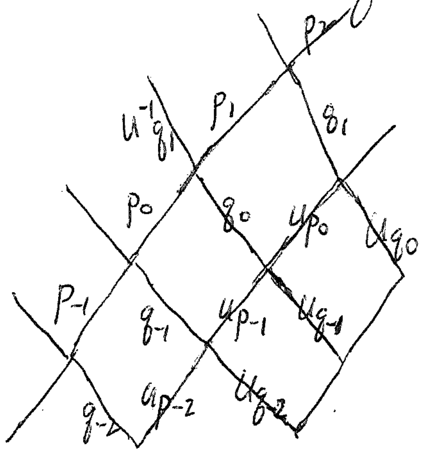
September 14, 1978

Problem: Existence of system $(\mathcal{H}, U, p_n, q_n)$ belonging to a Schur sequence $\{h_n\}$.




A possible procedure is to require $\mathcal{H}$ to have the orthonormal basis

$$\cdots, Up_{-2}, q_{-1}, p_0, U^{-1}q_1, U^{-1}p_2, \cdots$$

In terms of this basis we can define a new orthonormal basis

$$\cdots, Uq_{-2}, Up_{-1}, q_0, p_1, U^{-1}q_2, \cdots$$

using the matrices

$$(q_{-1} \;\; p_0) = (Up_{-1} \;\; q_0)\begin{pmatrix} -\bar{h}_0 & k_0 \\ k_0 & h_0 \end{pmatrix}$$

(In effect

$$p_0 - h_0 q_0 = k_0 Up_{-1} \qquad\qquad p_0 = k_0 Up_{-1} + h_0 q_0$$

$$-\bar{h}_0 p_0 + q_0 = k_0 q_{-1} \qquad\qquad k_0 q_{-1} = q_0 - \bar{h}_0(k_0 Up_{-1} + h_0 q_0)$$

$$= k_0^2 q_0 - k_0 \bar{h}_0 Up_{-1} \;)$$

and more generally

$$(U^{-n}q_{2n-1} \;\; U^{-n}p_{2n}) = (U^{-n+1}p_{2n-1} \;\; U^{-n}q_{2n})\begin{pmatrix} -\bar{h}_{2n} & k_{2n} \\ k_{2n} & h_{2n} \end{pmatrix}$$

In terms of the second basis we can get another orthonormal basis

$$\cdots\cdots, U^2p_{-2}, Uq_{-1}, Up_0, q_1, p_2, \cdots\cdots$$

using the matrices

$$(q_0 \ \ p_1) = (Up_0 \ \ q_1)\begin{pmatrix} -\bar{h}_1 & k_1 \\ k_1 & h_1 \end{pmatrix}$$

in general

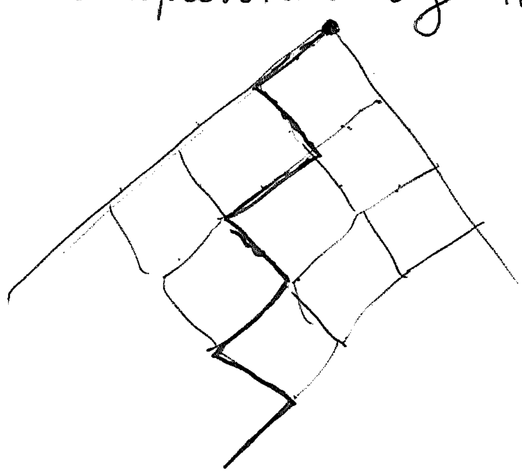
$$(U^{-n}q_{2n} \ \ U^{-n}p_{2n+1}) = (U^{-n+1}p_{2n} \ \ U^{-n}q_{2n+1})\begin{pmatrix} -\bar{h}_{2n+1} & k_{2n+1} \\ k_{2n+1} & h_{2n+1} \end{pmatrix}.$$

Now one can define the operator $U$ on $\mathcal{H}$ to be that transformation carrying the first orthonormal basis to the third orthonormal basis.

It's clear at this stage that we have defined the elements $p_n, q_n$ in $\mathcal{H}$. Moreover we have the basic recursion relation

$$\begin{pmatrix} p_n \\ q_n \end{pmatrix} = \frac{1}{k_n}\begin{pmatrix} 1 & h_n \\ \bar{h}_n & 1 \end{pmatrix}\begin{pmatrix} Up_{n-1} \\ q_{n-1} \end{pmatrix}.$$

The other thing to establish is that in the diagram each edge is perpendicular to edges in the shadow of its tail. The reason is that the shadow is spanned by the orthonormal basis below it

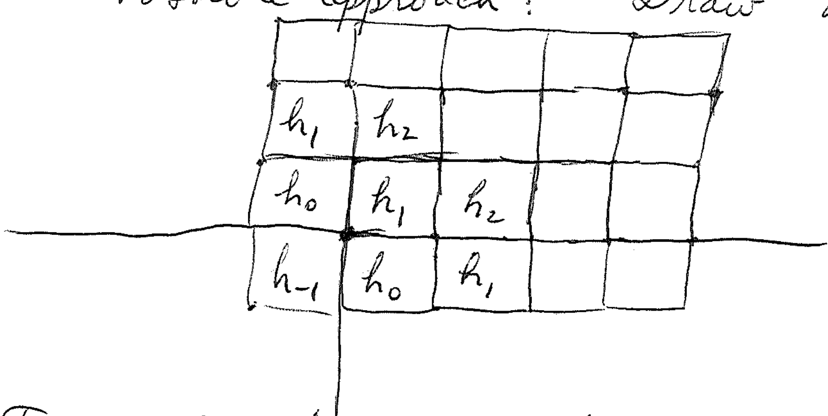


Notice that the unitary matrix

$$\begin{pmatrix} -\bar{h} & k \\ k & h \end{pmatrix} \qquad k = \sqrt{1-|h|^2}$$

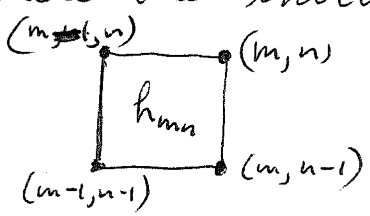
makes sense even for $|h| = 1$. Consequently I ought to be able to construct a pair $(\mathcal{H}, \mathcal{U})$ belonging to any sequence $\{h_n\}_{n\in\mathbb{Z}}$ with $|h_n| \le 1$.

Possible approach: Draw diagram




For each edge we get a generator for $\mathcal{H}$. Let $x_{mn}$ corresp to $(m-1,n) \text{———} (m,n)$ and $y_{mn}$ corresp. to $\begin{matrix}(m,n)\\ | \\ (m,n-1)\end{matrix}$

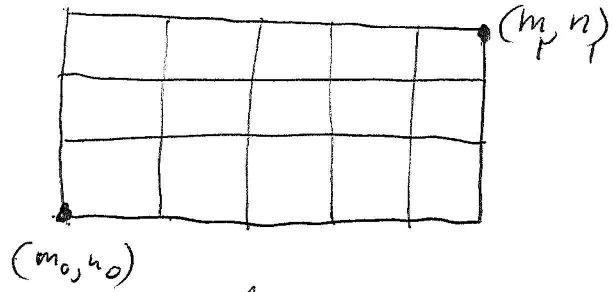
For each square one introduces a relation




which says that

$$\begin{pmatrix} y_{m-1,n} & x_{m,n} \end{pmatrix} = \begin{pmatrix} x_{m,n-1} & y_{mn} \end{pmatrix} \begin{pmatrix} -\bar{h}_{mn} & k_{mn} \\ k_{mn} & h_{mn} \end{pmatrix}$$

It is pretty clear that if we pick a rectangle




Then any path from $(m_0, n_0)$ to $(m_1, n_1)$ increasing for the product order gives a basis for the space $W_{(m_0,n_0)}^{(m_1,n_1)}$ spanned by the edges in this rectangle. Because the matrices are unitary we even get a well-defined inner product on $W_{(m_0,n_0)}^{(m_1,n_1)}$ by calling any of these bases orthonormal.

