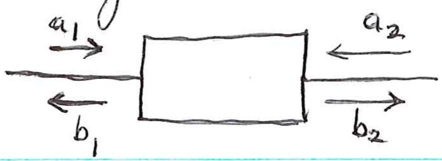
May 5, 1978:



$$\begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = [redacted] \underbrace{\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}}_{S} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$$

$S(z)$ is [redacted] analytic in $|z| \leq 1$ and unitary on the boundary $|z| = 1$, and of norm ≤ 1 inside S^1.

Example from Schroedinger equation: $-u'' + qu = k^2 u$ with q of compact support. Then any solution has the asymptotic description:

$$a_1 e^{ikx} + b_1 e^{-ikx} \longleftrightarrow a_2 e^{-ikx} + b_2 e^{ikx}$$

If A, B are defined by

$$Ae^{ikx} + Be^{-ikx} \longleftrightarrow e^{ikx}$$

$$\bar{B}e^{ikx} + \bar{A}e^{-ikx} \longleftrightarrow e^{-ikx}$$

(here $k \in \mathbb{R}$, otherwise $\overline{B(k)}$ should be replaced by $B(-k)$ [redacted]), then we have

$$\begin{aligned} a_1 &= Ab_2 + \bar{B}a_2 \\ b_1 &= Bb_2 + \bar{A}a_2 \end{aligned} \qquad \text{or} \qquad \begin{pmatrix} a_1 \\ b_1 \end{pmatrix} = \underbrace{\begin{pmatrix} A & \bar{B} \\ B & \bar{A} \end{pmatrix}}_{T} \begin{pmatrix} b_2 \\ a_2 \end{pmatrix}$$

So we have

$$T(k) = \begin{pmatrix} A(k) & B(-k) \\ B(k) & A(-k) \end{pmatrix} \qquad \begin{cases} \overline{A(-\bar{k})} = A(k) \\ \overline{B(-\bar{k})} = B(k) \end{cases}$$

and this is analytic for all k except possibly $k = 0, \infty$.

In this example the scattering matrix S is found to be

$$S(k) = \begin{pmatrix} \frac{B(k)}{A(k)} & \frac{1}{A(k)} \\ \frac{1}{A(k)} & -\frac{B(-k)}{A(k)} \end{pmatrix}$$

Now notice that this is analytic provided $A(k) \neq 0$. For $\mathrm{Im}(k) > 0$ this means that there are no bound states for the Schroedinger equation. So we have to assume no bound states if we want $S(k)$ to be analytic in the UHP.

$$\det S(k) = -\frac{B\bar{B}}{A^2} - \frac{1}{A^2} = -\frac{A\bar{A}}{A^2} = -\frac{\bar{A}}{A} = -\frac{A(-k)}{A(k)}$$

On the UHP $S(k)$ fails to be invertible when $A(-k) = 0$ which means that we have

$$\underset{\text{grows as } x \to -\infty}{B(k)e^{+ikx}} \longleftrightarrow \underset{\text{grows as } x \to +\infty}{e^{-ikx}}$$

The time dependendence of this solution is e^{+ikt}, so this represents a radiating state; you see waves travelling outward.

Example: Recall that a δ function potential leads to a transfer matrix of the form

$$\begin{pmatrix} e^{-ikd} & 0 \\ 0 & e^{ikd} \end{pmatrix} \underbrace{\begin{pmatrix} a & \bar{b} \\ b & \bar{a} \end{pmatrix}}_{\in SU(1,1)} \begin{pmatrix} e^{ikd} & 0 \\ 0 & e^{-ikd} \end{pmatrix} = \begin{pmatrix} a & \bar{b}e^{-2idk} \\ be^{2idk} & \bar{a} \end{pmatrix}$$

The corresponding scattering matrix is

$$S^{\circ} = \begin{pmatrix} \frac{\bar{b}}{a}e^{-2idk} & \frac{1}{a} \\ \frac{1}{a} & -\frac{be^{2idk}}{a} \end{pmatrix}$$

This blows up as $k \to +i\infty$ because we have not arranged for the incoming an outgoing spaces to be orthogonal.

