August 1, 1978: Press $W, W_x$, also $px, x$ for Dirac system

Let's consider a Dirac system $\mathbf{x}(u) = (i_k h x)(u)$ with $h$ real, smooth of compact support:

$$\frac{d}{dx}(u_1 + u_2) = (\frac{h}{i} - i_k \frac{h}{x})(u_1 + u_2)$$

$$\frac{d}{dx}(u_1 - u_2) = (i_k h \frac{h}{x})(u_1 - u_2)$$

Solutions of the corresponding wave equation

$$\frac{i}{2} \frac{\partial^2}{\partial t} = \left( \frac{-\partial}{\partial x} i \frac{h}{x} \right) \mathbf{\phi}$$

form a Hilbert space $H$ with one parameter unitary group. By means of the F.T.

$$\mathbf{\psi}(x, t) = \int e^{i k x} \mathbf{\psi}(x, k) dk / 2\pi$$

solutions of the wave equation correspond to solutions $\mathbf{\psi}(x, k)$ of the Dirac system. Because $h$ is real,

$$-\left( \frac{d}{dx} + h \right) \frac{d}{dx} - x_k (u_1 + u_2) = -i k \left( \frac{d}{dx} + h \right)(u_1 - u_2) = k^2 (u_1 + u_2)$$

$$-\frac{d^2}{dx^2} + (k^2 + h^2)$$

hence solutions $\mathbf{\psi}(x, k)$ of the Dirac system correspond to solutions of the Schrödinger equation with potential $q = h + h^2$, at least modulo a singularity at $k = 0$.

Let $\mathbf{\psi}(x, k)$ be a solution of the D-system. For $x > 0$ one has

$$u(x, k) = \begin{pmatrix} \alpha(k) e^{-ikx} \\ \beta(k) e^{-ikx} \end{pmatrix}$$

and we can define $\text{in}(u) = \alpha$, $\text{out}(u) = \beta$. 

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If we consider the solutions with asymptotic behavior
\[
\begin{pmatrix} 0 \\ e^{-ikx} \end{pmatrix} \leftrightarrow \begin{pmatrix} B e^{ikx} \\ A e^{-ikx} \end{pmatrix} \quad (\text{e}^{ikx}) \leftrightarrow \begin{pmatrix} \overline{A} e^{ikx} \\ \overline{B} e^{-ikx} \end{pmatrix}
\]

it follows (because \( e^{-ikx} \) decays as \( x \to -\infty \) for \( k \in \text{UHP} \)) that \( A \) is non-vanishing in the UHP. Hence we get
\[
\begin{pmatrix} 0 \\ e^{-ikx} \end{pmatrix} \leftrightarrow \text{cont} \begin{pmatrix} \frac{B}{A} e^{ikx} \\ e^{-ikx} \end{pmatrix}
\]

which clearly represents the solution \( \text{cont} \). Similarly
\[
\begin{pmatrix} e^{ikx} \\ e^{-ikx} \end{pmatrix} \leftrightarrow \begin{pmatrix} e^{ikx} \\ \frac{B}{A} e^{-ikx} \end{pmatrix}
\]

Next I want to consider the filtration of \( H \).

Put
\[
\mathcal{H}_{t_0, t} = (\text{out, in})^{-1} \left( e^{-ikt_0} H_+ \times e^{ikt_0} H_- \right)
\]

and suppose \( \bar{u} \in \mathcal{H}_{t_0, t} \). Then for \( x \gg 0 \) we have
\[
\text{in}(\bar{u}) = \alpha \in e^{-ikt_0} H_+ \quad \text{out}(\bar{u}) = \beta \in e^{ikt_0} H_-
\]

\[
\hat{u}_1(x, t) = \int e^{ikt_0} e^{ikt} e^{i2(k) e^{-ikx} dk/2\pi} = 0 \quad \text{for} \quad t - t_0 + x > 0
\]
\[
(t > t_0 - x)
\]

\[
\hat{u}_2(x, t) = \int e^{ikt_0} e^{ikt} e^{i\beta(k) e^{-ikx} dk/2\pi} = 0 \quad t + t_0 - x < 0
\]

But now let us use uniqueness in the Cauchy problem for the wave equation.
Now from characteristic theory one concludes that \( \hat{u}(x,t) = 0 \) for \( t < x - t_0 \). Conversely if \( \hat{u}(x,0) = 0 \) for \( x > t_0 \), then \( u \in \mathcal{H}_{t_0, t_0} \).

