Situation: For automorphic scattering [redacted] we obtained

$$R(k) = \frac{\hat{\zeta}(2-2s)}{\hat{\zeta}(2s)} \qquad s = \frac{1}{2} + \frac{k}{i}$$

$$= \frac{\hat{\zeta}(1+2ik)}{\hat{\zeta}(1-2ik)}$$

and in $\operatorname{Im} k > 0$ the singularities are a pole at $k = \frac{1}{2}i$ and zeroes on the line (by Riemann Hyp) $\operatorname{Im} k = \frac{1}{4}$. Now we sort of understand the pole as being due to a bound state. So the question is whether we might obtain something interesting by applying Marchenko-Faddeev to the scattering data $R(k)$, bound state at $k = \frac{1}{2}i$ with a suitable norming constant, to obtain a potential $q(x)$ on a half-line $x > a$. One problem with this is [redacted] that in the Jost representation for $R(k)$:

$$R(k) = \frac{B(k)}{A(k)} \qquad B(k) = A(-k) = \overline{A(\bar{k})}$$

$A(k)$ is analytic in the UHP with zeroes at the bound states. Hence $A(k)$ would vanish at $k = \frac{1}{2}i$ and would have poles [redacted] on $\operatorname{Im}(k) = -\frac{1}{4}$. Hence it seems that $A(k)$ could not be $\hat{\zeta}(1-2ik)$.

<u>November 18, 1978</u>

Lippmann-Schwinger integral equations:

Suppose we have two one-parameter groups of operators $U_0(t) = e^{iH_0t}$, $U(t) = e^{iHt}$ and we put $H = H_0 + V$. Assume the trajectories $e^{iH_0t}u$ and $e^{iHt}v$ are asymptotic as $t \to -\infty$. I want them to be strongly asymptotic - so that

$$v = \lim_{t \to -\infty} U(-t)U_0(t)u$$

$$u = \lim_{t \to -\infty} U_0(-t)U(t)v$$

For example if $\|U(t)v - U_0(t)u\| \to 0$ and U, U_0 are 1-par. unitary groups. One has

$$\frac{d}{dt} U_0(-t)U(t)v = U_0(-t)(-iH_0)U(t)v + U_0(-t)(iH)U(t)v$$

$$= U_0(-t)\,iV\,U(t)v$$

so

$$U_0(-t)U(t)v = u + \int_{-\infty}^{t} U_0(-t')\,iV\,U(t')v\,dt'$$

$$U(t)v = U_0(t)u + \int_{-\infty}^{t} U_0(t-t')\,iV\,U(t')v\,dt'$$

~~One has~~

$$\cancel{\frac{d}{dt}U(t)v = iHU(t)v}$$

$$\cancel{\left(\frac{d}{dt} - iH_0\right)U(t)v = iV\,U(t)v}$$

Now take Fourier transform of the last equation.

Put
$$\hat{v}(k) = \int_{-\infty}^{\infty} e^{-ikt}\, U(t)v\, dt \quad \text{and similarly for } \hat{u}.$$

$$\hat{v}(k) = \hat{u}(k) + \int_{-\infty}^{\infty} e^{-ikt} dt \int_{-\infty}^{t} e^{iH_0(t-t')}\, iV\, U(t')v\, dt'$$

$$\text{———} \int_{-\infty}^{\infty} dt' \left(\int_{t \geq t'} dt\, e^{-ikt}\, e^{iH_0(t-t')} \right) iV\, U(t')v$$

$$\text{———} \left(\int_{0}^{\infty} du\, e^{-ik(t'+u)}\, e^{iH_0 u} \right) \text{———}$$

$$\text{———} \int_{-\infty}^{\infty} dt'\; e^{-ikt'}\; \frac{-1}{iH_0 - i(k - i\varepsilon)}\, iV\; U(t')v$$

$$\therefore \quad \hat{v}(k) = \hat{u}(k) - \frac{1}{H_0 - (k - i\varepsilon)} V\, \hat{v}(k)$$

The explanation of the above: One wants to solve
$$\frac{d}{dt} U(t)v = iH\; U(t)v$$

with
$$U(t)v \sim U_0(t)u \quad \text{as } t \to -\infty.$$

Rewrite the DE
$$\left(\frac{d}{dt} - iH_0 \right) U(t)v = iV\; U(t)v$$

and use the Green's function solution of this
$$* \qquad U(t)v = U_0(t)u + \int_{-\infty}^{\infty} G^{-}(t,t')\; iV\, U(t')v\, dt'$$

which is adapted to the boundary condition at $t = -\infty$:

Thus

$$\bar{G}(t,t') = \begin{cases} 0 & t < t' \\ U(t-t') & t > t' \end{cases}$$

Then you take the Fourier transform of the integral equation * using the fact that convolution goes into product and that

$$\int_{-\infty}^{\infty} \bar{G}(t)\, e^{-ikt}\, dt = \int_{0}^{\infty} e^{-ikt+iH_0 t}\, dt$$

$$= \frac{1}{i} \frac{1}{k-H_0 - i\varepsilon}$$

where ε is an infinitesimal positive quantity. One obtains the Lippmann-Schwinger equation

$$\boxed{\hat{v}(k) = \hat{u}(k) + \frac{1}{k-H_0-i\varepsilon} V \hat{v}(k)}$$

Similarly if $U(t)v^+ \sim U_0(t)u$ at $t \longrightarrow +\infty$ we get

$$\boxed{\widehat{v^+}(k) = \hat{u}(k) + \frac{1}{k-H_0+i\varepsilon} V \widehat{v^+}(k)}$$

Review Lippmann–Schwinger: Let $u(t) = e^{iH_0 t} u(0)$, $v(t) = e^{iHt} v(0)$ be asymptotic (strongly) as $t \to +\infty$.

From

$$\left(\frac{d}{dt} - iH_0\right) v = i(H - H_0) v = i V v$$

$$\parallel$$

$$e^{iH_0 t} \frac{d}{dt}\left(e^{-iH_0 t} v\right)$$

we get

$$e^{-iH_0 t} v(t) = \underbrace{e^{-iH_0 t_1} v(t_1)}_{\downarrow \; u(0) \text{ as } t_1 \to +\infty} - \int_t^{t_1} e^{-iH_0 t'} i V v(t')\, dt'$$

or

$$v(t) = u(t) - \int_t^{\infty} e^{iH_0(t-t')} i V v(t')\, dt'$$

or

$$v(t) = u(t) + \int_{-\infty}^{\infty} G^+(t-t') V v(t')\, dt'$$

where

$$G^+(t-t') = \begin{cases} 0 & t > t' \\ \frac{1}{i} e^{+iH_0(t-t')} & t < t' \end{cases}$$

Take F.T.

$$\int_{-\infty}^{\infty} e^{-ikt} G^+(t)\, dt = \int_{-\infty}^{0} e^{-ikt} \frac{1}{i} e^{iH_0 t}\, dt = \frac{1}{i} \frac{1}{-ik + iH_0}$$

$$= \frac{1}{k - H_0 + i\varepsilon} \qquad \varepsilon > 0$$

$\therefore$ Get L-S equation

$$\hat{v}(k) = \hat{u}(k) + \frac{1}{k - H_0 + i\varepsilon} V \hat{v}(k)$$

But the basic equation is

$$v = u + G^+ V v$$

or $\qquad v = (I - G^+V)^{-1} u.$ Thus $(I - G^+V)^{-1}$ is essentially the Møller wave operator Ω^+:

$$\Omega^+ = \lim_{t \to +\infty} U(-t)\, U_0(t)$$

which associates to a free trajectory $u(t) = U_0(t)\, u(0)$ the perturbed trajectory $v(t) = U(t)\, v(0)$ asymptotic to it.