Finally if $h_{mn}$ depends only on [redacted] $m+n$:

$$h_{mn} = h_{m+n-1}$$

it is clear that on $\mathcal{H}$ we get a unitary operator $\mathcal{U}$ which on the vertices sends $(m,n)$ to $(m+1, n-1)$. Thus we have $(\mathcal{H}, \mathcal{U})$ and all that remains is to put

$$p_n = x_{n0}$$
$$q_n = y_{n0}$$

The rest requires checking.

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Recall

$$\begin{pmatrix} \mathcal{U}^{-n} p_n \\ q_n \end{pmatrix} = \underbrace{\theta(h_n \mathcal{U}^{-n}) \cdots \theta(h_1 \mathcal{U}^{-1})}_{T_{n0}(\mathcal{U})} \begin{pmatrix} p_0 \\ q_0 \end{pmatrix}$$

We showed that if $\sum_{n=1}^{\infty} |h_n| < \infty$, then $\lim_{n \to \infty} T_{n0}(z) = T_{\infty 0}(z)$

exists uniformly and is an invertible matrix of continuous functions on $S^1$.

We have the domination

$$\theta(h_n z^{-n}) = \frac{1}{k_n}\begin{pmatrix} 1 & h_n z^{-n} \\ \bar{h}_n z^n & 1 \end{pmatrix} \ll \frac{1}{k_n}\begin{pmatrix} 1 & |h_n| \\ |h_n| & 1 \end{pmatrix}$$

Recall that
$$\exp\begin{pmatrix} 0 & t \\ t & 0 \end{pmatrix} = \begin{pmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{pmatrix}$$

hence
$$\theta(h_n z^n) \ll \exp t_n \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$
where $\sinh(t_n) = \frac{|h_n|}{k_n}$

So we get the domination

$$T_{n0}(z) \ll \exp\left\{\left(\sum_{k=1}^{n} t_k\right)\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right\}.$$

Notice this domination is exact for $z = 1$ and $h_n > 0$ so therefore if we want any sort of estimated convergence we have to assume $\sum t_n < \infty$ which since

$$\sinh(t_n) \sim t_n \sim |h_n|$$

for $n$ small means that $\sum |h_n| < \infty$.

Note $\quad \sinh t \doteq t + \frac{t^3}{3!} = \frac{h}{(1-h^2)^{-1/2}} \doteq h(1 + \frac{1}{2}h^2)$

implies that $t = h + O(h^3)$.

$$\begin{pmatrix} U^{-n}p_n \\ q_n \end{pmatrix} = \Theta(h_n U^{-n}) \begin{pmatrix} U^{-n+1}p_{n-1} \\ q_{n-1} \end{pmatrix}$$

$$\begin{pmatrix} e_{out} \\ e_{in} \end{pmatrix} = T_{\infty,n}(U) \cdot \begin{pmatrix} U^{-n}p_n \\ q_n \end{pmatrix} \qquad \text{where}$$

$$T_{\infty,n}(z) = \lim_{N\to\infty} \underbrace{\Theta(h_N z^{-N}) \cdots \Theta(h_{n+1} z^{-n-1})}_{T_{Nn}(z)}$$

$$T_{n+m,n} = \Theta(h_{n+m} z^{-(n+m)}) \cdots \Theta(h_{n+1} z^{-(n+1)})$$

$$= \begin{pmatrix} z^{-n-1} & 0 \\ 0 & 1 \end{pmatrix} \Theta(h_{n+m} z^{-m+1}) \cdots \Theta(h_{n+1}) \begin{pmatrix} z^{n+1} & 0 \\ 0 & 1 \end{pmatrix}$$

So if we put

$$\Theta(h'_{m-1} z^{-m+1}) \cdots \Theta(h'_0) = \begin{pmatrix} \bar{A}_{m-1} & \bar{B}_{m-1} \\ B_{m-1} & A_{m-1} \end{pmatrix} \frac{1}{k_{m-1} \cdots k_0}$$

we have

$$\begin{pmatrix} 1 & h'_m z^{-m} \\ \bar{h}'_m z^m & 1 \end{pmatrix} \begin{pmatrix} \bar{A}_{m-1} & \bar{B}_{m-1} \\ B_{m-1} & A_{m-1} \end{pmatrix} = \begin{pmatrix} \bar{A}_m & \bar{B}_m \\ B_m & A_m \end{pmatrix}$$

or

$$A_m = A_{m-1} + \bar{h}'_m \bar{z}^m \bar{B}_{m-1}$$
$$B_m = B_{m-1} + \bar{h}'_m z^m \bar{A}_{m-1}$$

So inductively $A_m, B_m$ are polys. of degree $\leq m$ in $z$. If we assume $\sum |h_m| < \infty$, then we know $A_m, B_m$ have limits $A_\infty, B_\infty$ as $m \to \infty$. So we have

$$\tilde{T}_{\infty n}(z) = \begin{pmatrix} z^{-n-1} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \bar{A}_\infty & \bar{B}_\infty \\ B_\infty & A_\infty \end{pmatrix} \begin{pmatrix} z^{n+1} & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \bar{A}_\infty & z^{-n-1}\bar{B}_\infty \\ z^{n+1}B_\infty & A_\infty \end{pmatrix}$$

hence
$$\begin{pmatrix} e_{out} \\ e_{in} \end{pmatrix} = \begin{pmatrix} \bar{A}_\infty & z^{-n-1}\bar{B}_\infty \\ z^{n+1}B_\infty & A_\infty \end{pmatrix} \begin{pmatrix} u^{-n}p_n \\ q_n \end{pmatrix} \frac{1}{k_{n+1}\cdots\cdots}$$

or
$$\begin{pmatrix} u^{-n}p_n \\ q_n \end{pmatrix} = \frac{1}{k_{n+1}\cdots\cdots} \begin{pmatrix} A_\infty & -z^{-n-1}\bar{B}_\infty \\ -z^{n+1}B_\infty & \bar{A}_\infty \end{pmatrix} \begin{pmatrix} e_{out} \\ e_{in} \end{pmatrix}$$

or
$$\begin{pmatrix} u^{-n}\tilde{p}_n \\ \tilde{q}_n \end{pmatrix} = \begin{pmatrix} A_\infty & -z^{-n-1}\bar{B}_\infty \\ -z^{n+1}B_\infty & \bar{A}_\infty \end{pmatrix} \begin{pmatrix} e_{out} \\ e_{in} \end{pmatrix}$$

and so comparing with
$$\tilde{q}_n = (1-\beta_n)\, e_{in} - (z^n \alpha_n)\, e_{out}$$

we conclude
$$1-\beta_n = \bar{A}_\infty$$
$$\alpha_n = z B_\infty$$

So we see that knowledge of $\alpha_n, \beta_n$ is equivalent to knowledge of $T_{\infty,n}(z)$.

I have the hope now of showing that for $\hat{R} \in \ell^1$ we have $\{h_n\} \in \ell^1$. The idea is that I have good control over the convergence of $T_{\infty,n} \to I$ as $n \to \infty$ and hence ought to be able to control the $h_n$.

