Begin with a one-sided infinite J-matrix \( J = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \) with \( L^2 \subseteq \text{V} \), which we interpret as a bounded self-adjoint operator in \( L^2 = \text{V} \).

Then we consider functions \( Z \to \text{V} \), \( t \to u(t) \) such that

\[
Lu(t) = \frac{u(t+1) + u(t-1)}{2}
\]

Energy norm:

\[
E(t)(t) = \|u(t)\|^2 - \frac{1}{2} (u(t+1), u(t+1)) - \frac{1}{2} (u(t-1), u(t-1))
\]

\[
= \|u(t)\|^2 + \|u(t+1)\|^2 - (Lu(t), Lu(t)) - (u(t), u(t))
\]

\[
= \|u(t)\|^2 - \|Lu(t)\|^2 + \|u(t+1) - u(t)\|^2
\]

This is independent of \( t \).

Complete the space of solutions of (30) \( \Phi \) in the energy norm to obtain the space \( \text{H}_0 \). The map \( u(t) \to u(-t) \) is a symmetry of \( \text{H}_0 \), hence \( \text{H}_0 \) decomposes

\[
\text{H}_0 = \text{H}_0^{\text{even}} \oplus \text{H}_0^{\text{odd}}
\]

If \( u \in \text{H}_0^{\text{odd}}, \) i.e. \( u(-t) = -u(t) \), then

\[
E(u) = \frac{\|u(t) - u(-t)\|^2}{2} = \|u(t)\|^2
\]

hence

\[
\text{H}_0^{\text{odd}} \sim \text{V}
\]

If \( u \in \text{H}_0^{\text{even}}, \) i.e. \( u(t) = u(-t) \), then

\[
E(u) = \|u(t)\|^2 - \|Lu(t)\|^2
\]

and so \( \text{H}_0^{\text{even}} \) is isomorphic to \( \text{V} \) with the inner product \( \langle x, y \rangle = (I - L^2)x, y \). This won't be complete.
unless $L \leq 1 - \varepsilon$ and so under completion $H_0^{\omega}$ will become
enlarged; also eigenvectors for the eigenvalues $\pm 1$ will be killed.

Let's see what this construction gives for $V = L^2(d\mu)$, $d\mu$
a measure in $[-1, 1]$, $L$ multiplication by $x$. Then $H_0$
consists of pairs $(f, g)$ in $V$ with

$$
\|f, g\|^2 = \int |f|^2 d\mu + \int |(1-x^2)g|^2 d\mu
$$

Let $d\nu$ denote the measure on $S^1$ which is even and
such that

$$
\int f(x) d\nu = \int f(x) d\mu \quad \lambda = \cos \theta
$$

Then if I associate to $(f, g)$ the function $f(x) + ig(x)\sin \theta$
on $S^1$, we have

$$
\int |f + ig \sin \theta|^2 d\nu = \int \left[ |f|^2 + |g|^2 + (1-x^2) \right] d\mu
$$

and so in this way I can identify $H$ with $L^2(S^1, d\nu)$.

Note that $H^{\text{odd}}$ corresponds to the space of even functions on $S^1$.

So, my notation is losing.

Notice that

$$
u(t) = a(x)x^t + b(x)x^{-t}
$$

is a trajectory in $L^2(\mathbb{R}, d\mu)$. Better

$$
u(t) = \frac{zt - z^{-t}}{z - z^{-1}}
$$

is an odd trajectory with $\nu(1) = 1$, and

$$
u(0) = \frac{zt + z^{-t}}{2}
$$

is an even trajectory with $\nu(0) = 1$. Hence
\[ u(t) = f(\lambda) \frac{z^t - z^{-t}}{z - z^{-1}} + g(\lambda) \frac{z^t + z^{-t}}{2} \]

is the trajectory in \( L^2(\mathbb{R}, d\mu) \) whose odd part corresponds to \( f \) and whose even part corresponds to \( g \).

Write this
\[ u(t) = \frac{f}{\sin \theta} \sin t \theta + g \cos t \theta \]

Then
\[ u(t+1) = \frac{f}{\sin \theta} \left[ \sin t \theta \cos t \theta + \cos t \theta \sin t \theta \right] + g \left[ \cos t \theta \cos t \theta - \sin t \theta \sin t \theta \right] \]
\[ = \frac{f \cos \theta - g \sin^2 \theta}{\sin \theta} \sin t \theta + \left[ f + g \cos \theta \right] \cos t \theta \]

is the trajectory associated to
\[ (f + ig \sin \theta)(\cos \theta + i \sin \theta) = (f \cos \theta - g \sin^2 \theta) + i (f + g \cos \theta) \sin \theta \]

Therefore translation \( u(t) \rightarrow u(t+1) \) corresponds to multiplication by \( z \).

January 5, 1977

Let a \( T \)-matrix with \( L^2 \leq I \) and \( u : \mathbb{Z} \rightarrow \ell^2 \)
a solution of the discrete wave equation.

\[ Lu(t) = \frac{u(t+1) + u(t-1)}{2} \]

In order to describe \( u \) one can use eigenfunction expansions in both \( n \) and \( t \). First do for \( n \):

\[ u(n, t) = \int \varphi(n, \lambda) u(\lambda, t) d\mu(\lambda) \]

\[ u(n, t+1) + u(n, t-1) = 2u(n, t) \]
Then do for $t$:

$$u(\lambda, t) = \int_{-1}^{1} u(\lambda, z) \frac{d\theta}{2\pi}$$

whence $u(\lambda, z)$ satisfies

$$\frac{z + z^{-1}}{2} u(\lambda, z) = \lambda u(\lambda, z)$$

In other words $u(\lambda, z)$ is supported on $\lambda = \frac{z + z^{-1}}{2}$ which means effectively that it's a function (generalized perhaps) of $z$.