Since the scattering matrix is

$$S = (\Omega^+)^{-1} \Omega^-$$

one gets the formula

$$\det S(k) = \frac{\det(1 - G_k^+ V)}{\det(1 - G_k^- V)}$$

∃ problems here because $\Omega^\pm$ are not operator functions of k the way $1 - G_k^+ V$ is. See Dec. 14

November 20, 1978:

Schwinger's variational business. One has an integral equation

$$\psi = GK$$

where ψ is a given lowest mode wave function, G is a Green's function, and K is either a surface current or the field on an aperture. If

$$\frac{1}{X} = (\psi, K)$$

then X is the admittance of an equivalent circuit. The idea is that one really wants to compute X by choosing an approximation to K. If one uses the above formula for X, then the error matters. The point is instead to use the expression

$$X = \frac{(GK, K)}{(\psi, K)^2}$$

to compute X from an approximate K, because the latter expression is stationary when K is correct. In effect

$$\delta \frac{(GK,K)}{(\psi,K)^2} = \frac{(G\delta K, K) + (GK, \delta K)}{(\psi,K)^2} - (GK,K)\frac{2(\psi,\delta K)}{(\psi,K)^3}$$

Use G symm. and everything real.

$$= \frac{2}{(\psi,K)^2}\left\{ [redacted] \; (GK, \delta K) - \left(\frac{(GK,K)}{(\psi,K)}\psi, \delta K\right)\right\}$$

$$= 0 \iff GK = \frac{(GK,K)}{(\psi,K)}\psi$$

$$\iff GK \text{ is proportional to } \psi.$$

More generally assume only that G is hermitian.

Then
$$\delta \frac{(GK,K)}{|\psi,K|^2} = \frac{(GK,K)}{|(\psi,K)|^2}\left\{\frac{\delta(GK,K)}{(GK,K)} - \frac{\delta(\psi,K)}{(\psi,K)} - \frac{\delta(K,\psi)}{(K,\psi)}\right\}$$
$$= \frac{(GK,K)}{|(\psi,K)|^2}\left\{\frac{(G\delta K,K)+(GK,\delta K)}{(GK,K)} - \frac{(\psi,\delta K)}{(\psi,K)} - \frac{(\delta K,\psi)}{(K,\psi)}\right\}$$
$$= \frac{(GK,K)}{|\psi,K|^2}\, 2\left\{Re\left(\frac{GK}{(GK,K)},\delta K\right) - Re\left(\frac{\psi}{(\psi,K)},\delta K\right)\right\} = 0$$
$$\Longleftrightarrow \quad GK = \frac{(GK,K)}{(\psi,K)}\psi \quad \Longleftrightarrow \quad GK \text{ proportional to } \psi.$$

Example of a variational process is needed.

Green's function for $\Delta + k^2$ in $\mathbb{R}^3$: In spherical coordinates the basic infinitesimals are dr, $r\,d\varphi$, $r\sin\varphi\, d\theta$, that is
$$ds^2 = dr^2 + r^2 d\varphi^2 + r^2\sin^2\varphi\, d\theta^2$$
and an orthonormal frame is
$$\frac{\partial}{\partial r}, \quad \frac{1}{r}\frac{\partial}{\partial\varphi}, \quad \frac{1}{r\sin\varphi}\frac{\partial}{\partial\theta}$$
The volume element is $\quad dV = r^2\sin\varphi\, dr\, d\varphi\, d\theta$

and the Laplacian is
$$\frac{1}{r^2\sin\varphi}\frac{\partial}{\partial r} r^2\sin\varphi\frac{\partial}{\partial r} + \frac{1}{r^2\sin\varphi}\frac{\partial}{\partial\varphi}\frac{1}{r} r^2\sin\varphi\frac{1}{r}\frac{\partial}{\partial\varphi}$$
$$+ \frac{1}{r^2\sin\varphi}\frac{1}{r\sin\varphi}\frac{\partial}{\partial\theta} r^2\sin\varphi\frac{1}{r\sin\varphi}\frac{\partial}{\partial\theta}$$

$$\therefore \Delta = \frac{1}{r^2}\frac{\partial}{\partial r} r^2 \frac{\partial}{\partial r} + \frac{1}{r^2 \sin\varphi}\frac{\partial}{\partial \varphi}\text{[redacted]}\sin\varphi \frac{\partial}{\partial\varphi} + \frac{1}{r^2\sin^2\varphi}\frac{\partial^2}{\partial\theta^2}$$

$$= \frac{1}{r^2}\frac{\partial}{\partial r} r^2 \frac{\partial}{\partial r} + \frac{1}{r^2}(\text{Laplacian on } S^2)$$

~~Check: One knows that the eigenvalues of Δ on S^2 are $l(l-1)$ for $l = 0, 1, 2, \ldots$; constants and linear fns. are harmonic~~ [redacted] ~~, l is the degree of the spherical harmonic.~~
~~Better: A basis for harm~~

(I guess it is known that $r^l Y_l^m(\varphi, \theta)$ form a basis for the harmonic homogeneous polys. of degree l. Substituting in the above shows that eigenvalues of the Laplacian on S^2 are $-l(l+1)$. Consequently when the Schroedinger DE

$$(-\Delta + V(r))\psi\,\text{[redacted]} = \text{[redacted]}\,\lambda\psi$$

is separated in spherical coords, [redacted] for a component $\psi_l Y_l^m$ of angular momentum l one gets

$$\left\{-\frac{1}{r^2}\frac{d^2}{dr^2} + \frac{l(l+1)}{r^2} + V(r)\right\}\psi_l = \lambda\psi_l \,.)$$

The Green's function centered at O will be a radial function $u(r)$

$$0 = (\Delta + k^2)u = \left(\frac{1}{r^2}\frac{d}{dr} r^2 \frac{d}{dr} + k^2\right)u$$

$$r\left(\frac{1}{r^2}\frac{d}{dr} r^2 \frac{d}{dr}\right)\frac{1}{r} = \left(\frac{d}{dr} + \frac{1}{r}\right)\left(\frac{d}{dr} - \frac{1}{r}\right) = \frac{d^2}{dr^2} - \cancel{\frac{1}{r^2}} + \cancel{\frac{1}{r^2}}$$

$\therefore \quad u = \dfrac{e^{\pm ikr}}{r}$ ind. solutions. For $\operatorname{Im} k > 0$

it must decay so G is proportional to $\frac{e^{ikr}}{r}$.