For example: suppose that all $h_n > 0$. Then because we know that [redacted] we have convergence when $z=1$

$$T_{\infty,0}(1) = \lim_{N\to\infty} T_{N0}(1) = \prod_{n=1}^{\infty} \underbrace{\theta(h_n)}_{\exp\left(t_n \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right)}$$

where $\sinh(t_n) = \frac{h_n}{k_n}$

we can conclude that $\{h_n\} \in \ell^1$.

September 18, 1978

The goal is to prove that if $R(z) \in (\ell^1)^\wedge$, then $\sum_{n=1}^{\infty} |h_n| < \infty$. The method is to use domination. Let $\bar{R} = \sum a_n z^n$, and put $\bar{R}^\# = \sum |a_n| z^n$, and let $h_n^\#$ be the Schur sequence corresponding to [redacted] $R^\#$.* Then I want to show that $|h_n| \le h_n^\#$ and that $\sum_{n=1}^{\infty} h_n^\# < \infty$.

Recall that we have

$$\begin{pmatrix} u^{-n}\tilde{p}_n \\ \tilde{q}_n \end{pmatrix} = \begin{pmatrix} 1-\bar{\beta}_n & -z^{-n}\bar{\alpha}_n \\ -z^n\alpha_n & 1-\beta_n \end{pmatrix} \begin{pmatrix} e_{out} \\ e_{in} \end{pmatrix}$$

* It might not follow that $|R^\#| < 1$, however $h_n^\#$ should be defined for large $n$.

where $\alpha_n \in zH_+$, $\beta_n \in z^{-1}H_-$ satisfy the equations

$$P_+\left(z^{-n}\bar{R}(1-\beta_n)\right) = \alpha_n.$$

$$-\beta_n = P_-(z^n R \alpha_n)$$

Denote [redacted] with #'s the corresponding things for $R^\#$. [redacted] If $f = \sum f_n z^n$, $g = \sum g_n z^n$ are Laurent series we introduce the domination relation

$$f \prec g \iff |f_n| \le g_n$$

(This implies $g_n \ge 0$.) We propose to show that $1-\beta_n \prec 1-\beta_n^\#$ and $\alpha_n \prec \alpha_n^\#$.

Can suppose $n=0$ and drop the subscript. [redacted] $1-\beta$ satisfies

$$(1 - P_- R P_+ \bar{R})(1-\beta) = 1$$

and it has the Neumann series solution

$$1-\beta = 1 + P_- R P_+ \bar{R}\cdot 1 + (P_- R P_+ \bar{R})^2 \cdot 1 + \cdots$$

<u>September 19, 1978</u>



$$\alpha = P_+\left(\bar{R}(1-\beta)\right)$$
$$-\beta = P_-(R\alpha)$$

so $\alpha = P_+(\bar{R}) + (P_+\bar{R}P_-R)\alpha$ has the Neumann series soln

$$\alpha = P_+(\bar{R}) + (P_+\bar{R}P_-R)P_+(\bar{R}) + (P_+\bar{R}P_-R)^2(P_+\bar{R}) + \cdots$$

Suppose that $\bar{R} = \sum_{n\in\mathbb{Z}}' a_n z^n$. Then $P_+\bar{R} = \sum_{n\geq 1}' a_n z^n$.

Recall that

$$f \in H_- \implies P_+(\bar{R}f) = P_+\left((P_+\bar{R})f\right) + P_+\big(\underbrace{(\bar{R}-P_+\bar{R})}_{\in H_-}f\big)$$
$$= P_+\left((P_+\bar{R})f\right)$$

This calculation is completely formal:

$$P_+(\bar{R}f) = P_+\left(\sum' a_n z^n \cdot \sum f_n z^n\right)$$
$$= P_+\Big(\sum'\big(\sum_{i+j=n}' a_i f_j\big)\, z^n\Big) = \sum_{n>0}'\big(\sum_{i+j=n} a_i f_j\big) z^n$$

and if we have $f_n = 0$ for $n>0$ then we only use $a_i$, $i>0$. Similarly

$$f \in zH_- \implies P_+(\bar{R}f) = P_+\left(P_+(z^{-1}\bar{R})\, zf\right)$$

depends only on $zP_+(z^{-1}\bar{R}) = \sum_{n\geq 2}' a_n z^n$. From the Neumann series this tells me that

degree 1 coeff of $\alpha$ = $a_1$ + functions of $a_1, a_2, \ldots$

which is meaningless. However it is <u>linear</u> in $a_1$.

Let's use the $l^1$ norm:

$$\|f\| = \sum |f_n|$$

Then clearly we have for $f \in z^{-1}H_-$

$$\|P_+(\bar{R}f)\| = \|P_+((zP_+z^{-1}\bar{R})\cdot f)\| \leq \|zP_+z^{-1}\bar{R}\|\cdot\|f\|$$

and similarly for the operator $P_-R$. So provided

$$\|zP_+z^{-1}\bar{R}\| = \sum_{n\geq 2} |a_n| < 1$$

the Neumann series for $\alpha, \beta$ will converge in the $l^1$ norm. The same will be true for $\alpha^\#, \beta^\#$.

Next note that for any $f, g$ with $f \prec g$

$$P_+\bar{R}f = \sum_{n>0}\Big(\sum_{i+j=n} a_i f_j\Big)z^n \prec \sum_{n>0}\Big(\sum_{i+j=n} |a_i||f_j|\Big)z^n$$

$$\prec \sum_{n>0}\Big(\sum_{i+j=n} |a_i|\, g_j\Big)z^n = P_+\overline{R^\#}g$$

Hence compare terms of Neumann series:

$$P_+\bar{R} \prec P_+\overline{R^\#}$$

$$P_-(R\,(P_+\bar{R})) \prec P_-R^\# P_+\overline{R^\#}$$

etc, so we conclude therefore that

$$\alpha \prec \alpha^\#$$

$$1-\beta \prec 1-\beta^\#$$

Finally we relate the leading coeff. of $\alpha_n$ to $h_{n+1}$. The basic recursion relation yields

$$\begin{pmatrix} 1-\bar\beta_n & -z^{-n}\bar\alpha_n \\ -z^n\alpha_n & 1-\beta_n \end{pmatrix}\begin{pmatrix} e_{out} \\ e_{in} \end{pmatrix} = \begin{pmatrix} 1 & -h_{n+1}z^{-n-1} \\ -\bar h_{n+1} z^{n+1} & 1 \end{pmatrix}\begin{pmatrix} 1-\bar\beta_{n+1} & -z^{-n-1}\bar\alpha_{n+1} \\ -z^{n+1}\alpha_{n+1} & 1-\beta_{n+1} \end{pmatrix}\begin{pmatrix} e_{out} \\ e_{in} \end{pmatrix}$$

which in view of the uniqueness of the equations defining the $\alpha_n, \beta_n$ give

$$-z^n\alpha_n = -\bar h_{n+1} z^{n+1}(1-\bar\beta_{n+1}) - z^{n+1}\alpha_{n+1}$$

or

$$\alpha_n = z\left(\alpha_{n+1} + \bar h_{n+1}(1-\bar\beta_{n+1})\right)$$

which shows that

$$\boxed{\alpha_n = \bar h_{n+1} z + O(z^2)}$$

Therefore we conclude that $|h_{n+1}| \le h^{\#}_{n+1}$.

The last step will be to show $\sum h_n^{\#} < \infty$, so we can suppose $R = R^{\#}$ i.e. all $a_n \ge 0$. From the Neumann series we get estimates showing that $\alpha_n \to 0$, $1-\beta_n \to 1$ in $l^1$ norm as $n \to \infty$. Put

$$\tilde T_{n\infty} = \begin{pmatrix} 1-\bar\beta_n & -z^{-n}\bar\alpha_n \\ -z^n\alpha_n & 1-\beta_n \end{pmatrix}$$

so that we have

$$\tilde T_{n\infty} = \begin{pmatrix} 1 & -h_{n+1}z^{-n-1} \\ -h_{n+1}z^{n+1} & 1 \end{pmatrix} \tilde T_{n+1,\infty}$$

Recall

$$\exp t\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{pmatrix}$$

and let $t_n$ be such that $\sinh t_n = \frac{h_n}{k_n}$ $\cosh t_n = \frac{1}{k_n}$.

