August 1, 1978: Spaces $\mathcal{H}$, $\mathcal{H}_x$, also p_x, q_x for Dirac system

singular strings p. 175

Let's consider a Dirac system $\frac{d}{dx}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} ik & h \\ h & -ik \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$ with h real, smooth of compact support:

$$\frac{d}{dx}\begin{pmatrix} u_1 + u_1 \\ u_1 - u_2 \end{pmatrix} = \begin{pmatrix} h & ik \\ ik & -h \end{pmatrix}\begin{pmatrix} u_1 + u_2 \\ u_1 - u_2 \end{pmatrix}$$

[redacted] Solutions of the corresponding wave equation

$$\frac{1}{i}\frac{\partial \vec{\psi}}{\partial t} = \begin{pmatrix} \frac{1}{i}\frac{d}{dx} & ih \\ -ih & -\frac{1}{i}\frac{d}{dx} \end{pmatrix}\vec{\psi}$$

form a Hilbert space $\mathcal{H}$ with one parameter unitary group. By means of the F.T.

$$\vec{\psi}(x,t) = \int e^{ikt}\, \vec{u}(x,k)\, dk/2\pi$$

solutions of the wave equation correspond ~~to~~ solutions $\vec{u}(x,k)$ of the Dirac system. Because h is real [redacted] we have

$$\underbrace{-\left(\frac{d}{dx}+h\right)\left(\frac{d}{dx}-h\right)}_{-\frac{d^2}{dx^2}+(h^2+h')}(u_1+u_2) = -ik\left(\frac{d}{dx}+h\right)(u_1-u_2) = k^2(u_1+u_2)$$

hence solutions $\vec{u}(x,k)$ of the Dirac system [redacted] correspond to solutions $u(x,k)$ of the Schroedinger equation with potential $q = h' + h^2$, at least modulo a singularity at $k=0$

Let $\vec{u}(x,k)$ be a solution of the D-system. For $x \gg 0$ one has

$$u(\vec{x},k) = \begin{pmatrix} \alpha(k)e^{-ikx} \\ \beta(k)e^{-ikx} \end{pmatrix}$$

and we can define $\text{in}(u) = \alpha$, $\text{out}(u) = \beta$.

If we consider the solutions with asymptotic behavior

$$\begin{pmatrix} 0 \\ e^{-ikx} \end{pmatrix} \longleftrightarrow \begin{pmatrix} B e^{ikx} \\ A e^{-ikx} \end{pmatrix} \qquad \begin{pmatrix} e^{ikx} \\ 0 \end{pmatrix} \longleftrightarrow \begin{pmatrix} \bar{A} e^{ikx} \\ \bar{B} e^{-ikx} \end{pmatrix}$$

it follows (because e^{-ikx} decays as $x \to -\infty$ for $k \in UHP$) that A is non-vanishing in the UHP. Hence we get

$$\begin{pmatrix} 0 \\ \frac{1}{A} e^{-ikx} \end{pmatrix} \xleftrightarrow{e_{out}} \begin{pmatrix} \frac{B}{A} e^{ikx} \\ e^{-ikx} \end{pmatrix}$$

which clearly represents the solution e_{out}. Similarly

$$\begin{pmatrix} \frac{1}{\bar{A}} e^{ikx} \\ 0 \end{pmatrix} \xleftrightarrow{e_{in}} \begin{pmatrix} e^{ikx} \\ \overline{\left(\frac{B}{A}\right)} e^{-ikx} \end{pmatrix}$$

Next I want to consider the filtration of $\mathcal{H}$. Put

$$\mathcal{H}_{t_0, t_0} = (\text{out}, \text{in})^{-1} \left(e^{-ikt_0} H_- \times e^{-ikt_0} H_+ \right)$$

