January 4, 1978: Review discrete wave equation.

scattering + spectral measure for Bessel modified DE see p. 644

Begin with a one-sided infinite J-matrix L with $L^2 \leq I$, which we interpret as a bounded self-adjoint operator in $l^2 = V$. Then we consider all functions $\mathbb{Z} \to V$, $t \mapsto u(t)$ such that

$$(*) \qquad Lu(t) = \frac{u(t+1)+u(t-1)}{2}$$

Energy norm:

$$E(u)(t) = \|u(t)\|^2 - \tfrac{1}{2}(u(t+1), u(t-1)) - \tfrac{1}{2}(u(t-1), u(t+1))$$

$$= \|u(t)\|^2 + \|u(t+1)\|^2 - (u(t+1), Lu(t)) - (Lu(t), u(t+1))$$

$$= \|u(t)\|^2 - \|Lu(t)\|^2 + \|u(t+1) - Lu(t)\|^2$$

$$= \|u(t)\|^2 - \|Lu(t)\|^2 + \left\|\frac{u(t+1)-u(t-1)}{2}\right\|^2$$

This is independent of t.

Complete the space $\mathcal{H}_0$ of solutions of $(*)$ in the energy norm to obtain the space $\mathcal{H}$. The map $u(t) \mapsto u(-t)$ is a symmetry of $\mathcal{H}_0$, hence $\mathcal{H}_0$ decomposes

$$\mathcal{H}_0 = \mathcal{H}_0^{ev} \oplus \mathcal{H}_0^{odd}$$

If $u \in \mathcal{H}_0^{odd}$, i.e. $u(-t) = -u(t)$, then

$$E(u) = \left\|\frac{u(1)-u(-1)}{2}\right\|^2 = \|u(1)\|^2$$

hence

$$\mathcal{H}_0^{odd} \xrightarrow{\sim} V$$

If $u \in \mathcal{H}_0^{ev}$, i.e. $u(t) = u(t)$, then

$$E(u) = \|u(0)\|^2 - \|Lu(0)\|^2$$

and so $\mathcal{H}_0^{ev}$ is isomorphic to V with the inner product $\langle x, y \rangle = ((I - L^2)x, y)$. This won't be complete unless $L^2 \leqslant I - \varepsilon$ and so under completion $\mathcal{H}_0^{ev}$ will become enlarged; also eigenvectors for the eigenvalues ± 1 will be killed.

Let's see what this construction gives for $V = L^2(d\mu)$, $d\mu$ a measure in $[-1,1]$, $L =$ multiplication by λ. Then $\mathcal{H}_0$ consists of pairs $(f(\lambda), g(\lambda))$ in V with norm

$$\|(f,g)\|^2 = \int |f|^2 d\mu + \int (1-\lambda^2)|g|^2 d\mu$$

Let $d\nu$ denote the measure on S^1 which is even and such that

$$\int_{S^1} f(\lambda)\, d\nu = \int_{[-1,1]} f(\lambda)\, d\mu \qquad \lambda = \cos\theta$$

Then if I associate to (f,g) the function $f(\lambda) + i g(\lambda)\sin\theta$ on S^1 we have

$$\int |f + i g \sin\theta|^2 d\nu = \int \left[|f|^2 + |g|^2 (1-\lambda^2)\right] d\mu$$

and so in this way I can identify $\mathcal{H}$ with $L^2(S^1, d\nu)$. Note that $\mathcal{H}^{odd}$ corresponds to the space of even functions on S^1, so my notation is lousy.

Notice that

$$u(t) = a(z) z^t + b(z) z^{-t} \qquad b(z) = a(z^{-1})$$

is a trajectory in $L^2(\mathbb{R}, d\mu)$. Better

$$u(t) = \frac{z^t - z^{-t}}{z - z^{-1}}$$

is an odd trajectory with $u(1) = 1$, and

$$u(0) = \frac{z^t + z^{-t}}{2}$$

is an even trajectory with $u(0) = 1$. Hence

$$u(t) = f(\lambda)\,\frac{z^t - z^{-t}}{z - z^{-1}} + g(\lambda)\,\frac{z^t + z^{-t}}{2}$$

is the trajectory in $L^2(\mathbb{R}, d\mu)$ whose odd part corresponds to f and whose even part corresponds to g.

Write this

$$u(t) = \frac{f}{\sin\theta}\sin t\theta + g\cos t\theta$$

Then
$$u(t+1) = \frac{f}{\sin\theta}\left[\sin t\theta\cos\theta + \cos t\theta\sin\theta\right] + g\left[\cos t\theta\cos\theta - \sin t\theta\sin\theta\right]$$

$$= \frac{f\cos\theta - g\sin^2\theta}{\sin\theta}\sin t\theta + \left[f + g\cos\theta\right]\cos t\theta$$

is the trajectory associated to

$$\left(f + ig\sin\theta\right)\left(\cos\theta + i\sin\theta\right) = \left(f\cos\theta - g\sin^2\theta\right) + i\left(f + g\cos\theta\right)\sin\theta$$

Therefore translation $u(t) \mapsto u(t+1)$ corresponds to multiplication by z.

January 5, 1977

L a J-matrix with $L^2 \leq I$ and $u : \mathbb{Z} \to \ell^2$ a solution of the discrete wave equation.

$$Lu(t) = \frac{u(t+1) + u(t-1)}{2}$$

In order to describe u one can use ~~[illegible]~~ eigenfunction expansions in both n and t. First do for n:

$$u(n,t) = \int \varphi(n,\lambda)\, u(\lambda,t)\, d\mu(\lambda)$$

$$\frac{u(\lambda, t+1) + u(\lambda, t-1)}{2} = \lambda u(\lambda, t)$$

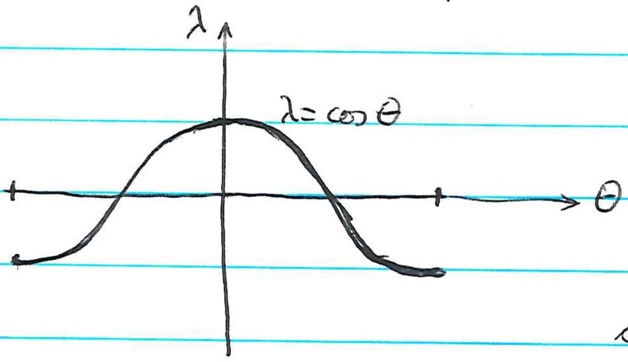
Then do for t:

$$u(\lambda,t) = \int_{S^1} z^t u(\lambda,z)\,\frac{d\theta}{2\pi}$$

whence $u(\lambda,z)$ satisfies

$$\frac{z+z^{-1}}{2}\,u(\lambda,z) = \lambda\,u(\lambda,z)$$

In other words $u(\lambda,z)$ is supported on $\lambda = \frac{z+z^{-1}}{2}$ which means effectively that it's a function (generalized perhaps) of z. [redacted]



