November 5, 1978

Consider on the line the Schroedinger equation

$$-u'' + qu = k^2 u$$

with $q \in C_0^\infty$. Consider the Green's function for the operator $L = \frac{d^2}{dx^2} + k^2$; it is

$$G_k(x,y) = \frac{e^{-ikx_<}\, e^{ikx_>}}{W(e^{-ikx}, e^{ikx})} = \frac{e^{ik|x-y|}}{2ik}$$

The original DE is

$$Lu = qu \qquad \text{or} \qquad u = Gqu,$$

hence the Schroedinger equation is equivalent to the integral equation

$$u = Gqu \quad \text{or}$$

$$u(x) = \int \frac{e^{ik|x-y|}}{2ik}\, q(y)\, u(y)\, dy$$

Assume the [redacted] operator Gq [redacted] is such that the Fredholm determinant of $I - Gq$ is defined. Actually since we more or less are assuming $\operatorname{Im} k > 0$ so that G is a bounded operator, it should be clear that the kernel of Gq is nice.

The point is that although neither L nor $L-q$ has[redacted] Fredholm determinants, the operator

$$L^{-1}(L-q) = I - Gq$$

does have one. I think Newton claims that this

Fredholm determinant is the function $A(k)$ defined by the scattering:

$$e^{-ikx} \longleftrightarrow A(k)e^{-ikx} + B(k)e^{ikx}$$

Some evidence for this claim. If k corresponds to a bound state, then $(I - Gq)u = 0$ has a non-zero L^2-solution and also $A(k) = 0$. (Recall k is in the UHP in order that G be defined). Also

$$\det(I - Gq) = 1 - \operatorname{tr}(Gq) + \cdots$$

(think of the Born series - where q is replaced by εq and the various powers of ε).

$$\operatorname{tr}(Gq) = \frac{1}{2ik}\int q(y)\,dy$$

On the other hand WKB gives the solution

$$u = e^{ikx}\left(a_0 + \frac{a_1}{k} + \frac{a_2}{k^2} + \cdots\right) \qquad (u'' + k^2 u) = qu$$

$$2ik\left(a_0' + \frac{a_1'}{k} + \frac{a_2'}{k^2} + \cdots\right) + \left(a_0'' + \frac{a_1''}{k} + \quad\right) = q\left(a_0 + \frac{a_1}{k} + \frac{a_2}{k^2}\right)$$

$$a_0' = 0 \qquad \therefore\ a_0 = 1$$

$$2i\,a_1' + a_0'' = q a_0 \qquad \therefore\ a_1' = \frac{q}{2i} \qquad a_1 = \int_{-\infty}^{x} \frac{q}{2i}$$

$$2i\,a_2' + a_1'' = q a_1$$

$$a_2' = \frac{i}{2}\frac{q'}{2i} + \frac{q}{2i}\int\frac{q}{2i} \qquad a_2 = \frac{q}{4} + \frac{1}{2}\left(\int_{-\infty}^{x}\frac{q}{2i}\right)^2$$

$$e^{ikx}\left(1+\frac{1}{2ik}\int_{-\infty}^{x} q + \frac{1}{k^2}\left(\frac{q}{4}-\frac{1}{8}\left(\int_{-\infty}^{x} q\right)^2\right)+\cdots\right.$$

So
$$u(x,k) \sim e^{-ikx}\left(1+\frac{i}{2k}\int_{-\infty}^{x} q + \frac{1}{k^2}\left(\frac{q}{4}-\frac{1}{8}\left(\int_{-\infty}^{x} q\right)^2\right)\right)$$

CHECK: $\left(1-\frac{q}{k^2}\right)^{-1/4}$ [redacted] $e^{-i\int\sqrt{k^2-q}\,dx} = \left(1+\frac{q}{4k^2}\right)e^{-ikx+\frac{i}{2k}\int^{x} q}$

So we have
$$A(k) \sim 1+\frac{i}{2k}\int_{-\infty}^{\infty} q - \frac{1}{8k^2}\left(\int_{-\infty}^{\infty} q\right)^2 + O\left(\frac{1}{k^3}\right)$$

Simpler case: Instead of working on $\mathbb{R}$, let's work [redacted] on $a \leq x \leq b$ with 0 boundary conditions at the end. Provided k stays away from [redacted] the eigenvalues of [redacted] $L = \frac{d^2}{dx^2}+k^2$ we know that G is well-defined, and we have a good
$$\det(I - Gq)$$
which vanishes [redacted] when k^2 is an eigenvalue for [redacted] $-\frac{d^2}{dx^2}+q$.

Newton's proof goes by explicit calculation: The Fredholm determinant is the series

$$1 - \int K(x,x)\,dx + \iint_{x_1 \geq x_2} \begin{vmatrix} K(x_1,x_1) & K(x_1,x_2) \\ K(x_2,x_1) & K(x_2,x_2) \end{vmatrix} dx_1\,dx_2 - \cdots$$

where K is the kernel for G_q namely

$$K(x,y) = \frac{e^{-ik|x-y|}}{2ik}\, q(y)$$

The first degree term is $-\frac{1}{2ik}\int_{-\infty}^{\infty} q(x)\,dx$. The 2nd degree term is

$$\iint_{x_1 \geq x_2} \frac{1}{(2ik)^2}\left(1 - e^{2ik|x_1 - x_2|}\right) q(x_1)\, q(x_2)\, dx_1\, dx_2$$

We can derive formulas for $A(k)$ by using the integral equation for $\phi(x,k)$:

$$\phi(x,k) = e^{-ikx} + \int_{-\infty}^{x} dx_1 \frac{\sin k(x-x_1)}{k}\, q(x_1)\, \phi(x_1, k)$$

Iterating gives

$$\frac{-e^{ik(x_1-x)}}{2i}$$

$$\phi(x,k) = e^{-ikx} + \int_{-\infty}^{x} dx_1 \frac{\sin k(x-x_1)}{k}\, q(x_1)\, e^{-ikx_1}$$

$$+ \int_{-\infty}^{x} dx_1 \int_{-\infty}^{x_1} dx_2 \frac{\sin k(x-x_1)\, \sin k(x_1-x_2)}{k^2}\, q(x_1)\, q(x_2)\, e^{-ikx_2}$$

Take x large and look at only the coeff of e^{-ikx} to get $A(k)$. One gets

$$A(k) = 1 - \frac{1}{2ik}\int q(x_1)\,dx_1 + \underbrace{\iint_{x_1 \geq x_2} dx_1\, dx_2 \frac{q(x_1)\, q(x_2)}{(2ik)^2}\left(-e^{+ik(x_1)}\right)\left(e^{ik(x_1-x_2)} - e^{-ik(x_1-x_2)}\right)}_{\frac{1}{(2ik)^2}\iint_{x_1 \geq x_2} dx_1\, dx_2\; q(x_1)\, q(x_2)\left(1 - e^{2ik(x_1-x_2)}\right)}$$

$$e^{-ikx_2}$$

which checks through 2nd degree.