Then we have

$$\tilde{T}_{n\infty}(1) = e^{-t_{n+1}\begin{pmatrix}0&1\\1&0\end{pmatrix}} T_{n+1,\infty}(1)\, k_{n+1}$$

or

$$\underbrace{T_{0n}(1)}\, \tilde{T}_{n\infty}(1) = \tilde{T}_{0\infty}(1)\; k_1 \cdots k_n$$

$$e^{t_1\begin{pmatrix}0&1\\1&0\end{pmatrix}} \cdots e^{t_n\begin{pmatrix}0&1\\1&0\end{pmatrix}} = e^{\left(\sum_{i=1}^{n} t_i\right)\begin{pmatrix}0&1\\1&0\end{pmatrix}}$$

Now the $t_i$'s are $\geqslant 0$, hence $\sum^{\infty} t_i$ is finite or infinite. All $k_n \leq 1$, hence the right side is bounded. Since $\tilde{T}_{n\infty}(1) \to I$ as $n \to \infty$, we conclude that $T_{0n}(1)$ remains bounded. So $\sum t_i < \infty$ and hence $t_i \to 0$ so that $h_i/t_i \to 1$ and hence $\sum h_i < \infty$.

<u>Summary</u>. We have established a 1-1 correspondence between functions $R(z)$ on $S^1$ with $|R(z)| \leq 1$ and with abs. convergent Fourier series, on one hand and with Schur sequences $\{h_n\}$ having $\sum_{n>>0} |h_n| < \infty$ on the other.

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Next problem is to see if I can recognize when $R$ comes from an $\ell^1$ sequence. Recall that when scattering holds in both directions (I think it suffices that $\{h_n\} \in \ell^2$, but certainly if $\{h_n\} \in \ell^1$) we have

$$\begin{pmatrix} e_{in}^- \\ e_{out}^- \end{pmatrix} = \underbrace{\begin{pmatrix} \frac{1}{T} & -\frac{R}{T} \\ -\frac{\bar{R}}{\bar{T}} & \frac{1}{\bar{T}} \end{pmatrix}}_{T_{-\infty,\infty}} \begin{pmatrix} e_{out} \\ e_{in} \end{pmatrix}$$

from which one derives the scattering matrix

$$T e_{in}^{-} = e_{out} - R e_{in} \qquad\qquad e_{out} = R e_{in} + T e_{in}^{-}$$

$$e_{out}^{-} = -\frac{\bar{R}}{\bar{T}}\left(R e_{in} + T e_{in}^{-}\right) + \frac{1}{\bar{T}} e_{in} = \overbrace{\frac{1-|R|^2}{\bar{T}}}^{=T} e_{in} - \frac{\bar{R}}{\bar{T}} T e_{in}^{-}$$

so

$$\begin{pmatrix} e_{out} \\ e_{out}^{-} \end{pmatrix} = \begin{pmatrix} R & T \\ T & -\frac{\bar{R}}{\bar{T}} T \end{pmatrix} \begin{pmatrix} e_{in} \\ e_{in}^{-} \end{pmatrix}$$

It might [redacted] be the case that [redacted] if the reflection coefficient for $e_{out}^{-}, e_{in}^{-}$ namely $-\frac{\bar{R}}{\bar{T}} T$ has $\ell^1$ Fourier transform, then $h_n \in \ell^1$ as $n \to -\infty$.

September 21, 1978

Suppose $R \in (\ell^1)^\wedge$ and $|R| < 1$ on $S^1$. I want to show that the Schur sequence $\{h_n\}$ associated to $R$ is in $\ell^1$. We know it is summable as $n \to +\infty$. Because $|R| \leq 1-\varepsilon$ on $S^1$ we know that

$$(\text{out}, \text{in}) : \mathcal{H} \longrightarrow L^2 \times L^2$$

is bijective, that $p_n, q_n$ exist for all $n$. If we define $T(z)$ by

$$T(z) = \exp\left\{ \int \frac{\zeta + z}{\zeta - z} \, \tfrac{1}{2} \log\left(1 - |R|^2\right) d\theta/2\pi \right\}$$

then we know that

$$\lim_{n \to \infty} \begin{pmatrix} U^{-n} p_n \\ q_n \end{pmatrix} = \begin{pmatrix} e_{in}^- \\ e_{out}^- \end{pmatrix} = \begin{pmatrix} \frac{1}{T} & -\frac{R}{T} \\ -\frac{\bar{R}}{\bar{T}} & \frac{1}{\bar{T}} \end{pmatrix} \begin{pmatrix} e_{out} \\ e_{in} \end{pmatrix}.$$

Justification: We know that $|T|^2 = 1 - |R|^2$ on $S^1$, hence $T, T^{-1}$ are bounded analytic in the disk. Look at

$$e_{out}^- = \frac{1}{\bar{T}} e_{in} - \frac{\bar{R}}{\bar{T}} e_{out}$$

and multiply by $\bar{T}(0)$ to get

$$\bar{T}(0) e_{out}^- = \underbrace{\frac{\bar{T}(0)}{\bar{T}}}_{\in 1 - z^{-1} H_-} e_{in} - \underbrace{\frac{\bar{T}(0) \bar{R}}{\bar{T}}}_{\in L^2} e_{out}$$

Notice also that $e_{out}^-$ is $\perp$ to $z^n e_{out}$ for all $n$ since $\text{out}(e_{out}^-) = 0$, and to $z^n e_{in}$ for $n < 0$. Therefore we see that $\bar{T}(0)e_{out}$ is the orthogonal projection of $e_{in}$ onto $\mathcal{H}_{-\infty} = (\text{out}, \text{in})^{-1}(0 \times H_+)$. So $\tilde{q}_{-\infty} \neq 0$ which we've seen implies $\lim_{n\to-\infty} q_n$ exists.

~~Another proof. If $\tilde{q}_{-\infty} = 0$, then the $U^{-1}$-invariant space generated by the $p_n, q_n$ is orthogonal to $\mathcal{H}_{-\infty}$~~

Suppose that we show the reflection coefficient $-\frac{\bar{R}}{\bar{T}}T$ relative to $\bar{e}_{out}, e_{in}$ has abs. conv. Fourier series. This follows from Wiener's theory which says that if $f \in \ell^{1\wedge}$ and $\Phi(z)$ is analytic on $f(S^1)$, then $\Phi \circ f \in \ell^{1\wedge}$.

So from $R \in \ell^{1\wedge} \Rightarrow \bar{R} \in \ell^{1\wedge} \Rightarrow |R|^2 \in \ell^{1\wedge}$ and $|R| \leq 1-\varepsilon \Rightarrow$ range $|R|^2$ is in the analyticity region for $\log(1-z) \Rightarrow \log(1-|R|^2) \in \ell^{1\wedge}$.

Now
$$f \longmapsto \int \frac{\zeta+z}{\zeta-z}\frac{1}{2}f\,\frac{d\theta}{2\pi} = \int\left(\frac{1}{2} + z\zeta^{-1} + z^2\zeta^{-2} + \cdots\right) f\,\frac{d\theta}{2\pi}$$
is essentially the $P_+$ operator which preserves $\ell^{1\wedge}$. Then exp preserves $\ell^{1\wedge}$ and so we conclude $T, T^{-1} \in \ell^{1\wedge}$, etc.