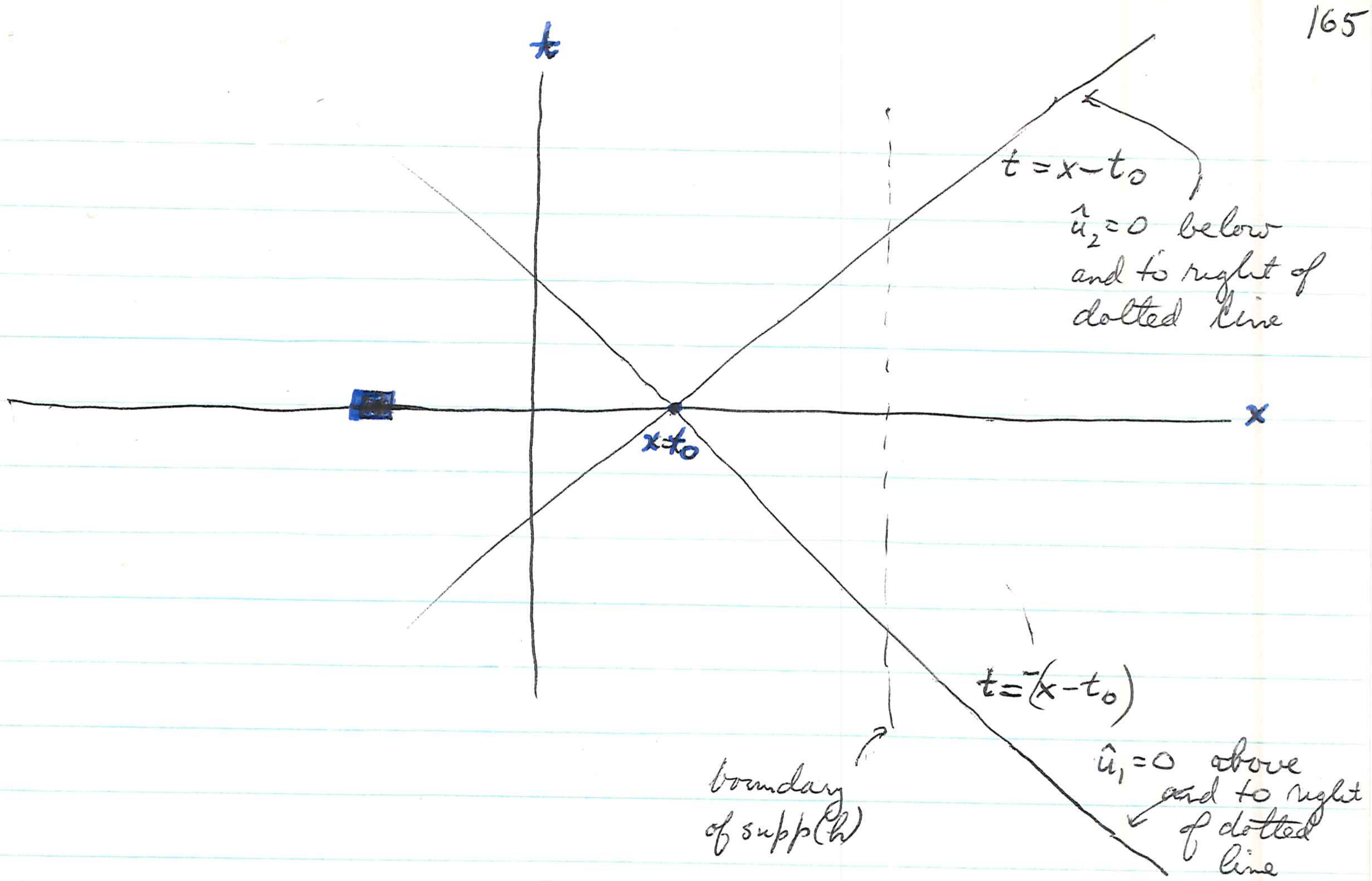
and suppose $\vec{u} \in \mathcal{H}_{t_0, t_0}$. Then for $x >> 0$ we have $\text{in}(\vec{u}) = \alpha \in e^{-ikt_0} H_+$, $\text{out}(\vec{u}) = \beta \in e^{ikt_0} H_-$

$$\hat{u}_1(x,t) = \int e^{ik(t-t_0)} e^{ikt_0} \alpha(k) e^{ikx} \, dk/2\pi = 0 \quad \text{for} \quad t - t_0 + x > 0$$

$(t > t_0 - x)$

$$\hat{u}_2(x,t) = \int e^{ik(t+t_0)} e^{-ikt_0} \beta(k) e^{-ikx} \, dk/2\pi = 0 \qquad t + t_0 - x < 0$$

But now let us use uniqueness in the Cauchy problem for the wave equation



Now from characteristic theory one concludes that $\hat{u}(x,t) = 0$ for [redacted] $|t| < x - t_0$. Conversely if $\hat{u}(x,0) = 0$ for $x > t_0$, then $u \in \mathcal{H}_{t_0, t_0}$.

Clearly it would be more appropriate to use x_0 instead of t_0. Note the above doesn't use h real.

<u>Idea:</u> [redacted] Think of $\mathcal{H}$ as being solutions of the Dirac system with a strange inner product. Introduce elements $\phi_i(y) \in \mathcal{H}$ which evaluate at the point y and $t=0$, the corresponding solution of the wave equation. Thus for $u \in \mathcal{H}$

$$(u, \phi_i(y)) = \hat{u}_i(y,0) = \int u_i(y,k)\, dk/2\pi$$

Then
$$\left(u, \frac{d\phi_1}{dy}(y)\right) = \int \frac{d}{dy} u_1(y,k)\, dk/2\pi$$
$$= \int \{ik\, u_1(y,k) + h(y) u_2(y,k)\}\, dk/2\pi$$
$$= (ik\, u, \phi_1(y)) + h(y)(u, \phi_2(y))$$
$$= (u, -ik\phi_1(y) + \overline{h(y)}\phi_2(y))$$

so formally at least
$$\frac{d\phi_1}{dy} = -ik\phi_1 + \bar{h}\,\phi_2$$
$$\frac{d\phi_2}{dy} = h\phi_1 + ik\,\phi_2$$

Actually it is clear that $\widehat{\phi_1(y)}(x,0) = \begin{pmatrix} \delta(x-y) \\ 0 \end{pmatrix}$ so that finding $\phi_1(y), \phi_2(y)$ formally amounts to solving the Cauchy problem for initial data given on the line $t=0$.

Here is how to obtain $\phi_1(y)$. Consider e_{in} which represents a δ impulse coming in along $x+t=0$, and hence $\hat{e}_{in}$ is supported for $x > -t$. Shift it to $U(-y)e_{in}$ to have δ coming in along $x+t-y=0$ so that the leading edge arives at $x=y$ when $t=0$. So now if you project onto $\mathcal{H}_{y,y}$, you kill the stuff above $x=y$ and are left with $\delta(x-y)$.