~~[illegible]~~

For 3rd degree terms we have on the $A(k)$ side

$$\frac{1}{(2ik)^3}\iiint\limits_{x_1 \geq x_2 \geq x_3} g(x_1)\,g(x_2)\,g(x_3)\underbrace{\left(-e^{ikx_1}\right)2i\sin k(x_1-x_2)\,2i\sin k(x_2-x_3)\,e^{-ikx_3}}$$

Put $\quad a = e^{ik(x_1-x_2)} \qquad b = e^{ik(x_2-x_3)} \qquad ab = e^{ik(x_1-x_3)}$

$$-ab\left(a-a^{-1}\right)\left(b-b^{-1}\right) = -(a^2-1)(b^2-1)$$

and on the determinant side the same integral with end term

$$-\begin{vmatrix} 1 & a & ab \\ a & 1 & b \\ ab & b & 1 \end{vmatrix} = -\left\{1 + a^2b^2 \cancel{+a^2b^2} - a^2 - b^2 - \cancel{a^2b^2}\right\}$$

$$= -(1-a^2)(1-b^2)$$

In general one uses the identity

$$\begin{vmatrix} 1 & a & ab & abc & \\ a & 1 & b & bc & \\ ab & b & 1 & c & \cdot \\ abc & bc & c & 1 & \vdots \\ & & \cdot & & \ddots \end{vmatrix} = (1-a^2)(1-b^2)(1-c^2)\cdots$$

which is clear if one subtracts $a\cdot$ 2nd row from the first row.

Note: We have seen that

$$A = \det(1-K).$$

Recall in the ζ-formalism one likes to work with

$$-\log \det(1-K) = \sum_{n\geq 1} \frac{1}{n} \operatorname{tr}(K^n)$$

Now note:

$$-\log \det(1-K) = \log \frac{1}{A} = \log T$$

where T is the transmission coefficient. Does this mean anything?

November 9, 1978

To understand $\quad \det\left(\left(k^2+\frac{d^2}{dx^2}\right)^{-1}\left(k^2+\frac{d^2}{dx^2}-q\right)\right) = \det(1-G_q)$

We have seen that on the line this determinant is equal to the coefficient $A(k)$.

Consider the half-line $0 \leq x < \infty$ with Dirichlet boundary condition $u(0)=0$. Put for $L_0 = -\frac{d^2}{dx^2}$ with boundary condition

$$\mathring{\phi}(x,k) = \frac{\sin kx}{k} \qquad \mathring{\psi}(x,k) = e^{ikx}$$

Then $\quad W(\mathring{\phi},\mathring{\psi}) = W\left(\frac{e^{ikx}-e^{-ikx}}{2ik}, e^{ikx}\right) = -1 \quad$ and so

$$G_k(x,x') = -\frac{\sin(kx_<)\,e^{ikx_>}}{k}$$

$$= \underbrace{\frac{e^{ik|x-x'|}}{2ik}}_{\text{G fn on line}} - \underbrace{\frac{e^{ik(x+x')}}{2ik}}_{\text{solution of } u''+k^2u=0}$$

Then $\quad \det(1-G_q) = 1 - \int_0^\infty \frac{1-e^{2ikx}}{2ik}\, q(x)\,dx + O(q^2)$

Now the ϕ-solution (satisfying $\phi(0)=0$, $\phi'(0)=1$) satisfies the integral equation

$$\phi(x,k) = \frac{\sin kx}{k} + \int_0^x dx_1 \frac{\sin k(x-x_1)}{k}\, q(x_1)\,\phi(x_1,k)\,dx_1$$

$$\therefore \quad \phi(x,k) = \frac{\sin kx}{k} + \int_0^x dx_1 \,\frac{\sin k(x-x_1)}{k}\, q(x_1)\, \frac{\sin kx_1}{k} + O(q^2)$$

To take the Wronskian with $\psi(x,k) = e^{ikx}$ $\quad x>>0 \quad$ one finds the coefficient of e^{-ikx} and multiplies by $2ik$

So
$$-W(\phi,\psi) = 1 + \int_0^\infty dx_1 \, (-2ik) \frac{-e^{ikx_1}}{2ik} q(x_1) \frac{e^{ikx_1} - e^{-ikx_1}}{2ik} + O(q^2)$$

$$= 1 - \int_0^\infty dx_1 \, q(x_1) \frac{1 - e^{2ikx_1}}{2ik} + O(q^2)$$

So assuming the calculation also works for the higher order terms, we find in this case

$$\det(1 - Gq) = -W(\phi,\psi) = -2ik\, A(k)$$

where
$$\phi(x,k) = A(k) e^{-ikx} + B(k) e^{ikx} \qquad x \gg 0.$$

<u>Note</u>: $\det(1-Gq)$ is an intrinsic quantity, however, Jost functions like $W(\phi,\psi)$ depend on the normalization chosen for ϕ; so it seems.

But an intrinsic quantity is

$$\frac{W(\phi,\psi)}{W(\phi^0,\psi^0)}$$

For the line we get

$$\frac{W(\phi,\psi)}{W(\phi^0,\psi^0)} = \frac{W(A(k)e^{-ikx} + B(k)e^{ikx}, e^{+ikx})}{W(e^{-ikx}, e^{ikx})} = A(k)$$

and for the Dirichlet half-line we get

$$\frac{W(Ae^{-ikx} + Be^{ikx}, e^{ikx})}{W\left(\frac{e^{ikx} - e^{-ikx}}{2ik}, e^{ikx}\right)} = \frac{A(k)\,2ik}{-\frac{1}{2ik}\cdot 2ik} = -2ik\, A(k)$$

So the general formula should be

$$\det\left(\left(k^2+\frac{d^2}{dx^2}\right)^{-1}\left(k^2+\frac{d^2}{dx^2}-q\right)\right) = \frac{W(\phi,\psi)}{W(\mathring{\phi},\mathring{\psi})}$$

