}{3} - k} + \frac{i}{\frac{2}{3} - k} \left( \sqrt{\frac{2}{3} + k} - \sqrt{2k} \right) + \sqrt{\frac{2}{3} + k} \mathring{V}(\zeta)
\]

and make \( \mathring{V} \) so the last two terms cancel:

\[
\mathring{V}(\zeta) = -\frac{i}{\frac{2}{3} - k} \left( 1 - \frac{\sqrt{2k}}{\sqrt{\frac{2}{3} + k}} \right)
\]

Hence

\[
\mathring{u}(\zeta, 0) = \frac{i \sqrt{2k}}{(\frac{2}{3} - k) \sqrt{\frac{2}{3} + k}}
\]
So it seems that the solution sought is
\[ \tilde{u}(x, y) = \int_{-\infty}^{\infty} e^{-i\xi x} e^{-\frac{k}{2} - \frac{k^2}{2} y} \frac{i\sqrt{2k}}{(i-k)\sqrt{\xi + k}} \, d\xi \]
and now the problem is to get this related to the form with the Fresnel integrals.
November 23, 1978:

Problem: suppose we have a plane wave coming in at an angle

\[ e^{-ik \mathbf{\hat{r}} \cdot \mathbf{\hat{n}}} \]

The plane wave is described by

\[ e^{-ik(x \cos \theta + y \sin \theta)} \]

(Physicists put \( \mathbf{\hat{r}} = k \mathbf{\hat{x}} \); then \( e^{ik \mathbf{\hat{x}} \cdot \mathbf{r}} \) describes a plane wave with wave length \( \lambda \) given by the distance between crests: \( |k| \lambda = 2\pi \) or \( \lambda = \frac{2\pi}{k} \).)

The problem is to solve the Helmholtz equation

\[ (\Delta + k^2) u = 0 \]

with boundary condition \( u = 0 \) on \( \mathbb{R} \times \mathbb{R} \) and the radiative boundary condition that \( u \approx e^{-ik \mathbf{\hat{x}} \cdot \mathbf{r}} + \text{outgoing waves} \).

Up to now the way I made sense of this is to suppose \( \text{Im} k > 0 \) and then to ask for a solution

\[ u = e^{-ik \mathbf{\hat{x}} \cdot \mathbf{r}} + \tilde{u} \]

where \( \tilde{u} \) decays far out — meaning that it is \( L^2 \). Then we solve the Dirichlet problem

\[
\begin{cases}
(\Delta + k^2) \tilde{u} = 0 \\
\tilde{u} = -e^{-ik \mathbf{\hat{x}} \cdot \mathbf{r}} & \text{on negative } x \text{ axis} \\
& = -e^{-ik \times \cos \theta} & x < 0, \ y = 0
\end{cases}
\]
Paradox: This last problem is symmetrical with respect to the $\chi$-axis.

Actually this is not a paradox. For example, suppose we want to solve the problem with $u=0$ on the $x$-axis. The solution is given in the UHP

$$u = e^{-ik\chi} - e^{ik\chi}$$

where $\chi$ is determined by reflection and by $u=0$ in the LHP. Then

$$\tilde{u} = \begin{cases} 
- e^{-ik\tilde{\chi}^*} = - e^{-ik(x\cos\theta - y\sin\theta)} & y > 0 \\
- e^{-ik\tilde{\chi}^*} = - e^{-ik(x\cos\theta + y\sin\theta)} & y < 0 
\end{cases}$$

if $\tilde{\chi} = (\cos\theta, \sin\theta)$, so $\tilde{u}$ is obviously symmetric around the $x$-axis.

To find $\tilde{u}$ we proceed as before

$$\tilde{u}(x,y) = \int e^{-ix\xi} \tilde{\varphi}(\xi, y) d\xi/2\pi$$

where

$$\tilde{\varphi}(\xi, y) = \varphi(\xi, 0) e^{-\sqrt{\xi^2 - k^2} y}$$

Continuity of $\frac{\partial \tilde{u}}{\partial y}(x, 0), \ x > 0$ gives

$$0 = \int e^{-ix\xi} \sqrt{\xi^2 - k^2} \varphi(\xi, 0) d\xi/2\pi \quad x > 0$$

Now

$$\tilde{u}(\xi, 0) = \begin{cases} 
- e^{-i\kappa \xi} & \xi < 0 \\
\varphi(\xi) & \xi > 0 
\end{cases}$$