$\therefore$
$$\phi_1(y) = pr_{\mathcal{H}_{y,y}}\left(U(-y)\, e_{in}\right)$$
$$\phi_2(y) = pr_{\mathcal{H}_{y,y}}\left(U(y)\, e_{out}\right)$$

So now it's clear that we have found the elements q_y, p_y essentially. Actually

$$q_y = pr_{\mathcal{H}_{y,0}}(e_{in}) = U(+\tfrac{y}{2})\, pr_{\mathcal{H}_{\frac{y}{2},\frac{y}{2}}}\left(U(-\tfrac{y}{2})\, e_{in}\right)$$

$$p_y = U(+\tfrac{y}{2})\, pr_{\mathcal{H}_{\frac{y}{2},\frac{y}{2}}}\left(U(+\tfrac{y}{2})\, e_{out}\right)$$

Also what's clear is that (p_y, q_y) isn't well-defined, but that the only reasonable interpretation is zero. This is because $\phi_1(y)$, $\phi_2(y)$ "see" orthogonal pieces of the Hilbert space $\mathcal{H}$.

August 2, 1978: deBranges spaces as ports

Suppose given a port $(\mathcal{H}_{[redacted]}, U(t), e_{out}, e_{in})$; when does it come from a deBranges space? Let $R(k)$ [redacted] be the response function belonging to the port. We know it is analytic and bounded by 1 in the UHP. Assume that R extends analytically across the real axis and that $|R(k)| = 1$ for k real. Then R extends to a meromorphic function on the k-plane, because

$$\frac{1}{R(k)} = \overline{R(\bar{k})}$$

extends R^{-1} to the LHP. [redacted] Thus R has zeroes only in the UHP and poles only in the LHP, the poles being the conjugates of the zeroes.

Let $\varphi(k)$ be an entire function having for its zeroes the zeroes of R. Then

$$R\frac{\varphi^{\#}}{\varphi}$$

is entire without zeroes [redacted] and hence is of the form e^{ig} with g entire. Moreover this function has modulus 1 on the real axis, hence g is real on the real axes, so $g = g^{\#}$. Thus

$$R\frac{\varphi^{\#}}{\varphi} = e^{ig} = e^{\frac{ig}{2}} \Big/ \left(e^{\frac{ig}{2}}\right)^{\#}$$

and so if we replace φ by $\varphi e^{ig/2}$ we get

$$R = \varphi / \varphi^{\#}$$

with φ entire. Next because $|R(k)|<1$ for $k \in UHP$, one has that $\varphi^\#$ is a de B function. (We exclude the trivial case R constant.)

Suppose E is a de Branges function, $S = E^\#/E$. Then the associated de Branges space $\mathcal{B}(E)$ is

$$\mathcal{B}(E) = EH_+ \cap E^\# H_- \subset L^2\left(\frac{dk}{2\pi|E|^2}\right)$$
$$\Big\downarrow S$$
$$\mathcal{H}_0 = H_+ \cap SH_- \subset L^2(dk/2\pi) = \mathcal{H}$$

I propose to work in the latter picture. $\mathcal{H}_0$ carries a symmetric operator A induced by the self-adjoint operator of multiplication by k on $L^2(dk/2\pi) = \mathcal{H}$. $\mathcal{D}_{A^*}$ clearly contains $g = pr_{\mathcal{H}_0}(h)$ with $h \in \mathcal{D}_k$ since

$$(g, Af) = (pr_{\mathcal{H}_0}(h), kf) = (h, kf) = (kh, f) = (pr_{\mathcal{H}_0}(kh), f)$$

for $f \in \mathcal{D}_A$; also $A^* g = pr_{\mathcal{H}_0}(k.h)$. Assuming that $\mathcal{D}_{A^*} = pr_{\mathcal{H}_0}(\mathcal{D}_k)$ which seems likely we can define

$$\text{out}, \text{in} : \mathcal{D}_{A^*} \longrightarrow \mathbb{C} \quad \text{by}$$

$$\text{in}(pr_{\mathcal{H}_0} h) = (h, e_{in}) = \int h(k)\, dk/2\pi$$
$$\text{out}(pr_{\mathcal{H}_0} h) = (h, e_{out}) = \int h(k)\overline{S(k)}\, dk/2\pi$$

Let's compute the response function for this port. We first find an element $\perp$ to $(A-\bar{\lambda})\mathcal{D}_A$, e.g. the point evaluator $J_{\bar{\lambda}}$.

[redacted]

Recall that for H_+ the point evaluator at $\lambda \in UHP$ is $\frac{i}{k-\bar\lambda}$, and for H_- [redacted] the point evaluator at $\bar\lambda$ is $-\frac{i}{k-\lambda}$. This follows from Cauchy

$$\left(f, -\frac{i}{k-\lambda}\right) = \int f \frac{i}{k-\bar\lambda} \frac{dk}{2\pi} = f(\bar\lambda)$$

Hence $-\frac{i\,S(k)}{k-\lambda}$ is the point evaluator for SH_- at $\bar\lambda$ so if this is projected onto $\mathcal{H}_0$ we get $J_{\bar\lambda}$. Unfortunately it is not in $\mathcal{D}_k$, so we remove from it $-\frac{i}{k+i}S(k)$ which is orthogonal to SH_- as $\frac{1}{k+i} \in H_+$. Thus we have

$$J_{\bar\lambda} = pr_{\mathcal{H}_0}\Big(\underbrace{\frac{-i\,S(k)}{k-\lambda} + \frac{i\,S(k)}{k+i}}_{h}\Big)$$

where $h \in$ [redacted] $\mathcal{D}_k$. Then we conclude

$$in(J_{\bar\lambda}) = (h, 1) = \frac{1}{2\pi i}\int S(k)\left\{\frac{1}{k-\lambda} - \frac{1}{k+i}\right\} dk$$

$$= S(\lambda)$$

$$out(J_{\bar\lambda}) = \frac{1}{2\pi i}\int S(k)\left\{\frac{1}{k-\lambda} - \frac{1}{k+i}\right\}\overline{S(k)}\, dk = 1$$

by Cauchy's thm.