Clearly it would be more appropriate to use \( \tau_0 \) instead of \( t_0 \). Note the above doesn't use \( h \) real.

**Idea:** Think of \( \mathcal{H} \) as being solutions of the Dirac system with a strange inner product. Introduce elements \( \phi_k(y) \in \mathcal{H} \) which evaluate at the point \( y \), the corresponding solution of the wave equation: Thus for \( u \in \mathcal{H} \)

\[
(u, \phi_k(y)) = \hat{u}(y,0) = \int_{\mathbb{R}} u(y,k) \, dk / 2\pi
\]
Then 
\[
(u, \frac{d\phi_1(y)}{dy}) = \int \frac{d}{dy} \phi_1(y, k) dk/2\pi
\]
\[
= \int \{ik u_1(y, k) + \bar{h}(y) u_2(y, k)\} dk/2\pi
\]
\[
= (ik u_1, \phi_1(y)) + \bar{h}(y) (u_2, \phi_2(y))
\]
\[
= (u_1, -ik \phi_1(y) + \bar{h}(y) \phi_2(y))
\]

so formally at least

\[
\frac{d\phi_1}{dy} = -ik \phi_1 + \bar{h} \phi_2
\]

\[
\frac{d\phi_2}{dy} = \bar{h} \phi_1 + ik \phi_2
\]

Actually it is clear that \(\phi_1(y)(x, 0) = (\delta(x-y))\)

so that finding \(\phi_1(y), \phi_2(y)\) formally amounts to solving the Cauchy problem for initial data given on the line \(t=0\).

Here is how to obtain \(\phi_1(y)\). Consider \(e_m\) which represents a \(\delta\) impulse coming in along \(x+t=0\), and hence \(\hat{e}_m\) is supported for \(x > -t\). Shift it to \(U(-y)e_m\) to have \(\delta\) coming in along \(x+t-y=0\) so that the leading edge arrives at \(x=y\) when \(t=0\). So now if you project onto \(H_{x,y}\), you kill the stuff above \(x=y\) and are left with \(\delta(x-y)\).

\[
\phi_1(y) = \text{pr}_{H_{x,y}} (U(-y)e_m)
\]

\[
\phi_2(y) = \text{pr}_{H_{x,y}} (U(y)e_{out})
\]
So now it's clear that we have found the elements $g_y, p_y$ essentially. Actually,

$$g_y = \text{pr}_{\mathcal{H}_0}^U (e_{\text{in}}) = U(+\frac{\theta}{2}) \text{pr}_{\mathcal{H}_0}^{\frac{\theta}{2}, \frac{\gamma}{2}} (U(-\frac{\theta}{2}) e_{\text{in}})$$

$$p_y = U(+\frac{\theta}{2}) \text{pr}_{\mathcal{H}_0}^{\frac{\theta}{2}, \frac{\gamma}{2}} (U(+\frac{\theta}{2}) e_{\text{out}})$$

Also what's clear is that $(p_y, g_y)$ isn't well-defined, but that the only reasonable interpretation is zero. This is because $\phi_1(y), \phi_2(y)$ "see" orthogonal pieces of the Hilbert space $\mathcal{H}$. 

Suppose given a port $(\mathcal{H}, \mathcal{U}(t), \mathcal{E}(t), \mathcal{E}_{in})$, when does it come from a deBranges space? Let $R(k)$ be the response function belonging to the port. We know it is analytic and bounded by 1 in the UHP. Assume that $R$ extends analytically across the real axis and that $|R(k)|=1$ for $k$ real. Then $R$ extends to a meromorphic function on the $k$-plane, because

$$\frac{1}{R(k)} = \overline{R(k)}$$

extends $R^{-1}$ to the LHP. Thus $R$ has zeroes only in the UHP and poles only in the LHP, the poles being the conjugates of the zeroes.