So take $u(\lambda,z)\frac{d\theta}{2\pi}$ to be a (signed) measure $h(z)\frac{d\theta}{2\pi}$ on the circle which one lifts up to the locus $\lambda = \frac{z+z^{-1}}{2}$. Then $u(\lambda,t)d\lambda$ can think of as the (signed) measure on $[-1,1]$ obtained by pushing $z^t h(z)\frac{d\theta}{2\pi}$ down via $\lambda = \cos\theta$; hence

$$u(\lambda,t)\,d\lambda = \frac{h(z)z^t\,d\lambda}{2\pi(-\sin\theta)} + \frac{h(z^{-1})z^{-t}\,d\lambda}{2\pi(\sin\theta)}$$

$$= \left\{\frac{h(z^{-1})z^{-t}}{\sin\theta} - \frac{h(z)z^t}{\sin\theta}\right\}\frac{d\lambda}{2\pi}$$

Hence

$$u(n,t) = \int_{-1}^{1} \varphi(n,\lambda)\left\{\frac{h(z^{-1})z^{-t}}{2\pi\sin\theta} - \frac{h(z)z^t}{2\pi\sin\theta}\right\}d\mu(\lambda)$$

Let's compare the preceding formula with yesterday's analysis of a path $t \longmapsto u(\lambda,t)$ in $L^2(d\mu)$ satisfying

$$\lambda u(\lambda,t) = \frac{u(\lambda,t+1)+u(\lambda,t-1)}{2}$$

We saw
$$u(\lambda,t) = f(\lambda)\frac{z^t - z^{-t}}{z - z^{-1}} + g(\lambda)\frac{z^t + z^{-t}}{2}$$
$$= \left(\frac{f(\lambda)}{z-z^{-1}} + \frac{i}{2i}g(\lambda)\right)z^t + \left(-\frac{f(\lambda)}{z-z^{-1}} + \frac{g(\lambda)}{2}\right)z^{-t}$$
$$= \left(\frac{f(\lambda) + ig(\lambda)\sin\theta}{2i\sin\theta}\right)z^t + \left(\frac{f(\lambda) - ig(\lambda)\sin\theta}{-2i\sin\theta}\right)z^{-t}$$

So it's clear that I want to represent solutions $u(n,t)$ to the wave equation for L close to L_0 in the form
$$u(n,t) = \int \varphi(n,\lambda)\frac{\alpha(z)}{\sin\theta} z^{-t}\,\frac{d\theta}{2\pi}$$

In effect if α is even: $\alpha = f(\lambda)$, then this is
$$u(n,t) = \int \varphi(n,\lambda)\, f(\lambda)\left(\frac{z^{-t}}{\sin\theta} - \frac{z^t}{\sin\theta}\right)\frac{1}{2}\,\frac{d\theta}{2\pi}$$
regular i.e. a Laurent poly.

and if α is odd, say $\alpha = g(\lambda)\sin\theta$, then also I get something regular. To be precise: If $\alpha(z) \in \mathbb{C}[z,z^{-1}]$, then
$$\frac{\alpha(z)z^{-t}}{\sin\theta} - \frac{\alpha(z^{-1})z^t}{\sin\theta} \in \mathbb{C}[z,z^{-1}]$$

Begin again: Let L be a J-matrix and $u(n,t)$ a solution of the discrete wave equation. Take the Fourier transform of $u(n,t)$
$$u(z) = \sum_{t\in\mathbb{Z}} u(n,t)z^t \qquad u(t) = \int_{S^1} z^{-t}u(z)\,\frac{d\theta}{2\pi}$$

Then because $Lu(t) = \dfrac{u(t+1)+u(t-1)}{2}$ we have

$$Lu(z) = \frac{z+z^{-1}}{2} u(z)$$

and hence $u(z)$ is a multiple of $\varphi(\lambda)$ $\lambda = (z+z^{-1})/2$, say

$$u(n,z) = \alpha(z)\,\varphi(n,\lambda).$$

Hence we have the representation

$$u(n,t) = \int_{S^1} z^{-t}\varphi(n,\lambda)\,\alpha(z)\,\frac{d\theta}{2\pi}$$

(The above is somewhat formal, however it might be interesting to see what becomes of the above when L had spectrum outside $[-1,1]$.).

We have

$$u(n,t) = \int \varphi(n,\lambda)\left(\frac{\alpha(z)z^{-t}+\alpha(z^{-1})z^{t}}{2}\right)\frac{d\theta}{2\pi}$$

$$\frac{\alpha(z)z^{-t}+\alpha(z^{-1})z^{t}}{2} = \frac{\alpha(z)+\alpha(z^{-1})}{2}\,\frac{z^{t}+z^{-t}}{2} - \frac{\alpha(z)-\alpha(z^{-1})}{2}\,(z-z^{-1})\,\frac{z^{t}-z^{-t}}{z-z^{-1}}$$

This equation shows that

$$\forall t \quad \frac{\alpha(z)z^{-t}+\alpha(z^{-1})z^{t}}{2} \in \mathbb{C}[z,z^{-1}] \iff (z-z^{-1})\alpha(z) \in \mathbb{C}[z,z^{-1}]$$

(better proof by linear equations), hence it perhaps nicer to use the representation

$$u(n,t) = \int_{S^1} z^{-t}\varphi(n,\lambda)\,\frac{h(z)}{\sin\theta}\,\frac{d\theta}{2\pi}.$$

January 6, 1978:

Suppose given a [redacted] Dirac system where ρ is is a sum of δ functions at the integers. I have seen that the basic [redacted] solution is of the form

$$\varphi(x,\lambda) = \begin{pmatrix} e^{i\lambda(x-n)} & \\ & e^{-i\lambda(x-n)} \end{pmatrix} [redacted] R(h_n) \begin{pmatrix} e^{i\lambda} & \\ & e^{-i\lambda} \end{pmatrix} \cdots R(h_1) \begin{pmatrix} e^{i\lambda} & \\ & e^{-i\lambda} \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