So take $u(\lambda, z) \frac{d\theta}{2\pi}$ to be a measure $h(z) \frac{d\theta}{2\pi}$ on the circle $\theta$ which one lifts up to the locus $\lambda = \frac{z + z^{-1}}{2}$. Then $u(\lambda, t) \frac{d\theta}{2\pi}$ can think of as the measure on $[-1, 1]$ obtained by pushing $\frac{\lambda h(z) \frac{d\theta}{2\pi}}{\sin \theta}$ down via $\lambda = \cos \theta$; hence

$$u(\lambda, t) \frac{d\lambda}{\sin \theta} = h(z) \frac{z^t}{2\pi} \frac{d\lambda}{\sin \theta} + h(z) \frac{z^{-t}}{2\pi} \frac{d\lambda}{\sin \theta}$$

$$= \left\{ \begin{array}{ll}
h(z) \frac{z^{-t}}{2\pi \sin \theta} & \\
h(z) \frac{z^t}{2\pi \sin \theta} & \end{array} \right\} \frac{d\lambda}{\sin \theta}$$

Hence

$$u(n, t) = \int_{-1}^{1} \varphi(n, \lambda) \left\{ \frac{h(z) z^t}{2\pi \sin \theta} - \frac{h(z) z^{-t}}{2\pi \sin \theta} \right\} d\mu(\lambda)$$

Let's compare the preceding formula with yesterday's analysis of a path $t \mapsto u(\lambda, t)$ in $L^2(d\mu)$ satisfying

$$\lambda u(\lambda, t) = u(\lambda, t+1) + u(\lambda, t-1)$$
We saw
\[ u(\lambda, t) = f(\lambda) \frac{z^t - z^{-t}}{z - z^{-1}} + g(\lambda) \frac{z^t + z^{-t}}{2} \]
\[ = \left( \frac{f(\lambda)}{z - z^{-1}} + \frac{i g(\lambda) \sin \theta}{2i \sin \theta} \right) z^t + \left( \frac{-f(\lambda) + i g(\lambda) \sin \theta}{-2i \sin \theta} \right) z^{-t} \]

So it's clear that I want to represent solutions \( u(n,t) \) to the wave equation for \( L \) close to \( L_0 \) in the form
\[ u(n,t) = \int \psi(n, \lambda) \frac{x(\lambda)}{\sin \theta} \frac{d\theta}{2\pi} \]

In effect if \( x \) is even: \( x = f(\lambda) \), then this is
\[ u(n,t) = \int \psi(n, \lambda) f(\lambda) \left( \frac{z^{-t}}{\sin \theta} - \frac{z^t}{\sin \theta} \right) \frac{d\theta}{2\pi} \]
regular, i.e. Laurent poly.

And if \( x \) is odd, say \( x = g(\lambda) \sin \theta \), then also \( x \)
got something regular. To be precise: if \( x(\lambda) \in \mathbb{C}[z, z^{-1}] \), then
\[ \frac{x(z) z^{-t}}{\sin \theta} - \frac{x(z^{-1}) z^t}{\sin \theta} \in \mathbb{C}[z, z^{-1}] \]

---

Begin again: let \( L \) be a J-matrix and \( u(n,t) \) a solution of the wave equation. Take the Fourier transform of \( u(n,t) \)
\[ u(\varepsilon) = \sum_{n} u(n,t) z^t \quad u(t) = \int u(\varepsilon) \frac{d\varepsilon}{2\pi} \]
Then because \( \text{Lu}(t) = \frac{u(t+1) + u(t-1)}{2} \) we have

\[
\text{Lu}(z) = \frac{z + z^{-1}}{2} u(z)
\]

and hence \( u(z) \) is a multiple of \( \varphi(\lambda) \) \( \lambda = (z + z^{-1})/2 \),

say

\[ u(n, z) = \varphi(z) \varphi(n, \lambda). \]

Hence we have the representation

\[
 u(n, t) = \int_{\mathbb{S}^1} z^{-t} \varphi(n, \lambda) \varphi(z) \frac{d\theta}{2\pi}
\]

(The above is somewhat formal, however it might be interesting to see what becomes of the above when \( \lambda \) had spectrum outside \([-1, 1]\).)

We have

\[
 u(n, t) = \int_{\mathbb{S}^1} \varphi(n, \lambda) \left( \frac{\varphi(z) - \varphi(z^{-1})}{2} \right) \frac{d\theta}{2\pi}
\]

\[
\frac{\varphi(z) - \varphi(z^{-1})}{2} = \frac{\varphi(z) + \varphi(z^{-1})}{2} \frac{z + z^{-1}}{2} - \frac{\varphi(z) - \varphi(z^{-1})}{2} \frac{z - z^{-1}}{2}
\]

This equation shows that

\[
\forall t \quad \frac{\varphi(z) + \varphi(z^{-1})}{2} \in \mathbb{C}[z, z^{-1}] \quad \iff \quad \varphi(z) \in \mathbb{C}[z, z^{-1}]
\]

(better proof by linear equations), hence it perhaps nicer to use the representation

\[
 u(n, t) = \int_{\mathbb{S}^1} z^{-t} \varphi(n, \lambda) \varphi(z) \frac{d\theta}{2\pi}
\]

\[
\frac{(z - z^{-1})}{2}
\]

\[
\frac{(z - z^{-1})}{2}
\]
Suppose given a Dirac system where $\lambda$ is a sum of $d$ functions at the integers. I have seen that the basic solution is of the form

$$\psi(x, \lambda) = \begin{pmatrix} e^{i\lambda(x-n)} \\ e^{-i\lambda(x-n)} \end{pmatrix} R(h_n) \begin{pmatrix} e^{i\lambda} \\ e^{-i\lambda} \end{pmatrix} R(h_1) \begin{pmatrix} e^{i\lambda} \\ e^{-i\lambda} \end{pmatrix} \cdots R(h_1) \begin{pmatrix} e^{i\lambda} \\ e^{-i\lambda} \end{pmatrix}$$

if $n < x < n+1$. Thus $\psi$ is discontinuous at integer points and

$$\psi(n^+, \lambda) = R(h_n) \psi(n^-, \lambda)$$

If $p_0, p_1, \ldots$ is the system of orthonormal polynomials on $S^1$ described by $h_1, h_2, \ldots$ then we have

$$\psi(n^+, \lambda) = \begin{pmatrix} z^{-n/2} p_n(z) \\ z^{n/2} p_n^*(z) \end{pmatrix}$$

where $z = e^{2i\lambda}$ and the deBroglie function at this point is

$$\eta(n^+, \lambda) = e^{i\lambda} p_n^*(e^{2i\lambda})$$

For example, if all $h_i = 0$, then

$$\psi(n^+, \lambda) = \begin{pmatrix} e^{i\lambda} \\ 0 \\ 0 \\ e^{-i\lambda} \end{pmatrix}$$

and the point evaluator is

$$T_\lambda(\lambda) = \frac{1}{2(\lambda - \bar{\lambda})} \left| \begin{array}{cc} e^{-i\lambda} & e^{-i\lambda} \\ e^{i\lambda} & e^{i\lambda} \end{array} \right| = \frac{\sin n(\lambda - \bar{\lambda})}{\lambda - \bar{\lambda}}$$
Suppose given a probability measure \( \nu \) on \( S^1 \) and let \( p_n \), \( h \) be as usual. We filter \( C[z, z^{-1}] \subset L^2(S^1, d\nu) \) symmetrically.