Let $\varphi(k)$ be an entire function having for its zeroes the zeroes of $R$. Then

$$\frac{\varphi}{\varphi^*}$$

is entire without zeroes and hence is of the form $e^{i\theta}$ with $\theta$ entire. Moreover this function has modulus 1 on the real axis, hence $\theta$ is real on the real axes, so $\theta = \theta^*$. Thus

$$\frac{\varphi^*}{\varphi} = e^{i\theta} = e^{i\frac{\theta}{2}}/(e^{i\frac{\theta}{2}})^*$$

and so if we replace $\varphi$ by $\varphi e^{i\frac{\theta}{2}}$ we get

$$R = \frac{\varphi^*}{\varphi}$$
Suppose $E$ is a de Branges function, $S = E^\dagger/E$. Then the associated de Branges space $B(E)$ is

$$B(E) = E H_+ \cap E^\dagger H_- \subset L^2(\frac{dk}{2\pi})$$

$$\mathfrak{H}_0 = H_+ \cap S H_- \subset L^2(\frac{dk}{2\pi}) = \mathfrak{H}$$

I propose to work in the letter picture, $\mathfrak{H}_0$ carries a symmetric operator $A$ induced by the self-adjoint operator of multiplication by $k$ on $L^2(\frac{dk}{2\pi}) = \mathfrak{H}$. $A^\dagger$ clearly contains $g = \text{pr}_{\mathfrak{H}_0}(k)$ with $k \in D_k$ since

$$\langle g, Af \rangle = \langle \text{pr}_{\mathfrak{H}_0}(k), kf \rangle = \langle k, kf \rangle = \langle k h, f \rangle = \langle \text{pr}_{\mathfrak{H}_0}(k h), f \rangle$$

for $f \in D_k$; also $A^\dagger = \text{pr}_{\mathfrak{H}_0}(k h)$. Assuming that $D_k^\dagger = \text{pr}_{\mathfrak{H}_0}(Q_k^\dagger)$ which seems likely we can define

$$\text{out}, \text{in} : D_k^\dagger \rightarrow C \text{ by}
$$

$$\text{in} (\text{pr}_{\mathfrak{H}_0} h) = (h, \text{in}) = \int h(k) \frac{dk}{2\pi}
$$

$$\text{out} (\text{pr}_{\mathfrak{H}_0} h) = (h, \text{out}) = \int h(k) S(k) \frac{dk}{2\pi}$$

Let's compute the response function for this port. We first find an element $1$ to $(A - I)\sigma_A$, e.g. the point evaluator $J_A$. 
Recall that for $H_+$ the point evaluator at $x$ in UHP is $\frac{i}{k-x}$, and for $H_-$ the point evaluator at $x$ is $\frac{i}{k+x}$. This follows from Cauchy

$$ (f, \frac{i}{k-x}) = \int f \frac{i}{k-x} \frac{dk}{2\pi} = f(\overline{x}) $$

This is the point evaluator for $SH_-$ at $x$, so if this is projected onto $H_0$ we get $J_\overline{x}$. Unfortunately it is not in $H_k$, so we remove from it $-\frac{i}{k+i} S(k)$ which is orthogonal to $SH_-$ as $\frac{i}{k+i} e^{H_+}$.

Thus we have

$$ J_\overline{x} = \text{proj}_{H_0} \left( \frac{-i S(k)}{k-x} + \frac{i S(k)}{k+i} \right) $$

where $x \in D_k$. Then we conclude

$$ \text{in} (J_\overline{x}) = (k, 1) = \frac{1}{2\pi i} \int S(k) \left\{ \frac{1}{k-x} - \frac{1}{k+i} \right\} dk = S(\overline{x}) $$

$$ \text{out} (J_\overline{x}) = \frac{1}{2\pi i} \int S(k) \left\{ \frac{1}{k-x} - \frac{1}{k+i} \right\} S(k) dk = 1 $$

by Cauchy's thm.

To summarize, I have seen that the deBranges space $B(E)$ can be considered as a port and that the associated response function is

$$ R(k) = \frac{E_k / E^\ast(k)}{E^\ast(k)} $$
Suppose $S(k)$ analytic and bounded by 1 in the UHP and that $|S(k)| = 1$ for $k$ real. Then I can form

$$H_0 = H_+ \cap SH_- \subset L^2(dk/2\pi) = H$$

with $U(t) = \text{multiplication by } e^{ikt}$. Because $S$ is analytic and bounded by 1 in the UHP we know that $SH_+ \subset H_+$, hence we have orthogonal decompositions

$$H = SH_- \oplus SH_+ = H_- \oplus (H_+ \cap SH_-) \oplus SH_+.$$

Let $A_0$ be the operator on $H_0$ induced by the self-adjoint operator of multiplication by $k$ on $H$. Thus $D_{A_0}$ consists of all $f \in H_0$ such that $k f \in H_0$. Then $A_0$ is a closed symmetric operator. Let $\lambda \in \text{UHP}$ and suppose $f \in H_0$ vanishes at $\lambda$. Then

$$\frac{f(k)}{k - \lambda} \in H_+, \quad \frac{S(k)f(k)}{k - \lambda} \in H_-$$

(5) This follows from what one knows about the operator $k - \lambda$ on $H_+$ and on $H_-$. Consequently we see that $\frac{f}{k - \lambda} \in H_0$, and we have established that $(A_0 - \lambda)D_{A_0}$ is of codim 1 in $H_0$. Similarly for $\lambda \in \text{LHP},$ so $A_0$ necessarily of type (1,1).