To summarize, I have seen that the deBranges space $B(E)$ [redacted] can be considered as a port [redacted] and that the associated response function is

$$R(k) = E^{\#}(k)/E^{[redacted]}(k)$$

Suppose [redacted] $S(k)$ analytic and bounded by 1 in the UHP and that $|S(k)|=1$ for k real. Then I can form

$$\mathcal{H}_0 = H_+ \cap SH_- \subset L^2(dk/2\pi) = \mathcal{H}$$

with $U(t)=$ multiplication by e^{ikt}. Because S is analytic and bounded by 1 in the UHP we know that $SH_+ \subset H_+$, hence we have [redacted] orthogonal decompositions

$$\mathcal{H} = SH_- \oplus SH_+ = H_- \oplus (H_+ \cap SH_-) \oplus SH_+ .$$

Let A_0 be the operator on $\mathcal{H}_0$ induced by the self-adjoint operator of multiplication by k on $\mathcal{H}$. Thus $\mathcal{D}_{A_0}$ consists of all $f \in \mathcal{H}_0$ such that $kf \in \mathcal{H}_0$. Then A_0 is a closed symmetric operator. Let $\lambda \in$ UHP and suppose $f \in \mathcal{H}_0$ vanishes at λ. Then [redacted]

$$\frac{f(k)}{k-\lambda} \in H_+ \quad , \quad \frac{\bar{S}(k)f(k)}{k-\lambda} \in H_-$$

([redacted] This follows from what one knows about the operator $k-\lambda$ on H_+ and on H_-.) Consequently we see that $\frac{f}{k-\lambda} \in \mathcal{H}_0$, and we have established that $(A_0-\lambda)\mathcal{D}_{A_0}$ is of codim 1 in $\mathcal{H}_0$. Similarly for $\lambda \in$ LHP, so A_0 necessarily of type (1,1).

The same might work for a general response function R. If $\mathcal{H}_0$ is the completion of $f_{out} + g_{in}$ with R-norm, then let as usual

$$\mathcal{H}_0 = (\text{out}, \text{in})^{-1}(H_- \times H_+)$$

and we define A_0 to be the operator on $\mathcal{H}_0$ induced by the self-adjoint operator A on $\mathcal{H}$ which generates the group $U(t)$. (Does it make sense to say A is given by multiplication by k?)

Let $\lambda \in UHP$ and let $h \in \mathcal{H}_0$ be such that $in(h) \in H_+$ vanishes at λ. Consider $(A-\lambda)^{-1}h$ which is well-defined because A is self-adjoint. In fact:

$$\frac{1}{i}(A-\lambda)^{-1}h = \int_{-\infty}^{0} e^{-i\lambda t} e^{iAt} h \, dt .$$

Then

$$out\,(A-\lambda)^{-1}h = \frac{1}{k-\lambda}\, out(h) \in H_-$$

$$in\,(A-\lambda)^{-1}h = \frac{1}{k-\lambda}\, in(h) \in H_+$$

so $(A-\lambda)^{-1}h \in \mathcal{H}_0$. Thus $(A-\lambda)^{-1}h$ is an element of $\mathcal{H}_0$ such that A applied to it stays in $\mathcal{H}_0$, so it is in $\mathcal{D}_{A_0}$. So it is now clear that A_0 has deficiency indices $(1,1)$.

Return to the case of an inner function S. We've just found a symmetric operator A_0 on $\mathcal{H}_0 = H_+ \cap SH_-$ of type $(1,1)$. Next we want the boundary values for an element of $\mathcal{D}_{A_0^*}$. (To simplify suppose for the moment that $\mathcal{D}_{A_0}$ is dense in $\mathcal{H}_0$).

Look at
$$\frac{S(k) - S(\lambda)}{k-\lambda}$$
where $\lambda \in UHP$. This is clearly in H_+ and

$$\overline{S(k)}\,\frac{S(k)-S(\lambda)}{k-\lambda} = \frac{1}{k-\lambda}\left\{1 - S(\lambda)\overline{S(k)}\right\} \in H_-$$

because $\overline{S(k)}$ admits the bounded analytic extension $\overline{S(\bar{k})}$ to LHP. Thus $\dfrac{S(k)-S(\lambda)}{k-\lambda} \in \mathcal{H}_0$

Also $\dfrac{1-\overline{S(\lambda)}\,S(k)}{k-\bar{\lambda}} \in H_+$ and

$$\overline{S(k)}\left\{\frac{1-\overline{S(\lambda)}\,S(k)}{k-\bar{\lambda}}\right\} = \frac{\overline{S(\bar{k})} - \overline{S(\lambda)}}{k-\bar{\lambda}} \in H_-$$

so $\dfrac{1-\overline{S(\lambda)}\,S(k)}{k-\bar{\lambda}} \in \mathcal{H}_0$. Also we have

$$\left(f, \frac{S(k)-S(\lambda)}{k-\lambda}\right) = \left(\bar{S}f, \frac{1}{k-\lambda}\right)_{H_-} - \underbrace{\left(f, \frac{1}{k-\lambda}\right)}_{=0 \text{ because } \frac{1}{k-\lambda} \in H_-}\overline{S(\lambda)}$$

$$= -i\,(\bar{S}f)(\bar{\lambda})$$

$$\left(f, \frac{1-\overline{S(\lambda)}\,S(k)}{k-\bar{\lambda}}\right) = \left(f, \frac{1}{k-\bar{\lambda}}\right) - S(\lambda)\underbrace{\left(\bar{S}f, \frac{1}{k-\bar{\lambda}}\right)}_{=0 \text{ because } \frac{1}{k-\bar{\lambda}} \in H_+}$$

$$= i f(\lambda)$$

Hence

$\dfrac{1}{i}\,\dfrac{S(k)-S(\lambda)}{k-\lambda}$ evaluates $\bar{S}f$ at $\bar{\lambda}$