So

$$\tilde{\varphi}(\xi, 0) = \frac{i}{\xi + \kappa} - \tilde{v}(\xi)$$

$\tilde{v}(\xi) = \int e^{i\kappa x} v(x) dx$
Suppose $a > 0$, i.e. $0 < \theta < \pi/2$, to begin with.

\[
\sqrt{\frac{1}{\epsilon^2} - k^2} \left( \frac{i}{\epsilon - ka} + \sqrt{i} \right) \quad \text{and in LHP.}
\]

\[
\sqrt{\frac{1}{\epsilon^2} + k^2} \left( \frac{i}{\epsilon + ka} + \sqrt{i} \right) \quad \text{and in UHP.}
\]

\[
\frac{\sqrt{k(1+a)}}{\sqrt{1-k^2} - \sqrt{k(1+a)}} + \frac{\sqrt{k(1+a)}}{\sqrt{1+k^2}} \sqrt{i}.
\]

Choose $\sqrt{i}$ to make the latter two terms cancel:

\[
\sqrt{i} = \frac{i}{\sqrt{1-k^2} - \sqrt{k(1+a)}} \left( 1 + \frac{\sqrt{k(1+a)}}{\sqrt{1+k^2}} \right)
\]

\[
\hat{u}(\frac{\pi}{2}, 0) = \frac{i}{\sqrt{1-k^2} - \sqrt{k(1+a)}} \frac{\sqrt{k(1+a)}}{\sqrt{1+k^2}}
\]

\[
\hat{u}(x, y) = \int_{-\infty}^{\infty} e^{ixy - \sqrt{1-k^2}y^2} \frac{i}{\sqrt{1-k^2} - \sqrt{k(1+a)}} \frac{\sqrt{k(1+a)}}{\sqrt{1+k^2}} \, dx / 2\pi.
\]

Problems:

1) What happens for $a < 0$?

2) Relate these formulas to the Fresnel integrals.

3) The scattering matrix?

Solution of 1) should be just a matter of moving the real $i$ axis a bit, because ka is between $-k$ and $k$. So the above formula for $\hat{u}(x, y)$ should be valid with the integration contour below ka.
In example if \( x < 0, y = 0 \) then the exponential decays for \( \text{Im}(z) > 0 \), so that if we push the contour vertically we pick up the residue as \( z = ka \), which is

\[ -e^{-ikax} \]

at it should be. But if \( x > 0, y = 0 \) we might want to push the contour downward.
Schwinger problem: solve Helmholtz for plane region

\[ y = \pi/2 \]

\[ x < 0, y = 0 \]

\[ y = -\pi/2 \]

with \( u = 0 \) on the lines and a given incoming wave form. The incoming wave should be a superposition of plane waves which vanishes on the boundary \( y = \pi/2 \).

Now

\[ e^{-i(\xi x + \eta y)} \quad \xi^2 + \eta^2 = k^2 \]

is the form of the plane wave, so the required form is

\[ e^{-i\xi x} (ae^{-i\eta y} + be^{i\eta y}) \quad \eta = \sqrt{\pi^2/2 - \xi^2} \]

at \( y = \pi/2 \)

\[ ae^{-i\eta \pi/2} + be^{i\eta \pi/2} = 0 \]

at \( y = -\pi/2 \)

\[ ae^{i\eta \pi/2} + be^{-i\eta \pi/2} = 0 \]

So we see that \( e^{-i\eta \pi} = e^{i\eta \pi} \) or \( e^{2\pi i \eta} = 1 \) so that \( \eta \in \mathbb{Z} \), in which case the form of the waves is

\[ e^{-i\xi x} \left( e^{i\eta \pi/2} e^{-i\eta y} - e^{-i\eta \pi/2} e^{i\eta y} \right) \]

or

\[ e^{-i\xi x} \sin \eta(\pi/2 - y) \]

where \( \eta = 1/2, 3/2, \ldots \)

\[ \xi^2 + \eta^2 = k^2 \]

From the form of the problem we see that it has period 2\( \pi \) in the \( y \)-direction, so I can solve
the Schwinger problem by using translation from the Sommerfeld problem.