At the origin we want $(k^2+\Delta)G = \delta$. For $k=0$ we know the appropriate G is $-\frac{1}{4\pi r}$ $\therefore$

$$\iiint_{|\vec{x}|\le \varepsilon} \Delta\left(-\frac{1}{4\pi r}\right) dV = \iint_{|\vec{x}|=\varepsilon} \nabla\left(-\frac{1}{4\pi r}\right)\cdot\hat{n}\, dS = \iint_{|x|=\varepsilon} \frac{1}{4\pi r^2}\, dS = 1.$$

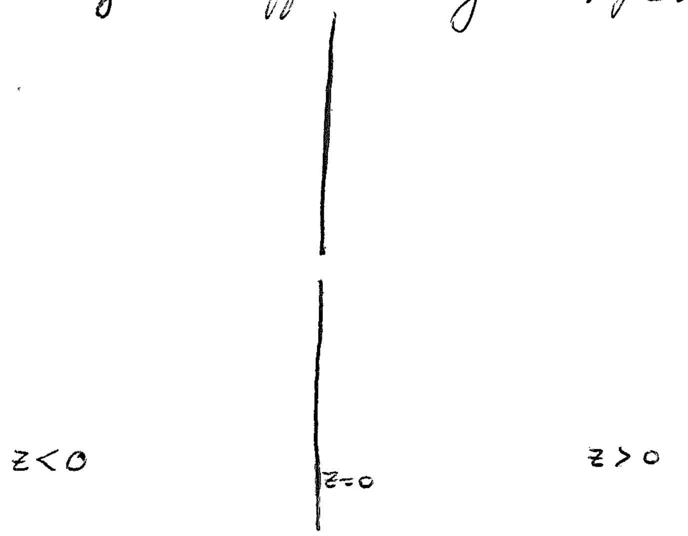
$\therefore$

$$\boxed{G_k(r) = -\frac{1}{4\pi}\frac{e^{ikr}}{r}}$$

On the other hand the Fourier transform gives

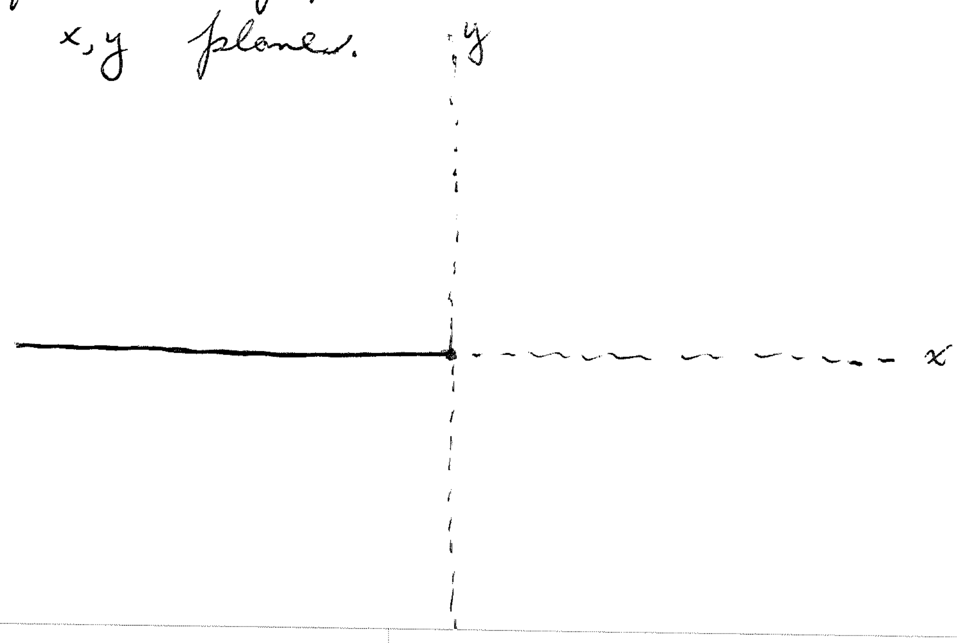
$$G_k(r) = \iiint e^{i\vec{k}\cdot\vec{\xi}} \frac{1}{k^2-|\vec{\xi}|^2} \frac{d\vec{\xi}}{(2\pi)^3}$$

Levine-Schwinger on diffraction by an aperture in an infinite plane.



One solves the Helmholtz equation $(\Delta + k^2)u = 0$ with $u = 0$ on the plane, both u, $\frac{\partial u}{\partial z}$ should be continuous across the aperture. The other boundary condition says that for $z < 0$ the solution consists of an incoming plane wave + outgoing waves and that for $z > 0$ it consists of outgoing waves.

It would be nice to understand precisely the meaning of incoming and outgoing. Let us consider a simpler example, the Sommerfeld diffraction problem: by an infinite half-plane, which we take to be $x \leq 0, y = 0$ in the x, y plane.



We want to solve $\frac{\partial^2\psi}{\partial t^2} = \Delta\psi$ with $\psi = 0$ on the half-line with

$$\psi(x,y,t) = f(x+t) \qquad t \geq 0 \qquad [redacted]$$

f supported in $\mathbb{R}_{\geq 0}$. Assume the solution exists. Then it will be a [redacted] superposition of plane wave solutions: [redacted]

$$\psi(x,y,t) = \int e^{-ikt}\, u(x,y,k)\, dk/2\pi$$

where $u(x,y,k)$ satisfies $(\Delta + k^2)u = 0$, $u = 0$ on half-line.
and

$$u(x,y,k) = e^{-ikx} + \text{reflected wave } \tilde{u}(x,y,k)$$

By causality, [redacted]

$$\int e^{-ikt}\, \tilde{u}(x,y,k)\, dk/2\pi = 0 \qquad t < \sqrt{x^2+y^2}$$

(at least if $x > 0$ (otherwise distance from obstacle)), hence $\tilde{u}(x,y,k)$ should be "made up" of e^{ikr} with $r \geq \sqrt{x^2+y^2}$. Thus for $\operatorname{Im} k > 0$ the function $\tilde{u}(x,y,k)$ should be exponentially decaying as one goes to ∞. Actually in this case if one considers $\tilde{u} = u - e^{-ikx}$, it is a solution of Helmholtz with a Dirichlet boundary [redacted] condition $= -e^{-ikx}$ on the half-line and 0 at ∞, so one expects the existence of $\tilde{u}$ to come from [redacted] elliptic theory. It would seem this works quite generally.