Be more careful. Suppose we have a δ-function potential supported at $x = 0$.

$$-u'' + c\,\delta(x)\,u = k^2 u$$

Integrate over $[-\varepsilon, \varepsilon]$ and let $\varepsilon \to 0$

$$-\left[u'\right]_{0-}^{0+} + c\,u(0) = 0$$

So if $Ae^{ikx} + Be^{-ikx} \longleftrightarrow e^{ikx}$ then u should be continuous at 0:

$$A + B = 1$$

and its derivative jumps by $c\,u(0)$:

$$-\left[Aik + B(-ik)\right] + ik = c \qquad \text{or}$$

$$A - B = 1 - \frac{c}{ik}$$

So

$$A = 1 - \frac{c}{2ik} \qquad B = \frac{c}{2ik}$$

and so

$$T = \begin{pmatrix} 1 - \frac{c}{2ik} & -\frac{c}{2ik} \\ \frac{c}{2ik} & 1 + \frac{c}{2ik} \end{pmatrix}$$

$$S = \begin{pmatrix} \dfrac{\frac{c}{2ik}}{1 - \frac{c}{2ik}} & \dfrac{1}{1 - \frac{c}{2ik}} \\ \dfrac{1}{1 - \frac{c}{2ik}} & \dfrac{\frac{c}{2ik}}{1 - \frac{c}{2ik}} \end{pmatrix} = \begin{pmatrix} \dfrac{c}{2ik - c} & \dfrac{2ik}{2ik - c} \\ \dfrac{2ik}{2ik - c} & \dfrac{c}{2ik - c} \end{pmatrix}$$

The scattering matrix has a pole where

$$k = \frac{c}{2i}$$

which for $c > 0$ is in the lower half-plane, hence there are no bound states when $c > 0$, which is of course intuitively clear as positive potentials have no bound states.

Another method which yields the same formula for T is to take a square well-potential and let it approach $c \cdot \delta(x)$. Incidently this gives

$$T = \begin{pmatrix} 1 & 1 \\ ik & -ik \end{pmatrix}^{-1} \begin{pmatrix} 1 & 0 \\ -c & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ ik & -ik \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix}^{-1} \begin{pmatrix} 1 & 0 \\ -\frac{c}{k} & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix}$$

which probably expresses the fact u' jumps by $cu(0)$ translated to the e^{ikx}, e^{-ikx} description.

Here's a problem with the above examples which makes them confusing: You are looking at classical scattering matrices instead of the ones which result from 2-ports in the abstract sense.

Another example. Suppose we take a segment and use the natural exponentials normalized at the ends:

$$e^{-ikm} e^{ikx} \longleftrightarrow e^{ik(x-m)} \qquad m > 0$$

$$\therefore \quad A = e^{-ikm} \qquad B = 0 \qquad \text{so}$$

$$T(k) = \begin{pmatrix} e^{-ikm} & 0 \\ 0 & e^{+ikm} \end{pmatrix}$$

Note that $\mathrm{Im}(k) > 0 \Rightarrow$ $T(k)$ ~~[illegible]~~ expands disk

In the previous example note that $\begin{pmatrix}1 & 1\\ +i & -i\end{pmatrix}(z) = \frac{1}{i}\frac{z+1}{z-1} = \lambda$ is the inverse of the Cayley transform

$$z = -\frac{1}{2i}\begin{pmatrix}-i & -1\\ -i & 1\end{pmatrix}(\lambda) = \frac{-i\lambda-1}{-i\lambda+1} = \frac{\lambda-i}{\lambda+i}$$

so it maps the disk to the UHP. Also

$$\begin{pmatrix}1 & 0\\ -\frac{c}{k} & 1\end{pmatrix}(\lambda) = \frac{1}{-\frac{c}{k}+\frac{1}{\lambda}}$$

in UHP if k is.

shrinks the LHP, so $T(k)$ expands the disk for $\mathrm{Im}(k)>0$.

Suppose

$$T = \begin{pmatrix}e^{-ikm} & 0\\ 0 & e^{+ikm}\end{pmatrix} \qquad m>0$$

then

$$S = \begin{pmatrix}0 & e^{+ikm}\\ e^{+ikm} & 0\end{pmatrix}$$

which goes to ∞ as $k \to +i\infty$.

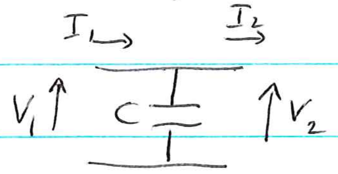
So it seems that for $S(z)$ to be analytic for $|z|<1$ corresponds to $T(z)$ expanding the disk.

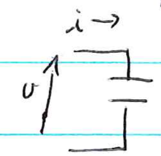
May 6, 1978.

