July 10, 1978

Let A be a closed d.d. symmetric operator on $\mathcal{H}$ without self-adjoint component and having indices 1,1. I propose to connect A up to a trans. line. What this means is that we will construct a self-adjoint operator $\tilde{A}$ on the Hilbert space

$$\tilde{\mathcal{H}} = L^2(-\infty,0) \oplus \mathcal{H} \oplus L^2(0,\infty)$$

which extends A on $\mathcal{H}$ and the operator A' on $L^2(-\infty,0)\oplus L^2(0,\infty) = L^2(-\infty,\infty)$ given by $i\frac{d}{dx}$ on the domain consisting of L^2 functions f which are absolutely continuous on $\mathbb{R}-\{0\}$ and have $f' \in L^2$. We've seen that $\tilde{A}$ is determined by the subspace $\mathcal{D}_{\tilde{A}}$:

$$\mathcal{D}_A \oplus \mathcal{D}_{A'} \subset \mathcal{D}_{\tilde{A}} \subset \mathcal{D}_{A^*} \oplus \mathcal{D}_{A'^*}$$

which is such that

$$\mathcal{D}_{\tilde{A}}/\mathcal{D}_A \oplus \mathcal{D}_{A'} \subset W_A \oplus W_{A'}$$

is maximal isotropic. Moreover a symplectic isomorphism

$$\theta : W_A \xrightarrow{\sim} W_{A'}$$

determines such an isotropic subspace and hence an $\tilde{A}$. So suppose θ is chosen - this ~~constitutes~~ the connection of the port $(\mathcal{H},A)$ to the trans. line - and let $U(t) = e^{i\tilde{A}t}$ be the associated one-parameter unitary group belonging to $\tilde{A}$. On $L^2(0,\infty)$ and $L^2(-\infty,0)$ one has

$$(U(t)f)(x) = e^{-t\frac{d}{dx}} f(x) = f(x-t)$$

provided this is defined.

The next thing to do is compute the scattering function belonging to $\tilde{\mathcal{H}}$.

Because any $f \in \mathcal{D}_{\tilde{A}}$ yields absolutely continuous functions on $(-\infty, 0]$, $[0, \infty)$ we can define δ_i, δ_{-i} by

$$(f, \delta_i) = f(0^+)$$
$$(f, \delta_{-i}) = f(0^-)$$

Technically we use a "rigged" Hilbert space with $\mathcal{H}_0 = \tilde{\mathcal{H}}$, $\mathcal{H}_1 = \mathcal{D}_{\tilde{A}}$, and then $\delta_i, \delta_{-i} \in \mathcal{H}_{-1}$. Then one has

$$in(f)(x) = \left(f, \mathcal{U}(x)\delta_i\right)$$
$$out(f)(x) = \left(f, \mathcal{U}(x)\delta_{-i}\right)$$

I want to compute in and out when $f \in \mathcal{H}$. Recall

$$\begin{tikzcd}
\mathcal{D}_{\tilde{A}} \arrow[r, hook] \arrow[d] & \mathcal{D}_{A^*} \times \mathcal{D}_{A'^*} \arrow[d] \\
\Gamma_\theta \arrow[r, hook] & W \times W'
\end{tikzcd}
\qquad
\begin{tikzcd}
f \arrow[r, mapsto] & (j^*f, j'^*f) \arrow[d, mapsto] \\
 & (\overline{j^*f}, \overline{j'^*f})
\end{tikzcd}$$

hence $\theta(\overline{j^*f}) = \overline{j'^*f}$, where $\overline{}$ denotes W image. In other words if we want $f(0^+), f(0^-)$, we can take $\theta(\overline{j^*f})$.

?

<u>Conjecture:</u> To get $S(k)$ take the line $\mathrm{Ker}(A^*-k)$ in $\mathcal{D}_{A^*}$ and consider its image under

$$\mathcal{D}_{A^*} \longrightarrow W \overset{\sim}{—} W' \xrightarrow{\sim} \mathbb{C}^{\oplus 2}$$

and take the ratio $\dfrac{0^- \text{ component}}{0^+ \text{ component}}$.

One way of seeing that this should be the case is to consider an eigenfunction (generalized sense) for $\tilde{A}$. If k is non-real, then we let ψ_k be the eigenfunction which on $[0,\infty)$ is e^{-ikx}. Then on $(-\infty, 0]$ it should be $S(k)e^{-ikx}$ and in $\mathcal{H}$ ψ_k projects to an element of $\mathrm{Ker}(A^*-k)$ with the boundary values $S(k)$ at 0^- and 1 at 0^+.

June 11, 1978

Suppose given a port $(\mathcal{H}, A)$ together with

$$\mathcal{D}_{A^*}/\mathcal{D}_A \longrightarrow \mathbb{C}^2$$
$$f \longmapsto \begin{pmatrix} f(0^+) \\ f(0^-) \end{pmatrix}$$

such that

$$\frac{1}{i}\{(A^*f,f) - (f,A^*f)\} = |f(0^+)|^2 - |f(0^-)|^2$$

(Think of $A = i\frac{d}{dx}$ on $L^2(a,b)$). Then we get a self-adjoint operator [redacted] $\tilde{A}$

$$\tilde{\mathcal{H}} = L^2(-\infty,0) \oplus \mathcal{H} \oplus L^2(0,\infty)$$

extending $A \oplus$ the minimal operator A' defined by $i\frac{d}{dx}$ on the 2 L^2-spaces, whose domain $\mathcal{D}_{\tilde{A}}$ consists of triples (f,h,g) with $h \in \mathcal{D}_{A^*}$, $f,g \in \mathcal{D}_{A'^*}$ (i.e. f,g are abs. continuous L^2 whose derivatives are also in L^2) such that $f(0-) = h(0-)$, $h(0+) = g(0+)$.