In fact, for potential scattering to get the solution $u(x,k)$ with a certain incoming behavior e^{-ikx} one works with $\tilde{u}(x,k) = u(x,k) - e^{-ikx}$ which satisfies $(\Delta + k^2)\tilde{u} = qu$ and hence is found by solving the equation (Lippmann-Schwinger).

$$u(x,k) - e^{-ikx} = \int G_k(x,x') \, g(x') \, u(x',k) \, dx' .$$

So now let's solve the Dirichlet problem or Helmholtz equation by introducing parabolic coordinates.

$$(x+iy) = \tfrac{1}{2}(\xi + i\eta)^2$$

$$x = \tfrac{1}{2}(\xi^2 - \eta^2) \qquad y = \xi\eta$$

$$dx = \xi\, d\xi - \eta\, d\eta \qquad dy = \eta\, d\xi + \xi\, d\eta$$

$$ds^2 = dx^2 + dy^2 = (\xi^2+\eta^2)(d\xi^2 + d\eta^2)$$

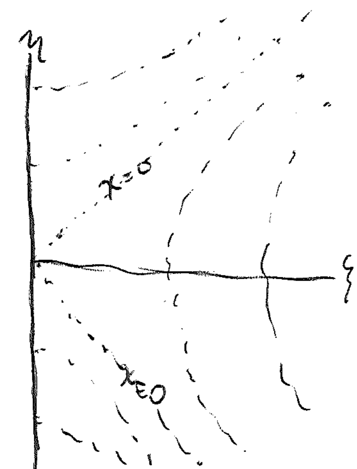
An orthonormal frame is $\frac{1}{\sqrt{\xi^2+\eta^2}}\frac{\partial}{\partial \xi}$, $\frac{1}{\sqrt{\xi^2+\eta^2}}\frac{\partial}{\partial \eta}$
and

$$dx\,dy = (\xi^2+\eta^2)\, d\xi\, d\eta$$

so

$$\Delta = \frac{1}{\xi^2+\eta^2} \frac{\partial}{\partial \xi} \frac{1}{\sqrt{\ }} (\xi^2+\eta^2) \frac{1}{\sqrt{\ }} \frac{\partial}{\partial \xi} + \text{same for } \eta$$

$$= \frac{1}{\xi^2+\eta^2}\left(\frac{\partial^2}{\partial \xi^2} + \frac{\partial^2}{\partial \eta^2}\right)$$



So the Helmholtz equation becomes

$$\left\{\left(\frac{\partial^2}{\partial \xi^2} + \frac{\partial^2}{\partial \eta^2}\right) + k^2(\xi^2+\eta^2)\right\} u = 0$$

which separates. We work in the domain $\xi > 0$; the boundary curve $\xi = 0$, $\eta \in \mathbb{R}$ corresponds to the half-line $x \leq 0$, $y = 0$.

Let us first find $u = e^{-ikx} + \tilde{u}$ so that we want $\tilde{u}(x,0) = -e^{-ikx}$ for $x \leq 0$. In parabolic coordinates this becomes

[redacted] $\tilde{u}(0,\eta) = -e^{ik\frac{\eta^2}{2}}$

Try $\tilde{u}(\xi,\eta) = f(\xi)\,g(\eta)$ whence $g(\eta) = e^{ik\frac{\eta^2}{2}}$ up to a constant. Then

$$\left(\frac{\partial^2}{\partial\eta^2} + k^2\eta^2\right) e^{ik\eta^2/2} = e^{ik\eta^2/2}\left\{\left(\frac{\partial}{\partial\eta} + ik\eta\right)^2 + k^2\eta^2\right\}(1)$$

$$= e^{ik\eta^2/2}\left\{\frac{\partial^2}{\partial\eta^2} + 2ik\frac{\partial}{\partial\eta} - k^2\eta^2 + ik + k^2\eta^2\right\}(1)$$

$$= ikg$$

Hence f must satisfy

$$\left(\frac{\partial^2}{\partial\xi^2} + k^2\xi^2 + ik\right)f = 0$$

$$= \left(\frac{\partial^2}{\partial\xi^2} + k^2\xi^2 + ik\right)e^{-ik\xi^2/2}\left(e^{ik\xi^2/2}f(\xi)\right)$$

$$= e^{-ik\xi^2/2}\left(\frac{\partial^2}{\partial\xi^2} - 2ik\xi\frac{\partial}{\partial\xi}\right)\left(e^{ik\xi^2/2}f(\xi)\right)$$

so

$$\frac{d}{d\xi}\, e^{ik\xi^2/2}f(\xi) = c_1 e^{ik\xi^2}$$

$$f(\xi) = c_1 e^{-ik\xi^2/2}\int_\infty^\xi e^{ik\xi'^2}\,d\xi' + c_2 e^{-ik\xi^2/2}$$

Now we want f to vanish as $\xi \to \infty$, so $c_2 = 0$ and so

$$\tilde{u}(\xi,\eta) = c_1\left(e^{-ik\xi^2/2}\int_\infty^\xi e^{ik\xi'^2}\,d\xi'\right)e^{ik\eta^2/2}$$

Put $\quad \mathrm{Erf}(x) = \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-x^2} dx \quad$ and recall $\quad \int_0^\infty e^{-x^2} dx = \frac{\sqrt{\pi}}{2}$

hence if $s = k/i$ as usual

$$\int_\xi^\infty e^{ik\xi'^2} d\xi' = \int_\xi^\infty e^{-s\xi'^2} d\xi' = \frac{1}{\sqrt{s}} \int_{\sqrt{s}\,\xi}^\infty e^{-\xi'^2} d\xi'$$

$$= \frac{\sqrt{\pi}}{2\sqrt{s}} \mathrm{Erf}(\sqrt{s}\,\xi)$$

So $\quad \tilde{u}(\xi,\eta) = -e^{-ik\xi^2/2}\, \mathrm{Erf}(\sqrt{s}\,\xi)\; e^{ik\eta^2/2}$

Now $\quad x = \frac{\xi^2 - \eta^2}{2} \qquad y = \xi\eta$

$$2x\xi^2 = \xi^4 - y^2 \qquad \xi^4 - 2x\xi^2 - y^2 = 0$$

$$\xi^2 = x \pm \sqrt{x^2 + y^2} \qquad \xi = \sqrt{x + \sqrt{x^2+y^2}} = \sqrt{x+r}$$

So it appears that the desired solution is

$$u(x,y,k) = e^{-ikx}\left\{1 - \mathrm{Erf}(\sqrt{s}\sqrt{x+r})\right\}$$

$$= e^{-ikx} \frac{2}{\sqrt{\pi}} \int_0^{\sqrt{s}\sqrt{x+r}} e^{-u^2} du$$

where $s = k/i$. But you want to move k to the real axis from the UHP and so you obtain a Fresnel integral instead of Erf.

Try Wiener-Hopf approach to the Sommerfeld problem: To solve $(\Delta + k^2)\tilde{u} = 0$ in the plane $- \mathbb{R}_{\leq 0}$ with $\tilde{u}(x,0) = -e^{-ikx}$ for $x<0$. Here $\operatorname{Im} k > 0$ and $\tilde{u}$ is to be zero at ∞. There are two approaches depending on whether we concentrate on $\tilde{u}(x,0)$ for $x>0$ or the discontinuity in $\frac{\partial \tilde{u}}{\partial y}(x,0)$ for $x<0$.