Yesterday I saw an example of a T-matrix was

$$\begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix}^{-1} \begin{pmatrix} 1 & 0 \\ -\frac{c}{\lambda} & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix} \qquad c > 0$$

and that it has the property of expanding the disk for $\mathrm{Im}(\lambda) > 0$. Recall the 2-port





$$Cv = q \qquad\qquad$$

$$C\frac{dv}{dt} = i \qquad \tilde{V} \overset{\text{def}}{=} iV$$

$$CVi\omega = I \qquad \frac{\tilde{V}}{I} = \frac{1}{C\omega}$$

$$\begin{pmatrix} \tilde{V}_1 \\ I_1 \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & 0 \\ C\omega & 1 \end{pmatrix}}_{} \begin{pmatrix} \tilde{V}_2 \\ I_2 \end{pmatrix}$$

shrinks UHP for $\mathrm{Im}\,\omega < 0$

$\therefore$ expands UHP for $\mathrm{Im}(\omega) > 0$.

Hence 2-ports will give T-matrices [redacted] expanding UHP for $\mathrm{Im}(\omega) > 0$. Corresponding disk-form is

$$\frac{1}{2}\begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix}\begin{pmatrix} 1 & 0 \\ C\omega & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 - iC\omega & -i \\ 1 + iC\omega & i \end{pmatrix}\begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} 2 - iC\omega & -iC\omega \\ iC\omega & 2 + iC\omega \end{pmatrix}$$

$$T(\omega) = \begin{pmatrix} 1 - \frac{iC\omega}{2} & -\frac{iC\omega}{2} \\ \frac{iC\omega}{2} & 1 + \frac{iC\omega}{2} \end{pmatrix}$$

Put $C = 2$ and [redacted] shift from ω to $z = \frac{\omega - i}{\omega + i}$

$$\omega = \frac{1}{i}\,\frac{z+1}{z-1} \qquad 1 - i\omega = 1 - \frac{z+1}{z-1} = \frac{-2}{z-1}$$

$$1 + i\omega = 1 + \frac{z+1}{z-1} = \frac{2z}{z-1}$$

$$\therefore \quad T(z) = \frac{1}{z-1}\begin{pmatrix} -2 & -z-1 \\ z+1 & 2z \end{pmatrix}$$

analytic & invertible for all $z \neq 1$ including $z = \infty$

$$S(z) = \begin{pmatrix} \frac{z+1}{-2} & \frac{z-1}{-2} \\ \frac{z-1}{-2} & -\frac{z+1}{2} \end{pmatrix}$$

or

$$-S(z) = \begin{pmatrix} \frac{z+1}{2} & \frac{z-1}{2} \\ \frac{z-1}{2} & \frac{z+1}{2} \end{pmatrix}$$

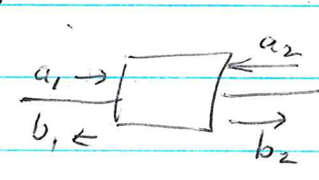
analytic for all $z \neq \infty$
$\det(-S) = z$
not invertible for $z = 0$.

$S(z)$ analytic for $|z| \leq 1$ and of $\| \; \| \leq 1$ on S^1

$$\Rightarrow \quad |(S(z)u, v)| \leq \|S(z)u\|\,\|v\| \leq \|u\|\,\|v\|$$

for $|z| = 1$ and hence for all $|z| \leq 1$ by maximum modulus. Thus $|S(z)| \leq 1$ in the disk. If $S(z)$ is an operator in a finite-dimensional space and $\|S(z)\| = 1$ at an interior point z_0 then we can find unit vectors (u,v) with $(S(z)u, v) = 1$ at z_0, hence $(S(z)u, v) = 1$ for all z, hence $S(z)u = v$. On the unit circle $S(z)$ carries $\langle u\rangle^\perp$ into $\langle v\rangle^\perp$ and this follows for all z by Cauchy. Thus if one can't split off a line on which $S(z)$ is constant we must have $\|S(z)\| < 1$ for $|z| < 1$.

Notice also that for $\begin{pmatrix} a_1 \\ b_1 \end{pmatrix} = T\begin{pmatrix} b_2 \\ a_1 \end{pmatrix}$



we have
$$|a_1|^2 - |b_1|^2 \geqslant |b_2|^2 - |a_2|^2 \qquad \text{for } |z|<1$$
because $\|S(z)\| \leqslant 1$ and $\begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = S\begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$. Thus for $|z|<1$ T expands the unit disk.