The final thing is to show $p_n, q_n$ are exactly what we would expect if we started with the pair $e_{in}, \bar{e}_{out}$ and did orthogonal projection. But we have
$$\begin{pmatrix} U^{-n}\tilde{p}_n \\ \tilde{q}_n \end{pmatrix} = \begin{pmatrix} 1-\bar{\beta}_n & -z_1^{-n}\bar{\alpha}_n \\ -z^n\alpha_n & 1-\beta_n \end{pmatrix}\begin{pmatrix} e_{out} \\ e_{in} \end{pmatrix}$$

$$= \begin{pmatrix} 1-\bar\beta_n & -\bar z^n \bar\alpha_n \\ -z^n\alpha_n & 1-\beta_n \end{pmatrix} \begin{pmatrix} \frac{1}{\bar T} & \frac{R}{T} \\ \frac{\bar R}{\bar T} & \frac{1}{T} \end{pmatrix} \begin{pmatrix} e\text{[redacted]}_{in} \\ e^-_{out} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1-\bar\beta_n - z^{-n}\bar R\bar\alpha_n}{\bar T} & \frac{(1-\bar\beta_n)R - \bar z^n\bar\alpha_n}{T} \\ \frac{(1-\beta_n)\bar R - z^n\alpha_n}{\bar T} & \frac{1-\beta_n - z^n R\alpha_n}{T} \end{pmatrix} \begin{pmatrix} e^-_{in} \\ e^-_{out} \end{pmatrix}$$

But recall that

$$in(\tilde g_n) = in((1-\beta_n)e_{in} - z^n\alpha_n e_{out}) = (1-\beta_n) - z^n R\alpha_n \in H_+$$

$$out(\tilde g_n) = (1-\beta_n)\bar R - z^n\alpha_n \in z^n H_-$$

Hence we see that

$$\boxed{\tilde g_n = \gamma_n e^-_{out} + z^n\delta_n e\text{[redacted]}\,in \qquad \text{with } \gamma_n \in H_+,\ \delta_n \in H_-}$$

Recall :

$$L^2(S^1) \xleftarrow{out^-} \mathcal{H} \xrightarrow{out} L^2(S^1)$$

$$\begin{array}{ccc} f e_{out} + g e_{in} & \longmapsto & f + g\bar R \\ \shortparallel & & \\ (f+g\bar R)e_{out} + g(e_{in} - \bar R e_{out}) & & \end{array}$$

$$\begin{array}{ccc} 1 & \longmapsto & \frac{e_{in} - \bar R e_{out}}{\bar T} \\ g\bar T & \longleftarrow\!\!\shortmid & f e_{out} + g e_{in} \end{array}$$

Thus

$$out^-(f e_{out} + g e_{in}) = g\bar T$$

$$in^-(f e_{out} + g e_{in}) = f T$$

so one sees that

$$out^{-}(\tilde{q}_n) = (1-\beta_n)\bar{T} \quad \in H_-$$
$$in^{-}(\tilde{q}_n) = -z^n\alpha_n T \quad \in z^{n+1}H_+$$

and consequently $\tilde{q}_n \perp (zH_+)e^-_{out} + (z^nH_-)e^-_{in}$
But
$$\tilde{q}_n = \gamma_n(0)\,e^-_{out} + \underbrace{(\gamma_n - \gamma_n(0))\,e^-_{out}}_{\in zH_+} + \underbrace{(z^n\delta_n)\,e^-_{in}}_{\in z^nH_-}.$$

* Hence we conclude that $\gamma_n(0) \neq 0$ (for otherwise $\tilde{q}_n \perp \tilde{q}_n$ ~~and~~ $\tilde{q}_n = 0$), and hence $\tilde{q}_n/\gamma_n(0)$ is the orthogonal projection of $e^-_{out}$ for the other direction filtration. Hence $q_n$ and similarly $p_n$ is obtainable by orthogonal projection from the reflection coefficient $-\frac{\bar{R}}{\bar{T}}T$) and so we can conclude $h_n \in \ell'$ as $n \to -\infty$.

* The above argument omits showing that $\gamma_n(0) > 0$. But we have

$$\begin{pmatrix} 1 & -h_nU^{-n} \\ -\bar{h}_nU^n & 1 \end{pmatrix}\begin{pmatrix} U^{-n}\tilde{p}_n \\ \tilde{q}_n \end{pmatrix} = \begin{pmatrix} 1 & -h_nU^{-n} \\ -\bar{h}_nU^n & 1 \end{pmatrix}\begin{pmatrix} \bar{\gamma}_n & z^{-n}\bar{\delta}_n \\ z^n\delta_n & \gamma_n \end{pmatrix}\begin{pmatrix} e^-_{in} \\ e^-_{out} \end{pmatrix}$$

so
$$z^{n-1}\delta_{n-1} = -z^n\bar{h}_n\bar{\gamma}_n + z^n\delta_n \qquad z^{-1}\delta_{n-1} = \delta_n - \bar{h}_n\bar{\gamma}_n$$
$$\gamma_{n-1} = -\bar{h}_n\bar{\delta}_n + \gamma_n$$

Hence $h_n\gamma_n(0) = \bar{\delta}_n(0)$ and $\gamma_{n-1}(0) = -\bar{h}_nh_n\gamma_n(0) + \gamma_n(0)$.
Thus
$$\gamma_{n-1}(0) = (1-|h_n|^2)\,\gamma_n(0)$$

As $n \to \infty$ clearly $\gamma_n \to \frac{1}{T}$ and since $T(0) > 0$ we can conclude $\gamma_n(0) > 0$.

Suppose we start with an $R$ with $|R| \le 1-\varepsilon$ on $S^1$. In this case we know that the maps

$$L^2 \times L^2 \longrightarrow \mathcal{H} \xrightarrow{(out, in)} L^2 \times L^2$$
$$f, g \longmapsto (f e_{out} + g e_{in}) \longmapsto (f + g\bar{R}, fR + g)$$

are bijective, in fact, topological isomorphisms. Consequently we know that the subspace

$$(z^{n+1} H_+) e_{out} + (z^{-1} H_-) e_{in}$$

is closed, hence that $\exists!\, z^n\alpha_n \in z^{n+1} H_+$, $\beta_n \in z^{-1}H_-$ such that

$$e_{in} - (\beta_n e_{in} + z^n \alpha_n e_{out}) \in \left[(z^{n+1}H_+)e_{out} + (z^{-1}H_-)\right]^{\perp} = \mathcal{H}_n$$

We also get this from the standard equations for $\alpha_n, \beta_n$ since the Neumann series converges as $\|R\|_\infty \le 1-\varepsilon$. Since it is clear that $\bigcup \mathcal{H}_n$ is dense in $\mathcal{H}$ we have

$$\tilde{g}_n \longrightarrow e_{in} \quad \text{as } n \to \infty.$$

Next from

$$T(z) = \exp\left\{ \int \frac{\zeta + z}{\zeta - z} \frac{1}{2} \ln(1 - |R|^2)\, d\theta/2\pi \right\}$$

and $|T|^2 = 1 - |R|^2$ we know that

$$T, \frac{1}{T} \quad \text{are bdd analytic in } |z| < 1$$

but more importantly $T, \frac{1}{T} \in H^\infty_+$ so they act on $L^2$. Consequently the matrix appearing in

$$\begin{pmatrix} e_{in}^{-} \\ e_{out}^{-} \end{pmatrix} = \begin{pmatrix} \frac{1}{T} & -\frac{R}{T} \\ -\frac{\bar{R}}{\bar{T}} & \frac{1}{\bar{T}} \end{pmatrix} \begin{pmatrix} e_{out} \\ e_{in} \end{pmatrix}$$

is invertible in the $L^\infty$ sense, hence we can deduce as before that

$$\tilde{q}_n = \text{[redacted]} \ \gamma_n e_{out}^{-} + z^n \delta_n e_{in}^{-} \qquad \begin{matrix} \gamma_n \in H_+ \\ \delta_n \in H_- \end{matrix} ,$$

and so conclude that $q_n$ results from projection at the other end.