Proof: Use transitivity for the determinant to reduce to proving

$$\det\left((L-q)^{-1}(L-q-\delta q)\right) = \frac{W(\phi+\delta\phi,\ \psi+\delta\psi)}{W(\phi,\psi)}$$

with δq a first order infinitesimal. This reduces to checking the first order integrals above, e.g.

$$\det\left((L-q)^{-1}(L-q-\delta q)\right) = 1 - \mathrm{tr}\left((L-q)^{-1}\delta q\right) .$$

Interesting point with the ζ-scattering: We know what incoming waves look like for large negative times – they are functions of y alone. It seems that the incoming and outgoing subspaces are automatically orthogonal, hence the scattering matrix ought to be analytic and bounded by 1 in the UHP. But the scattering matrix is

$$\frac{\hat{\zeta}(2-2s)}{\hat{\zeta}(2s)} = \frac{\hat{\zeta}\left(1-\frac{2k}{i}\right)}{\hat{\zeta}\left(1+\frac{2k}{i}\right)}$$ and it has a pole at $k=\frac{1}{2}i$.

November 10, 1978

automorphic wave equation:

Recall $\Delta = y^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)$ on the UHP. One has

$$\Delta y^s = s(s-1)y^s$$

and [redacted] if we put $s = \frac{1}{2} + \frac{k}{i}$, then

$$s(s-1) = \left(\tfrac{1}{2} - ik\right)\left(-\tfrac{1}{2} - ik\right) = -k^2 - \tfrac{1}{4}$$

Thus $\left(\Delta + \frac{1}{4}\right) y^{\frac{1}{2} \pm ik} = -k^2 y^{\frac{1}{2} \pm ik}$. The automorphic wave equation is

$$\star \qquad \frac{\partial^2 u}{\partial t^2} = \left(\Delta + \tfrac{1}{4}\right) u$$

and it has plane wave solutions

$$e^{ikt}\, y^{\frac{1}{2} \pm ik} = y^{\frac{1}{2}} e^{ik(t \pm \log y)}$$

[redacted] hence solutions independent of x of the form

$$\int y^{\frac{1}{2}} e^{ik(t \pm \log y)} \alpha(k)\, dk/2\pi = y^{1/2}\, \hat{\alpha}(t \pm \log y).$$

Now one takes suitable solutions of $\star$ and makes them into a Hilbert space by means of the energy norm. Put $L = -\Delta - \frac{1}{4}$, then

$$E(u) = \left\| \frac{\partial u}{\partial t} \right\|^2 + (Lu, u)$$

$$\frac{d}{dt} E(u) = (u_t, u_{tt}) + (u_{tt}, u_t) + (Lu_t, u) + (Lu, u_t)$$

provided $u_{tt} = -Lu$.

We are going to be interested in solutions which are invariant under $\Gamma = SL_2(\mathbb{Z})$, and then $\|u\|^2$ will be defined by integrating over a fundamental domain. Suitable solutions will be of compact support mod Γ. In particular any u periodic of period 1 in x supported in $1 < y < N$ for some N will have finite energy norm.

In particular if $\hat{\alpha} \in C_0^\infty(\mathbb{R}_{>1})$, then

$$u(z,t) = y^{1/2}\, \hat{\alpha}(t \pm \log y)$$

works for small $|t|$.

Let $\mathcal{H}$ be the Hilbert space you get for the wave equation on UHP/Γ. Then solutions u such that

$$u = y^{1/2} f(t + \log y) \qquad \text{for } t \leq 0$$

for some $f \in C_0^\infty(\mathbb{R}_{>1})$ close up to form a kind of incoming subspace, whereas those solutions v such that

$$v = y^{1/2} g(t - \log y) \qquad \text{for } t \geq 0$$

form an outgoing subspace. Call this D_{in}, D_{out}. I want to check carefully that these subspaces are orthogonal for the energy norm.

$$u_t = y^{1/2} f'(t + \log y) \qquad\qquad v_t = y^{1/2} g'(t - \log y)$$

$$\left(u_t\Big|_{t=0}, v_t\Big|_{t=0}\right) = \iint_{UHP/\Gamma} y\, f'(\log y)\, g'(-\log y)\, \frac{dx\,dy}{y^2}$$

$$= \int_1^\infty f'(\log y)\, g'(-\log y)\, \frac{dy}{y^2}$$

$$-(Lu,v)_{t=0} = \iint_{UHP/\Gamma} \left(y^2\frac{\partial^2}{\partial y^2}+\frac{1}{4}\right)\left(y^{1/2}f(\log y)\right)\cdot y^{1/2}g(-\log y)\,\frac{dx\,dy}{y^2}$$

$$= \int_1^\infty \left(y^2\left(\frac{\partial}{\partial y}+\frac{1/2}{y}\right)^2+\frac{1}{4}\right)f(\log y)\cdot g(-\log y)\,\frac{dy}{y}$$

$$\left(y^2\left\{\frac{\partial^2}{\partial y^2}+\frac{1}{y}\frac{\partial}{\partial y}+\frac{1/4-1/2}{y^2}\right\}+\frac{1}{4}\right)f(\log y).$$

$$= \int_1^\infty y\frac{\partial}{\partial y}\left(y\frac{\partial}{\partial y}f(\log y)\right)\cdot g(-\log y)\,\frac{dy}{y}$$

$$= -\int_1^\infty y\frac{\partial}{\partial y}\left(f(\log y)\right)\,y\frac{\partial}{\partial y}\left(g(-\log y)\right)\,\frac{dy}{y} \qquad y\frac{\partial}{\partial y}f(\log y)=f'(\log y)$$

$$= -\int_1^\infty f'(\log y)\cdot\left(-g'(-\log y)\right)\,\frac{dy}{y}$$

$$\therefore\quad (u_t,v_t)_{t=0}+(Lu,v)_{t=0} = 0$$

Next consider carefully the computation of the scattering coefficient. You start with the eigenfunction y^s, $s=\frac{1}{2}-ik$ and make it invariant under Γ:

$$y^s + \sum_{-\frac{d}{c}\in\mathbb{Q}} \frac{y^s}{|cz+d|^{2s}} \qquad (c,d)=1,\ c>0$$

The reflection coefficients is obtained by integrating over an x-period and looking what happens as $y \to +\infty$.