$i\,\dfrac{1-\overline{S(\lambda)}\,S(k)}{k-\bar{\lambda}}$ evaluates f at λ

In the general case of an $R(k)$ bounded by 1 and analytic in the UHP we consider

$$\alpha = \frac{i}{k-\bar\lambda}\, e_{in} - \frac{i\,\overline{R(\lambda)}}{k-\bar\lambda}\, e_{out}$$

$$in(\alpha) = \frac{i}{k-\bar\lambda}\{1 - \overline{R(\lambda)}\,R(k)\} \in H_+$$

$$out(\alpha) = i\,\frac{\overline{R(\bar k)} - \overline{R(\lambda)}}{k-\bar\lambda} \in H_-$$

So this element belongs to $\mathcal{H}_0$. Also if

$$\beta = \frac{1}{i(k-\lambda)}\left(e_{out} - R(\lambda)\, e_{in}\right)$$

then
$$in(\beta) = \frac{R(k)-R(\lambda)}{i(k-\lambda)} \in H_+$$

$$out(\beta) = \frac{1}{i(k-\lambda)}\left(1 - R(\lambda)\overline{R(\bar k)}\right) \in H_-$$

so $\beta \in \mathcal{H}_0$.

Furthermore for all $h \in \mathcal{H}_0$ we have

$$\left(h, \frac{i}{k-\bar\lambda}(e_{in} - \overline{R}(\lambda)\, e_{out})\right) = \left(in(h), \frac{i}{k-\bar\lambda}\right) - R(\lambda)\underbrace{\left(out\, h, \frac{i}{k-\bar\lambda}\right)}_{=0}$$

$$= in(h)(\lambda)$$

$$\left(h, \frac{1}{i(k-\lambda)}(e_{out} - R(\lambda)\, e_{in})\right) = \left(out\, h, \frac{1}{i(k-\lambda)}\right) - \overline{R(\lambda)}\underbrace{\left(in\, h, \frac{1}{i(k-\lambda)}\right)}_{=0}$$

$$= out(h)(\bar\lambda)$$

Problem: I am trying to understand what might be the continuous analogue of the Schur representation. What I want to do is to see if the formulas

$$p_x = pr_{\mathcal{H}_x}(u(x)e_{out})$$

$$q_x = pr_{\mathcal{H}_x}(e_{in})$$

can be extended to mean something for an arbitrary port. From de Branges theory we can see that the filtration $\mathcal{H}_x$ is not fine enough in general.

I shall want to use the viewpoint of strings. Recall the transformation formulas

$$\frac{d}{dx}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} ik & h \\ h & -ik \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$$

$$\frac{d}{dx}\begin{pmatrix} u_1+u_2 \\ \frac{u_1-u_2}{i} \end{pmatrix} = \left\{ k\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} + h\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right\}\begin{pmatrix} u_1+u_2 \\ \frac{u_1-u_2}{i} \end{pmatrix}$$

If $\frac{\varphi'}{\varphi} = h$, e.g. $\varphi = exp\left(\int^x h\right)$, then

$$(*) \quad \frac{d}{dx}\begin{pmatrix} \frac{u_1+u_2}{\varphi} \\ \varphi\frac{u_1-u_2}{i} \end{pmatrix} = k\begin{pmatrix} 0 & -\varphi^{-2} \\ \varphi^2 & 0 \end{pmatrix}\begin{pmatrix} \frac{u_1+u_2}{\varphi} \\ \varphi\frac{u_1-u_2}{i} \end{pmatrix}$$

The equation of motion for a string of density ρ when time is separated out is

$$-\frac{d^2w}{dx^2} = k^2\rho w$$

or $\quad \frac{d}{dx}\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = k\begin{pmatrix} 0 & -1 \\ \rho & 0 \end{pmatrix}\begin{pmatrix} w_1 \\ w_2 \end{pmatrix}$

with $w = w_1 \quad w_2 = -\frac{1}{k}\frac{dw_1}{dx}$

hence putting $dt = \rho^{1/2} dx$ for the independent variable we get an equation of the form ⊛ with $\varphi^2 = \rho^{1/2}$. In other words ⊛ describes the motion of a string with independent variable the time signals travel.

The next thing is to consider a string with a singularity.

Example 1. Take a uniform string and put a point mass m at $x=0$. The motion at frequency k is described by
$$\frac{d^2w}{dx^2} = -k^2 w \qquad \text{for } x \neq 0$$

At $x=0$, the slope change $\left.\frac{dw}{dx}\right]_{0-}^{0+}$ represents the vertical force on the point mass (recall tension $=1$) which is $m\frac{\partial^2 w(0)}{\partial t^2} = -mk^2 w(0)$. So we get
$$\left.\frac{dw}{dx}\right]_{0-}^{0+} = -mk^2 u(0)$$