Note that multiplying T by a scalar ξ of modulus 1 on S^1 changes S as follows
$$S = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \longmapsto \begin{pmatrix} \alpha & \xi\beta \\ \xi^{-1}\gamma & \delta \end{pmatrix}$$

Suppose $S(z)$ is rational in z and analytic in $|z| \leqslant 1$. Then β and γ are rational functions analytic in $|z| \leqslant 1$ with
$$|\beta|^2 = 1 - |\alpha|^2 = |\gamma|^2 \qquad \text{on } S^1.$$

So it follows that $p = \frac{\beta}{\gamma}$ is a rational function of z of modulus 1 on S^1, and hence is a product of Blaschke factors
$$\frac{z-h}{1-\bar{h}z} \qquad |h|<1$$
or their inverses. Suppose $p(h) = 0$. Then I take ξ^{-1} to be this Blaschke factor. ξ^{-1} is analytic inside $|z| \leqslant 1$ so $\xi^{-1}\gamma$ remains analytic, and also
$$\xi\beta \qquad \beta(h) = (p\gamma)(h) = 0.$$
is analytic because . For the new scattering matrix we have
$$\frac{\xi\beta}{\xi^{-1}\gamma} = \xi^2 \cdot \frac{\beta}{\gamma}$$

So in this way by scalar multiplication we can change [redacted] the order of zeroes of $p = \frac{\beta}{\gamma} = \det T$ to either 0 or 1. [redacted] If $p(h) = \infty$, take ζ to be $\frac{z-h}{1-\bar{h}z}$. Then ζ is analytic [redacted] in the closed disk, so is $\zeta\beta$. And also

$\zeta^{-1}\gamma$ [redacted] is analytic because $\gamma(h) =$ [redacted] 0.

Similarly we can make the order of the poles of p either 0 or 1.

May 7, 1978:

Suppose that f is an analytic function for $|z| < 1+\varepsilon$ and g is another such function such that $|f| = |g|$ for $|z| = 1$ and such that f, g have the same zeros counted with multiplicity for $|z| < 1$. From $|f| = |g|$ on S^1 we see f, g have same zeroes on S^1, hence $\frac{f}{g}$ is analytic for $|z| \leq 1$ and of modulus 1 on S^1 so

[redacted] $$\left|\frac{f}{g}\right| \leq 1$$

in the closed disk. Interchanging f, g we see that $\frac{f}{g}$ is a constant of modulus 1.

Next note that f has finitely many zeroes [redacted] in [redacted] $D = \{|z| < 1\}$, so there is a finite [redacted] Blashke product p with $h = \frac{f}{p}$ analytic ~~on~~ $\bar{D}$ and without zeroes in D. So

$$f = ph$$

is the canonical factoring of f into an inner factor p and the outer factor h. When f is rational, so are p and h.

[redacted]

May 9, 1978.

We describe a [redacted] 2-port by a rational matrix $T(z)$ such that for $|z| \leq 1$ it expands the disk and for $|z| \geq 1$ it contracts the disk, and we consider only z for which $T(z)$ is defined and invertible. We know that T corresponds to a scattering matrix $S = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ which is analytic for $|z| \leq 1$ by the formula[redacted]

$$T(z) = \begin{pmatrix} \frac{1}{\gamma} & -\frac{\delta}{\gamma} \\ \frac{\alpha}{\gamma} & \frac{\beta\gamma - \alpha\delta}{\gamma} \end{pmatrix} \qquad \det T = \frac{\beta}{\gamma}$$

from which we see immediately that the singularities of $T(z)$ for $|z| \leq 1$ are those z such that either $\gamma(z) = 0$ or $\beta(z) = 0$.

To find the singularities outside of S^1 we consider the inverse of S: Put

$$S^{-1} = \begin{pmatrix} \alpha_1 & \beta_1 \\ \gamma_1 & \delta_1 \end{pmatrix} = \frac{1}{\alpha\delta - \beta\gamma} \begin{pmatrix} \delta & -\beta \\ -\gamma & \alpha \end{pmatrix}$$

and recall that because S is unitary for $|z| = 1$ we have

$$S^{-1} = S^* = \begin{pmatrix} \bar{\alpha} & \bar{\gamma} \\ \bar{\beta} & \bar{\delta} \end{pmatrix}$$

on S^1. Hence by Schwarz reflection

$$S^{-1} = \begin{pmatrix} \overline{\alpha(z^*)} & \overline{\gamma(z^*)} \\ \overline{\beta(z^*)} & \overline{\delta(z^*)} \end{pmatrix} \qquad z^* = \frac{1}{\bar{z}}$$

for all z. This shows S^{-1} is analytic for $|z| \geq 1$. In terms of the entries of S^{-1} the matrix T can be written

$$T(z) = \begin{pmatrix} \frac{\beta_1\gamma_1 - \alpha_1\delta_1}{\gamma_1} & \frac{\alpha_1}{\gamma_1} \\ -\frac{\delta_1}{\gamma_1} & \frac{1}{\gamma_1} \end{pmatrix} \qquad \det T = \frac{\beta_1}{\gamma_1}$$