\( F_n = \text{span of } z^{-n}, z^{-n+1}, \ldots, z^n, \quad W_n = F_n \cap F_{n-1} \). We saw that \( W_n \) is spanned by \( z^{-n} p_n, z^n p_n^* \); however these units vectors are not orthogonal.

\[
\begin{align*}
    l_n &= \text{leading coeff. of } p_n = \text{const. term of } z^n p_n^* > 0 \\
    z p_{n-1} &= k_n p_n - h_n z^{n-1} p_{n-1}^* \\
    k_n p_n(0) &= h_n l_{n-1} \\
    k_{n-1} &= k_n \frac{h_n}{l_n} \\
    (p_n, z^n p_n^*) &= p_n(0) (1, z^n p_n^*) = \frac{p_n(0)}{l_n} \frac{(z^n p_n^*, z^n p_n^*)}{l_n} = \frac{p_n(0)}{l_n} \\
    &= \frac{k_n l_{n-1}}{k_n l_n} = h_n
\end{align*}
\]
Let $d\nu$ be a probability measure on $S^1$ and let $h_1, h_2, \ldots$ be the sequence of fiber parameters. Let us consider the vector space $V$ of finite support column vectors $(e_n)_{n \in \mathbb{Z}}$ with the usual inner product in which the basis vectors $e_n$ are orthonormal. I propose to define a unitary operator in $V$.

Take two copies of $L^2(S^1, d\nu)$ which I denote $L^2(S^1, d\nu)$ and $L^2(S^1, d\nu)$. I will describe orthonormal bases in both. Let $F_n L^2(S^1, d\nu), n=0, 1, \ldots$ be spanned by $z^{-n}, z^{-n+1}, \ldots, z^n$ and put

$$W_{2n} = F_{2n+1} \oplus F_{2n-1}$$

Then $W_{2n}$ has the basis $z^{-n} p_{2n}, z^{n} p_{2n}$ of finite vectors which are not orthogonal in general since

$$(z^{-n} p_{2n}, z^{n} p_{2n}^*) = (p_{2n}, z^{2n} p_{2n}^*) = h_{2n}$$

However there is a unique basis of the form $g_{2n}, g_{2n}^*$ where

$$g_{2n} = a z^{-n} p_{2n} + b z^{n} p_{2n}^*$$

and $a > 0$

Let $F_{2n} L^2(S^1, d\nu) \cong 1/2$ be spanned by $z^{-n+1/2}, \ldots, z^{n-1/2}$ for $n > 1$, and $W_{2n-1} = F_{2n} - F_{2n-2}$. Then $W_{2n-1}$ is spanned by $z^{-n} p_{2n-1}$ and $z^{n-1} p_{2n-1}^*$

and similarly we can find a unique orthonormal basis of the form $g_{2n-1}, g_{2n}^*$ as above.
So now map $V$ into $L^2(S^1 \, dv) \oplus L^2(S^1 \, dv) \varepsilon^{1/2}$ by sending $e_n$ to $q_n$ for $n \geq 0$ and $e_n$ to $q^{-n}$ for $n \leq 0$. Of course $q_0 = 1$. Since the basis is orthonormal, multiplication by $\varepsilon^{1/2}$ is a given by a unitary operator. Problem: Describe carefully the unitary matrix belonging to this operator $\varepsilon^{1/2}$. 
January 8, 1978:

Suppose we review the Bessel $K$-function

\[ K_\nu(x) = \int_0^\infty e^{-\nu t} \cos x t \frac{dt}{t} \]

\[
\frac{d}{dr} K_\nu = -\frac{1}{2} K_{\nu+1} - \frac{1}{2} K_{\nu-1}
\]

\[
\frac{s}{r} K_\nu = \frac{1}{2} K_{\nu+1} - \frac{i}{2} K_{\nu-1}
\]

\[
\left(\frac{d}{dr} + \frac{s}{r}\right) K_\nu = -K_{\nu-1} \quad \left(\frac{d}{dr} - \frac{s}{r}\right) K_\nu = -K_{\nu+1}
\]

\[
\left(\frac{d}{dr} - \frac{s-1}{r}\right)\left(\frac{d}{dr} + \frac{s}{r}\right) K_\nu = K_\nu \quad \text{or} \quad \left(\frac{d^2}{dr^2} + \frac{1}{r} \frac{d}{dr} - \frac{s^2-1}{r^2}\right) K_\nu = 0
\]

\[
\left(-\left(r \frac{d}{dr}\right)^2 + r^2\right) K_\nu = (-s^2) K_\nu
\]

Hence $K_\nu(e^{-x})$ satisfies the DE

\[
\left(-\frac{d^2}{dx^2} + e^{-2x}\right) u = x^2 u
\]

and moreover it decays as $x \to -\infty$. Hence we can take it as a $\varphi(x, \lambda)$ and determine its spectral measure. As $x \to +\infty$ the potential $e^{-2x}$ decays to zero hence we have

\[
K_{\nu}(e^{-x}) \sim A(\lambda)e^{-i\lambda x} + B(\lambda)e^{i\lambda x}
\]

\[
B(\lambda) = A(-\lambda) = A^\#(\lambda)
\]

$A(\lambda)$ is holomorphic for $\Re{\lambda} > 0$.