$$\int_0^1 \frac{y^s}{\underset{(c^2y^2+(cx+d)^2)^s}{|cz+d|^{2s}}}\,dx = \frac{y^s}{c^{2s}\,y^{2s}} \int_0^1 \frac{dx}{\Big(1+\underbrace{\big(\frac{x}{y}+\frac{d}{cy}\big)}_{q}^2\Big)^s} \qquad dq = \frac{dx}{y}$$

$$= \frac{y^{1-s}}{c^{2s}} \int_{\frac{1}{y}\frac{d}{c}}^{\frac{1}{y}(1+\frac{d}{c})} \frac{dq}{(1+q^2)^s}$$

As $y \to +\infty$ the integral tends to zero unless $1+\frac{d}{c} \geq 0$ and $\frac{d}{c} \leq 0$. For $c=1$ we have two limiting integrals:

$$d=0 \qquad \int_0^\infty$$

$$d=-1 \qquad \int_{-\infty}^0$$

whose sum gives $\int_{-\infty}^{\infty}$. For $c>1$ we have the limiting integral $\int_{-\infty}^{\infty}$ for each $-d$ relatively prime to c with

$$0 < -\frac{d}{c} < 1.$$

The number of these is $\varphi(c)$ (Euler fn). Hence the reflection coefficient for $Re(s) \gg 0$ so there is no convergence problem is

$$\sum_{c=1}^{\infty} \frac{\varphi(c)}{c^{2s}} \int_{-\infty}^{\infty} \frac{dq}{(1+q^2)^s}$$

Since $\sum_{d|n} \varphi(d) = n$ we have $\zeta(2s)\sum_{n=1}^{\infty} \frac{\varphi(n)}{n^{2s}} = \sum \frac{n}{n^{2s}} = \zeta(2s-1)$

and so the reflection coefficient is

$$R(k) = \frac{\zeta(2s-1)}{\zeta(2s)} \cdot \int_{-\infty}^{\infty} \frac{dq}{(1+q^2)^s} \qquad s = \tfrac{1}{2} - ik$$

The integral, which should be easily expressible using Γ-factors, is convergent for $Re(s) > \frac{1}{2}$ and has the value

$$\int_{-\infty}^{\infty} \frac{dq}{1+q^2} = \arctan(q)\Big|_{-\infty}^{\infty} = \pi \neq 0.$$

at $s = 1$.

But $\zeta(2s-1)$ has a simple pole at $s = 1$ which corresponds to $k = \frac{1}{2} i$.

Thus the reflection coefficient has a singularity in the UHP even though incoming and outgoing spaces are orthogonal. This is the <u>paradox</u> to be resolved.

<u>Question:</u> Are there bound states, i.e. positive eigenvalues for $\Delta + \frac{1}{4}$ which make the energy norm fail to be positive?

Use Green's formula to bound Δ from above

$$\iint \Delta u \cdot v \, \frac{dx\,dy}{y^2} + \iint (u_x v_x + u_y v_y)\, dx\,dy$$

$$= (u_{xx} v + u_{yy} v + u_x v_x + u_y v_y)\, dx\,dy$$

$$= \{(u_x v)_x + (u_y v)_y\}\, dx\,dy$$

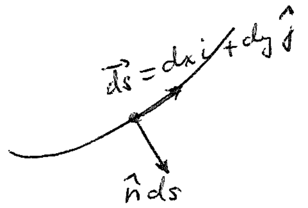
$$= d(u_x v\, dy - u_y v\, dx)$$

hence integrating over a region D:

$$\iint_D \Delta u \cdot v \,\frac{dx\,dy}{y^2} + \iint_D (u_x v_x + u_y v_y)\, dx\,dy$$

$$= \int_{\partial D} v \frac{\partial u}{\partial n}\, ds$$

where



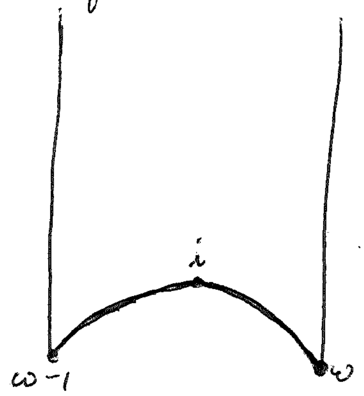
$$\hat{n}\,ds = dy\,\hat{\imath} - dx\,\hat{\jmath}$$

$$\frac{\partial u}{\partial n} ds = \nabla u \cdot \hat{n}\,ds = u_x\,dy - u_y\,dx$$

Note that this is the same in non-Euclidean metric because

$$\nabla u = (y u_x,\ y u_y) \qquad \vec{ds} = \left(\frac{dx}{y},\ \frac{dy}{y}\right)$$

So now consider a u invariant under $\Gamma = PSL_2(\mathbb{Z})$ and D the usual fundamental domain, and $v = u$.



Because u is invariant under $z \longmapsto z+1$ the two vertical integrals are the same except $\frac{\partial u}{\partial n} = \frac{\partial u}{\partial x}$ on one and $\frac{\partial u}{\partial n} = -\frac{\partial u}{\partial x}$ on the other; hence they cancel.

Similarly because u is invariant under the reflection through i the integrals $\omega - 1$ to i and i to ω cancel. Thus we get

$$\iint_D \Delta u \cdot u \,\frac{dx\,dy}{y^2} = - \iint_D (|u_x|^2 + |u_y|^2)\, dx\,dy \leq 0.$$

Thus the spectrum of $\Delta + \frac{1}{4}$ is in $[-\infty, \frac{1}{4}]$, and we have seen the continuous spectrum is $(-\infty, 0]$, so there remains the possibility that $L = -(\Delta + \frac{1}{4})$ has a negative eigenvalue.