[redacted] if $n < x < n+1$. Thus φ is discontinuous at integer points and

$$\varphi(n^+,\lambda) = R(h_n)\varphi(n^-,\lambda)$$

If $p_0, p_1, \ldots$ is the system of orthonormal polys on S^1 described by $h_1, h_2, \ldots$ then we have

$$\varphi(n^+,\lambda) = \begin{pmatrix} z^{-n/2} p_n(z) \\ z^{n/2} p_n^*(z) \end{pmatrix}$$

where $z = e^{2i\lambda}$ and the de B. function at this point is

$$\varphi_2(n^+,\lambda) = e^{in\lambda} p_n^*(e^{2i\lambda})$$

For example, if all $h_i = 0$, then

$$\varphi(n^+,\lambda) = \begin{pmatrix} e^{in\lambda} & 0 \\ 0 & e^{-in\lambda} \end{pmatrix}$$

and the point evaluator is

$$J_\alpha(\lambda) = \frac{i}{2(\lambda-\bar\alpha)} \begin{vmatrix} e^{-in\lambda} & e^{-in\bar\alpha} \\ e^{in\lambda} & e^{in\bar\alpha} \end{vmatrix} = \frac{\sin n(\lambda-\bar\alpha)}{\lambda-\bar\alpha}$$

Suppose given a probability measure $d\nu$ on S^1 and let p_i, h_i be as usual. We filter $\mathbb{C}[z,z^{-1}] \subset L^2(S^1, d\nu)$ symmetrically $F_n =$ span of $z^{-n}, z^{-n+1}, \dots, z^n$, $W_n = F_n \ominus F_{n-1}$. We saw that W_n is spanned by $z^{-n}p_{2n}, z^n p_{2n}^*$; however these units vectors are not orthogonal.

$$\ell_n = \text{leading coeff. of } p_n = \text{const. term of } z^n p_n^* > 0$$

$$z p_{n-1} = k_n p_n - h_n z^{n-1} p_{n-1}^*$$

$$k_n p_n(0) = h_n \ell_{n-1} \qquad k_n = \sqrt{1-|h_n|^2} > 0$$

$$\ell_{n-1} = k_n \ell_n$$

$$(p_n, z^n p_n^*) = p_n(0)(1, z^n p_n^*) = \frac{p_n(0)}{\ell_n}(z^n p_n^*, z^n p_n^*) = \frac{p_n(0)}{\ell_n}$$

$$= \frac{h_n \ell_{n-1}}{k_n \ell_n} = h_n$$

Let $d\nu$ be a probability measure on S^1 and let $h_1, h_2, \ldots$ be the associated sequence of Schur parameters. Let us consider the vector space V of finite support column vectors $(a_n)_{n\in\mathbb{Z}}$ with the usual inner product in which the basis vectors e_n are orthonormal. I propose to define a unitary operator in V.

Take two copies of $L^2(S^1, d\nu)$ which I denote $L^2(S^1,d\nu)$ and $L^2(S^1,d\nu)z^{1/2}$. I will describe orthonormal bases in both. Let $F_{2n+1} \subset L^2(S^1, d\nu)$ $n=0,2,\ldots$ be spanned by $z^{-n}, z^{-n+1}, \ldots, z^n$ and put

$$W_{2n} = F_{2n+1} \ominus F_{2n-1} \qquad n=1,2,\ldots$$

Then W_{2n} has the basis $z^{-n}p_{2n}, z^n p_{2n}^*$ of units vectors which are not orthog. in general since

$$\left(z^{-n}p_{2n}, z^n p_{2n}^*\right) = \left(p_{2n}, z^{2n}p_{2n}^*\right) = h_{2n}$$

However there is a unique orth. basis of the form q_{2n}, q_{2n}^* where

$$q_{2n} = a z^{-n}p_{2n} + b z^n p_{2n}^*$$

and $a > 0$

Let $F_{2n} \subset L^2(S^1, d\nu)z^{1/2}$ be spanned by $z^{-n+\frac{1}{2}}, \ldots, z^{n-\frac{1}{2}}$ for $n \geq 1$, and $W_{2n-1} = F_{2n} - F_{2n-2}$. Then W_{2n-1} is spanned by

$$z^{\frac{-n+1}{2}} p_{2n-1} \quad \text{and} \quad z^{n-\frac{1}{2}} p_{2n-1}^*$$

and similarly we can find a unique orthonormal basis of the for q_{2n-1}, q_{2n-1}^* as above.

So now map V into $L^2(S^1, d\nu) \oplus L^2(S^1, d\nu) z^{1/2}$ by sending e_n to q_n for $n \geq 0$ and e_n to q_{-n}^* for $n \leq 0$. Of course $q_0 = 1$. Since the basis is orthonormal, multiplication by $z^{1/2}$ is a given by a unitary matrix ~~operator~~. Problem: Describe carefully the unitary matrix belonging to this operator $z^{1/2}$.

January 8, 1978:

Suppose we review the Bessel K-function

$$K_s(r) = \int_0^\infty e^{-\frac{r}{2}(t+t^{-1})}\, t^s \frac{dt}{t}$$

$$\frac{d}{dr} K_s = -\frac{1}{2} K_{s+1} - \frac{1}{2} K_{s-1}$$

$$\frac{s}{r} K_s = \frac{1}{2} K_{s+1} - \frac{1}{2} K_{s-1}$$

$$\left(\frac{d}{dr} + \frac{s}{r}\right) K_s = -K_{s-1} \qquad \left(\frac{d}{dr} - \frac{s}{r}\right) K_s = -K_{s+1}$$

$$\left(\frac{d}{dr} - \frac{s-1}{r}\right)\left(\frac{d}{dr} + \frac{s}{r}\right) K_s = K_s \quad \text{or} \quad \left(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \frac{s^2}{r^2} - 1\right) K_s = 0$$

$$\left(-\left(r\frac{d}{dr}\right)^2 + r^2\right) K_s = (-s^2) K_s$$

Hence $K_{i\lambda}(e^{-x})$ satisfies the DE

$$\left(-\frac{d^2}{dx^2} + e^{-2x}\right) u = \lambda^2 u$$

and moreover it decays as $x \to -\infty$. Hence we can take it as a $\varphi(x,\lambda)$ and determine its spectral measure. As $x \to +\infty$ the potential e^{-2x} decays to zero hence we have

$$K_{i\lambda}(e^{-x}) \sim A(\lambda) e^{-i\lambda x} + B(\lambda) e^{i\lambda x}$$

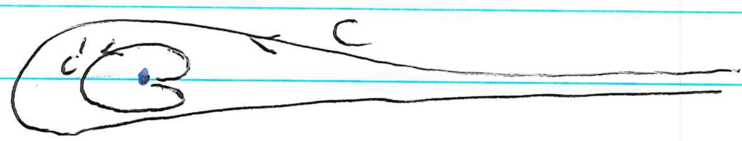
$$B(\lambda) = A(-\lambda) = A^\#(\lambda)$$

$A(\lambda)$ is holomorphic for $\operatorname{Im} \lambda > 0$.