This shows the singularities of T for $|z| \geq 1$ are those z such that $\gamma_1(z) = 0$ or $\beta_1(z) = 0$. Since

$$\gamma_1(z) = \overline{\beta(z^*)}$$

the zeroes of γ_1 are the reflections thru S^1 of the zeroes of β. So it's clear that one has:

Prop: Let $T(z)$ be the transfer matrix belonging to the scattering matrix $S(z) = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$. Then the poles of $T(z)$ [redacted] are the roots of γ in $\bar{D}$ [redacted] and the reflections of the roots of β [redacted] in $\bar{D}$. The poles of $T(z)^{-1}$ are the roots of β in $\bar{D}$ and the reflections of the roots of γ in $\bar{D}$.

Paradox: If f is a finite Blaschke product, i.e. a rational function with $|f| = 1$ on ∂D, then multiplying [redacted] T by f [redacted] doesnot affect the fractional linear transformation associated to T, hence fT should also be a transfer matrix. But this changes S as follows

$$S = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \longmapsto \begin{pmatrix} \alpha & f\beta \\ f^{-1}\gamma & \delta \end{pmatrix}$$

and this can introduce singularities to S inside [redacted] ∂D. So something is wrong.

I still do not understand what a transfer matrix is. Somehow those T coming from a scattering matrix S are not all the possible T one should consider.

Consider again potential scattering

$$Ae^{ikx} + Be^{-ikx} \longleftrightarrow e^{ikx}$$

$$\bar{B}e^{+ikx} + \bar{A}e^{-ikx} \longleftrightarrow e^{-ikx}$$

$$a_1 e^{ikx} + b_1 e^{-ikx} \longleftrightarrow b_2 e^{ikx} + a_2 e^{-ikx}$$

$$T = \begin{pmatrix} A & \bar{B} \\ B & \bar{A} \end{pmatrix}$$

~~This T is an entire function of k. If the potential is a finite sum of δ functions supported at integral points, the T is Laurent polynomial matrix in $e^{ik} = z$, hence $T(z)$ has singularities only at $z = 0, \infty$.~~ [redacted] ~~Such a matrix has the shrinking property.~~

Unfortunately the kind of T has possibly singularities at $k = 0$.

Suppose $T(z)$ is matrix of Laurent polynomials whose inverse is also (hence $\det T(z) = c z^n$ some $c \neq 0$ and n). Suppose that for $0 < |z| \leq 1$, $T(z)$ expands the disk and for $1 \leq |z| < \infty$ it contracts the disk. Then I can put [redacted] $z = e^{i\lambda}$ and obtain [redacted] a Nevanlinna matrix (or its inverse).

May 10, 1977

Let P denote a hermitian form of signature $(+,-)$ on a 2-dimensional complex vector space V. The isotropic lines for P form a circle in the Riemann sphere $\mathbb{P}V$ dividing it into open disks on which P is >0 and <0.

Let Q be another such form. If $P<Q$, then clearly $P(v)>0 \implies Q(v)>0$ so that the P-positive disk is contained in the Q-positive disk. In fact the P-circle is contained in the interior Q-positive disk.

Conversely suppose the P-positive closed disk is contained in the open Q-positive disk. I have seen that by a limiting process of successive reflections two non-intersecting circles in the Riemann sphere determine a pair of points which is placed at $z=0$ and $z=\infty$ then the circles become concentric circles around zero. So in this way we can identify $V=\mathbb{C}^2$ so that P is given by the matrix $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and Q is given by a real diagonal matrix $\begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}$. The P-circle is $|z|=1$. We have

$$|z|^2 - 1 \geq 0 \implies \lambda|z|^2 + \mu > 0$$

hence $\lambda > 0$, $\mu < 0$ and $\lambda > -\mu$. $Q-P$ is the form:

$$\begin{pmatrix} z \\ 1 \end{pmatrix}^* (Q-P) \begin{pmatrix} z \\ 1 \end{pmatrix} = (\lambda - 1)|z|^2 + (1+\mu)$$

so $Q>P$ when $\lambda > 1$ and $-\mu < 1$. So I we assume in addition that $\det(Q) = \lambda\mu = -1$ or

$$\lambda \cdot (-\mu) = 1$$

then we should have $Q>P$. To be sure we are given

$\lambda > 0$, $\lambda > (-\mu)$ and $\lambda \cdot (-\mu) = 1$. It follows that $\lambda > 1$ and $(-\mu) = \lambda^{-1} < 1$ so indeed $Q > P$.