We can Fourier transform to the k-space model

$$\tilde{\mathcal{H}} = H_- \oplus \mathcal{H} \oplus H_+ \qquad f(x) = \int e^{-ikx}\hat{f}(k)\frac{dk}{2\pi}$$

whence $\mathcal{D}_{\tilde{A}}$ consists of (f,h,g) with

$$h(0^+) = \int g(k)\frac{dk}{2\pi} \qquad h(0^-) = \int f(k)\frac{dk}{2\pi}$$

We have elements $\delta_i, \delta_{-i} \in \mathcal{D}_{\tilde{A}}^*$ which represent these boundary values:

$$(f,\delta_i) = f(0^+) \qquad (f,\delta_{-i}) = f(0^-) \qquad \forall f \in \mathcal{D}_{\tilde{A}}$$

$$g \longmapsto (g, U(x)\delta_i)$$

$$\tilde{\mathcal{H}} \underset{\longleftarrow}{\xrightarrow{\text{out}}} L^2(\mathbb{R})$$

$$\int f(x)U(x)\delta_i\,dx \longleftarrow\!\shortmid\; f(x)$$

How to ascribe meaning to these formulas. If $g \in \mathcal{D}_{\tilde{A}}$ then $(g, U(x)\delta_i)$ is an absolutely continuous function of x, as one sees by using the $L^2(0,\infty)$ tail. Then

$$\text{out}(g) = (g, U(x)\delta_i)$$

is a well-defined map from $\mathcal{D}_{\tilde{A}}$ to $L^2(\mathbb{R})$ which is of norm ≤ 1, and hence extends to $\tilde{\mathcal{H}}$ by continuity.

To compute the scattering we want the map

$$f(x) \longmapsto \int f(x)U(x)\delta_i\,dx \longmapsto \text{out}\int f(x)U(x)\delta_i\,dx$$

$$\int f(x')\big(U(x'-x)\delta_i, \delta_{-i}\big)dx' \overset{\text{formally}}{=} \left(\int f(x')U(x')\delta_i\,dx', U(x)\delta_{-i}\right)$$

to

$$S(k) = \int (\delta_i, U(x)\delta_{-i})e^{ikx}\,dx$$

(formal calculation)

$$= \Bigg(\underbrace{\int e^{-ikx}U(x)\delta_i\,dx}_{\psi(k)}, \delta_{-i}\Bigg)$$

Here $\psi(k) = e^{-ikx}$ for $x > 0$ (i.e. in $L^2(0,\infty)$), and $\psi(k)$ is a formal eigenfunction for $\tilde{A}$. Specifically $\psi(k)$ is orthogonal to $(\tilde{A}-\bar{k})g$ where $g \in \mathcal{D}_{\tilde{A}}$ and g has compact support.

Assuming that A has no self-adjoint components we know that $\tilde{\mathcal{H}}$ is generated by the elements $f\delta_i + g\delta_{-i}$ as $f, g \in L^2\left(\frac{dk}{2\pi}\right)$ where

$$f\delta_i = \int \check{f}(x)\, U(x)\, \delta_i\, dx \qquad \check{f}(x) = \int e^{-ikx} f(k)\, \frac{dk}{2\pi}$$

$$= \int \frac{dk}{2\pi} \int e^{-ikx} U(x)\, \delta_i\, dx \cdot f(k) = \int f(k)\, \psi(k)\, \frac{dk}{2\pi}$$

Also we can identify $\mathcal{H}$ with the inverse image of $H_+ \times H_-$ under

$$\tilde{\mathcal{H}} \xrightarrow{(\mathrm{out},\, \mathrm{in})} L^2\left(\frac{dk}{2\pi}\right) \times L^2\left(\frac{dk}{2\pi}\right)$$

and we can filter $\tilde{\mathcal{H}}$ by $F_x \tilde{\mathcal{H}} = (\mathrm{out}, \mathrm{in})^{-1}\left(H_+ \times e^{-ikx} H_-\right)$. One has $F_x \tilde{\mathcal{H}} \subset \mathcal{H}$ for $x \leq 0$.

Let $T(t)$, $t \geq 0$ be the contraction semigroup on $\mathcal{H}$ induced by $U(t) = e^{it\tilde{A}}$. We've seen that we get a filtration on $\mathcal{H}$ by considering $F^t\mathcal{H} = \{ h \mid \|T(t)h\| = \|h\| \}$. Equivalently $F^t\mathcal{H}$ consists of h such that $U(t)h \in \mathcal{H}$, i.e. such that $\mathrm{in}(U(t)h) = e^{ikt}\, \mathrm{in}(h) \in H_-$, hence

$$F^t\mathcal{H} = (\mathrm{out}, \mathrm{in})^{-1}\left(H_+ \times e^{-ikt} H_-\right) = F_{-t}\tilde{\mathcal{H}}.$$

(Formulas for reference: If $T(t) = e^{tB}$, then $A \subset \frac{1}{i} B \subset A^*$. The form

$$\frac{1}{i}\left\{ (A^*u, u) - (u, A^*u) \right\} = |u(0^+)|^2 - |u(0^-)|^2$$

restricted to $\mathcal{D}_B$ is $-(Bu, u) - (u, Bu)$ which we know is ≥ 0. This is consistent with

$$\mathcal{D}_B = \{ u \in \mathcal{D}_{A^*} \mid u(0^-) = 0 \}. \;)$$

Next stage is to [redacted] understand de Branges spaces.