The same might work for a general response function $R$. If $H_0$ is the completion of $f_{\text{out}} + g_{\text{in}}$ with $R$-norm, then let as usual

$$\hat{H}_0 = (\text{out}, \text{in})^{-1}(H_- \times H_+)$$
and we define $A_0$ to be the operator on $H_0$ induced by the self-adjoint operator $A$ on $H$ which generates the group $U(A)$. (Does it make sense to say $A$ is given by multiplication by $k$?)

Let $\lambda \in \text{UHP}$ and let $h \in H_0$ be such that $\text{in}(h) \in H_+$ vanishes at $\lambda$. Consider $(A-\lambda)^{-1}h$ which is well-defined because $A$ is self-adjoint. In fact:

$$\frac{1}{\lambda} (A-\lambda)^{-1} h = \int_{-\infty}^{0} e^{-i \lambda t} e^{i A t} h dt.$$ 

Then

$$\text{out} (A-\lambda)^{-1} h = \frac{1}{k-\lambda} \text{out}(h) \in H_-$$

$$\text{in} (A-\lambda)^{-1} h = \frac{1}{k-\lambda} \text{in}(h) \in H_+$$

so $(A-\lambda)^{-1} h \in H_0$. Thus $(A-\lambda)^{-1} h$ is an element of $H_0$ such that $A$ applied to it stays in $H_0$, so it is in $D_{A_0}$. So it is now clear that $A_0$ has deficiency indices $(1,1)$.

Return to the case of an inner function $S$. We've just found a symmetric operator $A_0$ on $H_0 = H_+ \cap S H_-$ of type $(1,1)$. Next we want the boundary values for an element of $D_{A_0}$. (To simplify suppose for the moment that $D_{A_0}$ is dense in $H_0$.)

Look at

$$\frac{S(k) - S(\lambda)}{k-\lambda}$$

where $\lambda \in \text{UHP}$. This is clearly in $H_+$ and
\[
\frac{S(k) - S(\lambda)}{k - \lambda} = \frac{1}{k - \lambda} \left\{ 1 - S(\lambda) \overline{S(k)} \right\} \in \mathbb{H}_- \\
\]

because \( S(k) \) admits the bounded analytic extension \( \overline{S(k)} \) to LHP. Thus \( \frac{S(k) - S(\lambda)}{k - \lambda} \in \mathbb{H}_0 \)

Also \( \frac{1 - S(\lambda) S(k)}{k - \lambda} \in \mathbb{H}_+ \) and \( \frac{S(k) - S(\lambda)}{k - \lambda} \in \mathbb{H}_- \)

20 \( \frac{1 - S(\lambda) S(k)}{k - \lambda} \in \mathbb{H}_0 \)

Also we have

\[
(f, \frac{S(k) - S(\lambda)}{k - \lambda}) = (\overline{sf}, \frac{1}{k - \lambda})_H_- - (f, \frac{1}{k - \lambda}) S(\lambda) = 0 \quad \text{because} \quad \frac{1}{k - \lambda} \in \mathbb{H}_-
\]

\[
(f, \frac{1 - S(\lambda) S(k)}{k - \lambda}) = (f, \frac{1}{k - \lambda}) - S(\lambda) (\overline{sf}, \frac{1}{k - \lambda}) = i f(\lambda) \quad \text{because} \quad \frac{1}{k - \lambda} \in \mathbb{H}_+ 
\]

Hence

\[
\overline{sf} \quad \text{evaluates at} \quad \lambda
\]

\[
i \frac{1 - S(\lambda) S(k)}{k - \lambda} \quad \text{evaluates} \quad f \quad \text{at} \quad \lambda
\]
In the general case of an \( R(k) \) bounded by 1 and analytic in the UMP we consider

\[
\chi = \frac{i}{k-\lambda} e_{\text{in}} - \frac{i R(\lambda)}{k-\lambda} e_{\text{out}}
\]

\[
in(\chi) = \frac{i}{k-\lambda} \{1 - R(\lambda)R(k)\} \in H_+
\]

\[
\text{out}(\chi) = i \frac{R(\lambda) - R(\lambda)}{k-\lambda} \in H_-
\]

so this element belongs to \( H_0 \). Also if

\[
\beta = \frac{1}{i(k-\lambda)} (e_{\text{out}} - R(\lambda)e_{\text{in}})
\]

then

\[
in(\beta) = \frac{R(k) - R(\lambda)}{i(k-\lambda)} \in H_+
\]

\[
\text{out}(\beta) = \frac{1}{i(k-\lambda)} (1 - R(\lambda)R(k)) \in H_-
\]

so \( \beta \in H_0 \).