Next consider the case where the closed P-positive disk is contained in the closed Q-positive disk but the P-circle and Q-circle are tangent at one point.

May 11, 1978.

Let V be a 2-dimensional complex vector space. If P is a hermitian form on V with signature $(+,-)$, then the isotropic lines for P form a circle in $\mathbb{P}V$ which determines P up to a real non-zero multiplicative constant. To see this one can suppose P is the form on $\mathbb{C}^2$ given by $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ whence the circle in $\mathbb{P}\mathbb{C}^2$ = Riemann sphere is described by

$$\begin{pmatrix} z \\ 1 \end{pmatrix}^* \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} z \\ 1 \end{pmatrix} = |z|^2 - 1 = 0.$$

If the hermitian form Q given by $\begin{pmatrix} a & b \\ \bar{b} & d \end{pmatrix}$ has the same circle, then for all z with $|z| = 1$ one has

$$\begin{pmatrix} z \\ 1 \end{pmatrix}^* \begin{pmatrix} a & b \\ \bar{b} & d \end{pmatrix} \begin{pmatrix} z \\ 1 \end{pmatrix} = a + \bar{z}b + \bar{b}z + d = 0$$

hence $b = 0$, $a + d = 0$ so that Q is a multiple of P, say $Q = aP$, $a \in \mathbb{R}^\circ$. Notice also that $a > 0$ iff the P-positive disk, i.e. the set of z with

$$\begin{pmatrix} z \\ 1 \end{pmatrix} P \begin{pmatrix} z \\ 1 \end{pmatrix}^* > 0$$

is the same as the Q-positive disk.

Now suppose P, Q are two hermitians forms of signature $(+,-)$

on V. The associated circles can be disjoint, tangent, or intersecting in 2 points.

Case 1: The P, Q circles are disjoint. We know that two disjoint circles in the Riemann sphere can be made concentric circles about the origin by a fractional linear transformation. (Successively reflect to find the points that should be sent to $0, \infty$). Suppose then the P, Q circles are resp. $|z| = 1$, $|z| = \rho$ and that the positive P-disk is $|z| > 1$. By a scalar change on V we can suppose P is given by $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. It follows that Q is given by

$$\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}$$

where $\frac{a}{d} = -\rho$. This is a basic canonical form for the pair (P, Q) under the action of $\mathrm{Aut}(V)$.

If we know that the discriminant of Q relative to P, i.e. $\det(P^{-1}Q)$, is 1, then $ad = -1$. If also

$$\begin{pmatrix} z \\ 1 \end{pmatrix}^* P \begin{pmatrix} z \\ 1 \end{pmatrix} \geq 0 \Rightarrow \begin{pmatrix} z \\ 1 \end{pmatrix}^* Q \begin{pmatrix} z \\ 1 \end{pmatrix} = a|z|^2 + d \geq 0$$

then we have $a > 0$ and $a + d = a - \frac{1}{a} > 0$, so $a > 1$ and hence

$$P = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \leq \begin{pmatrix} a & 0 \\ 0 & -\frac{1}{a} \end{pmatrix} = Q.$$

Case 2: The P, Q circles are tangent. We use a fractional linear transformation to put the point of tangent at ∞ and to make the P-circle the real axis, and the P-positive disk the LHP. Since

$$\frac{1}{i} \begin{pmatrix} z \\ 1 \end{pmatrix}^* \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} z \\ 1 \end{pmatrix} = \frac{\bar{z} - z}{i} = -2\,\mathrm{Im}\, z$$

P must be a positive multiple of $\frac{1}{i}\begin{pmatrix}0&1\\-1&0\end{pmatrix}$ which I can assume to be one by a scalar change of variable.