Suppose that $S(k) = \dfrac{\overline{E(\bar{k})}}{E(k)}$ where $E(k)$ is an entire function which doesn't vanish in the closed UHP. Then $|S(k)| < 1$ in the UHP $\Rightarrow$ E de B function. Then

$$H_+ \cap SH_- \xrightarrow{\sim} EH_+ \cap E^{\#} [redacted] H_-$$

$$g \longmapsto Eg = f \quad \text{with} \quad \|f\|^2 = \int |f|^2 \frac{dk}{2\pi |E|^2}$$

So clearly? f is entire and $\dfrac{f}{E} \in H_+$, $\dfrac{f}{E^{\#}} \in H_-$.
Remaining condition comes from the fact that for $g \in H_+$ one has

$$g(k_0) = \int_0^\infty e^{ik_0 x} \check{g}(x)\, dx \underset{\substack{\uparrow \\ \text{Perseval}}}{=} \left(g, \widehat{e^{-i\bar{k}_0 x}}\right)$$

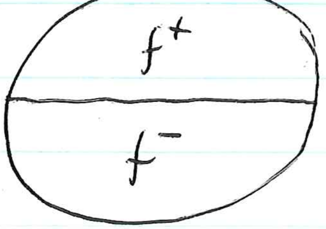
$$\widehat{e^{-i\bar{k}_0 x}} = \int_0^\infty e^{-i(k-\bar{k}_0)x}\, dx = \frac{i}{k-\bar{k}_0}$$

$$\|e^{-i\bar{k}_0 x}\|^2 = \int_0^\infty e^{-2(\operatorname{Im} k_0)x}\, dx = \frac{1}{2\operatorname{Im} k_0}$$

hence

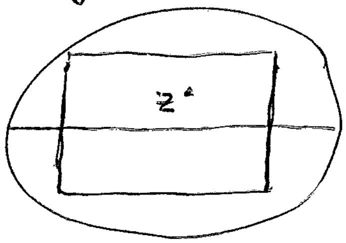
$$|g(k_0)| \le \|g\| \cdot \frac{1}{\sqrt{2\operatorname{Im} k_0}}$$

Here's why f has to be analytic on real axis. Suppose given f^+ analytic in [redacted] the upper half disk and f^- analytic in the lower half disk. Assume f^+, f^- have vertical limits at real points almost everywhere and that these coincide a.e.



Finally assume that [redacted] $f^+(x+iy) \to f^+(x)$ as $y \downarrow 0$ in L^1_{loc}, similarly for f^-. Then you can

Conclude f^+ f^- [redacted] fit together to give an analytic function on the disk. You can define f via Cauchy integral



$$f(z) = \frac{1}{2\pi i}\int_{\square} \frac{f(\zeta)}{\zeta - z}\, d\zeta$$

[redacted] Take vertical sides to be good for vertical [redacted] convergence, then f will be continuous on the rectangle and so the integral will be an analytic fn. of z in the rectangle. If $Im(z) > 0$ then we can push the bottom of the rectangle to the real axis and then just ~~above~~ and [redacted] use Cauchy to get $f^+(z)$.

July 12, 1978

Review discrete scattering. Let's begin with $(\mathcal{H}, V)$ choose $u_i \perp \mathcal{D}_V$, $u_{-i} \perp V\mathcal{D}_V$ and form $\tilde{\mathcal{H}}$ together with

$$\tilde{\mathcal{H}} \xrightarrow{(\text{out},\text{in})} L^2(S^1) \times L^2(S^1)$$

$$fu_i + gu_{-i} \longmapsto (Sf+g,\ f+\bar{S}g)$$

where
$$S(z) = \sum_n (u_i, U^n u_{-i}) z^n = ((1-zT^*)^{-1}u_i, u_{-i})$$

Recall we put
$$F_n = (\text{out},\text{in})^{-1}(H_+ \times z^n H_-)$$

and
$$p_n = pr_{F_n}(z^n u_i)/\text{norm const}$$
$$q_n = pr_{F_n}(u_{-i})/\text{norm. const.}$$

Then p_n is a unit vector spanning $F_n \ominus F_{n-1}$ and q_n is a unit vector spanning $F_n \ominus zF_{n-1}$. In the present situation $F_0 = \mathcal{H}$ and $p_n = z^n u_i$, $q_n = u_{-i}$ for $n \geq 0$. The Schur parameters are $h_n = (p_n, q_n)$ and one has

$$\begin{array}{ccc} F_{n-1} \subset & F_n & \\ & \cup & \\ & zF_{n-1} & \end{array} \qquad \begin{aligned} p_n - h_n q_n &= k_n z p_{n-1} \\ q_n - \bar{h}_n p_n &= k_n q_{n-1} \end{aligned} \qquad k_n = \sqrt{1-|h_n|^2}$$

or
$$\begin{pmatrix} p_n \\ q_n \end{pmatrix} = \frac{1}{\sqrt{1-|h_n|^2}} \begin{pmatrix} 1 & h_n \\ \bar{h}_n & 1 \end{pmatrix} \begin{pmatrix} z p_{n-1} \\ q_{n-1} \end{pmatrix}$$