<u>Summary</u>: If $\|R\|_\infty < 1$, then the inverse problem has a canonical solution which is the same from either end. <u>Question</u>: Is this the only Schur sequence with reflection coefficient $R$?

September 23, 1978.

Forward problem shows that given a Schur sequence (by this I mean a sequence $\{h_n\}$ defined on a segment of $\mathbb{Z}$ such that $|h_n|<1$ in the interior and $|h_n|=1$ at the ends of the segment) which is $l^2$ as $n\to\infty$, then we get a reflection coeff. $R(z)$ which is an element of $L^\infty(S^1)$ of norm $\leq 1$. The inverse problem in the largest sense consists of [redacted] finding all Schur sequences [redacted] belonging to a given reflection coefficient $R$.

To get some idea of what happens, let's restrict attention to Schur sequences $\{h_n\}_{n\geq 0}$ with $h_0=1$ and [redacted] $\sum' |h_n|^2<\infty$. We already have a 1-1 correspondence between $\{h_n\}_{n\geq 0}$ with $h_0=1$ and probability measures $d\nu$ on $S^1$. The condition $\sum'|h_n|^2<\infty$ means that $\frac{d\nu}{d\theta}=\frac{1}{|\varphi|^2}$ where $\frac{1}{\varphi}$ is an outer function in $H_+$ of norm $\leq 1$. [redacted] This is equivalent to $\ln\frac{d\nu}{d\theta}\in L^1$ for one knows that

$$\frac{1}{\varphi(z)}=\exp\left\{\int\frac{\zeta+z}{\zeta-z}\left(\frac{1}{2}\ln\frac{d\nu}{d\theta}\right)d\theta/2\pi\right\}$$

The corresponding reflection coeff. is

$$R=\frac{\bar\varphi}{\varphi}.$$

~~So the idea seems to be that one considers~~
~~$\varphi \geq 0$ with $\int \varphi \frac{d\theta}{2\pi} = 1$ and $\ln\varphi \in L^1$~~

So what happens is that if we restrict to measures $d\nu$ abs. cont. wrt. $d\theta$, then we look for $\frac{1}{\varphi} \in H_+ \cap RH_-$ satisfying the "reality" condition

$$R\overline{\frac{1}{\varphi}} = \frac{1}{\varphi}$$

+ which are outer. Then $d\nu = \frac{1}{|\varphi|^2}\frac{d\theta}{2\pi}$ after $\frac{1}{\varphi}$ is normalized [redacted] to have norm 1.

Example: Suppose $S$ analytic on $S^1$ and $|S|=1$ and that $S$ has degree 0, whence we know [redacted] $H_+ \cap SH_-$ is spanned by [redacted] an outer function $\frac{1}{\psi}$ with

$$S = \frac{\bar{\psi}}{\psi}$$

$\frac{1}{\psi}$ is unique, up to sign, if [redacted] it is required to have norm 1. Put $R = z^n S$. Then $H_+ \cap RH_-$ is spanned by $\frac{1}{\psi}, \frac{z}{\psi}, \ldots, \frac{z^n}{\psi}$ and we get required $\frac{1}{\varphi}$ of the form $\frac{p}{\psi}$ where $p$ is a poly. of degree $n$ such that $z^n\bar{p} = p$ and whose zeroes lie on $S^1$, e.g.

$$p(z) = (\bar{\zeta}_1^{-1} z + \zeta_1)\cdots(\bar{\zeta}_n^{-1} z + \zeta_n)$$

This example extends to the case where $S$ is continuous and $H_+ \cap SH_-$ is one-dimensional, which I think always occurs if $S \in \ell^1{}^\wedge$. (Because $\ln S$, which is nicely defined as $\deg S = 0$, should be in $\ell^1{}^\wedge$ by Wiener, hence $\ln S$ is the imaginary part of the bdry values of a bdd analytic function in $|z|<1$, so $S = \frac{\bar{\varphi}}{\varphi}$ where $\varphi, \frac{1}{\varphi} \in H_\infty$.)

Problem: Suppose $R: S^1 \to S^1$ analytic of degree 0, e.g. take $R(z) = 1$. Then we have a unique $\ell^1$ system $\{h_n\}$ with reflection coefficient $R$, ~~[illegible]~~ and moreover it starts in degree $n = 0$: $|h_0| = 1$, in fact $h_0$ is related to

$$\exp \int \log S \; d\theta/2\pi .$$

On the other hand we have seen how to produce lots of Schur systems with reflection coeff. $R$ which start in degree $n > 0$. Question: Does any Schur system belonging to $R$ have to have a beginning? Can I rule out infintely many $h_n$ as $n \to -\infty$?

<u>September 24, 1978:</u>

Regard a Schur system as obtained by coupling two ports together.

$$\begin{pmatrix} e_{in}^- \\ e_{out}^- \end{pmatrix} = \underbrace{\begin{pmatrix} \frac{1}{T} & -\frac{R}{T} \\ -\overline{\left(\frac{R}{T}\right)} & \overline{\left(\frac{1}{T}\right)} \end{pmatrix}}_{T_{-\infty,\infty} \;=\; T_{-\infty,0}\cdot T_{0\infty}} \begin{pmatrix} e_{out} \\ e_{in} \end{pmatrix}$$

Let
$$T_{-\infty,0} = \begin{pmatrix} \frac{1}{T_-} & -\frac{R_-}{T_-} \\ -\left(\frac{R_-}{T_-}\right)^- & \overline{\frac{1}{T_-}} \end{pmatrix}$$

Here $R_-$ denotes the reflection coefficient for the port containing $p_n, q_n$ for $n \leq 0$. We have

$$T_{0,\infty} = \begin{pmatrix} \left(\frac{1}{T_-}\right)^- & +\frac{R_-}{T_-} \\ +\left(\frac{R_-}{T_-}\right)^- & \frac{1}{T_-} \end{pmatrix} = \Theta(h_0)\,\Theta(h_{-1}z)\,\Theta(h_{-2}z^2)\cdots\cdots$$

so that $R(z) = T_{0,-\infty}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \Theta(h_0)\begin{pmatrix} z & 0 \\ 0 & 1 \end{pmatrix}\Theta(h_{-1})\begin{pmatrix} z & 0 \\ 0 & 1 \end{pmatrix}\cdots\cdots$

We have

$$T_{0,\infty} = \Theta(-h_1 z^{-1})\,\Theta(-h_2 z^{-2})\cdots\cdots\cdots = \begin{pmatrix} A & \bar{B} \\ B & \bar{A} \end{pmatrix}$$

where $A, B$ are analytic for $|z| < 1$, $A(0) > 0$, $B(0) = 0$, and $|A|^2 - |B|^2 = 1$ on $S^1$.