To be more precise take $\eta = 1$ so we have

$$e^{-i\frac{k^2}{2}}(e^{-i\frac{y}{2}} + e^{i\frac{y}{2}})$$

$\kappa^2 = 1 + \kappa^2 k^2$

suppose that $v(x, y) + e^{-ik'y - iy}$ is the solution of the Sommerfeld problem for the incoming wave $e^{-ik'y - iy}$, so that we know $v(x, -y) = v(x, y)$. Then

$$\sum_{n \in \mathbb{Z}} \{v(x, y + 2\pi n) - v(x, y - 2\pi n)\} + e^{-ik'x}(e^{-iy}e^{i\gamma})$$

should be the solution of Schwinger's problem.

But let's consider a more direct approach. To simplify let us look for solutions $v$ 2\pi-periodic in $y$, vanishing on the negative real axis, with incoming part $e^{-ik'y}$ where $\omega = (k\cos \theta, k\sin \theta)$ and $k\sin \theta \in \mathbb{Z}$. $v$ is to satisfy $(\Delta + \kappa^2) u = 0$ off $R < 0 + 2\pi \mathbb{Z}$. If we pass to $\tilde{u} = u - e^{-ik\omega y}$, we want to solve the Dirichlet equation with the given boundary data. Again take F.T. in $x$

$$\tilde{u}(x, y) = \int e^{-i\frac{x}{2}} \hat{u}(\xi, y) d\xi/2\pi$$

$$(\frac{d^2}{dy^2} + k^2 - \xi^2) \hat{u}(\xi, y) = 0$$

for $0 < y < 2\pi$
This has the solutions
\[
\hat{u}(x, y) = a e^{\gamma y} + b e^{-\gamma y} \quad \gamma^2 = \frac{x^2}{\pi^2} - k^2
\]

We want this to be continuous in \(y\) and the same at \(y = 0, 2\pi\) which gives
\[
a + b = \hat{u}(0, 0) \quad a e^{\gamma 2\pi} + b e^{-\gamma 2\pi} = \hat{u}(0, 0)
\]

so
\[
\hat{u}(x, y) = \hat{u}(0, 0) \frac{\cosh \gamma(\pi - y)}{\cosh \gamma \pi}
\]

\[
\hat{u}(x, y) = \int_{-\infty}^{0} \frac{dx}{2\pi} e^{-i \frac{x}{\pi}} \frac{\cosh \gamma(\pi - y)}{\cosh \gamma \pi} \hat{u}(0, 0) \quad \text{for} \quad 0 \leq x \leq 2\pi
\]

Now we want the derivative to be continuous across the positive \(x\)-axis, so by symmetry \(y \rightarrow -y\) the derivative must vanish
\[
\frac{d}{dy} \frac{\cosh \gamma(\pi - y)}{\cosh \gamma \pi} \bigg|_{y = 0} = -\gamma \tanh \gamma \pi
\]

So our integral equation becomes
\[
\int_{-\infty}^{0} \frac{dx}{2\pi} e^{-i \frac{x}{\pi}} \gamma \tanh \gamma \pi \hat{u}(0, 0) = 0 \quad x > 0
\]

This is a Wiener-Hopf equation like the previous one except that \(\gamma = \sqrt{\frac{x^2}{\pi^2} - k^2}\) is replaced by the meromorphic function \(\gamma \tanh \gamma \pi\), whose singularities
are simple occurring where
\[
\text{sinh}(\eta \pi) = 0 \quad \Rightarrow \quad \eta = \imath n \quad n \in \mathbb{Z}
\]
\[
\Rightarrow \quad \xi^2 - k^2 = -n^2 \quad \text{or} \quad \xi^2 = k^2 - n^2
\]
\[
\text{cosh}(\eta \pi) = 0 \quad \Rightarrow \quad \eta = \imath (n + \frac{1}{2}) \quad n \in \mathbb{Z}
\]
\[
\Rightarrow \quad \xi^2 = k^2 - (n + \frac{1}{2})^2
\]
Thus the singularities occur where \( \xi^2 = k^2 - \frac{n^2}{4} \quad n \in \mathbb{Z} \) which are located:

Notice what happens if \( \pi \) is replaced by \( \pi l \) and \( l \to \infty \)
\[
\eta \tanh(\pi l \eta) \to \eta
\]
\[
\xi^2 = k^2 - \left(\frac{n}{2l}\right)^2
\]
so that the curves on which the zeroes & poles are located for finite \( l \) become \( \mathbb{Z} \) cuts.