Furthermore, for all \( h \in H_0 \) we have

\[
(h, \frac{i}{k-\lambda}(e_{\text{in}} - R(\lambda)e_{\text{out}})) = (\text{in}(h), \frac{i}{k-\lambda}) \overline{R(\lambda)}(\text{out}(h), \frac{i}{k-\lambda}) = 0
\]

\[
(h, \frac{1}{i(k-\lambda)}(e_{\text{out}} - R(\lambda)e_{\text{in}})) = (\text{out}(h), \frac{1}{i(k-\lambda)}) - R(\lambda)(\text{in}(h), \frac{1}{i(k-\lambda)}) = 0
\]
Problem: I am trying to understand what might be the continuous analoge of the Schur representation. What I want to do is to see if the formulas

\[ p_x = p_x^a (Ux)e^u \]
\[ q_x = p_x^{au} (e^u) \]

can be extended to mean something for an arbitrary port. From de Branges' theory, we can see that the filtration \( H_x \) is not fine enough in general.

I shall want to use the viewpoint of strings. Recall the transformation formulas

\[ \frac{d}{dx} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} ik & -k \\ k & -ik \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \]

\[ \frac{d}{dx} \begin{pmatrix} u_1 + u_2 \\ u_1 - u_2 \end{pmatrix} = \begin{pmatrix} k & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} u_1 + u_2 \\ u_1 - u_2 \end{pmatrix} \]

\[ \frac{d}{dx} \begin{pmatrix} u_1 + u_2 \\ u_1 - u_2 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} u_1 + u_2 \\ u_1 - u_2 \end{pmatrix} \]

If \( \varphi = h \), e.g. \( \varphi = \exp(\int x^2) \), then

\[ \frac{d}{dx} \begin{pmatrix} \varphi \\ \varphi u_1 - u_2 \end{pmatrix} = k \begin{pmatrix} 0 & -\varphi^2 \\ \varphi^2 & 0 \end{pmatrix} \begin{pmatrix} \varphi \\ \varphi u_1 - u_2 \end{pmatrix} \]

The equation of motion for a string of density \( \rho \) when time is separated out is

\[ -\frac{d^2 w}{dx^2} = k^2 \rho \]

or

\[ \frac{d}{dx} \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix} = k \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix} \]

with \( w = \omega_1 \),

\[ \omega_2 = -\frac{i}{k} \frac{dw}{dx} \]
hence putting \( dt = \phi^{1/2} dx \) for the independent variable we get an equation of the form \( \ast \) with \( \phi^2 = \phi^{1/2} \).

In other words \( \ast \) describes the motion of a string with independent variable the time signals travel.

The next thing is to consider a string with a singularity.

**Example 1.** Take a uniform string and put a point mass \( m \) at \( x = 0 \). The motion at frequency \( \omega \) is described by 
\[
\frac{d^2 \psi}{dx^2} = -k^2 \psi \quad \text{for } x \neq 0
\]

At \( x = 0 \), the slope change \( \frac{d\psi}{dx} \bigg|_{0^-} \to 0^+ \) represents the vertical force on the point mass \( m \). This is \( m \frac{d^2 \psi}{dx^2} = -mk^2 \psi(0) \). So we get
\[
\frac{d\psi}{dx} \bigg|_{0^-} = -mk^2 \psi(0)
\]

So look at
\[
e^{-ikx} \longleftrightarrow Ae^{-ikx} + Be^{ikx}
\]

Continuity,
\[
1 = A + B
\]
\[
-mk^2 = -ikA + ikB - (-ik)
\]
\[
-1 + imk = -A + B
\]
\[
A = 1 - \frac{imk}{2}, \quad B = \frac{imk}{2}
\]

**Note:** Reflection coefficient: \( R = \frac{imk}{1 - imk} \) as \( k \to 0 \)
i.e. at high frequencies everything gets reflected which is most unlike potential scattering.