So we need the asymptotic behavior of $K_s(r)$ as $r \searrow 0$. Put

$$F_s(r) = \int_C e^{-\frac{r}{2}(t+t^{-1})} t^s \frac{dt}{t} \qquad G_s(r) = \int_{C'} e^{-\frac{r}{2}(t+t^{-1})} t^s \frac{dt}{t}$$

where C and C' are the contours



and t^s is defined with a cut along $R_{\geq 0}$ to be $e^{s \log t}$ for $t \geq 0$ on top of the cut. Then

$$F_s - G_s = (e^{2\pi i s} - 1) K_s$$

Also for $r > 0$ we can substitute $t = \frac{2}{r} u$

$$F_s(r) = \int_C e^{-u - \frac{r^2}{4u}} \left(\frac{r}{2}\right)^{-s} u^s \frac{du}{u}$$

$$= \left(\frac{r}{2}\right)^{-s} \int_C e^{-u} \sum_{n \geq 0} \frac{1}{n!} \left(-\frac{r^2}{4}\right)^n u^{s-n} \frac{du}{u}$$

convergence of series is nice because $|u| \geq 1$ on C

$$= \left(\frac{r}{2}\right)^{-s} \sum_{n \geq 0} \frac{1}{n!} \left(-\frac{r^2}{4}\right)^n \Gamma(s-n) \cdot (e^{2\pi i s} - 1)$$

It's obvious that $t \mapsto t^{-1}$ converts C into $-C'$ so that in the formula

$$K_s = \frac{F_s}{e^{2\pi i s} - 1} - \frac{G_s}{e^{2\pi i s} - 1}$$

the second term on the left should be obtained from the first by putting $-s$ in for s. Thus we have

$$K_s(r) = \left(\frac{r}{2}\right)^{-s} \sum_{n \geq 0} \frac{1}{n!} \left(-\frac{r^2}{4}\right)^n \Gamma(s-n) + \left(\frac{r}{2}\right)^{s} \sum_{n \geq 0} \frac{1}{n!} \left(-\frac{r^2}{4}\right)^n \Gamma(-s-n)$$

So $\quad K_s(r) \sim \Gamma(s)\left(\frac{r}{2}\right)^{-s} + \Gamma(-s)\left(\frac{r}{2}\right)^{s} \quad$ as $r \to 0$

$s = i\lambda \quad \lambda \in \mathbb{R}$

or

$$K_{i\lambda}(e^{-x}) = \Gamma(i\lambda) 2^{+i\lambda} e^{i\lambda x} + \Gamma(-i\lambda) 2^{-i\lambda} e^{-i\lambda x} \qquad x \to \infty$$

Thus $\quad A(\lambda) = \Gamma(-i\lambda) 2^{-i\lambda}$. Now recall the formula

$$\delta(x_0 - y) = \int_{-\infty}^{\infty} \varphi(x,\lambda)\varphi(y,\lambda) \underbrace{\frac{d\lambda}{2\pi A(\lambda) A(-\lambda)}}_{d\mu(\lambda)}$$

$$\frac{\sin \pi s}{\pi} = \frac{1}{\Gamma(s)\Gamma(1-s)} = \frac{1}{\Gamma(s)\Gamma(-s)(-s)} \quad \text{so}$$

$$\frac{1}{A(\lambda)A(-\lambda)} = \frac{1}{\Gamma(i\lambda)\Gamma(-i\lambda)} = \frac{(-i\lambda)}{\pi}\left(\frac{e^{-\pi\lambda} - e^{\pi\lambda}}{2i}\right) = \frac{\lambda}{2\pi}\left(e^{\pi\lambda} - e^{-\pi\lambda}\right)$$

So

$$d\mu(\lambda) = \left(e^{\pi\lambda} - e^{-\pi\lambda}\right) \frac{\lambda \, d\lambda}{(2\pi)^2}$$

$$\left(\frac{d}{dr} - \frac{s}{r}\right)\left(r^{1/2} K_{s-\frac{1}{2}}\right) = r^{1/2}\left(\frac{d}{dr} + \frac{1/2}{r} - \frac{s}{r}\right) K_{s-\frac{1}{2}} = -r^{1/2} K_{s+\frac{1}{2}}$$

$$\left(\frac{d}{dr} + \frac{s}{r}\right)\left(r^{1/2} K_{s+\frac{1}{2}}\right) = -r^{1/2} K_{s-\frac{1}{2}}$$

Consider the system $\quad \frac{d}{dx} u = \begin{pmatrix} i\lambda & e^x \\ e^x & -i\lambda \end{pmatrix} u \qquad -\infty < x < \infty$

If $r = e^x$ it becomes

$$r \frac{d}{dr} u = \begin{pmatrix} i\lambda & r \\ r & -i\lambda \end{pmatrix} u$$

or $$\left(\frac{d}{dr}-\frac{i\lambda}{r}\right)u_1 = u_2$$
$$\left(\frac{d}{dr}+\frac{i\lambda}{r}\right)u_2 = u_1$$

so we get the solution

$$u = \begin{pmatrix} r^{1/2}K_{s-\frac{1}{2}} \\ -r^{1/2}K_{s+\frac{1}{2}} \end{pmatrix} \qquad s = i\lambda$$

which decays as $x \to +\infty$. As $r \to 0$ we have

$$r^{1/2}K_{s-\frac{1}{2}} \sim r^{1/2}\Big[\underbrace{\Gamma(s-\tfrac{1}{2})2^{s-\frac{1}{2}}r^{-s+\frac{1}{2}}}_{\text{involves } h} + \Gamma(-s+\tfrac{1}{2})2^{-s+\frac{1}{2}}r^{s-\frac{1}{2}}\Big]$$

so $$\begin{pmatrix} r^{1/2}K_{s-\frac{1}{2}} \\ -r^{1/2}K_{s+\frac{1}{2}} \end{pmatrix} \sim \begin{pmatrix} \Gamma(\frac{1}{2}-s)\,2^{\frac{1}{2}-s}r^{s} \\ -\Gamma(\frac{1}{2}+s)\,2^{\frac{1}{2}+s}r^{-s} \end{pmatrix} \qquad r^s = e^{i\lambda x}$$