Under what conditions do we

get a scattering situation as $n \to -\infty$? This means that the limits

$$\lim_{n\to-\infty} g_n = v_{-i} \qquad \lim_{n\to-\infty} z^{-n} p_n = v_i$$

exist and that they give rise to a scattering description of $\tilde{\mathcal{H}}$.

Put $\quad \tilde{g}_n = pr_{F_n}(u_{-i})$. Then

$$\tilde{g}_n - (\tilde{g}_n, p_n) p_n = \tilde{g}_{n-1}$$

$$= \|\tilde{g}_n\| \left(g_n - \bar{h}_n p_n \right) = \|\tilde{g}_n\| k_n g_{n-1}$$

or

$$\|\tilde{g}_{n-1}\| = \|\tilde{g}_n\| k_n$$

$$\|\tilde{g}_{-m-1}\| = k_{-m} \|\tilde{g}_{-m}\|$$

or

$$\|\tilde{g}_{-m}\| = k_{-m+1} \|\tilde{g}_{-m+1}\| = k_{-m+1} k_{-m+2} \cdots k_0 \|\tilde{g}_0\|$$

$$= \prod_{j=0}^{m-1} k_{-j} = \prod_{j=0}^{m-1} \left(1 - |h_{-j}|^2\right)^{1/2}$$

Now I claim that if g_n converges, then $\prod_{j\geq 0} (1-|h_{-j}|^2)^{1/2}$ converges and conversely. The point is that $F_0 = \langle p_0, p_{-1}, p_{-2}, \ldots\ldots \rangle \oplus F_{-\infty}$ with $F_{-n} = \langle p_{-n}, \ldots \rangle \oplus F_{-\infty}$ so that the question is whether

$$\tilde{g}_{-\infty} = \lim_{n\to\infty} \tilde{g}_{-n} = pr_{F_{-\infty}}(u_{-i})$$

is non-zero or not. If non-zero, then the infinite product converges to $\|\tilde{g}_{-\infty}\|$. If zero, then $u_{-i} \in \langle p_0, p_{-1}, \ldots \rangle$ and so the g_n belong to this subspace, hence $v = \lim g_n$ is in this subspace. But $g_n \perp p_0, p_{-1}, \dots, p_{-n+1}$ so v would have to be $\perp$ to all the p_{-j}.

So we see that if $\sum_{j=0}^{\infty} |h_j|^2 < \infty$, then we get limits for $g_n, z^{-n} p_n$ as $n \to -\infty$. In fact

$$g_{-\infty} = pr_{F_{-\infty}}(u_{-i}) / \text{norm}.$$

Since

$$F_{-\infty} = \bigcap F_{-n} = (\text{out}, \text{in})^{-1}(H_+ \times \bigcap z^{-n} H_-) = (\text{out}, \text{in})^{-1}(H_+ \times 0)$$

we have $\text{in}(g_{-\infty}) = 0$ and $\text{out}(g_{-\infty}) \in H_+$

Notice also that for $m > 0$

$$(z^m g_{-\infty}, g_{-\infty}) = \lim_{n \to \infty} (z^m g_{-n}, g_{-n+m+1}) = 0$$

Recall the definition of $\tilde{\mathcal{H}}$ with

$$\|fu_i + gu_{-i}\|^2 = \|f + \bar{S}g\|^2 + \|g\|^2 - \|\bar{S}g\|^2$$

It follows that $fu_i + gu_{-i} = (f + \bar{S}g)u_i + g(u_{-i} - \bar{S}u_i)$ is an orthogonal decomposition, hence

$$\tilde{\mathcal{H}} = L^2(S^1)u_i \oplus \text{Ker}(\text{in})$$

where $\text{Ker}(\text{in})$ has the cyclic vector $u_{-i} - \bar{S}u_i$ and hence

$$L^2\left(S^1, (1 - |S|^2)\frac{d\theta}{2\pi}\right) \xrightarrow{\sim} \text{Ker}(\text{in})$$

$$g \longmapsto g(u_{-i} - \bar{S}u_i)$$

So now choose g so that $g(u_{-i} - \bar{S}u_i) = g_{-\infty}$

From the fact that $(z^m g_{-\infty}, g_{-\infty}) = \delta_m$ we conclude that

$$|g|^2(1-|S|^2) = 1 \qquad \text{a.e.}$$

Put $\varphi = \mathrm{out}(g_{-\infty})$ which we know belongs to H_+. Then we find

$$\varphi = \mathrm{out}\big(g(u_i - \bar{S}u_{-i})\big) = g(1-|S|^2)$$

hence

$$|\varphi|^2 = |g|^2(1-|S|^2)^2 = 1-|S|^2.$$

Consequently we have shown that there is a $\varphi \in H_+$ with $|\varphi|^2 = 1-|S|^2$ which one knows is equivalent to $\log(1-|S|^2) \in L^1$. One would also like to know that φ is an outer function. In any case we are lead to the following