Do we need the asymptotic behavior of $K_{\nu}(x)$ as $x \to 0$.
\[ F_s(t) = \int \frac{e^{-\frac{t}{2}(t+c)}}{t} t^s dt \quad \text{and} \quad G_s(t) = \int \frac{e^{-\frac{t}{2}(t-c)}}{t} t^s dt \]

where \( C \) and \( C' \) are the contours

and \( t^s \) is defined with a cut along \( R_{>0} \) to be \( e^{s \log t} \) for \( t > 0 \) on top of the cut. Then

\[ F_s - G_s = (e^{2\pi i s} - 1) K_s \]

Also for \( r > 0 \) we can substitute

\[ K_s = \frac{F_s}{e^{2\pi i s} - 1} - \frac{G_s}{e^{2\pi i s} - 1} \]

It's obvious that \( t \to t^{-1} \) converts \( C \) into \( -C' \) so that in the formula

\[ K_s(t) = \left( \frac{r}{2} \right)^s \sum_{n=0}^{\infty} \frac{1}{n!} \left( -\left( \frac{r^2}{4} \right)^n \Gamma(s-n) \right) \]

the second term on the left should be obtained from the first by putting \( s \) in for \( s \). Thus we have

\[ K_s(t) = \left( \frac{r}{2} \right)^s \sum_{n=0}^{\infty} \frac{1}{n!} \left( -\left( \frac{r^2}{4} \right)^n \Gamma(s-n) \right) + \left( \frac{r}{2} \right)^s \sum_{n=0}^{\infty} \frac{1}{n!} \left( -\left( \frac{r^2}{4} \right)^n \Gamma(-s-n) \right) \]
\[ K_\alpha(x) = \Gamma(\alpha)2^{-\alpha}e^{i\alpha x} + \Gamma(-\alpha)2^{\alpha}e^{-i\alpha x} \quad x \to \infty \]

Thus,

\[ A(\lambda) = \Gamma(-i\lambda)2^{-i\lambda}. \]

Now recall the formula

\[ \delta(x_0 - y) = \int_{-\infty}^{\infty} \varphi(x, \lambda) \varphi(y, \lambda) \frac{d\lambda}{2\pi A(\lambda)A(-\lambda)} \int \mu(\lambda) \]

\[ \frac{\sin \pi s}{\pi} = \frac{1}{\Gamma(s)\Gamma(1-s)} = \frac{1}{\Gamma(s)\Gamma(-s)} \quad s > 0 \]

\[ \frac{1}{A(\alpha)A(-\lambda)} = \frac{1}{\Gamma(i\lambda)\Gamma(-i\lambda)} = \frac{(-i\lambda)(e^{-\pi\lambda} - e^{\pi\lambda})}{2i} = \frac{\lambda(e^{\pi\lambda} - e^{-\pi\lambda})}{2\pi} \]

\[ \mu(\lambda) = (e^{\pi\lambda} - e^{-\pi\lambda}) \frac{1}{(2\pi)^2} \]

\[ \left( \frac{d}{dr} - \frac{s}{r} \right) \left( r^{\frac{1}{2}} K_{\frac{1}{2}-s} \right) = r^{\frac{1}{2}} \left( \frac{d}{dr} + \frac{s}{r} - \frac{s}{r^2} \right) K_{\frac{1}{2}-s} = -r^{\frac{1}{2}} K_{\frac{1}{2}+s} \]

\[ \left( \frac{d}{dr} + \frac{s}{r} \right) \left( r^{\frac{1}{2}} K_{\frac{1}{2}+s} \right) = -r^{\frac{1}{2}} K_{\frac{1}{2}-s} \]

Consider the system

\[ \frac{d}{dx} u = \begin{pmatrix} i\lambda & e^x \\ e^x & -i\lambda \end{pmatrix} u \quad -\infty < x < \infty \]

If \( z = e^x \) it becomes

\[ r \frac{d}{dr} u = \begin{pmatrix} i\lambda & z \\ z & -i\lambda \end{pmatrix} u \]

where

\[ K_\alpha(x) \sim \Gamma(s) \left( \frac{r^{-\alpha}}{x} \right)^{-\frac{s}{2}} + \Gamma(-s) \left( \frac{r^{-\alpha}}{x} \right)^{\frac{s}{2}} \quad \text{as} \quad x \to 0 \]
so we get the solution
\[ u = \begin{pmatrix} r^{\frac{1}{2}} K_{s-\frac{1}{2}} \\ -r^{\frac{1}{2}} K_{s+\frac{1}{2}} \end{pmatrix} \quad s = i\lambda \]

which decays as \( x \to +\infty \). As \( n \to 0 \) we have
\[ r^{\frac{1}{2}} K_{\frac{s-1}{2}} \sim r^{\frac{1}{2}} \left[ \Gamma(s-\frac{1}{2}) 2^{\frac{s-1}{2}} 2^{-s+\frac{1}{2}} \Gamma(-s+\frac{1}{2}) 2^{\frac{s+1}{2}} n^{-\frac{s-1}{2}} \right] \sim r^{\frac{1}{2}} \left( \Gamma\left(\frac{1}{2}-s\right) 2^{\frac{1}{2}-s} n^{-s} \right) \quad n^{s} = e^{i\beta x} \]

Now \( \varphi(x,\lambda) \sim \begin{pmatrix} A(\lambda)e^{i\beta x} \\ B(\lambda)e^{-i\beta x} \end{pmatrix} \quad x \to -\infty \quad \text{in } \lambda > 0 \]
then \( A(\lambda) = \Gamma\left(\frac{1}{2}-i\beta\right) 2^{\frac{1}{2}-i\beta} \) should be holom. \( \text{In } \lambda > 0 \).

The spectral measure for this function should be
\[ \frac{d\lambda}{2\pi |A(\lambda)|^2} = \frac{d\lambda}{2\pi} \frac{1}{\Gamma\left(\frac{1}{2}-i\beta\right) \Gamma\left(\frac{1}{2}+i\beta\right)} \]
\[ = \frac{d\lambda}{4\pi} \frac{\sin \pi \left(\frac{1}{2}-i\beta\right)}{\pi} = \frac{d\lambda}{4\pi} \cos \beta \pi \]
\[ = \frac{d\lambda}{8\pi^2} (e^{\pi \lambda} + e^{-\pi \lambda}) \]

Let's put the above system into debranges form.
The system is initially \( \frac{n}{r} \frac{du}{dr} = (i\lambda \frac{1}{r}) u \).
We use $z = \frac{w-i}{w+i}$ to relate $|z| < 1$ to $\text{Im} \, w > 0$.