Now $$\varphi(x,\lambda) \sim \begin{pmatrix} A(\lambda)e^{ix\lambda} \\ B(\lambda)e^{-ix\lambda} \end{pmatrix}$$ $x \to -\infty$, $\operatorname{Im}\lambda > 0$ $\Rightarrow$ $e^{ix\lambda}$ grows

then $A(\lambda) = \Gamma(\frac{1}{2}-i\lambda)\,2^{\frac{1}{2}-i\lambda}$ should be holom. $\operatorname{Im}\lambda > 0$. ✓

The spectral measure for this function should be

$$\frac{d\lambda}{2\pi|A(\lambda)|^2} = \frac{d\lambda}{2\pi}\,\frac{1}{2}\,\frac{1}{\Gamma(\frac{1}{2}-i\lambda)\Gamma(\frac{1}{2}+i\lambda)}$$
$$= \frac{d\lambda}{4\pi}\,\frac{\sin\pi(\frac{1}{2}-i\lambda)}{\pi} = \frac{d\lambda}{4\pi^2}\cos i\lambda\pi$$
$$= \frac{d\lambda}{8\pi^2}\left(e^{\pi\lambda}+e^{-\pi\lambda}\right)$$

Let's put the above system into deBranges form. The system is initially

$$r\frac{du}{dr} = \begin{pmatrix} i\lambda & r \\ r & -i\lambda \end{pmatrix}u.$$

We use $\quad z = \dfrac{w-i}{w+i} \quad$ to relate $|z|<1$ to $\operatorname{Im} w > 0$.

This means we put

$$\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 1 & -i \\ 1 & +i \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$

$$\frac{1}{2i}\begin{pmatrix} i & i \\ -1 & 1 \end{pmatrix}\begin{pmatrix} i\lambda & r \\ r & -i\lambda \end{pmatrix}\begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix} = \frac{1}{2i}\begin{pmatrix} i & i \\ -1 & 1 \end{pmatrix}\begin{pmatrix} i\lambda+r & \lambda+ir \\ -i\lambda+r & \lambda-ir \end{pmatrix}$$

$$= \begin{pmatrix} r & \lambda \\ -\lambda & -r \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} \lambda & r \\ r & \lambda \end{pmatrix}$$

So we get $\quad r\dfrac{dv}{dr} = \begin{pmatrix} r & \lambda \\ -\lambda & -r \end{pmatrix} v$

Then we take S to be a solution matrix for $\lambda = 0$

$$S = \begin{pmatrix} e^{r} & 0 \\ 0 & e^{-r} \end{pmatrix}$$

and put $\quad v = Sw.$

$$r\begin{bmatrix} e^{r}w_1' + e^{r}w_1 \\ e^{-r}w_2' - e^{-r}w_2 \end{bmatrix} = \begin{pmatrix} r & \lambda \\ -\lambda & -r \end{pmatrix}\begin{pmatrix} e^{r}w_1 \\ e^{-r}w_2 \end{pmatrix} = \begin{pmatrix} re^{r}w_1 + \lambda e^{-r}w_2 \\ -\lambda e^{r}w_1 - re^{-r}w_2 \end{pmatrix}$$

or

$$\frac{dw_1}{dr} = \lambda\,\frac{e^{-2r}}{r}\,w_2$$

$$\frac{dw_2}{dr} = -\lambda\,\frac{e^{2r}}{r}\,w_1$$

or

$$\boxed{\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\frac{dw}{dr} = \lambda\begin{pmatrix} \frac{e^{2r}}{r} & 0 \\ 0 & \frac{e^{-2r}}{r} \end{pmatrix} w}$$

What deB calls a "special Kummer" situation.

January 9, 1978

A heuristic explanation of the completeness relation:

Let $u(t) = e^{-iLt} v$ be the solution of the wave equation $i\frac{\partial}{\partial t} u = Lu$ starting at v. Take its Fourier transform

$$\hat{u}(\lambda) = \int e^{i\lambda t} u(t)\,dt = \int e^{i(\lambda - L)t} v\,dt$$

and break this into two parts:

$$\int_0^\infty e^{i(\lambda-L)t} v\,dt = -\frac{1}{i}(\lambda - L)^{-1} v \qquad \text{analytic for } \operatorname{Im}\lambda > 0$$

$$= -\frac{1}{i} G^+(\lambda)\, v$$

where $G^+(\lambda)$ is the limit of $(\lambda - L)^{-1}$ for λ real of the resolvent in the upper half plane. Similarly

$$\int_{-\infty}^0 e^{i(\lambda-L)t} v\,dt = +\int_0^\infty e^{-i(\lambda-L)t} v\,dt = -i(\lambda - L)^{-1} v \qquad \text{and for } \operatorname{Im}\lambda < 0$$

$$= -i\, G^-(\lambda)\, v$$

Then by Fourier inversion we have

$$v = \int_{-\infty}^\infty \hat{u}(\lambda)\frac{d\lambda}{2\pi} = \frac{-1}{2\pi i}\int_{-\infty}^\infty \left[G^+(\lambda) - G^-(\lambda)\right] v\,d\lambda$$

or

$$\boxed{\frac{-1}{2\pi i}\int_{-\infty}^\infty \left[G^+(\lambda) - G^-(\lambda)\right] d\lambda = I}$$

Put $g(t) = \int_{-\infty}^{\infty} e^{-i\lambda t} \rho(\lambda) \frac{d\lambda}{2\pi}$ so that by Fourier inversion

$$\rho(\lambda) = \int_{-\infty}^{\infty} e^{i\lambda t} g(t)\, dt = \int_0^{\infty} + \int_{-\infty}^{0}$$

$$\int_0^{\infty} e^{i\lambda t} g(t) dt = \int_{\hat\lambda \in \mathbb{R}} \rho(\hat\lambda) \frac{d\hat\lambda}{2\pi} \int_0^{\infty} e^{i(\lambda - \hat\lambda)t} dt = -\int \frac{\rho(\hat\lambda)}{\lambda - \hat\lambda} \frac{d\hat\lambda}{2\pi i} \qquad \text{anal in UHP}$$

$$\int_{-\infty}^{0} e^{i\lambda t} g(t)\, dt = \int \rho(\hat\lambda) \frac{d\hat\lambda}{2\pi} \int_{-\infty}^{0} e^{i(\lambda - \hat\lambda)t} dt = +\int \frac{\rho(\hat\lambda)}{\lambda - \hat\lambda} \frac{d\hat\lambda}{2\pi i} \qquad \text{anal in LHP.}$$

Thus

$$\rho(\lambda) = \frac{1}{2\pi i}\left[f^{+}(\lambda) + f^{-}(\lambda)\right]$$

where $f^{\pm}(\lambda)$ are the limiting values of the function

$$f(\lambda) = \int \frac{\rho(\hat\lambda)\, d\hat\lambda}{\lambda - \hat\lambda}$$