The Q-circle is a line $\operatorname{Im}(z) = a$, so Q must be a non-zero real multiple of

$$\frac{1}{i}\begin{pmatrix}1&0\\+ia&1\end{pmatrix}\begin{pmatrix}0&1\\-1&0\end{pmatrix}\begin{pmatrix}1&-ia\\0&1\end{pmatrix} = \frac{1}{i}\begin{pmatrix}1&0\\+ia&1\end{pmatrix}\begin{pmatrix}0&1\\-1&+ia\end{pmatrix} = \frac{1}{i}\begin{pmatrix}0&1\\-1&+2ia\end{pmatrix}$$

$$= \frac{1}{i}\begin{pmatrix}0&1\\-1&0\end{pmatrix} + \begin{pmatrix}0&0\\0&2a\end{pmatrix}$$

If $\det(P^{-1}Q) = 1$ we then have the canonical form

$$P = \frac{1}{i}\begin{pmatrix}0&1\\-1&0\end{pmatrix} \qquad \pm Q = \frac{1}{i}\begin{pmatrix}0&1\\-1&0\end{pmatrix} + \begin{pmatrix}0&0\\0&2a\end{pmatrix}.$$

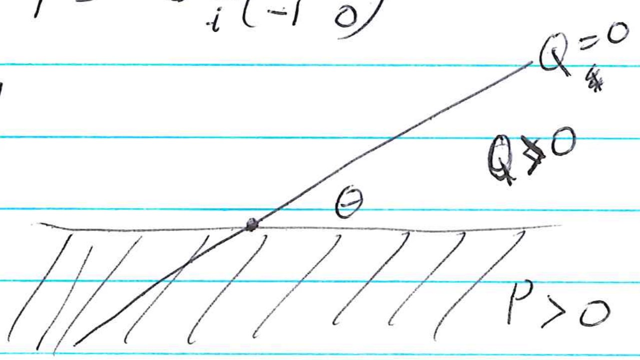
If we know the P-positive disk is contained in the Q-positive disk, then since

$$\begin{pmatrix}z\\1\end{pmatrix}^* P \begin{pmatrix}z\\1\end{pmatrix} = -2\operatorname{Im} z \qquad \begin{pmatrix}z\\1\end{pmatrix}^* Q \begin{pmatrix}z\\1\end{pmatrix} = \pm(-2\operatorname{Im} z + 2a)$$

we see the $+$ sign holds and also $a > 0$, hence $P \leq Q$.

<u>Case 3</u>: The P, Q circles intersect in 2 points. We put these points at $0, \infty$ and put the P-positive disk on the LHP and arrange that

$$P = \frac{1}{i}\begin{pmatrix}0&1\\-1&0\end{pmatrix}$$



If θ is the angle between the $P>0$, $Q>0$ half-planes, then we have Q must be a multiple of

$$\begin{pmatrix} e^{i\theta} & 0 \\ 0 & 1 \end{pmatrix} \frac{1}{i} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} e^{-i\theta} & 0 \\ 0 & 1 \end{pmatrix} = \frac{1}{i} \begin{pmatrix} 0 & e^{i\theta} \\ e^{-i\theta} & 0 \end{pmatrix}$$

Above calculation shows:

__Prop:__ Let P, Q be hermitian forms of signature $(+,-)$ on a 2-dimensional complex vector space V.

1) If $P \leq Q$, then the P-positive disk in $\mathbb{P}V$ is contained in the Q-positive disk.

2) If $\det(P^{-1}Q) = 1$, then the converse to 1) holds.

__Cor:__ Let $T \in GL_2(\mathbb{C})$ and $P = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

1) If $P \leq T^* P T$, then T as a fractional linear transformation of the Riemann sphere expands the unit disk: $|z| > 1 \Rightarrow |T(z)| > 1$.

2) If $|\det T| = 1$, then the converse to 1) holds.

__What is a 2-port?__ I give three equivalent descriptions.

1) Grassmannian description: Let $\mathbb{C}^4$ be equipped with the hermitian form

$$P : \begin{pmatrix} a_1 \\ a_2 \\ b_1 \\ b_2 \end{pmatrix} \longmapsto |a_1|^2 + |a_2|^2 - |b_1|^2 - |b_2|^2$$

Then a 2-port is a rational map

$$\mathbb{CP}^1 \longrightarrow \mathrm{Grass}_2(\mathbb{C}^4)$$
$$z \longmapsto W_z$$

such that

$$|z| \leq 1 \implies P \geq 0 \quad \text{on} \quad W_z$$
$$|z| \geq 1 \implies P \leq 0 \quad \text{———} \, .$$

2) Scattering matrix description: Recall that a Zariski open subset of $\mathrm{Grass}_2(\mathbb{C}^4)$ is given by subspaces

$$\Gamma_S = \left\{ \begin{pmatrix} a_1 \\ a_2 \\ b_1 \\ b_2 \end{pmatrix} \middle| \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = S \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \right\}$$

where S runs over the affine space of 2×2 matrices. This open set consists of all subspaces W intersecting

$$\left\{ \begin{pmatrix} 0 \\ 0 \\ * \\ * \end{pmatrix} \right\}$$

trivially. Those subspaces W which are isotropic for P are of the form Γ_S where $S \in U(2)$. Also any subspace W on which $P \geq 0$ is in the Zariski open set. Hence given a rational map $z \mapsto W_z$ as in 1) we get a rational map

$$z \mapsto S(z) \qquad \mathbb{CP}^1 \longrightarrow M_{2\times 2}(\mathbb{C})$$

such that $\Gamma_{S(z)} = W_z$. Moreover $S(z)$ has poles outside the unit circle and it has unitary values on S^1.