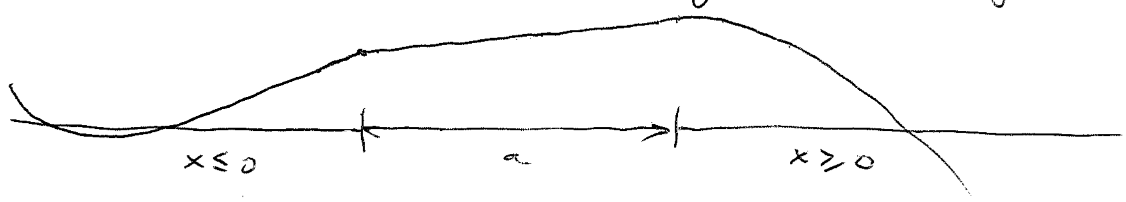
So look at
$$e^{-ikx} \longleftrightarrow Ae^{-ikx} + Be^{ikx}$$

contin.
$$1 = A + B$$
$$-mk^2 = -ikA + ikB - (-ik)$$
$$-1 + imk = -A + B$$
$$A = 1 - \frac{imk}{2} \qquad B = \frac{imk}{2}$$

NOTE: Reflection coefficient: $R = \dfrac{\frac{imk}{2}}{1 - imk/2} \sim -1$ as $k \to \infty$

i.e. at high frequencies everything gets reflected which is most unlike potential scattering.

Example 2. Take two uniform strings and attach them with a weightless segment of length a:



Then

$$e^{-ikx} \quad \text{has} \quad w\big|_{x=0^-} = 1, \quad \frac{dw}{dx}\Big|_{x=0^-} = -ik \quad \text{so} \quad w(0^+) = 1-(ik)a, \quad \frac{dw}{dx}(0^+) = -ik$$

and so if

$$e^{-ikx} \longleftrightarrow Ae^{-ikx} + Be^{ikx}$$

one has

$$1 - ika = A + B$$
$$-ik = -ikA + ikB$$
$$-1 = -A + B$$

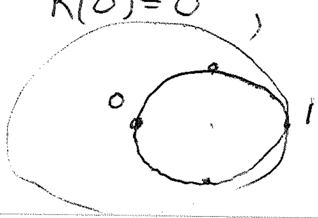
so

$$A = 1 - \frac{iak}{2} \qquad B = -\frac{iak}{2}$$

and

$$R = \frac{-\frac{iak}{2}}{1-\frac{iak}{2}} \approx 1 \quad \text{as } k \to \infty.$$

I ought to check that $|R|<1$ in the UHP.
But $R(0)=0$, $R(\infty)=1$, $R\left(\frac{2}{a}\right) = \frac{-i}{1-i} = \frac{-i(1+i)}{2} = \frac{1-i}{2}$



Clean.

Return to Ex. 1 and compute the wave equation solution corresponding to e_{in}:

$$\frac{1}{1+\frac{imk}{2}}\, e^{ikx} \longleftrightarrow e^{ikx} + \frac{-\frac{imk}{2}}{1+\frac{imk}{2}}\, e^{-ikx}$$

$$i\int e^{ik(t+x)} \frac{1}{1+\frac{imk}{2}} \frac{dk}{2\pi i} = \begin{cases} 0 & t+x<0 \\ i\,\dfrac{e^{i\frac{2i}{m}(t+x)}}{\frac{im}{2}} = \frac{2}{m} e^{-\frac{2}{m}(t+x)} & t+x>0 \end{cases}$$

pole at $-\frac{2}{mi} = \frac{2i}{m} \in UHP$

$$i\int e^{ik(t-x)} \underbrace{\frac{-\frac{imk}{2}}{1+\frac{imk}{2}}}_{\text{should write this } -1+\frac{1}{1+\frac{imk}{2}}} \frac{dk}{2\pi i} = \begin{cases} 0 & t-x<0 \\ \dfrac{i\, e^{-\frac{2}{m}(t-x)} \cdot 1}{\frac{im}{2}} = \frac{2}{m} e^{-\frac{2}{m}(t-x)} & t-x>0 \end{cases}$$

So for $x<0$ we see the transmitted wave

$$\frac{2}{m} e^{-\frac{2}{m}(t+x)} \qquad x>-t$$

and for $x>0$ we the reflected wave

$$-\delta(x-t) + \frac{2}{m} e^{-\frac{2}{m}(t-x)} \qquad t>x$$

Notice that if we try projecting this solution at time $t=x$ on the subspace having support in $(-\infty, x]$, then for $x<0$ we get zero. This shows that

$$q_x = pr_{\mathcal{H}_x}(e_{in})$$

gives $q_x = 0$ for $x<0$.

August 10, 1978

Suppose $S(k) = \frac{k-i\alpha}{k+i\alpha}$ with $\alpha > 0$. Then

$$\mathcal{H}_0 = H_+ \cap SH_-$$
$$\downarrow$$
$$\mathcal{B}(E) = EH_+ \cap E^{\#}H_-$$

where $E(k) = k + i\alpha$. [redacted] Since $E(k)$ has deg 1 we know $\mathcal{B}(E)$ is one-dimensional spanned by 1. Thus $\mathcal{H}_0$ is 1-dimensional spanned by $\frac{1}{k+i\alpha} = S\frac{1}{k-i\alpha}$.

But I thought $\mathcal{H}_0$ was a port in a natural way [redacted] leading to the response function S. However since $\mathcal{H}_0$ is 1-dimensional $\mathcal{D}_A = 0$, [redacted] so the line $(A-\bar{\lambda})\mathcal{D}_A^{\perp}$ is identically $= \mathcal{H}_0$, so the response function is constant.