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$$\left(\Delta+\tfrac{1}{4}\right)u \cdot u \,\frac{dx\,dy}{y^2} = \left(y^2\frac{\partial^2}{\partial y^2} + \frac{1}{4} + y^2\frac{\partial^2}{\partial x^2}\right)u \cdot u\,\frac{dx\,dy}{y^2}$$

$$= \left(y^2\left(\frac{\partial}{\partial y} + \frac{1/2}{y}\right)^2 + \frac{1}{4} + y^2\frac{\partial^2}{\partial x^2}\right)\underbrace{y^{-1/2}u}_{\tilde{u}} \cdot \underbrace{y^{-1/2}u}_{\tilde{u}}\,\frac{dx\,dy}{y}$$

$$= \left(\left(y\frac{\partial}{\partial y}\right)^2 + \left(y\frac{\partial}{\partial x}\right)^2\right)\tilde{u}\cdot\tilde{u}\,\frac{dx\,dy}{y}$$

$$= \left(\frac{\partial}{\partial y}\,y\frac{\partial}{\partial y} + \frac{\partial}{\partial x}\,y\frac{\partial}{\partial x}\right)\tilde{u}\cdot\tilde{u}\;dx\,dy$$

$$d\left(-y\frac{\partial\tilde{u}}{\partial y}\,\tilde{u}\,dx + y\frac{\partial\tilde{u}}{\partial x}\,\tilde{u}\,dy\right) = \left\{+\frac{\partial}{\partial y}\left(y\frac{\partial\tilde{u}}{\partial y}\,\tilde{u}\right) + \frac{\partial}{\partial x}\left(y\frac{\partial\tilde{u}}{\partial x}\,\tilde{u}\right)\right\}dx\,dy$$

so

$$= \left(\Delta+\tfrac{1}{4}\right)u\cdot u\,\frac{dx\,dy}{y^2} + \left(\left(y\frac{\partial\tilde{u}}{\partial y}\right)^2 + \left(y\frac{\partial\tilde{u}}{\partial x}\right)^2\right)\frac{dx\,dy}{y}$$

so

$$\iint_D \left(-\Delta-\tfrac{1}{4}\right)u\cdot u\,\frac{dx\,dy}{y^2} = \iint_D \left(y\frac{\partial\tilde{u}}{\partial y}\right)^2 + \left(y\frac{\partial\tilde{u}}{\partial x}\right)^2\,\frac{dx\,dy}{y}$$

$$= \int_{\partial D} -y\frac{\partial\tilde{u}}{\partial y}\,\tilde{u}\,dx + y\frac{\partial\tilde{u}}{\partial x}\,\tilde{u}\,dy$$

$$\int_{\partial D} \underbrace{-y\,\frac{\partial(y^{-1/2}u)}{\partial y}\,y^{-1/2}u}_{-\frac{\partial u}{\partial y}u + \frac{1}{2}yy^{-3/2}y^{-1/2}u^2}\,dx + y\,\frac{\partial(y^{-1/2}u)}{\partial x}\,y^{-1/2}u\,dy$$

$$= \int_{\partial D} \underbrace{-u\frac{\partial u}{\partial y}\,dx + u\frac{\partial u}{\partial x}\,dy}_{u\frac{\partial u}{\partial n}\,ds} + \int_{\partial D} \frac{1}{2}u^2\,\frac{dx}{y}$$

We saw this integral vanishes when u satisfies the boundary conditions on the fundamental domain. So we get

$$\iint_D \left(-\Delta - \tfrac{1}{4}\right)u\cdot u\,\frac{dx\,dy}{y^2} = \iint_D \left(\left(y\,\frac{\partial(y^{-1/2}u)}{\partial y}\right)^2 + \left(y\,\frac{\partial(y^{-1/2}u)}{\partial x}\right)^2\right)\frac{dx\,dy}{y}$$

$$-\ \frac{1}{2}\int u^2\,\frac{dx}{y}$$

doesn't prove $-\Delta - \frac{1}{4} \geq 0$.

Look: $u = 1$ is square-integrable over D, hence Δ has the discrete eigenvalue 0. Thus $L = -\Delta - \frac{1}{4}$ has the discrete eigenvalue $-\frac{1}{4} = \left(\frac{1}{2}i\right)^2$ and so this explains the simple pole in the reflection coefficient at $s = 1$ or $k = \frac{1}{2}i$. $\therefore$ Paradox is resolved.

Since $k = \frac{1}{2}i$ is the only pole for the reflection coefficient in UHP, I would expect that $-\frac{1}{4}$ is the only negative eigenvalue for L.

Set $t = \frac{1}{1+q^2}$ or $q = \left(\frac{1}{t}-1\right)^{1/2} = (1-t)^{1/2}t^{-1/2}$.

$2q\,dq = -t^{-2}dt$

$$\int_{-\infty}^{\infty} \frac{dq}{(1+q^2)^s} = 2\int_0^{\infty} \frac{dq}{(1+q^2)^s} = 2\int_1^0 t^s \,\frac{-t^{-2}dt}{2(1-t)^{1/2}t^{-1/2}}$$

$$= \int_0^1 t^{s-3/2}(1-t)^{-1/2}dt = B\left(s-\tfrac{1}{2}, \tfrac{1}{2}\right) = \frac{\Gamma\left(s-\frac{1}{2}\right)\pi^{1/2}}{\Gamma(s)}$$

On "tunnel" solutions - these are somehow related to the eigenvalue $k=0$ and foul up the wave equation in dimension 1 just like bound states. Let's consider a radial Schroedinger equation and the associated wave equation. When you form the energy Hilbert space, you consider Cauchy data of compact support in x, i.e. $u(x,t)$ which have compact support in x for any time. This is not the same as looking at solutions which are nice (say rapidly decreasing) in t. If $u(x,t)$ is in the latter case we can take its F.T. and obtain a representation

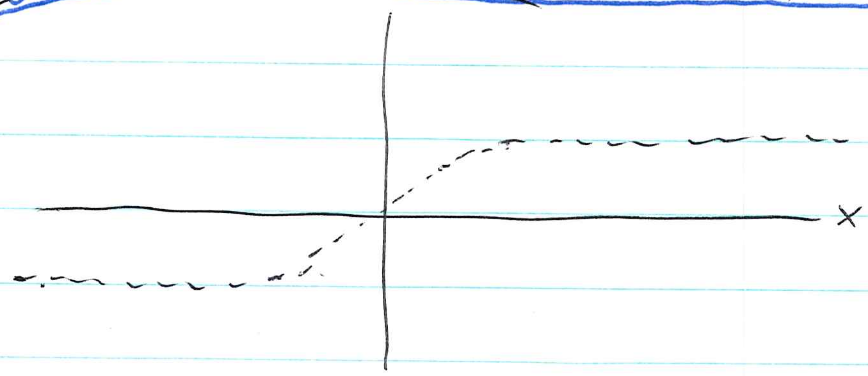
$$u(x,t) = \int e^{ikt}\,\phi(x,k)\,\alpha(k)\,dk/2\pi$$

where $\alpha(k)$ is a nice function of k.

Example: Consider $-u'' = k^2 u$ on $0 \le x < \infty$ with the boundary condition $u'(0) = 0$. The general solution of the wave equation is

$$u(x,t) = f(x+t) + [redacted] f(-x+t)$$

with $f(x)$ defined for all x. If f has the graph



then $u(x,t)$ has compact support for any time t, but for [redacted] x fixed u is constant and $\neq 0$ for $t \gg 0$ and $t \ll 0$. In particular we get a counterexample to [redacted] decay of solutions.