More generally suppose

$$g(t) = \int e^{-i\lambda t} d\mu(\lambda) \qquad \overline{g(t)} = g(-t)$$

is the Fourier transform of a measure. Then

$$\int_0^{\infty} e^{i\lambda t} g(t)\, dt = f^{+}(\lambda) \qquad \text{analytic in UHP}$$

$$\int_{-\infty}^{0} e^{i\lambda t} g(t)\, dt = f^{-}(\lambda) \qquad \text{" " LHP}$$

and

$$\overline{f^{-}(\lambda)} = \int_{-\infty}^{0} e^{-i\bar\lambda t} g(-t)\, dt = \int_0^{\infty} e^{i\bar\lambda t} g(t)\, dt = f^{+}(\bar\lambda)$$

so

$$f^{-}(\lambda) = \overline{f^{+}(\bar\lambda)}$$

$f^{\pm}(\lambda)$ boundary values of $\quad f(\lambda) = i\int \frac{d\mu(\hat{\lambda})}{\lambda - \hat{\lambda}}$

Maybe the important thing to notice is that [redacted] the [redacted] Stieltjes transform of the measure, which is what de B, Krein play with, is essentially that [redacted] part of the Fourier transform of the measure for $t > 0$ which is what enters into the Gelfand-Levitan equation.

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Consider a J-matrix eigenvalue problem

1) $$Lu = (aT + b + T^{-1}a)u = \lambda u$$

on the space of column vectors $u = (u_i)$, $0 \le i \le \ell+1$ satisfying given real boundary conditions at the ends:

2) $$\frac{u_1}{a_0 u_0} = \frac{\alpha_1}{\alpha_2} \quad \text{and} \quad \frac{u_{\ell+1}}{a_\ell u_\ell} = \frac{\beta_1}{\beta_2}$$

I can write 1) in the form

$$\begin{pmatrix} u_{n+1} \\ a_n u_n \end{pmatrix} = \begin{pmatrix} 0 & -\frac{1}{a_n} \\ a_n & 0 \end{pmatrix} \begin{pmatrix} u_n \\ -a_n u_{n+1} \end{pmatrix} = \begin{pmatrix} 0 & -\frac{1}{a_n} \\ a_n & 0 \end{pmatrix} \begin{pmatrix} u_n \\ (b_n - \lambda)u_n + a_{n-1}u_{n-1} \end{pmatrix}$$

$$= \begin{pmatrix} 0 & -\frac{1}{a_n} \\ a_n & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ b_n & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\lambda & 1 \end{pmatrix} \begin{pmatrix} u_n \\ a_{n-1}u_{n-1} \end{pmatrix}$$

and I recall that $\begin{pmatrix} 1 & 0 \\ -\lambda & 1 \end{pmatrix}$ shrinks the UHP for $\operatorname{Im}\lambda > 0$. Recall Green's formula:

$$\left[(Lu)v - uLv\right]_n = W(u,v)_n - W(u,v)_{n-1}$$

where $$W(u,v)_n = \begin{vmatrix} u_{n+1} & v_{n+1} \\ a_n u_n & a_n v_n \end{vmatrix}$$

Hence we have

$$\sum_{n=1}^{\ell} \left[(Lu)\bar{v} - uL\bar{v}\right]_n = W(u,\bar{v})_\ell - W(u,\bar{v})_0$$

for arbitrary u, v given over the interval $0 \le n \le \ell+1$.
Now you have to be careful to get a good Hillbert space

and self-adjoint operator. We obviously want to use the space of column vectors $(u_1, \ldots, u_n)$ with the usual inner product, so it is necessary to be able to extend any such thing uniquely to a u given over $[0, l+1]$ satisfying the boundary conditions. Hence we must have $\alpha_1 \neq 0$ and $\beta_2 \neq 0$, so it seems.

Define $\varphi(n,\lambda)$ for $0 \le n \le l+1$ to be the solution of 1) in degrees $1 \le n \le l$, i.e. satisfying

$$a_n \varphi(n+1,\lambda) + b_n \varphi(n,\lambda) + a_{n-1}\varphi(n-1,\lambda) = \lambda \varphi(n,\lambda)$$

for $1 \le n \le l$ starting with the values

$$\begin{pmatrix} \varphi(1,\lambda) \\ a_0 \varphi(0,\lambda) \end{pmatrix} = \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix}$$

Similarly define $\psi(n,\lambda)$ $\quad 0 \le n \le l+1$ using

$$\begin{pmatrix} \psi(l+1,\lambda) \\ a_l \psi(l,\lambda) \end{pmatrix} = \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix}$$

The choice of φ determines a spectral measure

$$d\mu(\lambda) = \sum_{i=1}^{n} \frac{\delta(\lambda - \lambda_i)}{\|\varphi_{\lambda_i}\|^2}$$

whose Stieltjes transform

$$\int \frac{d\mu(\bar\lambda)}{\lambda - \bar\lambda}$$

is a rational function with poles at the eigenvalues.

Another way of proceeding is to choose a solution $\tilde{\varphi}$ of $Lu = \lambda u$, or rather a set of boundary values $\binom{\gamma_1}{\gamma_2}$ at $n=0$, and let $\tilde{\varphi}$ be the resulting solution, such that $W(\varphi, \tilde{\varphi}) = 1$. Then we can define a function $m(\lambda)$ by

$$m\varphi + \tilde{\varphi} = \frac{\psi}{W(\varphi, \psi)}$$

$\tilde{\varphi}$ is unique up to a real constant. The question is which $\tilde{\varphi}$ makes $m =$ the Stieltjes transform.

$$\begin{pmatrix} \varphi(1,\lambda) & \tilde{\varphi}(1,\lambda) \\ a_0\varphi(0,\lambda) & a_0\tilde{\varphi}(1,\lambda) \end{pmatrix} (m) = \begin{pmatrix} \alpha_1 & \gamma_1 \\ \alpha_2 & \gamma_2 \end{pmatrix} (m) = \frac{\psi(1,\lambda)}{a_0\psi(0,\lambda)}$$

so
$$m(\lambda) = \begin{pmatrix} \gamma_2 & -\gamma_1 \\ -\alpha_2 & \alpha_1 \end{pmatrix} \left(\frac{\psi(1,\lambda)}{a_0\psi(0,\lambda)} \right)$$

$$= \frac{\gamma_2 \psi(1,\lambda) - \gamma_1 a_0 \psi(0,\lambda)}{-\alpha_2 \psi(1,\lambda) + \alpha_1 a_0 \psi(0,\lambda)}$$

where the denominator is $W(\varphi, \psi)$.