Denote by $G_k(\vec{r};\vec{r}\,')$ the Green's function for the plane. One has $G_k(\vec{r},\vec{r}\,') = G_k(|\vec{r}-\vec{r}\,'|)$ where $G_k(r)$ satisfies

$$\left(\frac{1}{r}\frac{d}{dr} r \frac{d}{dr} + k^2\right) u = 0 \qquad r>0$$

or

$$\left(\left(r\frac{d}{dr}\right)^2 + k^2 r^2\right) u = 0$$

which is essentially Bessel's DE of order 0, except for kr being substituted for r. If we need this we can find it later.

First method is to use the Green's function for the half-plane to determine u off the line in terms of u on the line. This Green's function is obtained by reflection

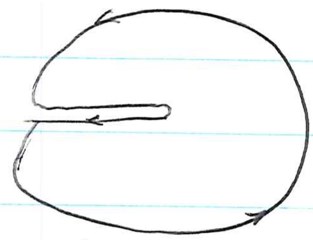
$$G^*(x,y;x',y') = G(x,y;x',y') - G(x,y;x',-y')$$

Using Green's formula $\iint (u\Delta v - v\Delta u) = \int\left(u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n}\right)$ one gets

$$\tilde{u}(x',y') = \int_{-\infty}^{\infty} \tilde{u}(x,0)\frac{\partial G^*}{-\partial y}(x,0;x',y')\,dx \qquad \text{for } y'>0$$

The [redacted] same formula holds, with a minus, for $y' < 0$ because $\frac{\partial}{\partial n} = -\frac{\partial}{\partial y}$. ~~and the integration goes from $+\infty$ to $-\infty$.~~ (flux is a sum: $ds = dx$). The condition to be satisfied is that $\frac{\partial \tilde{u}}{\partial y}$ is continuous on $y=0, x>0$.

?

Apply Green's formula to 

in order to express $\tilde{u}$ in terms of the discontinuity of $\frac{\partial \tilde{u}}{\partial y}$ along $\mathbb{R}_{\leq 0}$.

$$\tilde{u}(x',y') = \iint \left(\tilde{u}\Delta G - (\Delta\tilde{u})G\right) = \int \left(\tilde{u}\frac{\partial G}{\partial n} - \frac{\partial \tilde{u}}{\partial n} G\right)$$

$$\tilde{u}(x',y') = -\int_{-\infty}^{0} \left\{ \frac{\partial \tilde{u}}{\partial y}(x,0^+) - \frac{\partial \tilde{u}}{\partial y}(x,0^-) \right\} G(x,0;x',y')\, dx$$

If we know $\tilde{u}$ on $\mathbb{R}$, then we can compute its value off the real axis using the formula at the bottom of the preceding page. By symmetry one necessarily has $\tilde{u}(x',y') = \tilde{u}(x',-y')$ and hence the only way for $\frac{\partial \tilde{u}}{\partial y}$ to be continuous across $x \geq 0, y = 0$ is for it to vanish. Thus we get the integral equation

$$\int_{-\infty}^{\infty} \tilde{u}(x,0) \frac{\partial}{\partial y'} \left\{ \frac{\partial G^*}{\partial y}(x,0;x',y') \right\}_{y'=0} dx = 0 \quad \text{for } x'>0$$

which if we put $K(x,x') = \frac{\partial^2 G^*}{\partial y' \partial y}(x,0;x',0)$ becomes a Wiener–Hopf equation

$$\int_{-\infty}^{0} e^{-ikx} K(x,x')\,dx = \int_{0}^{\infty} \tilde{u}(x,0)\,K(x,x')\,dx \qquad x'>0.$$

We can calculate $K(x,x')$ as follows:

$$G(x,y;x',y') = G\left(\sqrt{(x-x')^2+(y-y')^2}\right)$$

$$G^*(x,y,x',y') = G\left(\sqrt{(x-x')^2+(y-y')^2}\right) - G\left(\sqrt{(x-x')^2+(y+y')^2}\right)$$

$$\frac{\partial G^*}{\partial y}(\quad) = \frac{G'(\sqrt{})}{\sqrt{}}(y-y') - \frac{G'(\sqrt{})}{\sqrt{}}(y+y')$$

$$\therefore \frac{\partial G^*}{\partial y}(x,0,x',y') = -2y'\,\frac{G'\left(\sqrt{(x-x')^2+y'^2}\right)}{\sqrt{(x-x')^2+y'^2}}$$

Since this vanishes at $y'=0$ we get

$$K(x,x') = \frac{\partial^2 G^*}{\partial y'\partial y}(x,0,x',y') = -2\,\frac{G'(|x-x'|)}{|x-x'|}$$

Since $G(r)$ behaves like $\frac{1}{2\pi}\log(r)$ as $r\to 0$, this kernel has singularity $-\frac{1}{\pi}\frac{1}{r^2}$, which indicates that we don't really know it. So there should be a Fourier transform approach which makes things clearer.

Solve the Helmholtz equation in the UHP using F.T. in x:

$$u(x,y) = \int e^{-i\xi x}\,\hat{u}(\xi,y)\,d\xi/2\pi$$

Then $(\Delta+k^2)u=0$ yields $\left(-\xi^2+\frac{d^2}{dy^2}+k^2\right)\hat{u}=0.$

We want the solution decaying as $y\to+\infty$, so

$$\hat{u}(\xi,y) = \hat{u}(\xi,0)\, e^{-\left(\sqrt{\xi^2-k^2}\right) y}$$

The point is that because $\operatorname{Im} k > 0$, $k^2 \notin \mathbb{R}_{\geq 0}$, so $\xi^2 - k^2 \neq 0$ for ξ real, and so there is a unique branch for $(\xi^2-k^2)^{1/2}$ asymptotic to $|\xi|\left(1-\frac{k^2}{\xi^2}\right)^{1/2}$ for ξ large. So the solution of Helmholz is given by

$$u(x,y) = \int e^{-i\xi x}\, e^{-\sqrt{\xi^2-k^2}\, y}\, \hat{u}(\xi,0)\, d\xi/2\pi$$

Now we are interested in

$$\frac{\partial u}{\partial y}(x,0) = -\int e^{-i\xi x} \sqrt{\xi^2-k^2}\, \hat{u}(\xi,0)\, d\xi/2\pi$$

$$= \int K(x-x')\, u(x,0)\, dx$$

where

$K(x)$ has the F.T. $-\sqrt{\xi^2-k^2}$

Rough check: Compute F.T. of $\operatorname{sgn}(\xi)$

$$\int_{-\infty}^{\infty} e^{-i\xi x} e^{-\varepsilon|\xi|} \operatorname{sgn}(\xi)\, d\xi/2\pi$$

$$= \int_0^{\infty} e^{-i\xi x - \varepsilon\xi}\, d\xi/2\pi - \int_{-\infty}^{0} e^{-i\xi x + \varepsilon \xi}\, d\xi/2\pi$$

$$= \frac{1}{2\pi}\frac{1}{\varepsilon + ix} - \frac{1}{2\pi}\frac{1}{\varepsilon - ix} = \frac{1}{\pi}\frac{-ix}{x^2+\varepsilon^2}$$

$$= \frac{1}{i\pi}\, P.\frac{1}{x}$$

Differentiating given

$$\frac{1}{i\pi} P\left(-\frac{1}{x^2}\right) = \int -i|\xi| e^{-ix\xi} \, d\xi/2\pi$$

or

$$\int e^{-ix\xi} |\xi| \, d\xi/2\pi = \frac{1}{\pi} P\left(-\frac{1}{x^2}\right)$$ which agrees

with our earlier <u>analysis of</u> $K(x)$.