This means we put

$$(u_1) = (1-i)(v_1)$$

$$(u_2) = (1+i)(v_2)$$

Simplifying,

$$\frac{1}{2i} \left( i - 1 \right) \begin{pmatrix} \lambda + i & 1 - i \\ \lambda - i & 1 + i \end{pmatrix} = \frac{1}{2i} \left( i - 1 \right) \begin{pmatrix} i \lambda + 1 & 2 + i \\ -i \lambda + 1 & 2 - i \end{pmatrix} = \begin{pmatrix} \lambda & \lambda \\ -\lambda & -\lambda \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \lambda & \lambda \\ -\lambda & -\lambda \end{pmatrix}$$

Do we get

$$\frac{r \, dv}{dr} = \begin{pmatrix} \lambda & \lambda \\ -\lambda & -\lambda \end{pmatrix} v$$

Then we take $S$ to be a solution matrix for $\lambda = 0$

$$S = \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}$$

and put $v = Sw$.

$$r \begin{pmatrix} e^{2w_1} + e^{2w_2} \\ e^{2w_1} e^{-2w_2} + e^{2w_2} \end{pmatrix} = \begin{pmatrix} \lambda & \lambda \\ -\lambda & -\lambda \end{pmatrix} \begin{pmatrix} e^{2w_1} \\ e^{-2w_2} \end{pmatrix} = \begin{pmatrix} re^{2w_1} + re^{2w_2} \\ -re^{2w_1} - re^{2w_2} \end{pmatrix}$$

or

$$\frac{dw_1}{dr} = \lambda \frac{e^{-2r}}{r} w_2$$

$$\frac{dw_2}{dr} = -\lambda \frac{e^{2r}}{r} w_1$$

or

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \frac{dw}{dr} = \lambda \begin{pmatrix} \frac{e^{2r}}{r} & 0 \\ 0 & \frac{e^{-2r}}{r} \end{pmatrix} w$$

What deB calls a "special Kummer" situation.
A heuristic explanation of the completeness relation:

Let $u(t) = e^{-iLt} v$ be the solution of the wave equation $i \frac{\partial}{\partial t} u = Lu$ starting at $v$. Take its Fourier transform

$$u(\lambda) = \int e^{i\lambda t} u(t) \, dt = \int e^{i(\lambda - L)t} v \, dt$$

and break this into two parts:

$$\int_0^\infty e^{i(\lambda - L)t} v \, dt = -\frac{i}{L} (\lambda - L)^{-1} v$$

analytic for Im $\lambda > 0$

$$= -\frac{i}{L} G^+(\lambda) v$$

where $G^+(\lambda)$ is the limit of $(\lambda - L)^{-1}$ for $\lambda$ real of the resolvent in the upper half plane. Similarly

$$\int_{-\infty}^0 e^{i(\lambda - L)t} v \, dt = -\int_0^\infty e^{-i(\lambda - L)t} v \, dt = -i (\lambda - L)^{-1} v$$

for Im $\lambda < 0$

$$= -i G^-(\lambda) v$$

Then by Fourier inversion we have

$$v = \int_{-\infty}^\infty u(\lambda) \frac{d\lambda}{2\pi} = -\frac{L}{2\pi i} \int_{-\infty}^\infty \left[ G^+(\lambda) - G^-(\lambda) \right] v \, d\lambda$$

or

$$-\frac{1}{2\pi i} \int_{-\infty}^\infty \left[ G^+(\lambda) - G^-(\lambda) \right] d\lambda = I$$
Put \( g(t) = \int e^{-i\lambda t} \mu(\lambda) \frac{d\lambda}{2\pi} \) so that by Fourier inversion
\[
\hat{f}(\lambda) = \int e^{i\lambda t} g(t) dt = \int_{-\infty}^{\infty} e^{i\lambda t} g(t) dt = \int_{-\infty}^{\infty} e^{i(\lambda - \lambda') t} d\lambda' \quad \text{and in UHP}
\]
\[
\int_{-\infty}^{\infty} e^{i\lambda t} g(t) dt = \int_{-\infty}^{\infty} \hat{f}(\lambda) \frac{d\lambda}{2\pi} \int_{-\infty}^{\infty} e^{i(\lambda - \lambda') t} d\lambda' = \int \frac{f(\lambda)}{\lambda - \lambda'} \frac{d\lambda}{2\pi i} \quad \text{and in LHP.}
\]
Thus
\[
\hat{f}(\lambda) = \frac{1}{2\pi i} \left[ f^+(\lambda) + f^-(\lambda) \right]
\]
where \( f^\pm(\lambda) \) are the limiting values of the function
\[
f(\lambda) = \int \frac{f(\lambda)}{\lambda - \lambda'} \frac{d\lambda}{2\pi i}
\]

More generally suppose
\[
g(t) = \int e^{-i\lambda t} \mu(\lambda) \quad \text{and} \quad \overline{g(t)} = g(-t)
\]
is the Fourier transform of a measure. Then
\[
\int_{-\infty}^{\infty} e^{i\lambda t} g(t) dt = f^+(\lambda) \quad \text{analytic in UHP}
\]
\[
\int_{-\infty}^{\infty} e^{i\lambda t} g(t) dt = f^-(\lambda) \quad \text{in LHP}
\]
and
\[
\overline{f^-(\lambda)} = \int_{-\infty}^{\infty} e^{-i\lambda t} g(-t) dt = \int_{-\infty}^{\infty} e^{i\lambda t} g(t) dt = f^+(\lambda)
\]
so
\[
f^-(\lambda) = \overline{f^+(\lambda)}
\]
\[ f^+(\lambda) \quad \text{boundary values of} \quad f(\lambda) = i \int \frac{d\mu(\xi)}{\lambda - \xi} \]

Maybe the important thing to notice is that the Stieltjes transform of the measure, which is what de B. Klein play with, is essentially that part of the Fourier transform of the measure for \( t > 0 \) which is what enters into the Gelfand-Levitan equation.
Consider a J-matrix eigenvalue problem

1) \[ Lu = (aT + b + T^T)u = \lambda u \]
on the space of column vectors \( u = (u_i) \), \( 0 \leq i \leq l + 1 \)
satisfying given real boundary conditions at the ends:

2) \[ \frac{u_i}{a_i u_0} = \frac{\alpha_i}{\alpha_2} \quad \text{and} \quad \frac{u_{l+1}}{a_{l+1} u_l} = \frac{\beta_i}{\beta_2} \]

I can write 1) in the form

\[
\begin{pmatrix}
  u_{n+1} \\
  a_n u_n \\
\end{pmatrix}
= \begin{pmatrix}
  0 & -\frac{1}{a_n} \\
  \frac{1}{a_n} & 0 \\
\end{pmatrix}
\begin{pmatrix}
  u_n \\
  a_n u_{n+1} \\
\end{pmatrix}
= \begin{pmatrix}
  0 & -\frac{1}{a_n} \\
  \frac{1}{a_n} & 0 \\
\end{pmatrix}
\begin{pmatrix}
  u_n \\
  a_n u_{n+1} \\
\end{pmatrix}
= \begin{pmatrix}
  0 & 1 & 0 \\
  \frac{1}{a_n} & b_n & 1 \\
\end{pmatrix}
\begin{pmatrix}
  u_n \\
  a_n u_{n+1} \\
\end{pmatrix}
\]

and I recall that \( \begin{pmatrix}
  0 & 1 \\
  1 & 0 \\
\end{pmatrix} \) shrinks the UHP for \( \text{Im} \lambda > 0 \).

Recall Green's formula:

\[ [(Lu)\nu - u LV]_n = W(u, v)_n - W(u, v)_{n+1} \]

where \( W(u, v)_n = \left| \begin{array}{cc}
  u_{n+1} & v_{n+1} \\
  a_n u_n & a_n v_n \\
\end{array} \right| \)

Hence we have

\[
\sum_{n=1}^l [(Lu)\tilde{v} - u L \tilde{v}]_n = W(u, \tilde{v})_l - W(u, \tilde{v})_0
\]

for arbitrary \( u, \tilde{v} \) given over the interval \( 0 \leq n \leq l \).

Now you have to be careful to get a good Hilbert space.
and self-adjoint operator. We obviously want to use the space of vectors \((v_1, \ldots, v_n)\) with the usual inner product, so it is necessary to be able to extend any such thing uniquely to a \(u\) given over \([0, l+1]\) satisfying the boundary conditions. Hence we must have \(\alpha_i \neq 0\) and \(\beta_i \neq 0\), so it seems.

Define \(\psi(n, \lambda)\) for \(0 \leq n \leq l+1\) to be the solution of (1) in degree \(1 \leq n \leq l\), i.e., satisfying

\[a_n \psi(n+1, \lambda) + b_n \psi(n, \lambda) + a_{n-1} \psi(n-1, \lambda) = 2 \psi(n, \lambda)\]

for \(1 \leq n \leq l\), starting with the values

\[
\begin{pmatrix}
\psi(0, \lambda) \\
(\alpha_0 \psi(0, \lambda))
\end{pmatrix} =
\begin{pmatrix}
\alpha_1 \\
\alpha_2
\end{pmatrix}
\]

Similarly define \(\psi(n, \lambda)\) for \(0 \leq n \leq l+1\) using

\[
\begin{pmatrix}
\psi(l+1, \lambda) \\
(\alpha_l \psi(l, \lambda))
\end{pmatrix} =
\begin{pmatrix}
\beta_1 \\
\beta_2
\end{pmatrix}
\]

The choice of \(\psi\) determines a spectral measure

\[d \mu(\lambda) = \sum_{i=1}^{n} \frac{\delta(\lambda - \lambda_i)}{\|\psi_i\|^2}\]

where Stieltjes transform

\[
\int \frac{d \mu(\lambda)}{\lambda - \tau}
\]

is a rational function with poles at the eigenvalues.
Another way of proceeding is to choose a solution \( \tilde{\varphi} \) of \( Lu = \lambda u \), or rather a set of boundary values \( (\tilde{\gamma}_1, \tilde{\gamma}_2) \) at \( h = 0 \), and let \( \tilde{\varphi} \) be the resulting solution, such that \( W(\varphi, \tilde{\varphi}) = 1 \). Then we can define a function \( m(\lambda) \) by

\[
m(\lambda) = \frac{\tilde{\varphi} \varphi - \lambda \varphi(0)}{W(\varphi, \tilde{\varphi})}
\]

\( \tilde{\varphi} \) is unique up to a real constant. The question is which \( \tilde{\varphi} \) makes \( m = \) the Stieltjes transform.

\[
\begin{pmatrix}
\varphi(1, \lambda) & \tilde{\varphi}(1, \lambda) \\
\varphi(0, \lambda) & \tilde{\varphi}(0, \lambda)
\end{pmatrix}
\begin{pmatrix}
\alpha_1 \\
\alpha_2
\end{pmatrix}
= \begin{pmatrix}
\gamma_1 \\
\gamma_2
\end{pmatrix}
\]

\[
m(\lambda) = \begin{pmatrix}
\alpha_2 & -\alpha_1 \\
-\alpha_2 & \alpha_1
\end{pmatrix}
\begin{pmatrix}
\varphi(1, \lambda) \\
\varphi(0, \lambda)
\end{pmatrix}
\]

\[
= \frac{\alpha_2 \varphi(1, \lambda) - \alpha_1 \alpha_0 \varphi(0, \lambda)}{-\alpha_2 \varphi(1, \lambda) + \alpha_1 \alpha_0 \varphi(0, \lambda)}
\]

where the denominator is \( W(\varphi, \tilde{\varphi}) \).

Now \( \varphi(1, \lambda) \) is a poly of degree \( \lambda - 1 \), and \( \varphi(0, \lambda) \) has degree \( \lambda \), since \( \lambda \neq 0 \), \( W(\varphi, \tilde{\varphi}) \) has degree \( \lambda \), as it must because there are \( \lambda \)-eigenvalues. But also the Stieltjes transform vanishes at \( \lambda = \infty \), so thus we want \( \varphi(1, \lambda) = 0 \) and \( \varphi(0, \lambda) = \alpha_0 \lambda \).