We have seen that the energy of

$$u(x,t) = \int e^{ikt} \phi(x,k) \alpha(k)\, dk/2\pi$$

is

$$E(u) = [redacted] \int |A\alpha|^2 k^2 \, dk/2\pi$$

Put $\frac{1}{2}$ in the definition of $E(u)$.

where $\phi(x,k) = A(k) e^{-ikx} + B(k) e^{ikx}$ $\quad x \gg 0$

In this case [redacted] $\phi(x,k) = \cos kx$, $A(k) = \frac{1}{2}$.

It seems clear to me that the energy Hilbert space should be isomorphic to

$$L^2(\mathbb{R},$$ [redacted] $$|A|^2 k^2 \, dk/2\pi) \qquad U(t) = e^{ikt}$$

and that the in and out representations should be $\alpha \longmapsto \sqrt{2}\, A k \alpha$, $\sqrt{2}\, B k \alpha$. When $A(k)$ is analytic at $k=0$, which means that $\phi(x,0)$ is constant for large x (this is somehow what Guillemin + Melrose call a "tunnel" solution), then the energy Hilbert space has $\alpha(k)$ with singularity $\frac{1}{k}$ at $k=0$, and the corresponding solution $u(x,t)$ doesn't [redacted] decay. [redacted]

Another case: Take wave equation on the line $(q=0)$ whose general solution is $u(x,t) = f(x-t) + g(x+t)$. The energy norm of this solution is

$$E(u) = \frac{1}{2}\{\|-f'+g'\|^2 + \|f'+g'\|^2\} =$$ [redacted] $$\|f'\|^2 +$$ [redacted] $$\|g'\|^2$$

In particular taking [redacted] u to have compact support in x, whence from

$$\frac{\partial u}{\partial x}(x,0) = f'(x) + g'(x)$$

$$\frac{\partial u}{\partial t}(x,0) = -f'(x) + g'(x)$$

we see f' and g' have compact support, hence we get an isometry $u \longmapsto (f', g')$ from the energy Hilbert space into $L^2 \times L^2$. Note that $\{f' \in L^2$ as f runs over $C_0^\infty\}$ is dense. It follows that the energy Hilbert space is isomorphic to $L^2 \times L^2$.

November 12, 1978:

Recall the Laplace transform method for solving the initial value problem

$$\frac{d^2u}{dt^2} = -Lu \qquad u(0)=u_0\,,\ \frac{du}{dt}(0)=u_0' \ .$$

Multiply by $e^{ikt} = e^{-st}$ $\left(s = \frac{k}{i} \text{ so } \operatorname{Re} s>0 \Longleftrightarrow \operatorname{Im} k>0 \right)$.
and integrate from 0 to ∞.

$$\int_0^\infty e^{-st}\frac{d^2u}{dt^2}\,dt = \int_0^\infty d\left(e^{-st}\frac{du}{dt} + se^{-st}u\right) + \int_0^\infty s^2e^{-st}\,dt$$

or $$\boxed{\mathcal{L}\left(\frac{d^2u}{dt^2}\right) = s^2\mathcal{L}(u) - u_0' - su_0}$$ [redacted]

so $$s^2\mathcal{L}(u) - u_0' - su(0) = -L\,\mathcal{L}(u)$$

$$(L+s^2)\,\mathcal{L}(u) = u_0' + su_0$$ [redacted]

or $$\mathcal{L}(u) = (L+s^2)^{-1}(u_0'+su_0)$$

or $$u(t) = \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} ds\, e^{st}(L+s^2)^{-1}(u_0'+su_0)$$

In Fourier transform terms put $s=-ik$ and you get

$$\boxed{u(t) = \int_{-\infty+ia}^{\infty+ia} dk/2\pi\, e^{-ikt}(L-k^2)^{-1}(u_0'-iku_0)}$$

Here a is positive enough to lie above the singularities of the resolvent. Now to get at the [redacted] decay of

$u(t)$ in time we want to bring the integration contour down as far as possible.

Notice that the operator $(L-k^2)^{-1}$ doesn't seem to have an obvious analytic continuation to the LHP, but that it's kernel

$$G_k(x,y) = \frac{\phi(x_<,k)\,\psi(x_>,k)}{W(\phi,\psi)} \qquad \left(\text{e.g. } \frac{e^{ik|x-y|}}{2ik}\right)$$

does. This means that the resolvent doesn't have an analytic continuation as an operator on L^2, however, it does have one as an operator from C_0^∞ to C^∞. So now it is clear somewhat why the singularities of $G_k(x,y)$ in the LHP (= non-physical sheet) affect local decay of solutions.

Another point also becomes clear: The kernel $1 - G_k(x,y)\,q(y)$ can have a determinant, in fact it does, when q has compact support, or more generally dies faster than any exponential.

Let's look at the wave equation $\frac{\partial^2 u}{\partial t^2} = (\Delta + \frac{1}{4})u$ in the box $0 \le x \le 1$, $y \ge a$ with periodic boundary condition $u(x+1,y) = u(x,y)$ and the Neumann condition $\frac{\partial u}{\partial y}(x,y) = 0$ on $y = a$. The boundary condition gives us a bound state $u = 1$.

Separate variables x, y to solve $(\Delta + \frac{1}{4})u = -k^2 u$

$$u(x,y) = e^{2\pi i m x}\, y^{1/2}\, V_m(y)$$

$$y^{-1/2}\left\{ y^2 \frac{\partial^2}{\partial y^2} + y^2 (2\pi i m)^2 + \frac{1}{4} \right\} y^{1/2}\, V_m = -k^2\, V_m$$

$$\left\{\left(y\frac{\partial}{\partial y}\right)^2 - 4\pi^2 m^2 y^2\right\} v = -k^2 v$$

Recall $K_s(r)$ satisfies $\left\{\left(r\frac{d}{dr}\right)^2 - r^2\right\}u = +s^2 u$, hence we see that v_m is proportional to

[redacted] $K_{ik}(2\pi|m|y)$

[redacted] if it decays as $y \to \infty$. This is for $m \neq 0$. Thus

$$u(x,y) = e^{2\pi i m x}\, y^{1/2} K_{ik}(2\pi|m|y)$$

will be an eigenfunction for $\Delta + \frac{1}{4}$ with eigenvalue $-k^2$ provided $\frac{\partial u}{\partial y} = 0$ for $y = a$.

$$0 = \frac{\partial}{\partial y}\left(y^{1/2} K_{ik}(2\pi|m|y)\right) = \frac{1}{2} y^{-1/2} K_{ik}(2\pi|m|y) + y^{1/2}\, 2\pi|m|\, K'_{ik}(2\pi|m|y)$$

so $$K_{ik}(2\pi|m|a) + 4\pi|m|a\, K'_{ik}(2\pi|m|a) = 0$$

is the requirement that k should satisfy. From what we know about Bessel, we know the possible k's are real. ? NOT CLEAR - ONLY k^2 HAS TO BE REAL.