So let's return to the integral equation of Wiener-Hopf type which we want to solve.

$$\int_{-\infty}^{\infty} K(x-x') u(x',0) \, dx' = 0 \qquad \text{for } x>0$$

where $u(x',0) = -e^{-ikx}$ for $x<0$. Let's put $v(x) = u(x,0)$ for $x>0$ and $v=0$ for $x<0$. Then

$$\hat{v}(\xi) = \int_0^{\infty} e^{i\xi x} v(x) \, dx$$

is analytic in the UHP, whereas

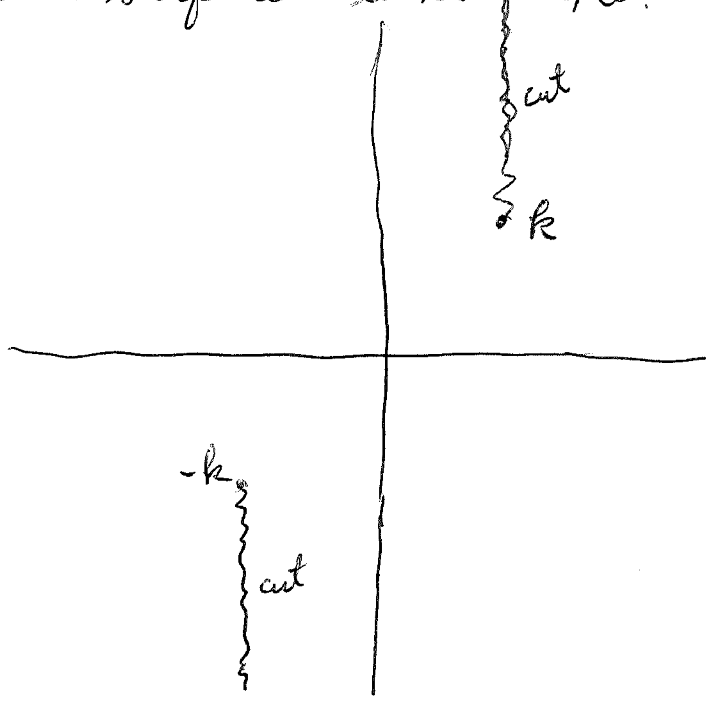
$$\int_{-\infty}^{0} -e^{-ikx} e^{i\xi x} \, dx = \frac{i}{\xi - k}$$

is analytic in the LHP. We want to determine $\hat{v}$ so that

$$\underset{\substack{\| \\ \sqrt{\xi^2-k^2}}}{\hat{K}(\xi)} \left\{ \frac{i}{\xi-k} + \hat{v}(\xi) \right\} \quad \text{analytic in LHP.}$$

Note that $\hat{K}$ is analytic near the real axis, so its

Riemann surface looks like:



Now $\sqrt{\xi^2-k^2} = \sqrt{\xi+k}\,\sqrt{\xi-k}$ where the latter is analytic and invertible in the LHP so our condition is

$$\sqrt{\xi+k}\,\frac{i}{\xi-k} + \underbrace{\sqrt{\xi+k}\,\hat{v}(\xi)}_{\text{anal UHP}} \quad \text{analytic LHP}$$

If first term were already analytic in UHP we could take $\hat{v}(\xi)$ to make the sum $=0$. Instead write

$$\underbrace{\frac{i\sqrt{2k}}{\xi-k}}_{\text{anal LHP}} + \underbrace{\frac{i}{\xi-k}\left(\sqrt{\xi+k}-\sqrt{2k}\right) + \sqrt{\xi+k}\,\hat{v}(\xi)}_{\text{anal UHP}}$$

and make $\hat{v}$ so the last two terms cancel:

$$\hat{v}(\xi) = \frac{-i}{\xi-k}\left(1-\frac{\sqrt{2k}}{\sqrt{\xi+k}}\right)$$

whence $$\hat{u}(\xi,0) = \frac{i\sqrt{2k}}{(\xi-k)\sqrt{\xi+k}}$$

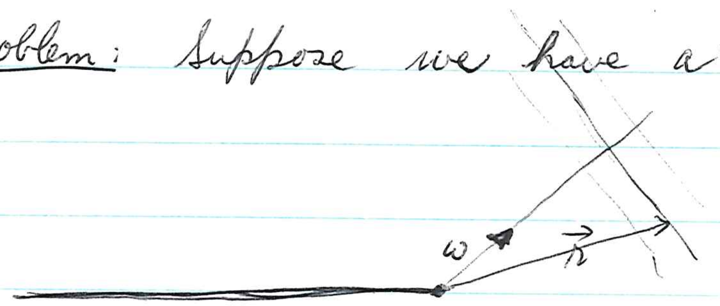
So it seems that the solution sought is

$$\tilde{u}(x,y) = \int_{-\infty}^{\infty} e^{-i\xi x}\, e^{-\sqrt{\xi^2-k^2}\,y}\, \frac{i\sqrt{2k}}{(\xi-k)\sqrt{\xi+k}}\, d\xi/2\pi$$

and now the problem is to get this [redacted] related to the form with the Fresnel integrals.

November 23, 1978.

<u>Problem</u>: Suppose we have a plane wave coming in at an angle



The plane wave is described by $e^{-ik\vec{\omega}\cdot\vec{r}}$

$$= e^{-ik(x\cos\theta + y\sin\theta)}$$

(Physicists put $\vec{k} = k\vec{\omega}$; then $e^{i\vec{k}\cdot\vec{r}}$ describes a plane wave with wave length λ given by the distance between crests: $|k|\lambda = 2\pi$ or $\lambda = \frac{2\pi}{k}$.)
The problem is to solve the Helmholtz equation

$$(\Delta + k^2)u = 0$$

with boundary condition $u = 0$ on $R_{\leq 0}$ and the radiative boundary condition that

$$u \quad e^{-ik\vec{\omega}\cdot\vec{r}} + \text{outgoing waves.}$$

Up to now the way I made sense of this is to suppose $\operatorname{Im} k > 0$ and then to ask for a solution

$$u = e^{-ik\vec{\omega}\cdot\vec{r}} + \tilde{u}$$

where $\tilde{u}$ decays far out - meaning that it is L^2. Then we solve the Dirichlet problem

$$\begin{cases} (\Delta + k^2)\tilde{u} = 0 \\ \tilde{u} = -e^{-ik\vec{\omega}\cdot\vec{r}} & \text{on negative } x \text{ axis} \\ \quad = -e^{-ikx\cos\theta} & x < 0, \ y = 0 \end{cases}$$

Paradox: This last problem is symmetrical with respect to the ~~x~~ [redacted] -axis.