Another proof: We have the formula

\[
G(n, n', \lambda) = \frac{\varphi(n, \lambda) \varphi(n', \lambda)}{W(\varphi, \tilde{\varphi})}
\]
valid for $1 \leq n, n' \leq l$ because $(\lambda - l)^{-2}$ makes sense only in the Hilbert space in question. Put $n = n' = 1$ and you get
\[\alpha_1 \frac{\psi(1, \lambda)}{W(1, \lambda)} = \int \frac{\alpha_2 \, d\mu(\lambda)}{\lambda - \lambda'}\]
\[\int \frac{d\mu(\lambda)}{\lambda - \lambda'} = \frac{1}{\alpha_1} \frac{\psi(1, \lambda)}{W(1, \lambda)}\]

So it appears that when working with $J$-matrices on a half-line $0 \leq n < \infty$, of the possible boundary conditions at $0$, there is a singular one, namely $\psi_0 = 0$. If the $\psi$ solution is defined by boundary values away from this singular case, then one gets a definite analytic function of $\lambda$ from the boundary values of $\psi$, and this function is the Stieltjes transform of the measure.

Let's look now at \( Lu = -u'' + q \psi = \lambda u \) where $q$ dies fast enough as $x \to \infty$ so scattering calculations are valid. \( \psi(x, \lambda) \sim e^{i \sqrt{\lambda} x} \) \( \lambda \in \text{UHP} \) for $\lambda \not\in \mathbb{R}$

\[ \psi(x, \lambda) = A(\lambda) e^{-i \sqrt{\lambda} x} + B(\lambda) e^{+i \sqrt{\lambda} x} \]

Assuming no bound states the spectral measure is
\[ d\mu(\lambda) = \begin{cases} \frac{1}{2\pi} \frac{d\sqrt{\lambda}}{A(\sqrt{\lambda}) A(-\sqrt{\lambda})} & \text{if } \lambda > 0 \\ 0 & \text{if } \lambda < 0 \end{cases} \]
Calculate the example \( g = 0 \), \( \varphi(x, \lambda) = \cos \sqrt{\lambda} x \)

Then \( \varphi(x, \lambda) = \cos \sqrt{\lambda} x = \frac{1}{2} (e^{-i\sqrt{\lambda} x} + e^{i\sqrt{\lambda} x}) \)

so \( A(\sqrt{\lambda}) = \frac{1}{2} \) and \( d\mu(\lambda) = \frac{1}{2\pi} \frac{d\sqrt{\lambda}}{\lambda} = \frac{2}{\pi} \frac{d\sqrt{\lambda}}{\sqrt{\lambda}} \) on \( \mathbb{R}_{\geq 0} \).

The Stieltjes transform is

\[
\frac{2}{\pi} \int_0^\infty \frac{d\sqrt{\lambda}}{\lambda - \lambda} = \frac{2}{\pi} \int_0^\infty \frac{dt}{\lambda - t^2}
\]

which is perfectly well-defined. Put \( \lambda = -s^2 \)

where \( s > 0 \). Then

\[
\frac{2}{\pi} \int_0^\infty \frac{dt}{\lambda - t^2} = -\frac{2}{\pi} \int_0^\infty \frac{s \, dt}{s^2 + t^2} = -\frac{2}{\pi} \frac{1}{s} \int_0^\infty \frac{dt}{1 + t^2} = -\frac{1}{s}
\]

hence recalling we choose \( \sqrt{\lambda} \) to be in the UHP,

\[
f(\lambda) = -\frac{1}{i\sqrt{\lambda}} = -\frac{i}{\sqrt{\lambda}}
\]

(Check if \( \lambda < 0 \), this should be real and \( < 0 \).)

Next consider \( \varphi(x, \lambda) = \alpha_1 \cos \sqrt{\lambda} x + \alpha_2 \sin \sqrt{\lambda} x \)

\[
= \left( \frac{\alpha_1}{\sqrt{\lambda}} - \frac{\alpha_2}{\lambda / 2i} \right) e^{-i\sqrt{\lambda} x} + \left( \frac{\alpha_1}{\sqrt{\lambda}} + \frac{\alpha_2}{\lambda / 2i} \right) e^{i\sqrt{\lambda} x}
\]

so that \( A(\lambda) = \frac{1}{2} \left( \frac{\alpha_1}{\sqrt{\lambda}} + \frac{i}{\lambda} \frac{\alpha_2}{\sqrt{\lambda}} \right) \) and

\[
d\mu(\lambda) = \frac{2}{\pi} \frac{d\sqrt{\lambda}}{\left( \frac{\alpha_1}{\sqrt{\lambda}} + \frac{i}{\lambda} \frac{\alpha_2}{\sqrt{\lambda}} \right)^2} = \frac{2}{\pi} \frac{d\sqrt{\lambda}}{\frac{\alpha_1^2}{\lambda} + \frac{\alpha_2^2}{\lambda}}
\]

so the Stieltjes transform of the measure is defined for \( \alpha_1 \neq 0 \).
Look at \( \alpha_1 = 0, \alpha_2 = 1 \) more carefully

\[
d\mu(\lambda) = \frac{2}{\pi} \lambda^{-1/2} d\sqrt{\lambda} \quad \text{for} \quad \lambda > 0
\]

\[
\int \frac{d\mu(\lambda)}{\lambda - \lambda^2} = \frac{2}{\pi} \int_0^\infty \frac{t^2 dt}{\lambda - \lambda^2} = \frac{2}{\pi} \int_0^\infty \frac{t^2 dt}{\lambda - t^2}
\]

not convergent although its imaginary part is.