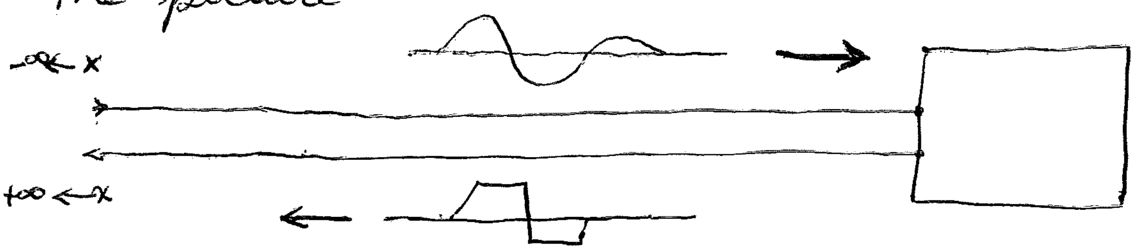
<u>Conjecture</u>: There is natural 1-1 correspondence between null-set equivalence classes of measurable functions $S : S^1 \longrightarrow \mathbb{C}$ such that $|S(z)| \leq 1$ and $\log(1-|S|^2)$ is integrable on one hand and L^2 sequences $\{h_n, n \in \mathbb{Z}\}$ with $|h_n| < 1$ for all n. Moreover we should have Szegö's formula

$$\prod_{n\in\mathbb{Z}} (1-|h_n|^2) = \exp \int \log(1-|S|^2)\,\frac{d\theta}{2\pi}$$

and S is analytic in the disk iff $h_n = 0$ for $n>0$.

July 15, 1978

Motivation will be provided by the notion of a port (or 1-port) from electrical engineering. This is an electrical device represented by the box in the picture



which can be connected to a transmission line represented by the two lines on the left. The transmission line carries signals coming in and going out from the port ~~propagating to the left and to the right~~ with unit speed. It will be convenient to imagine ~~think~~ the incoming signal as travelling along the upper line which is coordinatized by $-\infty < x \leq 0$ so that $-x$ is the distance from the port. The incoming signal then has the form $f(x-t)$, $x \leq 0$ for some function f. Similarly the lower line is coordinatized by $0 \leq x < \infty$ with x = distance from the port, and it carries the outgoing signal ~~which has the form~~ $g(x-t)$, $x \geq 0$.

Suppose now that for $t < 0$ there is zero signal on the outgoing line and in the box and that a signal $f(x-t)$, $x \leq 0$ is coming in. with

$f(x)=0$ for $x>0$.

Suppose now that one has a δ-function signal $\delta(x-t)$, $x\le 0$ coming into the port and that for $t<0$ the port and the outgoing line are unexcited. Then at $t=0$ this δ-function signal enters the port and one sees a reflected signal $K(x-t)$, $x\geq 0$ going out with $K(x)=0$ for $x>0$. By linearity if the system is unexcited for $t\ll 0$ except for an incoming signal $f(x-t)$, $x\le 0$, then the reflected signal is given by $g(x-t)$, $x\geq 0$ where

(*) $$g(x) = \int K(x-x')f(x')\,dx'.$$

If we use the Fourier transform to analyze the signals into frequencies.

$$f(x) = \int e^{-ikx}\hat{f}(k)\,dk/2\pi$$

$$\hat{f}(k) = \int e^{ikx} f(x)\,dx$$

then the relation (*) between the input f and the output g becomes

$$\hat{g}(k) = \hat{K}(k)\hat{f}(k).$$

~~Note that $\hat{f}(k) = \int e^{ikx} f(x)\,dx$ is analytic in the LHP.~~

The function $\hat{K}(k)$ has an analytic extension to the LHP because $K(x)=0$ for $x>0$. It is called the reflection coefficient or response function for the port.

In the following it will be more convenient to work with $R(k) = \overline{\hat{K}(k)}$ which extends analytically to the UHP.

Next we want to present the mathematical description of a port. We shall discrete ports where x, t run over $\mathbb{Z}$ and the signals on the line are given by sequences f_n, $n \in \mathbb{Z}$. The Fourier transform becomes

$$\hat{f}(z) = \sum_{n \in \mathbb{Z}}{}' f_n z^n$$

$$f_n = \int \hat{f}(z) z^{-n} \, d\theta / 2\pi$$

where $z = e^{i\theta}$ and the integration is over the unit circle $S^1 = \{z \mid |z| = 1\}$. In this case the response function $R(z)$ will be analytic in the disk $|z| < 1$.