Recall that we can interpret $p_n, q_n$ $n \geq 0$ as coming from a port. Specifically one writes the recursion relations in the form

$$\begin{pmatrix} q_0 \\ p_0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & z^{-1} \end{pmatrix} \theta(-\bar{h}_1) \begin{pmatrix} 1 & 0 \\ 0 & z^{-1} \end{pmatrix} \theta(-\bar{h}_2) \cdots\cdots \theta(-\bar{h}_n) \begin{pmatrix} q_n \\ u^{-n} p_n \end{pmatrix}$$

and the reflection coefficient is

$$R_+(z) = z\,\theta(-\bar{h}_1)\, z\, \theta(-\bar{h}_2) \cdots\cdots$$

One has

$$\begin{pmatrix} 1 & 0 \\ 0 & z^{-1} \end{pmatrix} \theta(-\bar{h}_1) \begin{pmatrix} 1 & 0 \\ 0 & z^{-1} \end{pmatrix} \cdots = \text{flip } T_{0,\infty}$$

where flip means conjugation by $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Hence

$$\text{flip}(T_{0,\infty}) = \begin{pmatrix} \frac{1}{T^+} & \frac{R_+}{T_+} \\ \frac{R_+}{T_+} & \frac{1}{T_+} \end{pmatrix}$$

and so $$T_{0,\infty} = \begin{pmatrix} \frac{1}{T_+} & \left(\frac{R_+}{T_+}\right)^- \\ \frac{R_+}{T_+} & \left(\frac{1}{T_+}\right)^- \end{pmatrix}$$

$\therefore\ A = \frac{1}{T_+}$

$B = \frac{R_+}{T_+}$

Up to now I been somewhat sloppy about these matrices. However the point is that we already know that a port with scattering corresponds to an analytic function $R(z)$ of modulus $<1$ in $|z|<1$ such that $\ln(1-|R|^2) \in L^1(S^1)$. Therefore when $R_+$ and $R_-$ are given (and $R_+(0)=0$) we get a Schur system with scattering in both directions.

Next we know we have

$$T_{-\infty,\infty} = \begin{pmatrix} \frac{1}{T} & -\frac{R}{T} \\ -\left(\frac{R}{T}\right)^- & \left(\frac{1}{T}\right)^- \end{pmatrix} = T_{-\infty,0} \cdot T_{0,\infty} = \begin{pmatrix} \frac{1}{T_-} & -\frac{R_-}{T_-} \\ -\left(\frac{R_-}{T_-}\right)^- & \left(\frac{1}{T_-}\right)^- \end{pmatrix} \begin{pmatrix} A & \bar{B} \\ B & \bar{A} \end{pmatrix}$$

Let me suppose now that $R$ is given and is nice say analytic and of modulus $<1$ on $S^1$. Let us look at possible $R_+, R_-$ yielding $R$. Solve for $R_-$

$$T_{0,-\infty} = \begin{pmatrix} \left(\frac{1}{T_-}\right)^- & \frac{R_-}{T_-} \\ \left(\frac{R_-}{T_-}\right)^- & \frac{1}{T_-} \end{pmatrix} = \begin{pmatrix} A & \bar{B} \\ B & \bar{A} \end{pmatrix} \begin{pmatrix} \left(\frac{1}{T}\right)^- & \frac{R}{T} \\ \left(\frac{R}{T}\right)^- & \frac{1}{T} \end{pmatrix}$$

$$\begin{cases} T\dfrac{1}{T_-} = BR + \bar{A} \\[2ex] T\dfrac{R_-}{T_-} = AR + \bar{B} \end{cases}$$

Let's try to use these equations to prove uniqueness.

<u>Example:</u> $R = 0$, whence we want

$$\frac{1}{T_-} = \overline{\left(\frac{1}{T_+}\right)} \qquad \frac{R_-}{T_-} = \overline{\left(\frac{R_+}{T_+}\right)}$$

This is to be interpreted as a.e. on $S^1$. This forces $\overline{T_+} = T_-$ a.e. But $T_+, T_-$ are in $L^\infty$ so this can happen only when $T_+, T_-$ are constant, hence also $R_\pm$, whence since $R_+(0) = 0$, we must have $R_\pm = 0$.

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Suppose $\mathcal{H}$ is a Hilbert space with unitary operator $U$ and $\mathcal{H}_0$ is a subspace. The biinvariant subspace of $\mathcal{H}$ spanned by $\mathcal{H}_0$ ~~is determined by~~ the sequence of contraction operators

$$T_n = i^* U^n i$$

where $i : \mathcal{H}_0 \hookrightarrow \mathcal{H}$ is the embedding. In effect one obtains the biinvariant subspace by completing $\bigoplus_n U^n \mathcal{H}_0$ wrt the inner product for which

$$(U^m h_1, U^{m'} h_2) = (T_{m-m'} h_1, h_2)$$

Conversely given a sequence of contraction operators $T_n$, $n \in \mathbb{Z}$ $T_{-n} = T_n^*$, $T_0 = I$ on $\mathcal{H}_0$ when can we tell when they come from a triple $(\mathcal{H}, U, i)$? Form the operator-valued function

$$\Theta(z) = I + 2\sum_{n=1}^{\infty} z^n T_n \qquad (\text{converges for } |z|<1)$$

so that

$$Re\,\Theta(z) = \sum_{n=-1}^{\infty} (\bar{z})^{-n} T_n + I + \sum_{n=1}^{\infty} z^n T_n$$

If the $T_n$ come from $(\mathcal{H}, U, i)$ then

$$Re\,\Theta(z) = \sum_{n \geq 0}{}' \bar{z}^n i^* U^{-n} i + \sum_{n \geq 1}{}' z^n i^* U^n i$$

$$= i^* \left\{ (1 - \bar{z}U^{-1})^{-1} + zU(1 - zU)^{-1} \right\} i$$

$$= i^* (1 - \bar{z}U^{-1})^{-1} \left\{ (1 - zU) + (1 - \bar{z}U^{-1}) zU \right\} (1 - zU)^{-1} i$$

$$= (1 - |z|^2) \cdot \; i^* (1 - \bar{z}U^{-1})^{-1} (1 - zU)^{-1} i \;\; \geq 0$$

This necessary condition is also sufficient. To see one can replace $T_n$ by $r^{|n|} T_n$ and let $r \uparrow 1$, hence we can suppose $\|T_n\| \leq r^{|n|}$, so these series make sense for $|z| = 1$. So by passing to the boundary one has

$$Re\,\Theta(z) = \sum_{n \in \mathbb{Z}}{}' z^n T_n \;\; \geq 0 \qquad \text{for } |z| = 1.$$

Let $f(z) = \sum{}' f_n z^n$

$\in \bigoplus_n z^n \mathcal{H}_0$

Then

$$(f,f) = \sum_{m,n}{}' (z^m f_m, z^n f_n) = \sum_{m,n}{}' (T_{m-n} f_m, f_n)_{\mathcal{H}_0}$$

$$= \int \Big( \sum_{p \in \mathbb{Z}} z^{-p} T_p \sum_m{}' z^m f_m, \sum_n{}' z^n f_n \Big)_{\mathcal{H}_0} \, d\theta/2\pi \geq 0$$

I see that above I should have had

$$\Theta(z) = I + 2 \sum_{n=1}^{\infty} z^n T_n^*$$

so a few minor changes are required.

September 25, 1978

[redacted] Problem: To calculate the matrix spectral measure for a Schur system [redacted] in terms of the reflection coefficients $R_+, R_-$. Suppose to simplify that the system is $\ell^1$.