**Example 2.** Take two uniform strings and attach them with a weightless segment of length $a$.

Then

\[ e^{-ikx} \text{ has } \left. \frac{dw}{dx} \right|_{x=0} = -ik \quad \text{and} \quad \left. \frac{dw}{dx} \right|_{x=a} = 1 \]

and so if

\[ e^{-ikx} \overset{\sim}{\longrightarrow} Ae^{-ikx} + Be^{ikx} \]

one has

\[ 1 - ik = A + B \]
\[ -ik = -ikA + ikB \]
\[ -1 = -A + B \]

so

\[ A = 1 - \frac{ik}{2} \]
\[ B = -\frac{ik}{2} \]

and

\[ R = \frac{-\frac{ik}{2}}{1 - \frac{ik}{2}} \overset{\sim}{\longrightarrow} 1 \quad \text{as} \quad k \to \infty. \]

I ought to check that $|R| < 1$ in the UHP. But

\[ R(0) = 0, \quad R(\infty) = 1, \quad R(\frac{\pi}{a}) = \frac{-i}{1-i} = \frac{-i(1+i)}{2} = \frac{1-i}{2} \]

\[ \text{Check.} \]
Return to Eq. 1 and compute the wave equation solution corresponding to \( e_{in} \):

\[
i \int e^{\frac{ik(t+x)}{1+\frac{ik}{2}}} dk = \begin{cases} \frac{2i}{m} & t+x < 0 \\ \frac{2}{m} e^{\frac{-2}{m}(t-x)} & t+x > 0 \end{cases}
\]

\[
\int e^{\frac{-ik(t-x)}{1+\frac{ik}{2}}} dk = \begin{cases} 0 & t-x < 0 \\ \frac{2}{m} e^{\frac{-2}{m}(t-x)} & t-x > 0 \end{cases}
\]

Should write this: 
\[-1 + \frac{1}{1 + \frac{ik}{2}}\]

So for \( x < 0 \) we see the transmitted wave:

\[
\frac{2}{m} e^{\frac{-2}{m}(t+x)} \quad \quad x > -t
\]

And for \( x > 0 \) we see the reflected wave:

\[-8(x-t) + \frac{2}{m} e^{\frac{-2}{m}(t-x)} \quad t > x\]

Notice that if we try projecting this solution at time \( t = x \) on the subspace having support in \((x, \infty)\), then for \( x < 0 \) we get zero. This shows that

\[
g_x = p^2 q_x(e_{in})
\]

Gives \( g_x = 0 \) for \( x < 0 \).
August 10, 1978

Suppose \( S(k) = \frac{k-ix}{k+ix} \) with \( x > 0 \). Then

\[
H_0 = H_+ \cap \overline{S} H_-
\]

\[
B(E) = E H_+ \cap E \overline{H}_-
\]

where \( E(k) = k + ix \). Since \( E(k) \) has deg 1 we know \( B(E) \) is one-dimensional spanned by 1. Thus \( H_0 \) is one-dimensional spanned by \( \frac{1}{k + ix} = S \frac{1}{k-ix} \).

But I thought \( H_0 \) was a part in a natural way leading to the response function \( S \). However since \( H_0 \) is one-dimensional \( \Delta = 0 \), so the line \( (A-\mathcal{I}) \Delta^+ \) is identically \( \overline{H}_0 \), so the response function is constant.

August 12, 1978: What goes wrong with the preceding example is that the formula

\[
\phi(E) (\mathcal{H}_0(k)) = (\mathcal{H}_0, \mathcal{E}_m) \quad k \in D_k
\]

doesn't make sense.

For example if \( \mathcal{X}, \mathcal{Y} \in \text{null} \)

Then \( \mathcal{X} = -iS(k) + iS(k) \)