July 16, 1978

$$\mathcal{H} \xrightarrow{(out,\, in)} L^2(S^1)^2$$

$$f e_{out} + g e_{in} \longmapsto (f + \bar{R} g,\ R f + g)$$

$$\mathcal{H}_n = (out, in)^{-1}(z^{-n} H_- \times H_+)$$

$$\tilde{p}_n = pr_{\mathcal{H}_n}(U^{-n} e_{out}) \qquad \tilde{q}_n = pr_{\mathcal{H}_n}(e_{in})$$

If $\mathcal{H}_{n+1} < \mathcal{H}_n$, one knows $\exists\, x \in \mathcal{H}_n$ with $out(x) \in z^{-n}H_- - z^{-n-1}H_-$. Hence

$$(x, \tilde{p}_n) = (x, U^{-n} e_{out}) = (out(x), z^{-n}) \neq 0$$

$$(\mathcal{H}_{n+1}, \tilde{p}_n) = (\mathcal{H}_{n+1}, U^{-n} e_{out}) = (out(\mathcal{H}_{n+1}), z^{-n}) = 0$$

and so $\tilde{p}_n$ is a non-zero vector of $\mathcal{H}_n \ominus \mathcal{H}_{n+1}$, and we can define

$$p_n = \tilde{p}_n / \|\tilde{p}_n\|$$

Similarly when $U\mathcal{H}_{n+1} < \mathcal{H}_n$, $\tilde{q}_n \neq 0$ and we can define q_n.

Try to understand when the above doesn't work. I believe everything works beautifully when $\log(1 - |R|^2) \in L^1$, but you should check this carefully. Later.

For now suppose that R is analytic and that $\mathcal{H}_n > \mathcal{H}_{n+1}$ for all n, i.e. that the Schur sequence is infinite. We have the recursion relations.

$$\tilde{p}_n - h_n \tilde{q}_n = U \tilde{p}_{n+1}$$
$$\tilde{q}_n - \bar{h}_n \tilde{p}_n = \tilde{q}_{n+1}$$

where $h_n = (p_n, q_n)$. Also $\|\tilde{q}_{n+1}\| = \sqrt{1-|h_n|^2}\,\|\tilde{q}_n\|$ so that

$$\|\tilde{q}_n\|^2 = \|\tilde{p}_n\|^2 = \prod_{j=0}^{n-1} (1-|h_j|^2)$$

Other forms of the recursion relations are

$$\begin{pmatrix} \tilde{p}_n \\ \tilde{q}_n \end{pmatrix} = \begin{pmatrix} 1 & h_n \\ \bar{h}_n & 1 \end{pmatrix} \begin{pmatrix} U & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \tilde{p}_{n+1} \\ \tilde{q}_{n+1} \end{pmatrix}$$

$$\begin{pmatrix} p_n \\ q_n \end{pmatrix} = \frac{1}{\sqrt{1-|h_n|^2}} \begin{pmatrix} 1 & h_n \\ \bar{h}_n & 1 \end{pmatrix} \begin{pmatrix} U & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} p_{n+1} \\ q_{n+1} \end{pmatrix}$$

Since the p_n are orthonormal we have

$$e_{in} = q_0 = \bar{h}_0 p_0 + \bar{h}_1 \tilde{p}_1 + \cdots + \bar{h}_n \tilde{p}_n + \tilde{q}_{n+1}$$

and hence if we put $\mathcal{H}_\infty = \bigcap \mathcal{H}_n = (\text{out}, \text{in})^{-1}(0 \times H_+)$

$$\tilde{q}_\infty = \lim \tilde{q}_n = pr_{\mathcal{H}_\infty}(e_{in})$$

I saw before that

$$\tilde{q}_\infty \neq 0 \iff \prod_{j\geq 0} (1-|h_j|^2) \quad \text{converges}$$
$$(\text{i.e. } \sum |h_j|^2 < \infty)$$

$$\iff q_\infty = \lim_{n\to\infty} q_n \quad \text{exists}$$

In fact $\|\tilde{q}_\infty\|^2 = \prod_{j \geq 0} (1 - |h_j|^2)$, and we have $q_\infty = \tilde{q}_\infty / \|\tilde{q}_\infty\|$

when $\|\tilde{q}_\infty\| > 0$. Similarly the limit

$$\tilde{t} = \lim_{n \to \infty} U^n \tilde{p}_n = pr_{\cap U^n \mathcal{H}_n} (e_{out})$$

exists, and we have when $\prod (1 - |h_j|^2) > 0$ that $t = \lim U^n p_n$, where $t = \tilde{t} / \|\tilde{t}\|$.

Now let's apply in to the recursion relation

$$(*) \quad \begin{pmatrix} R \\ 1 \end{pmatrix} = in \begin{pmatrix} p_0 \\ q_0 \end{pmatrix} = \begin{pmatrix} 1 & h_0 \\ \bar{h}_0 & 1 \end{pmatrix} \begin{pmatrix} z & 0 \\ 0 & 1 \end{pmatrix} \cdots \begin{pmatrix} z & 0 \\ 0 & 1 \end{pmatrix} in \begin{pmatrix} \tilde{p}_n \\ \tilde{q}_n \end{pmatrix}$$

From the Schur process we have

$$\frac{in \, \tilde{p}_n}{in \, \tilde{q}_n} = R_n$$

and for a given z with $|z| < 1$, $R(z)$ is the unique number such that if $\left\{ \begin{pmatrix} x_n \\ y_n \end{pmatrix} \right\}$ is a solution of

$$\begin{pmatrix} x_n \\ y_n \end{pmatrix} = \begin{pmatrix} 1 & h_n \\ \bar{h}_n & 1 \end{pmatrix} \begin{pmatrix} z & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix}$$

with $y_0 = 1$ and $\left| \dfrac{x_n}{y_n} \right| < 1$ for all n, then $x_0 = R(z)$.