[redacted] The measure we are after is the matrix measure $d\nu$ such that

$$\| f(u) p_0 + g(u) q_0 \|^2 = \int \begin{pmatrix} f \\ g \end{pmatrix}^* d\nu \begin{pmatrix} f \\ g \end{pmatrix}$$

So for the pair $e_{out}, e_{in}$ instead of $p_0, q_0$ we get

$$\| f e_{out} + g e_{in} \|^2 = \| (e_{out}, e_{in}) \begin{pmatrix} f \\ g \end{pmatrix} \|^2$$

Actually it would be better to write $i_{out}(f)$ instead of $f e_{out}$, for then

$$\left( (i_{out}, i_{in}) \begin{pmatrix} f \\ g \end{pmatrix}, (i_{out}, i_{in}) \begin{pmatrix} f \\ g \end{pmatrix} \right)$$

$$= \left( (i_{out}, i_{in})^* (i_{out}, i_{in}) \begin{pmatrix} f \\ g \end{pmatrix}, \begin{pmatrix} f \\ g \end{pmatrix} \right)$$

$$= \left( \begin{pmatrix} i_{out}^* i_{out} & i_{out}^* i_{in} \\ i_{in}^* i_{out} & i_{in}^* i_{in} \end{pmatrix} \begin{pmatrix} f \\ g \end{pmatrix}, \begin{pmatrix} f \\ g \end{pmatrix} \right)$$

$$= \int \begin{pmatrix} f \\ g \end{pmatrix}^* \begin{pmatrix} 1 & \bar{R} \\ R & 1 \end{pmatrix} \begin{pmatrix} f \\ g \end{pmatrix} d\theta/2\pi$$

so the measure for this pair is $\begin{pmatrix} 1 & \bar{R} \\ R & 1 \end{pmatrix} d\theta/2\pi$

We have

$$\begin{pmatrix}p_0\\ q_0\end{pmatrix} = \begin{pmatrix}A & \bar{B}\\ B & \bar{A}\end{pmatrix}\begin{pmatrix}e_{out}\\ e_{in}\end{pmatrix} \qquad A = \frac{1}{T_+} \quad B = \frac{R_+}{T_+}$$

$$\begin{pmatrix}\bar{e}_{in}\\ \bar{e}_{out}\end{pmatrix} = \begin{pmatrix}\frac{1}{T_-} & -\frac{R_-}{T_+}\\ \overline{-\frac{R_-}{T_-}} & \overline{\frac{1}{T_-}}\end{pmatrix}\begin{pmatrix}p_0\\ q_0\end{pmatrix}$$

or

$$\begin{pmatrix}p_0\\ q_0\end{pmatrix} = \begin{pmatrix}\left(\frac{1}{T_-}\right)^- & \frac{R_-}{T_-}\\ \left(\frac{R_-}{T_-}\right)^- & \frac{1}{T_-}\end{pmatrix}\begin{pmatrix}e_{in}^-\\ e_{out}^-\end{pmatrix} \qquad \begin{pmatrix}\bar{C} & D\\ \bar{D} & C\end{pmatrix}$$

so that

$$\| f(u)p_0 + g(u)q_0 \|^2 = \left( (p_0, q_0)\begin{pmatrix}f\\ g\end{pmatrix}, (p_0, q_0)\begin{pmatrix}f\\ g\end{pmatrix} \right)$$

$$= \left( (e_{out}, e_{in})\begin{pmatrix}A & B\\ \bar{B} & \bar{A}\end{pmatrix}\begin{pmatrix}f\\ g\end{pmatrix}, \ (e_{in}^- \ e_{out}^-)\begin{pmatrix}\left(\frac{1}{T_-}\right)^- & \left(\frac{R_-}{T_+}\right)^-\\ \frac{R_-}{T_-} & \frac{1}{T_-}\end{pmatrix}\begin{pmatrix}f\\ g\end{pmatrix} \right)$$

(You should use the reverse order so that

$$\begin{pmatrix}out(e_{in}^-) & out(e_{out}^-)\\ in(e_{in}^-) & in(e_{out}^-)\end{pmatrix} = \begin{pmatrix}\bar{T} & 0\\ 0 & T\end{pmatrix}$$

so

$$dv = \begin{pmatrix}\bar{A} & B\\ \bar{B} & A\end{pmatrix}\begin{pmatrix}\bar{T} & 0\\ 0 & T\end{pmatrix}\begin{pmatrix}\bar{C} & \bar{D}\\ D & C\end{pmatrix}$$

$$= \begin{pmatrix}\overline{ATC} + BTD & \overline{ATD} + BTC\\ \overline{BTC} + ATD & \overline{BTD} + ATC\end{pmatrix}$$

$$A = \frac{1}{T_+}, \quad B = \frac{R_+}{T_+}, \quad C = \frac{1}{T_-}, \quad D = \frac{R_-}{T_-}$$

Also you need

$$\begin{pmatrix} \frac{1}{T} & -\frac{R}{T} \\ -\overline{\left(\frac{R}{T}\right)} & \overline{\left(\frac{1}{T}\right)} \end{pmatrix} = \begin{pmatrix} C & -D \\ -\overline{D} & \overline{C} \end{pmatrix} \overset{T_{0\infty}}{\begin{pmatrix} A & \overline{B} \\ B & \overline{A} \end{pmatrix}}$$

$$\frac{1}{T} = CA - DB = \frac{1}{T_-}\frac{1}{T_+} - \frac{R_-R_+}{T_-T_+} = \frac{1 - R_-R_+}{T_-T_+}$$

or

$$\boxed{T = \frac{T_-T_+}{1-R_-R_+}} \qquad \text{so}$$

$$\overline{ATC} + BTD = \overline{\frac{1}{T_+}\frac{T_-T_+}{1-R_-R_+}\frac{1}{T_-}} + \frac{R_+}{T_+}\frac{T_-T_+}{1-R_-R_+}\frac{R_-}{T_-}$$

$$= \frac{1}{1-\overline{R_-R_+}} + \frac{R_-R_+}{1-R_-R_+} = \frac{1 - |R_-R_+|^2}{|1-R_-R_+|^2}$$

$$\overline{ATD} + BTC = \overline{\frac{1}{T_+}\frac{T_-T_+}{1-R_-R_+}\frac{R_-}{T_-}} + \frac{R_+}{T_+}\frac{T_-T_+}{1-R_-R_+}\frac{1}{T_-}$$

$$= \frac{\overline{R_-}}{1-\overline{R_-R_+}} + \frac{R_+}{1-R_-R_+}$$

$$= \frac{(\overline{R_-} + R_+) - \overline{R_-}R_+(R_- + \overline{R_+})}{|1-R_-R_+|^2}$$

This leads to the

<u>Conjecture:</u> In general if $|R_-R_+| \leq 1-\varepsilon$, then the spectral measure is absolutely continuous wrt $d\theta$.

<u>Example</u>: Take $R_- = 1$. Then one gets

$$\overline{ATD} + BTC = \frac{1}{1-\overline{R_+}} + \frac{R_+}{1-R_+} = \frac{1-R_+ + R_+ - |R_+|^2}{|1-R_+|^2} = \frac{1-|R_+|^2}{|1-R_+|^2}$$

and the same is true for the other entries. This is consistent with our previous formula

$$d\nu = \frac{1-|zR|^2}{|1-zR|^2}\, d\theta/2\pi$$

in the case $p_0 = q_0$.

Next question consists of fixing $R$ and trying to show that $R_+, R_-$ are uniquely determined when $|R| \leq 1-\varepsilon$, if this is true. Formula

$$R = \frac{R_- - \overline{R_+}}{1-R_-R_+}\,\frac{T_+}{\overline{T_+}} \qquad -\frac{\overline{R}}{\overline{T}}\,T = \frac{R_+ - \overline{R_-}}{1-R_-R_+}\,\frac{T_-}{\overline{T_-}}$$

$$= \begin{pmatrix} \left(\frac{1}{T_+}\right)^{-} & -\left(\frac{R_+}{T_+}\right)^{-} \\ -\frac{R_+}{T_+} & \frac{1}{T_+} \end{pmatrix} (R_-)$$