Then \( \mathcal{X} \in D_k \) and \( (\mathcal{X}, \mathcal{E}_m) = S(\mathcal{X}) - S(\mathcal{X}) \)
For example: Take $h = f_{-} + Sf_{+}$ with $f_{\pm} \in H_{\pm}$, then $H_{+} \cap \overline{S}H_{-} = H_{0}$. So if we take

$$h = \frac{1}{k-i} - \frac{k-ix}{k+i} \frac{1}{k+i}$$

then $h = O(\frac{1}{k^2})$ so $h \in D_{k}$, yet

$$(h, e^{ik}) = i \int_{k-i}^{k+i} \frac{1}{k-i} - \frac{k-ix}{k+i} \frac{dk}{2\pi i} = i$$

and, in UHP

so what this means is that I do not yet understand entrances and exits for a continuous port.
Suppose given a measure $d\nu$ on $S^1$ and let us form the associated sequence of orthonormal polynomials $\{p_0, p_1, \ldots, p_n, \ldots\}$, as well as $g_n = z^n p_n$, so that we have

\[
\begin{pmatrix} p_n \\ g_n \end{pmatrix} = \frac{1}{\sqrt{1 - |h_n|^2}} \begin{pmatrix} h_n \nu \chi_{\mu} & 0 \\ 1 \end{pmatrix} \begin{pmatrix} 1 \\ g_{n-1} \end{pmatrix}
\]

where $h_n = (p_n, g_n) = \int_{\mu} z^n p_n \ d\nu.$

We can also form the analytic function

\[
R(z) = \Theta(h_0)(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}) \Theta(h_1)(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}) \ldots
\]

and the port belonging to the sequence $h_0, h_1, \ldots$.

Let's begin with $\mathcal{H}_0 = L^2(S^1, d\mu)$ and $e_{\text{out}} = \mathcal{J}^{-1}$, $e_{\text{in}} = 1$, so that $U(e_{\text{out}}) = e_{\text{in}}$ where $U =$ multiplication by $\mathcal{J}$. This is the standard model for a port $(\mathcal{H}_0, V, e_{\text{out}}, e_{\text{in}})$ which has been terminated by the unitary operator $U$ which extends $V$ and carries $e_{\text{out}}$ to $e_{\text{in}}$. Let's compute the response function for this port. If $|z| < 1$, we need a vector $\mathcal{J}^{-1} (1-\mathcal{J}V) e_{\text{out}}$, since $e_{\text{out}} = \mathcal{J}^{-1} \mathcal{J} e_{\text{out}}$ such a vector is $(1-\mathcal{J}U^*)^{-1} e_{\text{out}} = (1-\mathcal{J}^*)^{-1} \mathcal{J}^{-1}$. Then

\[
R(z) = \frac{\langle (1-\mathcal{J}U^*)^{-1} e_{\text{out}}, e_{\text{in}} \rangle}{\langle (1-\mathcal{J}U^*)^{-1} e_{\text{out}}, e_{\text{out}} \rangle} = \frac{\int_{\mu} e_{\text{out}}^* \ d\nu}{\int_{\mu} e_{\text{out}}^2 \ d\nu}.
\]
Then
\[1 - z R(z) = \int \frac{1}{1 - z R(z)} \, dv, \quad 1 + z R(z) = \int \frac{1 + z R(z)}{1 - z R(z)} \, dv,\]

so
\[
\frac{1 + z R(z)}{1 - z R(z)} = \int \frac{1 + z^{-1}}{1 - z^{-1}} \, dv(v).
\]

The next thing is to relate the Schur system associated to this port with the orthogonal polynomials associated to \(dv\).

\[\mathcal{H}_0 = j^{-1}, \quad \mathcal{H}_1 = j^{-2}, \quad \mathcal{H}_2 = j^{-3}, \quad \mathcal{H}_3 = j^{-4} j^{-1}\]

Now
\[p_n = p_n(\mathcal{H}_0(\text{out})), \quad \tilde{q}_n = p_n(\mathcal{H}_0(\text{in})).\]

\[p_n = p_n(\mathcal{H}_0(j^{-2})) = j^{-2} - h_{-1} j^{-1}, \quad h_{-1} = \int j^{-1} \, dv.
\]

\[\tilde{p}_{-1} = j^{-1} - h_{-1} = e_{\text{out}} \circ h_{-1} e_{\text{in}}, \quad \tilde{q}_{-1} = p_n(\mathcal{H}_0(\text{in})) = 1 - h_{-1} j^{-1} = v_0 - h_{-1} p_0.
\]

Thus it is clear that \(p_n\) is obtained by orthonormalizing the sequence \(j^{-1}, \ldots, j^{-n-1}\) and consequently with minor changes the Schur system belonging to this port coincides with the system of orthogonal polynomials associated to \(dv\).
\[
\int_{n=1}^{\infty} f(z) = \frac{1+zR(z)}{1-zR(z)} = \int_{n=1}^{\infty} \frac{1+2\frac{1}{\sqrt{n}}}{1-2\frac{1}{\sqrt{n}}} \, d\nu(j)
\]

\[
\text{Im} f(z) = \text{Re} \left( \frac{1}{z} \right) = \int \frac{\text{Re}(1+zR(z)-zR(1-iz))}{|1-zR(z)|^2} \, d\nu(j) = \int \frac{1-|z|^2}{|1-zR(z)|^2} \, d\nu(j)
\]