Since both x_n, y_n can't vanish for a non-zero solution of the recursion relation, we see that

$$(in \, \tilde{q}_n)(z) \neq 0 \quad \text{for} \quad |z| < 1.$$

~~Next~~

$$z^n \, in(\tilde{p}_n) = in\left(U^n \tilde{p}_n\right) \longrightarrow in(\tilde{t}) = 0 \quad \text{because}$$

$$\tilde{t} \in \bigcap U^n H_n = (out, in)^{-1}(H_- \times 0)$$

Hence the sequence $in(\tilde{p}_n) \in H_+$ converges to zero, and so for any $|z| < 1$

$$in(\tilde{p}_n)(z) \longrightarrow 0$$

Now suppose that $\tilde{q}_\infty \neq 0$. As $out(\tilde{q}_\infty) = 0$, then $in(\tilde{q}_\infty)$ is a non-zero element of H_+ which is the limit of the $in(\tilde{q}_n)$ which don't vanish for $|z| < 1$. By Hurwitz

(*) $$in(\tilde{q}_\infty)(z) \neq 0 \qquad \text{for } |z| < 1.$$

Hence we see in this case that

$$R_n(z) = \frac{in\, \tilde{p}_n(z)}{in\, \tilde{q}_n(z)} \longrightarrow 0 \qquad \text{as } n \to \infty.$$

whereas in general it just remains of modulus < 1. I want to strengthen (*) to show $in(\tilde{q}_\infty)$ is an outer function.

But

$$in(\tilde{q}_n)(0) = (in\, \tilde{q}_n, 1) = (\tilde{q}_n, e_{in}) \simeq (\tilde{q}_n, pr_{H_n}(e_{in}))$$

$$= \|\tilde{q}_n\|^2$$

For $n = \infty$, $\|in(\tilde{q}_\infty)\| = \|\tilde{q}_n\|$. ?

July 17, 1978:

Lemma: Let $\varphi \in H_+^\infty$ be outer. Then

$$H_+ = \{ f \in L^2 \mid \varphi f \in H_+ \}$$

Proof: If $f \in L^2$ and $\varphi f \in H_+$, then

$$0 = (\overline{\varphi f}, \text{[redacted]} \, z H_+^{\text{[redacted]}}) = (\bar{f}, z \varphi H_+)$$

Because φ is outer, φH_+ is dense in H_+, hence $(\bar{f}, zH_+) = 0 \implies f \in H_+$.

So let us return to the [redacted] situation where R is analytic for $|z|<1$ with Schur parameters $h_0, h_1, \ldots$ (for all n). [redacted] Recall

$$\tilde{p}_n = pr_{\mathcal{H}_n}(U^{-n} e_{out}) \qquad \tilde{q}_n = pr_{\mathcal{H}_n}(e_{in})$$

$$p_n = \text{[redacted]} \, \tilde{p}_n / \|\tilde{p}_n\| \qquad q_n = \tilde{q}_n / \|\tilde{q}_n\|$$

$$\|\tilde{p}_n\| = \|\tilde{q}_n\| = \prod_{j=0}^{n-1} (1 - |h_j|^2)^{1/2}$$

Assume $\prod_{j \geq 0} (1 - |h_j|^2) > 0$ so that

$$\tilde{q}_\infty = \lim_{n\to\infty} \tilde{q}_n \; \text{[redacted]} = pr_{\mathcal{H}_\infty}(e_{in}) \neq 0$$

and $$q_\infty = \tilde{q}_\infty / \|\tilde{q}_\infty\| = \lim_{n\to\infty} q_n \quad \text{exists.}$$

We are going to be interested in the function

$$\varphi_i = in(q_\infty)$$

which belongs to H_+ as $\mathcal{H}_\infty = (out, in)^{-1}(0 \times H_+)$.

Notice $\text{out}(g_\infty) = 0$, so we want to understand the kernel of "out".

Go back to

$$\|f e_{out} + g e_{in}\|^2 = \|f\|^2 + 2\operatorname{Re}(f, \bar{R}g) + \|g\|^2$$
$$= \|f + \bar{R}g\|^2 + \|g\|^2 - \|\bar{R}g\|^2$$

This shows that

$$f e_{out} + g e_{in} = g(e_{in} - \bar{R}e_{out}) + (f + \bar{R}g)e_{out}$$

is the orthogonal decomposition of $f e_{out} + g e_{in}$ relative to $L^2(S^1)e_{out}$ and its orthogonal complement. Put another way, recall that out is essentially the projection onto the subspace $L^2(S^1)e_{out}$ and hence

$$f e_{out} + g e_{in} - (f + \bar{R}g)e_{out} = g(e_{in} - \bar{R}e_{out})$$

is the projection onto $\operatorname{Ker}(\text{out})$. So we see that $\operatorname{Ker}(\text{out})$ is a cyclic subspace with generator $e_{in} - \bar{R}e_{out}$. Clearly we get an isomorphism

$$L^2(S^1, (1 - |R|^2)d\theta/2\pi) \xrightarrow{\sim} \operatorname{Ker}(\text{out})$$
$$g \longmapsto g(e_{in} - \bar{R}e_{out})$$

because $\|g(e_{in} - \bar{R}e_{out})\|^2 = \|g\|^2 - \|\bar{R}g\|^2 = \int |g|^2(1 - |R|^2)\, d\theta/2\pi$