Next let's find the formula for the Stieltjes transform \( m(\lambda) \) when it exists in terms of the \( y \) boundary values at 0. The idea is to choose \( \hat{y} \) to have the singular boundary values, and then \( m(\lambda) \) is given by

\[
m(\lambda) \hat{y} + \hat{y} = \frac{y}{W(y, \hat{y})}
\]

\[
\begin{pmatrix} \alpha_1 & 0 \\ \alpha_2 & \frac{1}{\alpha_1} \end{pmatrix} m = \begin{pmatrix} y(0, \lambda) \\ y'(0, \lambda) \end{pmatrix}
\]

\[
m(\lambda) = \begin{pmatrix} \frac{1}{\alpha_1} & 0 \\ -\alpha_2 & \alpha_1 \end{pmatrix} \begin{pmatrix} y(0, \lambda) \\ y'(0, \lambda) \end{pmatrix}
\]

\[
m(\lambda) = \frac{1}{\alpha_1} \frac{y(0, \lambda)}{W(y, \hat{y})}
\]

Check this for \( y = \cos \sqrt{\lambda} x \), \( \hat{y} = e^{i\sqrt{\lambda} x} \)

\[
W(y, \hat{y}) = \frac{1}{2} 2i\sqrt{\lambda} = i\sqrt{\lambda}
\]

so

\[
m(\lambda) = \frac{1}{i\sqrt{\lambda}}
\]

which is what we computed the Stieltjes transform to be. Green's formula proof will yield the above, at least assuming the integral for \( G \) converges nicely.
Consider $Lu = -u'' + gu = \lambda^2 u$ \hspace{1cm} \text{(note the $\lambda^2$)}

and let $\varphi(x, \lambda)$ denote the solution starting with

\begin{align*}
\varphi(x, \lambda) &= \alpha_1 \\
\varphi'(x, \lambda) &= \alpha_2.
\end{align*}

Assuming the spectrum is in $\mathbb{R}_{\geq 0}$ it follows that $\varphi(x, 0)$ doesn't vanish for $x > 0$, so if we have the good case $\alpha_1 \neq 0$ we have a well-defined function

$p(x) = \frac{\varphi'(x, 0)}{\varphi(x, 0)}$

satisfying

$p'(x) = \frac{\varphi''(x, 0)}{\varphi(x, 0)} - \frac{(\varphi'(x, 0))^2}{\varphi(x, 0)^2} = \varphi - p^2.$

Hence I can factor the operator $L$

$L = -\frac{d^2}{dx^2} + \varphi = -\left(\frac{d}{dx} + p\right)\left(\frac{d}{dx} - p\right).$

Consider the system

\begin{align*}
\begin{cases}
\left(\frac{d}{dx} - p\right)u_1 = \lambda u_2 \\
\left(\frac{d}{dx} + p\right)u_2 = -\lambda u_1
\end{cases}
\end{align*}

Let $u$ satisfy $Lu = \lambda^2 u$, \hspace{1cm} \frac{u(0)}{u'(0)} = \frac{\alpha_1}{\alpha_2}$ a little

$u'(0) = \frac{\alpha_2}{\alpha_1} u(0)$.

Then provided $\lambda \neq 0$ we can put $u_1 = u$ and

$u_2 = \frac{1}{\lambda} \left(\frac{d}{dx} - p\right)u_1$
and we get a solution of the system satisfying the boundary condition \( u_2(0) = 0 \). If \( \lambda = 0 \) we put \( u_2 = u \) and \( u_2 = 0 \). Clearly \((u_1, u_2)\) is uniquely determined by \( u \), hence we get a one-one correspondence in the above way.

The system \((x)\) admits the symmetry \((u_1, u_2) \mapsto (-u_1, -u_2)\), hence the spectral measure is symmetric provided the boundary condition is preserved, which means we have either of the boundary conditions \( u_1(0) = 0 \) or \( u_2(0) = 0 \).

Note that if \( u \) is a \( l^2 \)-eigenfunction

\[
(u_2, u_2) = (u_2, \frac{1}{\lambda} \left( \frac{d}{dx} - p \right) u_2) = \frac{1}{\lambda} \left( -\frac{d}{dx} - p \right) (u_2, u_2)
\]

\[
= (u_1, u_1)
\]

if \( \lambda \neq 0 \) and that \((u_2, u_2) = 0\) if \( \lambda = 0 \). Thus the spectral measure for the two eigenvalue problems are probably the same. Also if \( d\mu(x) \) is even, then the Stieltjes transform can be written

\[
\int \frac{d\mu(x)}{\lambda - x} = \frac{1}{\lambda^2} \int \left( \frac{1}{\lambda - x} + \frac{1}{\lambda + x} \right) d\mu(x)
\]

\[
= \lambda \int \frac{d\mu(x)}{x^2 - \lambda^2}
\]

this integral exists.

hence there is a definite way of making sense out of the Stieltjes transform of the spectral measure for \((x)\) with
the boundary condition \( u_2(0) = 0 \). The question is how is \( \tilde{q} \) to be defined so that we have

\[
(1) \quad m(\lambda) \tilde{q}(\lambda) + \tilde{\varphi}(\lambda) = \frac{\varphi(\lambda)}{W(\varphi, \varphi)}
\]

with \( m(\lambda) \) the Stiles transform. What is \( \varphi \)?

\[
\varphi_1(x, \lambda) = \varphi(x, \lambda) \\
\varphi_2(x, \lambda) = \frac{1}{\lambda} (\varphi(x, \lambda) - p(x) \varphi(x, \lambda))
\]

hence

\[
\begin{pmatrix}
\psi_1(0, \lambda) \\
\psi_2(0, \lambda)
\end{pmatrix} = \begin{pmatrix}
\alpha_1 \\
0
\end{pmatrix}
\]

\[
\psi_1(x, \lambda) = \psi(x, \lambda) \\
\psi_2(x, \lambda) = \frac{1}{\lambda} (\psi(x, \lambda) - p(x) \psi(x, \lambda))
\]

\[
W(\tilde{q}, \tilde{\varphi}) = \begin{vmatrix}
\tilde{q} & \tilde{\varphi} \\
\frac{1}{\lambda}(\psi_1' \psi_2) & \frac{1}{\lambda}(\psi_1 \psi_2')
\end{vmatrix} = \frac{1}{\lambda} W(\varphi, \psi)
\]

so from the first component of (1)

\[
m(\lambda) \psi(x, \lambda) + \tilde{\varphi}(x, \lambda) = \frac{\lambda \psi(x, \lambda)}{W(\varphi, \psi)}
\]

It seems reasonable to expect this to agree with what was done for \( L \psi = \lambda^2 \psi \), hence \( \tilde{\varphi}(0, \lambda) = 0 \) and so we get

\[
m(\lambda) \alpha_1 = \lambda \int \frac{d\mu(\lambda)}{\lambda^2 - \lambda^2} = \int \frac{d\mu(\lambda)}{\lambda - \lambda^2}
\]

Principal value.