The above is for $m \neq 0$. If $m = 0$ we get

$$\left(y\frac{\partial}{\partial y}\right)^2 v = -k^2 v$$

which has the solutions y^{ik}, y^{-ik} so

$$u(x,y) = y^{1/2}\left(c_1 y^{ik} + c_2 y^{-ik}\right)$$

The boundary condition at $y = a$ gives

$$\frac{1}{2} y^{-1/2}\left(c_1 y^{ik} + c_2 y^{-ik}\right) + y^{1/2}\left(ikc_1 y^{ik-1} - ikc_2 y^{-ik-1}\right) = 0$$

at $y=a$. Too hard this way. Instead introduce $t = \log y$. The DE becomes

$$\frac{d^2}{dt^2} v = -k^2 v$$

and the boundary condition becomes

$$\frac{d}{dy}\left(e^{\frac{1}{2}t} v\right) = 0 \qquad \text{at} \quad t = \log a$$

or

$$\frac{d}{dt}\left(e^{\frac{1}{2}t} v\right) = 0 \qquad \text{"}$$

or

$$\frac{dv}{dt} + \frac{1}{2} v = 0 \qquad \text{at} \quad t = \log a$$

Since the boundary condition is real we know that that the spectrum consists of k such that $-k^2$ is real. If $-k^2 = a^2 > 0$, then the solutions of the DE are lin. comb. of e^{at}, e^{-at}, so the only one decaying is e^{-at} assuming $a>0$. Only $a = \frac{1}{2}$ gives a solution satisfying the boundary condition. So one gets the required bound state at $k = \frac{1}{2} i$ and otherwise a continuous spectrum with k real.

The next thing to do is to determine the eigenvalue distribution for the discrete spectrum. Thus for each $m = 1, 2, 3, 4, 5, 6, \dots$ we want to estimate the solutions k of

$$K_{ik}(2\pi m a) + 4\pi m a \, K'_{ik}(2\pi m a) = 0$$

This depends only on $2\pi m a$

We want to understand those k such that

$$K_{ik}(r) + 2r\frac{d}{dr}K_{ik}(r) = 0 \qquad \text{at} \quad r = r_0$$

Recall that $\phi(x,k) = K_{ik}(e^{-x})$ satisfies

$$\left\{-\frac{d^2}{dx^2} + e^{-2x}\right\}\phi = +k^2\phi$$

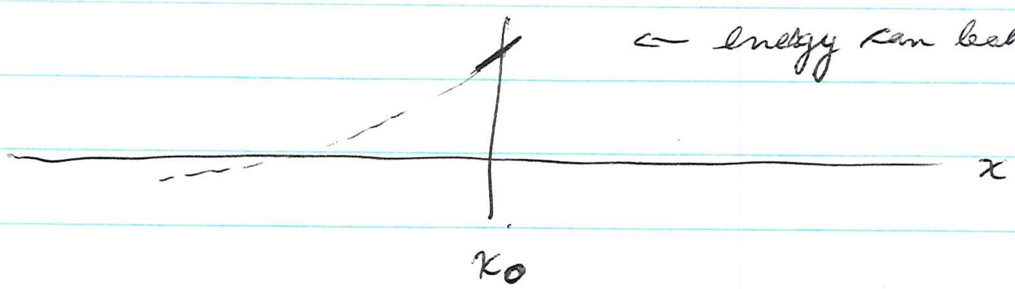
and that it has the asymptotic expansion as $x \longrightarrow +\infty$

$$K_{ik}(e^{-x}) \sim 2^{\frac{k}{i}}\Gamma\left(\tfrac{k}{i}\right)e^{-ikx} + 2^{ik}\Gamma(ik)e^{ikx}$$

We want to find those k such that

$$K_{ik}(e^{-x}) = 2\frac{d}{dx}K_{ik}(e^{-x}) \qquad \text{at } x = x_0 = -\log r_0$$

This boundary condition is not of the string type



so there is no obvious reason for the eigenvalue k^2 to be ≥ 0. In fact it seems likely that if $x_0 \gg 0$ then we should get close to the $k = \frac{1}{2}i$ bound state.

To estimate the number of eigenvalues we want to estimate the number of zeroes of $K_{ik}(e^{-x})$ for $x \leq -\log r_0$ (the error is $O(1)$), i.e. the number of zeroes of $K_{ik}(e^{x})$ for $x \geq \log r_0$. By the Milne formula the estimate is (see April 9, 1977)

$$N(k) \doteq \frac{1}{\pi}\int_{\log r_0}^{\log k} \sqrt{k^2 - e^{2x}}\, dx = \frac{1}{\pi}\int_{r_0}^{k} \sqrt{k^2-u^2}\,\frac{du}{u}$$

$$\doteq \frac{k}{\pi}\int_{\frac{r_0}{k}}^{1} \sqrt{1-u^2}\,\frac{du}{u}$$

But

$$\int \sqrt{1-u^2}\,\frac{du}{u} \underset{u=\cos\theta}{=} \int \sin\theta \,\frac{-\sin\theta\, d\theta}{\cos\theta} = \int \left\{\cos\theta - \frac{1}{\cos\theta}\right\} d\theta$$

$$= \sin\theta - \log(\sec\theta + \tan\theta)$$

$$= \sqrt{1-u^2} - \log\left(\frac{1}{u} + \frac{\sqrt{1-u^2}}{u}\right)$$

$$= \sqrt{1-u^2} - \log\left(u + \sqrt{1-u^2}\right) + \log u$$

$$N(k) \doteq \frac{k}{\pi}\left\{ \sqrt{1-u^2} - \log\left(u+\sqrt{1-u^2}\right) + \log u \right\}_{r_0/k}^{1}$$

$$= \frac{k}{\pi}\left\{ -\sqrt{1-\frac{r_0}{k^2}} + \log\left(\frac{r_0}{k} + \sqrt{1-\frac{r_0^2}{k^2}}\right) \log\frac{r_0}{k} \right\}$$

$$= \frac{k}{\pi}\left\{ -1 + O\left(\frac{1}{k^2}\right) + \frac{r_0}{k} + O\left(\frac{1}{k^2}\right) + \log\frac{k}{r_0} \right\}$$

$$= \frac{k}{\pi}\left(\log\frac{k}{r_0} - 1\right) + O(1)$$

(Check: recall that for the comparison with ζ one has $r_0 = 2\pi$ and also a factor of 2 because $\zeta(2s)$ is related to $\Gamma(s)$ or to K_s. Thus one would get $\frac{T}{2\pi}\left(\log\frac{T}{2\pi} - 1\right)$ as expected.)