[redacted]

August 12, 1978: What goes wrong with the preceding example [redacted] is that the formula

$$\underset{\text{b. value}}{\text{in}}\left(pr_{\mathcal{H}_0}(h)\right) = (h, e_{in}) \qquad h \in \mathcal{D}_k$$

doesn't make sense. ~~For example if $\lambda, \lambda' \in$ UHP then take $h = -\frac{iS(k)}{k-\lambda} + \frac{iS(k)}{k-\lambda'}$. Then $h \in \mathcal{D}_k$ and $(h, e_{in}) = S(\lambda) - S(\lambda')$ while~~ [redacted]

$$\cancel{\left(h, \frac{1}{k+i\alpha}\right) = \int \left(-\frac{i}{k-\lambda} + \frac{i}{k-\lambda'}\right) \frac{k-i\alpha}{k+i\alpha} \frac{1}{k-i\alpha} \frac{dk}{2\pi}}$$

$$\cancel{= \frac{1}{\lambda+i\alpha} - \frac{1}{\lambda'+i\alpha}} \quad ?$$

For example: Take $h = f_- + Sf_+$ with $f_+ \in H_+$, $f_- \in H_-$. Then $h \perp H_+ \cap SH_- = \mathcal{H}_0$. So if we take

$$h = \frac{1}{k-i} - \frac{k-i\alpha}{k+i\alpha}\frac{1}{k+i}$$

Then $h = O\left(\frac{1}{k^2}\right)$ so $h \in \mathcal{D}_k$, yet

$$(h, e_{in}) = i \int \frac{1}{k-i} - \underbrace{\frac{k-i\alpha}{k+i\alpha}\frac{1}{k+i}}_{\text{anal. in UHP}} \frac{dk}{2\pi i} = i$$

So what this means is that I do not yet understand entrances and exits for a continuous port.

August 21, 1978:

Suppose given a prob measure $d\nu$ on S^1 and let us form the associated sequence of orthonormal polynomials $1 = p_0, p_1, \ldots, p_n, \ldots$, as well as $q_n = z^n p_n^*$, so that we have

$$\begin{pmatrix} p_n \\ q_n \end{pmatrix} = \underbrace{\frac{1}{\sqrt{1-|h_n|^2}} \begin{pmatrix} 1 & h_n \\ \bar{h}_n & 1 \end{pmatrix}}_{\theta(h_n)} \begin{pmatrix} z & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} p_{n-1} \\ q_{n-1} \end{pmatrix}$$

where $h_n = (p_n, q_n) = \int z^n p_n^2 \, d\nu$. [redacted]

[redacted]

We can also form the analytic function

$$R(z) = \theta(-h_0) \begin{pmatrix} z & 0 \\ 0 & 1 \end{pmatrix} \theta(-h_1) \begin{pmatrix} z & 0 \\ 0 & 1 \end{pmatrix} \cdots\cdots$$

and the port belonging to the sequence $h_0, h_1, \ldots\ldots$

Let's begin with $\mathcal{H}_0 = L^2(S^1, d\mu)$ and $e_{out} = \zeta^{-1}$, $e_{in} = 1$, so that $U(e_{out}) = e_{in}$ where $U =$ multiplication by ζ. This is the standard model for a port $(\mathcal{H}_0, V, e_{out}, e_{in})$ which has been terminated by the unitary operator U which extends V and carries e_{out} to e_{in}. Let's compute the response function for this port. If $|z| < 1$, we need a vector $\perp (1 - \bar{z}V)\mathcal{D}_V$. Since $e_{out} = \zeta^{-1} \perp \mathcal{D}_V$ such a vector is $(1 - zU^*)^{-1} e_{out} = (1 - z\zeta^{-1})^{-1}\zeta^{-1}$. Then

$$R(z) = \frac{((1 - zU^*)^{-1} e_{out}, e_{in})}{((1 - zU^*)^{-1} e_{out}, e_{out})} = \frac{\int \frac{\zeta^{-1} d\nu}{1 - z\zeta^{-1}}}{\int \frac{d\nu}{1 - z\zeta^{-1}}}$$

Then
$$1 - zR(z) = \frac{1}{\int \frac{d\nu}{1 - z\zeta^{-1}}} \qquad 1 + zR(z) = \frac{\int \frac{1+z\zeta^{-1}}{1-z\zeta^{-1}}\,d\nu}{\int \frac{d\nu}{1-z\zeta^{-1}}}$$

so
$$\boxed{\frac{1+zR(z)}{1-zR(z)} = \int \frac{1+z\zeta^{-1}}{1-z\zeta^{-1}}\,d\nu(\zeta)}$$

The next thing is to relate the Schur system associated to this port with the orthogonal polynomials associated to $d\nu$.

$$\mathcal{H}_{-1} \perp \zeta^{-1}$$
$$\mathcal{H}_{-2} \perp \zeta^{-2}, \zeta^{-1}$$
$$\mathcal{H}_{-3} \perp \zeta^{-3}, \zeta^{-2}, \zeta^{-1}$$

Now
$$\tilde{p}_n = pr_{\mathcal{H}_n}(\zeta^n e_{out}) \qquad \tilde{q}_n = pr_{\mathcal{H}_n}(e_{in})$$

$$\tilde{p}_{-1} = pr_{\mathcal{H}_{-1}}(\zeta^{-2}) = \zeta^{-2} - h_{-1}\zeta^{-1} \qquad h_{-1} = \int \zeta^{-1}\,d\nu$$

$$\zeta\tilde{p}_{-1} = \zeta^{-1} - h_{-1} = \underset{\substack{\shortparallel \\ p_0}}{e_{out}} - h_{-1}\underset{q_0}{e_{in}}$$

$$\tilde{q}_{-1} = pr_{\mathcal{H}_{-1}}(e_{in}) = 1 - \bar{h}_{-1}\zeta^{-1} = q_0 - \bar{h}_{-1}p_0$$

Thus it is clear that p_{-n} is obtained by orthonormalizing the sequence $\zeta^{-1}, \dots, \zeta^{-n-1}$ and consequently with minor changes the Schur system belonging to this port coincides with the system of orthogonal polynomials associated to $d\nu$.