Actually this is not a paradox. For example, suppose we want to solve the similar problem with $u=0$ on the x-axis. The solution is given in the UHP

$$u = e^{-ik\vec{\omega}\cdot\vec{r}} - e^{ik\vec{\omega}^*\cdot\vec{r}}$$

where $\vec{\omega}^*$ is determined by reflection, and by $u=0$ in the LHP. Then

$$\tilde{u} = \begin{cases} -e^{ik\vec{\omega}^*\cdot\vec{r}} = -e^{-ik(x\cos\theta - y\sin\theta)} & y>0 \\ -e^{-ik\vec{\omega}\cdot\vec{r}} = -e^{-ik(x\cos\theta + y\sin\theta)} & y<0 \end{cases}$$

[redacted]

if $\vec{\omega} = (\cos\theta, \sin\theta)$, so $\tilde{u}$ is obviously symmetric around the x-axis.

So to find $\tilde{u}$ we proceed as before

$$\tilde{u}(x,y) = \int e^{-i\xi x}\, \hat{\tilde{u}}(\xi,y)\, d\xi/2\pi$$

where

$$\hat{\tilde{u}}(\xi,y) = \hat{\tilde{u}}(\xi,0)\, e^{-\sqrt{\xi^2-k^2}\,y}$$

Continuity of $\frac{\partial\tilde{u}}{\partial y}(x,0)$, $x>0$ gives [redacted]

$$0 = \int e^{-i\xi x}\sqrt{\xi^2-k^2}\;\hat{\tilde{u}}(\xi,0)\, d\xi/2\pi \qquad x>0$$

Now $\tilde{u}(x,0) = \begin{cases} -e^{-ikax} & x<0 \\ v(x) & x>0 \end{cases}$ $\qquad a = \cos\theta$

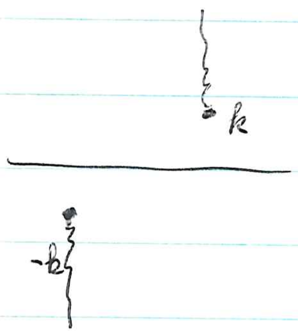
So

$$\hat{u}(\xi,0) = \frac{i}{\xi - ka} + \hat{v}(\xi) \qquad \hat{v}(\xi) = \int_0^\infty e^{i\xi x} v(x)\,dx$$

ANAL IN UHP

Suppose $a > 0$, i.e. $0 \le \theta < \pi/2$, to begin with.

We want



$$\sqrt{\xi^2 - k^2}\left(\frac{i}{\xi - ka} + \hat{v}(\xi)\right) \quad \text{anal in LHP.}$$

$$\underbrace{\sqrt{\xi + k}\left(\frac{i}{\xi - ka}\right)} + \sqrt{\xi + k}\,\hat{v}(\xi)$$

$$\underset{\text{anal in LHP}}{\frac{\sqrt{k(1+a)}\, i}{\xi - ka}} + \underset{\text{anal in UHP}}{\frac{\left(\sqrt{\xi + k} - \sqrt{k(1+a)}\right) i}{\xi - ka}} + \sqrt{\xi + k}\,\hat{v}(\xi)$$

Choose $\hat{v}$ to make the latter two terms cancel:

$$\hat{v}(\xi) = \frac{i}{\xi - ka}\left(-1 + \frac{\sqrt{k(1+a)}}{\sqrt{\xi + k}}\right)$$

$$\therefore \quad \hat{u}(\xi, 0) = \frac{i}{\xi - ka}\,\frac{\sqrt{k(1+a)}}{\sqrt{\xi + k}}$$

$$\therefore \quad \tilde{u}(x, y) = \int_{-\infty}^{\infty} e^{-i\xi x - \sqrt{\xi^2 - k^2}\,|y|}\,\frac{i}{\xi - ka}\,\frac{\sqrt{k(1+a)}}{\sqrt{\xi + k}}\, d\xi/2\pi$$

Problems: 1) What happens for $a \le 0$?
2) Relate these formulas to the Fresnel integrals.
3) The scattering matrix?

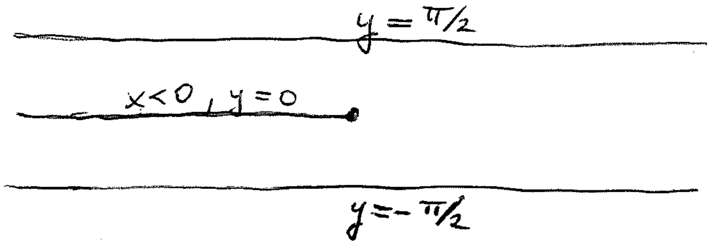
Solution of 1) should be just a matter of moving the real ξ axis a bit, because ka is between $-k$ and k. So the above formula for $\tilde{u}(x, y)$ should be valid with the integration contour below ka.

For example if $x<0$, $y=0$ then the exponential decays for $\mathrm{Im}(\xi)>0$, so that if we push the contour vertically we pick up the residue as $\xi = ka$, which is

$$\sim e^{-ikax}$$

at it should be. But if $x>0$, $y=0$ we might want to push the contour downward.

Schwinger problem: Solve Helmholz for plane region



with $u=0$ on the lines and a given incoming wave form. The incoming wave should be a [redacted] super-position of plane waves which vanishes on the boundary $y=\pm\pi/2$. Now

$$e^{-i(\xi x+\eta y)} \qquad \xi^2+\eta^2=k^2$$

is the form of the plane wave, so the required form is

$$e^{-i\xi x}\left(a e^{-i\eta y}+b\, e^{+i\eta y}\right) \qquad \eta=\sqrt{k^2-\xi^2}$$

at $y=\pi/2$ $\qquad a e^{-i\eta\pi/2}+b e^{i\eta\pi/2}=0$

" $y=-\pi/2$ $\qquad a e^{+i\eta\pi/2}+b e^{-i\eta\pi/2}=0$

So we see that $\quad e^{-i\eta\pi}=e^{i\eta\pi}$ or $e^{2\pi i\eta}=1$ so that $\eta\in\mathbb{Z}$, in which case the form of the waves is

$$e^{-i\xi x}\left(e^{i\eta\pi/2}e^{-i\eta y}-e^{-i\eta\pi/2}e^{i\eta y}\right)$$

or [redacted] $\boxed{e^{-i\xi x}\sin\eta\left(\frac{\pi}{2}-y\right)}$ $\quad$ where $\begin{cases}\eta=1,2,3,\dots\\ \xi^2+\eta^2=k^2\end{cases}$

From the form of the problem we see that it has period 2π in the y-direction. So I can solve the Schwinger problem by using translation from the Sommerfeld problem.