Now assume g_∞ exists, and let $g \in L^2(S^1, (1-|R|^2)d\theta/2\pi)$ be such that

$$g(e_{in} - \bar{R}e_{out}) = g_\infty$$

Because $(z^n q_\infty, q_\infty) = \delta_n$ we have

$$|g|^2(1-|R|^2) = 1 \qquad \text{a.e.}$$

Hence applying in we get

$$g(1-|R|^2) = in(q_\infty) = \varphi_i \in H_+$$

or

$$|\varphi_i|^2 = |g|^2(1-|R|^2)^2 = 1-|R|^2.$$

Because $\varphi_i \in H_+$ one knows that this implies

$$\log(1-|R|^2) \in L^1$$

and moreover

$$\varphi(z) = \exp\int \frac{\zeta+z}{\zeta-z}\log(1-|R|^2)^{1/2}\,d\theta/2\pi$$

is the unique outer function with

$$|\varphi|^2 = 1-|R|^2$$
$$\varphi(0) > 0.$$

We want now to show that $\varphi_i = \varphi$. We have a unitary isomorphism

$$\psi : L^2(S^1) \xrightarrow{\sim} \mathrm{Ker}(out)$$
$$1 \longmapsto \frac{1}{\varphi}(e_{in} - \bar{R}\,out)$$

whose composition with in sends 1 to $\frac{1}{\varphi}(1-|R|^2) = \varphi$. In view of the lemma on page 119, the inverse image of H_+ under $in \circ \psi$ is H_+, and so ψ induces an isom $\psi : H_+ \xrightarrow{\sim} (out, in)^{-1}(0 \times H_+) = \mathcal{H}_\infty$

Now $$\tilde{g}_\infty = pr_{\mathcal{H}_\infty}(e_{in}) = pr_{\mathcal{H}_\infty}(e_{in} - \bar{R}e_{out})$$
$$= pr_{\mathcal{H}_\infty}(\psi(\bar{\varphi})) = \psi\, pr_{H_+}(\bar{\varphi})$$
$$= \psi(\bar{\varphi}(0)) = \bar{\varphi}(0)\cdot \frac{e_{in} - \bar{R}e_{out}}{\bar{\varphi}}$$

So applying in we get
$$\|\tilde{g}_\infty\|\cdot \varphi_i = \bar{\varphi}(0)\,\frac{1-|R|^2}{\bar{\varphi}} = \bar{\varphi}(0)\cdot\varphi$$

But $$\|\tilde{g}_\infty\|\varphi_i(0) = (\text{in}\ \tilde{g}_\infty, 1) = (\tilde{g}_\infty, e_{in}) = (\tilde{g}_\infty, pr_{\mathcal{H}_\infty}(e_{in}))$$
$$= \|\tilde{g}_\infty\|^2 = \prod_{j\geq 0}(1-|h_j|^2)$$

Since $\varphi(0) > 0$ we conclude that $\varphi(0) = \varphi_i(0) = \|\tilde{g}_\infty\|$ and that $\varphi = \varphi_i$. Also we get the Szego formula

$$\prod_{j\geq 0}(1-|h_j|^2) = \varphi(0)^2 = \exp\int \log(1-|R|^2)\, d\theta/2\pi .$$

Next stage is to remove assumption that R is analytic. Suppose R measurable on S^1 of modulus ≤ 1 a.e. and that $\log(1-|R|^2) \in L^1$, so that there is an outer function φ with
$$|\varphi|^2 = 1 - |R|^2$$

Hence we have a direct sum decomposition of $\mathcal{H}$

$$
\begin{array}{ccccc}
L^2(S^1) & \xleftarrow{\ in^+\ } & \mathcal{H} & \xrightarrow{\ in\ } & L^2(S^1) \\
f\varphi & \longleftarrow\!\!\shortmid & fe_{out}+ge_{in} & \longmapsto & fR+g \\
1 & \longmapsto & \underbrace{\dfrac{e_{out}-Re_{in}}{\varphi}}_{\text{define: } e_{in}^{+}} \quad , & e_{in} \longleftarrow & 1
\end{array}
$$

Then the composition

$$
\begin{array}{ccccc}
L^2(S^1) & \longrightarrow & \mathcal{H} & \xrightarrow{\ out\ } & L^2(S^1) \\
g & \longmapsto & g\,\dfrac{e_{out}-Re_{in}}{\varphi} & \longmapsto & g\bar{\varphi}
\end{array}
$$

carries H_- densely into H_-. Thus

$$\bigcap U^n \mathcal{H}_n = (out, in)^{-1}(H_- \times 0) \xrightarrow{\ out\ } H_-$$

is dominant (i.e. has a dense image). Similarly we find that

$$\mathcal{H}_\infty = (out, in)^{-1}(0 \times H_+) \xrightarrow{\ in\ } H_+$$

is dominant. From this we can conclude

$$\mathcal{H}_n \xrightarrow{\ (out, in)\ } z^{-n} H_- \times H_+$$

is dominant because $\mathcal{H}_n \supset U^{-n} \bigcap U^m \mathcal{H}_m + \mathcal{H}_\infty$

Question: Is $\mathcal{H}_0 = \mathcal{H}_\infty + \bigcap U^n \mathcal{H}_n$?