Now
\[
\int \frac{1+z^{-1}}{1-z^{-1}} \, d\theta = \int \frac{1+z}{1-z} \, \frac{d\theta}{2\pi i} = \frac{z+z}{2} + \frac{(z)}{2} = 1
\]

hence we see that
\[
\frac{1-|z|^2}{|1-zR(z)|^2} \, d\theta \rightarrow 8(1-\gamma_0)
\]
as \(z \rightarrow \infty\) radially. Consequently as \(n \rightarrow 1\)

\[
\text{Im}(f(1,\gamma)) \frac{d\theta}{2\pi} \rightarrow d\nu(j).
\]

Now we know that \(R \in H^\infty\), hence \(R(1) \rightarrow R(1)\) a.e. as \(n \rightarrow 1\). Consequently if one knows that \(\int R(1)\)

stays away from 1 as \(J\) ranges over \(S^1\), then one sees that \(\text{Im}(f(j)) = f(1)\) is a bounded measurable fn.

Consequently, it follows that \(d\nu(1) = \text{Im}(f(j)) \frac{d\theta}{2\pi}\) is absolutely continuous with respect to Lebesgue measure.

Suppose \(R\) analytic for \(|z| \leq 1\) and that \(\int R(1) = 1\) but that \(|R| \not= 1\) on \(S^1\). Then \(f(z)\) maps the unit disk to the UHP and it extends meromorphically to the closed unit circle with poles at each such point \(\gamma_0\). Such a pole has to be simple because \(\text{Im} \gamma_0\) in the disk, and the residue is positive. In this case, \(d\nu\) will have singular part concentrated at these poles.
August 24, 1975

It might be the case that the sequence \[ h_n \] is uniquely determined by \( R \) when one knows \( |R| < 1 - \varepsilon \). Some examples supporting this are as follows:

1) Consider a sequence \( h_n, n \geq 0 \) with \( |h_0| = 1 \). This means we are looking at a terminated port. From Aug 22 we know the associated spectral measure is absolutely continuous w.r.t. Lebesgue measure, assuming \( |R| < 1 - \varepsilon \). Hence the scattering determines the whole picture.

2) Consider a Schrödinger equation \(-u'' + V u = k^2 u\) with \( V \) of compact support having no bound states. Then when we calculate the scattering

\[ e^{-ikx} \leftrightarrow A e^{-ikx} + B e^{ikx} \]

it can happen that \( A \) has a simple pole at \( k = 0 \). In fact this happens when \( 1 \leftrightarrow \) non-constant linear function, which gives us different Dirac systems with the same Schröd. equation and hence the same scattering. But in this case we have \( A \sim \frac{c}{k} \quad B \sim \frac{-c}{k} \quad \) as \( k \to 0 \) so that

\[ R = \frac{B}{A} \to -1 \quad \text{as} \quad k \to 0. \]

When \( A, B \) are analytic at \( k = 0 \), one has from \( |A|^2 - |B|^2 = 1 \) that \( |R| < 1 \).

Here's an example of a doubly-infinite Schur system with point spectrum. Suppose that \( h_0, V_0, u, \varepsilon \)
is a port such that upon extending $V$ to the unitary operator $U$ with $U(e_{out}) = e_{in}$ we get the discrete eigenvalue $z_0$. This means there is a $g \in D_V$ such that

$$U(e_{out} + g) = e_{in} + Vg = z_0(e_{out} + g)$$

(or equivalently that $e_{in} - z_0 e_{out} \in (V - z_0)D_V$). Take another such port with primes and put

$$H = H_0 \oplus H'_0$$

$$U = \text{unitary operator extending } V \oplus V' \text{ with } U(e_{out}) = e'_{in}, \quad U(e'_{out}) = e_{in}.$$ Intuitively we get the ports connected together by a unit length of line:

![Diagram](image)

But then

$$U(e_{out} + g + e'_{out} + g') = (e'_{in} + Vg + e_{in} + V'g')$$

$$= z_0 \left( e_{out} + g + e'_{out} + g' \right)$$

and so $U$ has point spectrum. This shows that I can connect together two ports with scattering and obtain bound states.