To be more precise take $\eta = 1$ so we have the incoming wave

$$e^{-ik'x}\left(e^{-iy}+e^{iy}\right) \qquad k^2 = 1 + \text{[redacted]}\, k'^2$$

Suppose that $V(x,y) + e^{-ik'x-iy}$ is the solution of the Sommerfeld problem for the incoming wave $e^{-ik'x-iy}$, so that we know $V(x,-y) = V(x,y)$. [redacted]

then

$$\sum_{n\in\mathbb{Z}} \left\{ V(x, y+2\pi n) - V(x, \pi - y \text{ [redacted] } 2\pi n) \right\} + e^{-ik'x}\left(e^{-iy}+e^{iy}\right)$$

should be the solution of Schwinger's problem.

But let's consider a more direct approach. To simplify let us look for solutions u 2π-periodic in y, vanishing on the negative real axis, with incoming part $e^{-ik\vec{\omega}\cdot\vec{r}}$ where $k\vec{\omega} = (k\cos\theta, k\sin\theta)$ and $k\sin\theta \in \mathbb{Z}$. u is to satisfy $(\Delta + k^2)u = 0$ off $\mathbb{R}_{\leq 0} + 2\pi\mathbb{Z}\hat{j}$. If we pass to $\tilde{u} = u - e^{-ik\vec{\omega}\cdot\vec{r}}$, we want [redacted] to solve the Dirichlet equation with the given boundary data. Again take F.T. in x

$$\tilde{u}(x,y) = \int e^{-i\xi x}\, \hat{\tilde{u}}(\xi, y)\, d\xi/2\pi$$

$$\left(\frac{d^2}{dy^2} + k^2 - \xi^2\right)\hat{\tilde{u}}(\xi, y) = 0 \qquad \text{for} \quad 0 < y < 2\pi$$

This has the solutions

$$\hat{\tilde{u}}(\xi, y) = a e^{\eta y} + b e^{-\eta y} \qquad \eta^2 = \xi^2 - k^2$$

We want this to be continuous in y and the same at $y = 0, 2\pi$ which gives

$$a + b = \hat{\tilde{u}}(\xi, 0)$$
$$a e^{\eta 2\pi} + b e^{-\eta 2\pi} = \hat{\tilde{u}}(\xi, 0)$$

so

$$\hat{\tilde{u}}(\xi, y) = \hat{\tilde{u}}(\xi, 0) \frac{\cosh \eta(\pi - y)}{\cosh \eta\pi}$$

$$\therefore \quad \tilde{u}(x, y) = \int_{-\infty}^{\infty} \frac{d\xi}{2\pi} e^{-i\xi x} \frac{\cosh \eta(\pi - |y|)}{\cosh \eta\pi} \hat{\tilde{u}}(\xi, 0) \qquad \text{for } |y| \le 2\pi$$

Now we want the $\frac{\partial \tilde{u}}{\partial y}$ to be continuous across the positive x-axis, so by symmetry $y \mapsto -y$ the derivative must vanish

$$\frac{d}{dy} \frac{\cosh \eta(\pi - y)}{\cosh \eta\pi}\bigg|_{y=0} = \frac{\sinh \eta(\pi - y) \cdot (-\eta)}{\cosh \eta\pi}\bigg|_{y=0} = -\eta \tanh \eta\pi$$

So our integral equation becomes

$$\int_{-\infty}^{\infty} \frac{d\xi}{2\pi} e^{-i\xi x} \, \eta \tanh(\eta\pi) \, \hat{\tilde{u}}(\xi, 0) = 0 \qquad x > 0.$$

This is a Wiener-Hopf equation like the previous one except that $\eta = \sqrt{\xi^2 - k^2}$ is replaced by the meromorphic (in ξ) function $\eta \tanh(\eta\pi)$, whose singularities are simple occurring where

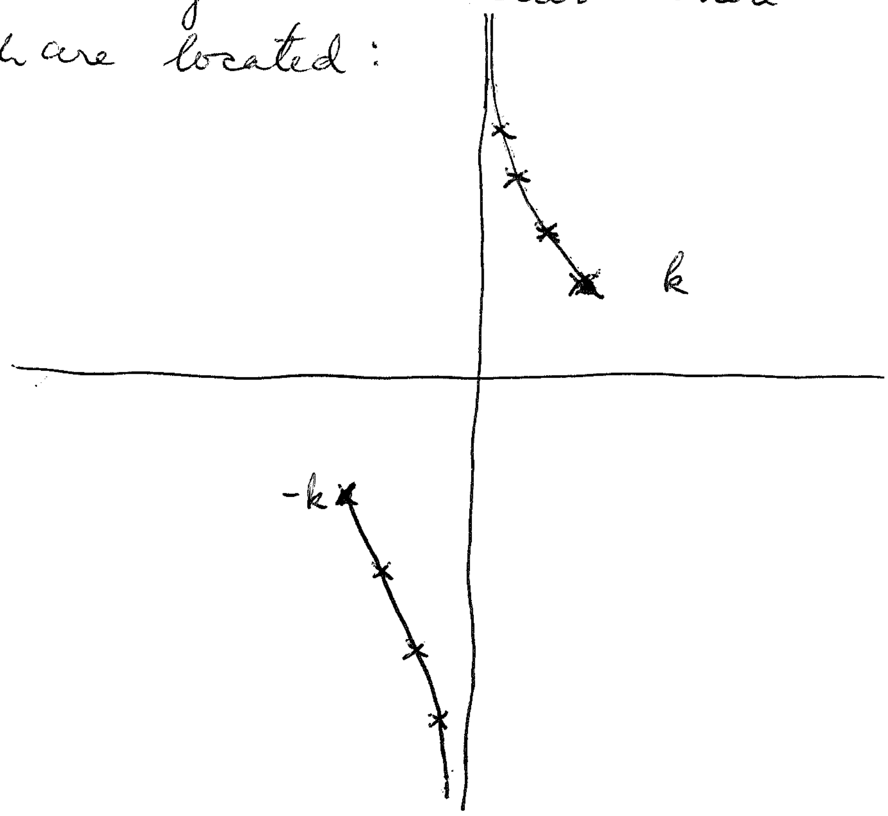
$$\sinh(\eta\pi) = 0 \quad \Rightarrow \quad \eta = in \qquad n \in \mathbb{Z}$$

$$\Rightarrow \quad \xi^2 - k^2 = -n^2 \qquad \text{or} \qquad \xi^2 = k^2 - n^2$$

$$\cosh(\eta\pi) = 0 \quad \Rightarrow \quad \eta = i\left(n + \tfrac{1}{2}\right) \qquad n \in \mathbb{Z}$$

$$\Rightarrow \quad \xi^2 = k^2 - \left(n + \tfrac{1}{2}\right)^2$$

Thus the singularities occur where $\xi^2 = k^2 - \frac{n^2}{4}$ $\quad n \in \mathbb{Z}$
which are located:



Notice what happens if π is replaced by πl
and $l \to \infty$

$$\eta \tanh \pi l \eta \longrightarrow \eta$$

$$\xi^2 = k^2 - \left(\frac{n}{2l}\right)^2$$

so that the curves on which the zeroes & poles are
located for finite l become [redacted] cuts.