August 22, 1978.

$$\frac{1}{i} f(z) = \frac{1+zR(z)}{1-zR(z)} = \int \frac{1+z\,\zeta^{-1}}{1-z\,\zeta^{-1}}\, d\nu(\zeta)$$

$$\operatorname{Im} f(z) = \operatorname{Re}\left(\frac{1}{i} f(z)\right) = \int \frac{\operatorname{Re}(1+z\zeta^{-1}-\bar{z}\zeta-|z|^2)}{|1-z\zeta^{-1}|^2}\, d\nu(\zeta) = \int \frac{1-|z|^2}{|1-z\zeta^{-1}|^2}\, d\nu(\zeta)$$

Now $\displaystyle\int \frac{1+z\zeta^{-1}}{1-z\zeta^{-1}} \frac{d\theta}{2\pi} = \int \frac{\zeta+z}{\zeta-z} \frac{d\zeta}{2\pi i \zeta} = \frac{z+z}{z} + \left(\frac{z}{-z}\right) = 1$

hence we see that $\displaystyle\frac{1-|z|^2}{|1-z\zeta^{-1}|^2} \frac{d\theta}{2\pi} \longrightarrow \delta(\zeta-\zeta_0)$

as $z \to \zeta_0$ radially. Consequently as $r \nearrow 1$

$$\operatorname{Im}(f(r\zeta)) \frac{d\theta}{2\pi} \longrightarrow d\nu(\zeta).$$

Now we know that $R \in H^\infty$, hence $R(r\zeta) \longrightarrow R(\zeta)$ a.e. as $r \nearrow 1$. Consequently if one knows that $\zeta R(\zeta)$ stays away from 1 as ζ ranges over S^1, then one sees that $\lim_{r\nearrow 1} f(r\zeta) = f(\zeta)$ is a bounded measurable fn. Consequently, it follows that $d\nu(\zeta) = \operatorname{Im}(f(\zeta))\frac{d\theta}{2\pi}$ is absolutely continuous with respect to Lebesgue measure.

Suppose R analytic for $|z| \leq 1$ and that $\zeta_0 R(\zeta_0) = 1$ but that $|R|$ not identically 1 on S^1. Then $f(z)$ maps the unit disk to the UHP and it extends meromorphically to the closed unit circle with poles at each such point ζ_0. Such a pole has to be simple because $\operatorname{Im} f > 0$ in the disk, and the residue is positive. In this case $d\nu$ will have singular part concentrated at these poles.

August 24, 1978:

It might be the case that the sequence $\{h_n\}$ is uniquely determined by R when one knows $|R| \leq 1-\varepsilon$. Some examples supporting this are as follows.

1) Consider a sequence $\{h_n, n \geq 0\}$ with $|h_0| = 1$. This means we are looking at a terminated port. From Aug. 22 we know the associated spectral measure is absolutely continuous wrt Lebesgue measure assuming $|R| \leq 1-\varepsilon$. Hence the scattering determines the whole picture.

2) Consider a Schroedinger equation $-u'' + qu = k^2 u$ with q of compact support having no bound states. Then when we calculate the scattering

$$e^{-ikx} \longleftrightarrow Ae^{-ikx} + Be^{ikx}$$

it can happen that A has a simple pole at $k=0$. In fact this happens when $1 \longleftrightarrow$ non-constant linear function, which gives us different Dirac systems with the same Schroed. equation and hence the same scattering. But in this case we have $A \sim \frac{c}{k}$ $B \sim -\frac{c}{k}$ as $k \to 0$ so that

$$R = \frac{B}{A} \longrightarrow -1 \qquad \text{as } k \longrightarrow 0.$$

When A, B are analytic at $k=0$, one has from $|A|^2 - |B|^2 = 1$ that $|R| < 1$.

Here's an example of a doubly-infinite Schur system with point spectrum. Suppose that $(\mathcal{H}_0, V, e_{out}, e_{in})$

is a port such that upon extending V to the unitary operator with $U(e_{out}) = e_{in}$ we get the discrete eigenvalue z_0. This means there is a $g \in \mathcal{D}_V$ such that

$$U(e_{out} + g) = e_{in} + Vg = z_0(e_{out} + g)$$

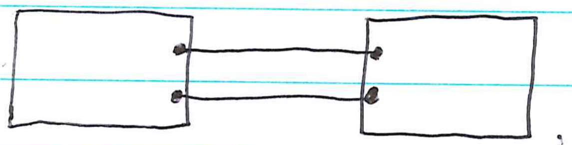
(or equivalently that $e_{in} - z_0 e_{out} \in (V - z_0)\mathcal{D}_V$). Take another such port with primes and put

$$\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_0'$$

U = unitary operator extending $V \oplus V'$ with

$$U(e_{out}) = e_{in}' \qquad U(e_{out}') = e_{in}.$$

Intuitively we get the ports connected together by a unit length of line:



But then

$$U(e_{out} + g + e_{out}' + g') = (e_{in}' + Vg + e_{in} + V'g')$$

$$= z_0(e_{out} + g + e_{out}' + g')$$

and so U has point spectrum. This shows that I can connect together two ports with scattering and obtain bound states.