Now $\psi(1,\lambda)$ is a poly of degree $l-1$, and $\psi(0,\lambda)$ has degree l. Since $\alpha_1 \neq 0$, $W(\varphi,\psi)$ has degree l, as it must because there are l eigenvalues. But also the Stieltjes transform vanishes at $\lambda = \infty$, so then we want $\gamma_1 = 0$, and so $\gamma_2 = \alpha_1^{-1}$.

Another proof: We have the formula

$$G(n,n';\lambda) = \frac{\varphi(n_<,\lambda)\,\psi(n_>,\psi)}{W(\varphi,\psi)}$$

valid for $1 \le n, n' \le l$ because $(\lambda - L)^{-1}$ makes sense only in the Hilbert space in question. Put $n = n' = 1$ and you get

$$\alpha_1 \frac{\psi(1,\lambda)}{W(\varphi,\psi)} = \int \frac{\alpha_1^2 \, d\mu(\hat{\lambda})}{\lambda - \hat{\lambda}}$$

or

$$\int \frac{d\mu(\hat{\lambda})}{\lambda - \hat{\lambda}} = \frac{1}{\alpha_1} \frac{\psi(1,\lambda)}{W(\varphi,\psi)}$$

So it appears that when working with J-matrices on a half-line $0 \le n < \infty$, of the possible boundary conditions at 0, there is a singular one namely $u_1 = 0$. If the φ solution is defined by boundary values away from this singular case, then one gets a definite analytic function of λ off $\operatorname{Im} \lambda \neq 0$ using the 0-boundary values of ψ, and this function is the Stiejes transform of the measure.

Let's look now at $Lu = -u'' + qu = \lambda u$ where q dies fast enough as $x \longrightarrow \infty$ so scattering calculations are valid.

$$\psi(x,\lambda) \sim e^{i\sqrt{\lambda}\, x} \qquad \begin{array}{l} \sqrt{\lambda} \in UHP \\ \text{for } \lambda \notin \mathbb{R}_{\ge 0} \end{array}$$

$$\varphi(x,\lambda) \simeq A(\sqrt{\lambda}) e^{-i\sqrt{\lambda}\, x} + B(\sqrt{\lambda}) e^{+i\sqrt{\lambda}\, x}$$

Assuming no bound states the spectral measure is

$$d\mu(\lambda) = \begin{cases} \dfrac{1}{2\pi} \dfrac{d\sqrt{\lambda}}{A(\sqrt{\lambda})\, A(-\sqrt{\lambda})} & \lambda > 0 \\ 0 & \lambda < 0 \end{cases}$$

Calculate the example $q=0$, $\varphi(x,\lambda) = \cos\sqrt{\lambda}\,x$

Then
$$\varphi(x,\lambda) = \cos\sqrt{\lambda}\,x = \tfrac{1}{2}\left(e^{-i\sqrt{\lambda}x} + e^{i\sqrt{\lambda}x}\right)$$

so $A(\sqrt{\lambda}) = \frac{1}{2}$ and $d\mu(\lambda) = \frac{1}{2\pi}\,\frac{d\sqrt{\lambda}}{\frac{1}{4}} = \frac{2}{\pi}\,d\sqrt{\lambda}$ on $\mathbb{R}_{\geq 0}$

The Stieljes transform is
$$f(\lambda) = \frac{2}{\pi}\int_0^\infty \frac{d\sqrt{\hat{\lambda}}}{\lambda - \hat{\lambda}} = \frac{2}{\pi}\int_0^\infty \frac{dt}{\lambda - t^2}$$

which is perfectly well-defined. Put $\lambda = -s^2$ where $s>0$. Then
$$\frac{2}{\pi}\int_0^\infty \frac{dt}{\lambda - t^2} = -\frac{2}{\pi}\int_0^\infty \frac{s\,dt}{s^2+t^2s^2} = -\frac{2}{\pi}\frac{1}{s}\int_0^\infty \frac{dt}{1+t^2} = -\frac{1}{s}$$

hence recalling we choose $\sqrt{\lambda}$ to be in the UHP.
$$f(\lambda) = -\frac{1}{\sqrt{-\lambda}} = -\frac{i}{\sqrt{\lambda}}$$

(Check if $\lambda \leq 0$, this should be real and <0.)

Next consider
$$\varphi(x,\lambda) = \alpha_1\cos\sqrt{\lambda}\,x + \alpha_2\frac{\sin\sqrt{\lambda}\,x}{\sqrt{\lambda}}$$
$$= \left(\frac{\alpha_1}{2} - \frac{\alpha_2}{\sqrt{\lambda}\,2i}\right)e^{-i\sqrt{\lambda}x} + \left(\frac{\alpha_1}{2} + \frac{\alpha_2}{\sqrt{\lambda}\,2i}\right)e^{i\sqrt{\lambda}x}$$

so that $A(\lambda) = \frac{1}{2}\left(\alpha_1 + \frac{i}{\sqrt{\lambda}}\alpha_2\right)$ and
$$d\mu(\lambda) = \frac{2}{\pi}\,\frac{d\sqrt{\lambda}}{\left|\left(\alpha_1 + \frac{i}{\sqrt{\lambda}}\alpha_2\right)\right|^2} = \frac{2}{\pi}\,\frac{d\sqrt{\lambda}}{\alpha_1^2 + \frac{\alpha_2^2}{\lambda}}$$

So the Stieljes transform of the measure is defined for $\alpha_1 \neq 0$.

Look at $\alpha_1 = 0$, $\alpha_2 = 1$ more carefully

$$d\mu(\lambda) = \frac{2}{\pi} \lambda \, d\sqrt{\lambda} \qquad \text{for } \lambda \geq 0$$

$$\int \frac{d\mu(\hat\lambda)}{\lambda - \hat\lambda} = \frac{2}{\pi} \int_0^\infty \frac{\hat\lambda \, d\sqrt{\hat\lambda}}{\lambda - \hat\lambda} = \frac{2}{\pi} \int_0^\infty \frac{t^2 \, dt}{\lambda - t^2}$$

not convergent although its imaginary part is.