So we conclude that the growth of discrete eigenvalues for fixed $m, a > 0$ is given by

$$N_m(k) = \frac{k}{\pi}\left(\log \frac{k}{2\pi m a} - 1\right) + O(1)$$

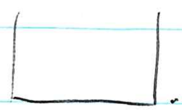
Now you want to find a way to combine these results for different m. Rough idea would be to use the above expressions for $m \le \frac{k}{2\pi}$ (Take $a=1$). and sum. You get

$$\frac{k}{\pi} \sum_{m \le \frac{k}{2\pi}} \left(\log \frac{k}{2\pi} - 1 - \log m\right) \qquad \log n! = (n+\tfrac{1}{2})\log n - n + \tfrac{1}{2}\log 2\pi$$

$$= \frac{k}{\pi}\left\{ \frac{k}{2\pi}\left(\log \frac{k}{2\pi} - 1\right) - \log\left(\frac{k}{2\pi}\right)! \right\}$$

$$\left(\frac{k}{2\pi} + \frac{1}{2}\right)\log \frac{k}{2\pi} - \log \frac{k}{2\pi} + \frac{1}{2}\log 2\pi$$

$$= -\frac{k}{2\pi}\log\left(\frac{k}{2\pi}\right) + O(k)$$ which is of the wrong sign, so you learn nothing.

Compute the scattering for  .

$y=a$

$$\Delta y^s = s(s-1)y^s \qquad s = \tfrac{1}{2} - ik \qquad s(s-1) + \tfrac{1}{4} = \left(\tfrac{1}{2} - ik\right)\left(-\tfrac{1}{2} - ik\right) + \tfrac{1}{4} = -k^2$$

$$\left(\Delta + \tfrac{1}{4}\right) y^s = \left(s(s-1) + \tfrac{1}{4}\right) y^s = -k^2 y^s$$

The reflection coefficient $R(k)$ is determined by requiring $u = y^{\frac{1}{2} - ik} + R(k) y^{\frac{1}{2} + ik}$

to satisfy the boundary condition $\frac{\partial u}{\partial y} = 0$ at $y = a$.

$$\left(\tfrac{1}{2} - ik\right) a^{-\frac{1}{2} - ik} + R(k)\left(\tfrac{1}{2} + ik\right) a^{-\frac{1}{2} + ik} = 0$$

so

$$R(k) = -\frac{\frac{1}{2}-ik}{\frac{1}{2}+ik}\, a^{-2ik} = \frac{k+\frac{1}{2}i}{k-\frac{1}{2}i}\, a^{-2ik}$$

Can we use Guillemin-Duistermaat (+ Melrose) to understand the distribution of discrete eigenvalues for $\Delta + \frac{1}{4}$ in terms of the lengths of closed geodesics?

November 14, 1978

Duistermaat - Guillemin: relating spectrum of Δ to lengths of closed geodesics. This work starts with Hörmander work on the asymptotics of eigenvalues. Let P be a positive pseudo-differential operator of order 1, and consider the wave equation $i\frac{\partial u}{\partial t} = Pu$ whose solution is given by the operator e^{-itP}. Let $K(x,y,t)dy$ be the Schwartz kernel for e^{-itP}. Hörmander calculates the singularities of $K(x,y,t)dy$ for t near 0; this then gives the singularities of

$$\sum e^{-it\lambda_n} = \operatorname{tr} e^{-itP} = \int K(x,x,t)\,dx$$

for t near zero, and then a simple Tauberian argument gives the asymptotic behavior for λ_n.

Ex. 1: Take $X = S^1$ and $P = \frac{1}{i}\frac{d}{dx}$ (This isn't ≥ 0, but proceed anyway. Then

$$\left(e^{-itP}f\right)(x) = \left(e^{-t\frac{d}{dx}}f\right)(x) = f(x-t)$$

$$= \int_{S^1} \delta(x-y-t)\,f(y)\,dy$$

so $K(x,y,t) = \delta(x-y-t)\,dy$. Then

$$\operatorname{tr}\left(e^{-itP}\right) = \int K(x,x,t)\,dx = \int_{S^1} \delta(-t)\,dx = 2\pi\,\delta(t)$$

On the other hand the eigenfunctions for P are

$$P e^{imx} = m e^{imx} \qquad m \in \mathbb{Z}$$

so

$$\operatorname{tr}(e^{-itP}) = \sum_{m \in \mathbb{Z}} e^{-imt}$$

and hence we obtain the formula (Poisson summation)

$$\delta(t) = \frac{1}{2\pi} \sum_{m \in \mathbb{Z}} e^{-imt} \qquad \text{on } S^1.$$

Ex. 2: $X = S^1$, P = pos. square root of $-\frac{d^2}{dx^2}$.

$$(Pf)(x) = \int e^{-i\xi x}\, \xi\, \hat{f}(\xi)\, d\xi/2\pi$$

$$= \frac{1}{2\pi} \sum_{n \in \mathbb{Z}} |n| \hat{f}(n)\, e^{+inx} \qquad \hat{f}(n) = \int_{S^1} e^{-inx} f(x)\, dx$$

Here the eigenfunctions are given by

$$P e^{imx} = |m| e^{imx}$$

so that

$$\operatorname{tr}(e^{-itP}) = 1 + 2\sum_{m=1}^{\infty} e^{-imt} = 1 + 2\frac{e^{-it}}{1-e^{-it}} = \frac{1+e^{-it}}{1-e^{-it}}$$

It's possible that this formula omits a contribution of $\delta(0)$ – after all $\operatorname{tr}(e^{-itP})$ is really a distribution.