Next let's find the formula for the Stieltjes transform $m(\lambda)$ when it exists in terms of the ψ boundary values at 0. The idea is to choose $\tilde\varphi$ to have the singular boundary values, and then $m(\lambda)$ is given by

$$m(\lambda)\varphi + \tilde\varphi = \frac{\psi}{W(\varphi,\psi)}$$

$$\begin{pmatrix} \alpha_1 & 0 \\ \alpha_2 & \frac{1}{\alpha_1} \end{pmatrix} m = \frac{\psi(0,\lambda)}{\psi'(0,\lambda)}$$

or

$$m(\lambda) = \begin{pmatrix} \frac{1}{\alpha_1} & 0 \\ -\alpha_2 & \alpha_1 \end{pmatrix} \left(\frac{\psi(0,\lambda)}{\psi'(0,\lambda)} \right)$$

$$\boxed{m(\lambda) = \frac{1}{\alpha_1} \cdot \frac{\psi(0,\lambda)}{W(\varphi,\psi)}}$$

Check this for $\varphi = \cos\sqrt{\lambda}\,x \qquad \psi = e^{i\sqrt{\lambda}\,x}$

$$W(\varphi,\psi) = \frac{1}{2} 2i\sqrt{\lambda} = i\sqrt{\lambda}$$

so

$$m(\lambda) = \frac{1}{i\sqrt{\lambda}}$$

which is what we computed the Stieltjes transform to be. Green's formula proof will yield the above at least assuming the integral for G converges nicely.

Consider $\quad Lu = -u'' + qu = \lambda^2 u$ $\quad$ on $0 \leq x < \infty$ (note the λ^2)
and let $\varphi(x,\lambda)$ denote the solution starting with

$$\begin{cases} \varphi(0,\lambda) = \alpha_1 \\ \varphi'(0,\lambda) = \alpha_2 . \end{cases}$$

Assuming the spectrum is in $\mathbb{R}_{>0}$ it follows that $\varphi(x,0)$ doesn't vanish for $x>0$, so if we have the good case $\alpha_1 \neq 0$ we have a well-defined function

$$p(x) = \frac{\varphi'(x,0)}{\varphi(x,0)}$$

satisfying

$$p'(x) = \frac{\varphi''(x,0)}{\varphi(x,0)} - \frac{(\varphi'(x,0))^2}{\varphi(x,0)^2} = q - p^2 .$$

Hence I can factor the operator L

$$L = -\frac{d^2}{dx^2} + q = -\left(\frac{d}{dx} + p\right)\left(\frac{d}{dx} - p\right).$$

Consider the system

$$(*) \quad \begin{cases} \left(\frac{d}{dx} - p\right) u_1 = +\lambda u_2 \\ \left(\frac{d}{dx} + p\right) u_2 = -\lambda u_1 \end{cases}$$

Let u satisfy $\quad Lu = \lambda^2 u , \quad \frac{u(0)}{u'(0)} = \frac{\alpha_1}{\alpha_2}$ or better

$$u'(0) = \frac{\alpha_2}{\alpha_1} u(0).$$

Then provided $\lambda \neq 0$ we can put $u_1 = u$ and

$$u_2 = \frac{1}{\lambda}\left(\frac{d}{dx} - p\right) u_1$$

and we get a solution of the system satisfying the boundary condition $u_2(0)=0$. If $\lambda = 0$ we put $u_1 = u$ and $u_2 = 0$. Clearly (u_1, u_2) is uniquely determined by u, hence we get a one-one correspondence in the above way.

The system $(*)$ admits the symmetry $\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \mapsto \begin{pmatrix} u_1 \\ -u_2 \end{pmatrix}$ $\lambda \mapsto -\lambda$, hence the spectral measure is symmetric provided the boundary condition is preserved, which means we have either of the boundary conditions $u_1(0)=0$ or $u_2(0)=0$

Note that if u is a l^2-eigenfunction

$$(u_2, u_2) = \left(u_2, \frac{1}{\lambda}\left(\frac{d}{dx} - p\right)u_1\right) = \frac{1}{\lambda}\left(\left(-\frac{d}{dx} - p\right)u_2, u_1\right)$$

$$= (u_1, u_1)$$

if $\lambda \neq 0$ and that $(u_2, u_2) = 0$ if $\lambda = 0$. Thus the spectral measures for the two eigenvalue problems are probably the same. Also if $d\mu(\lambda)$ is even, then its Stieljes transform can be written

$$\int \frac{d\mu(\hat{\lambda})}{\lambda - \hat{\lambda}} = \frac{1}{2}\int \left(\frac{1}{\lambda - \hat{\lambda}} + \frac{1}{\lambda + \hat{\lambda}}\right) d\mu(\hat{\lambda})$$

$$= \lambda \int \frac{d\mu(\hat{\lambda})}{\lambda^2 - \hat{\lambda}^2}$$ this integral exists

hence there is a definite way of making sense out of the Stieljes transform of the spectral measure for $(*)$ with the boundary condition $u_2(0) = 0$. The question is how is $\tilde{\varphi}$ to be defined so that we have

$$(1) \qquad m(\lambda)\,\vec{\varphi}(\lambda) + \vec{\tilde{\varphi}}(\lambda) = \frac{\vec{\psi}(\lambda)}{W(\vec{\varphi},\vec{\psi})}$$

with $m(\lambda)$ the Stieltjes transform. What is φ?

$$\varphi_1(x,\lambda) = \varphi(x,\lambda)$$
$$\varphi_2(x,\lambda) = \frac{1}{\lambda}\left(\varphi'(x,\lambda) - p(x)\varphi(x,\lambda)\right)$$

hence

$$\begin{pmatrix} \varphi_1(0,\lambda) \\ \varphi_2(0,\lambda) \end{pmatrix} = \begin{pmatrix} \alpha_1 \\ 0 \end{pmatrix}$$

$$\psi_1(x,\lambda) = \psi(x,\lambda)$$
$$\psi_2(x,\lambda) = \frac{1}{\lambda}\left(\psi'(x,\lambda) - p(x)\varphi(x,\lambda)\right)$$

$$W(\vec{\varphi},\vec{\psi}) = \begin{vmatrix} \varphi & \psi \\ \frac{1}{\lambda}(\varphi' - p\varphi) & \frac{1}{\lambda}(\psi' - p\psi) \end{vmatrix} = \frac{1}{\lambda} W(\varphi,\psi)$$

So from the first component of (1)

$$m(\lambda)\,\varphi(x,\lambda) + \vec{\tilde{\varphi}}(x,\lambda)_1 = \frac{\lambda\psi(x,\lambda)}{W(\varphi,\psi)}$$

It seems reasonable to expect this to agree with what was done for $Lu = \lambda^2 u$, hence $\vec{\tilde{\varphi}}(0,\lambda)_1 = 0$ and so we get

$$m(\lambda)\,\alpha_1 = \lambda \int \frac{d\mu(\hat{\lambda})}{\lambda^2 - \hat{\lambda}^2} = \int \frac{d\mu(\hat{\lambda})}{\lambda - \hat{\lambda}}$$

Principal value