Conversely if one is given $z \mapsto S(z)$ analytic matrix-valued function on $|z| \leq 1$ with unitary values on $|z| = 1$, then one can extend $S(z)^{-1}$ analytically to $|z| \geq 1$ via

$$S(z)^{-1} = S(z^*)^* \qquad z^* = (\bar{z})^{-1}$$

Then one gets a holomorphic map $W: \mathbb{CP}^1 \to \mathrm{Grass}_2(\mathbb{C}^4)$ which one knows is algebraic (GAGA). The next point is that maximum-modulus implies that $\|S(z)\| \leq 1$ for $|z| \leq 1$. In effect given vectors a, b in $\mathbb{C}^2$ one has

$$\left|\big(S(z)a, b\big)\right| \leq \|S(z)a\| \cdot \|b\| = \|a\| \cdot \|b\|$$

for $|z| = 1$ because $S(z) \in U(2)$. Thus for $|z| \leq 1$ this $\leq$ holds by maximum modulus for all a, b so $\|S(z)\| \leq 1$. Similarly $\|S^{-1}(z)\| \leq 1$ for $|z| \geq 1$ and so it is clear that the holomorphism map W is a 2-port in the sense of defn. 1).

3) Transfer matrix description. Here we describe a Zariski-open subset of the grassmannian consisting of subspaces

$$\Gamma'_T = \left\{ \begin{pmatrix} a_1 \\ a_2 \\ b_1 \\ b_2 \end{pmatrix} \middle| \begin{pmatrix} a_1 \\ b_1 \end{pmatrix} = T \begin{pmatrix} b_2 \\ a_2 \end{pmatrix} \right\}$$

where T is a 2×2 matrix. We get all subspaces intersecting the subspace $\left\{ \begin{pmatrix} * \\ 0 \\ * \\ 0 \end{pmatrix} \right\}$ trivially.

If we have a rational map $z \mapsto W_z$ as in 1) which intersects this Zariski-open subset, then we get a rational map $z \mapsto T(z)$ of $\mathbb{CP}^1$ to matrices such that $W_z = \Gamma'_{T(z)}$.

The condition $P \geq 0$ on W_z for $|z| \leq 1$ says that provided $T(z)$ is defined one has

$$|a_1|^2 - |b_1|^2 \geq |b_2|^2 - |a_2|^2 \qquad \text{if } \begin{pmatrix} a_1 \\ b_1 \end{pmatrix} = T\begin{pmatrix} b_2 \\ a_2 \end{pmatrix}$$

or equivalently that

$$T^*\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} T \geq \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad (\leq \text{ for } |z| \geq 1)$$

Conversely if we are given a rational matrix function $T(z)$ such that

$$T^*\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} T \geq \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad \text{for } |z| \leq 1 \text{ such that } T(z) \text{ is def'd.}$$

$$\leq \qquad\qquad \geq \quad '' \quad .$$

then it is clear we get a 2-port in the sense of 1).

Note that not all 2 ports can be described by transfer matrices. The ones that can't be are those such that W_z contains a non-zero vector $\begin{pmatrix} a_1 \\ 0 \\ b_1 \\ 0 \end{pmatrix}$ for each z. Taking $|z| = 1$ this implies for

$$S(z) = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$$

we must have $\gamma = \beta = 0$, as $|a_1| = |b_1| \neq 0$. But then $\beta = \gamma = 0$ identically. So we see that the 2-ports without transfer matrices are those whose scattering matrices are diagonal.

So now I propose to study the set of 2-ports admitting transfer matrices, ie. the set of rational matrices $T(z)$ such that for z for which $T(z)$

are defined one has

$$T^*\begin{pmatrix}1&0\\0&-1\end{pmatrix}T \geq \begin{pmatrix}1&0\\0&-1\end{pmatrix} \qquad \text{if } |z|\leq 1$$

$$\leq \qquad \text{if } |z|\geq 1$$

It is clear that [redacted] it suffices to assume [redacted] these condition for a dense set of z. The set of these transfer matrices forms a monoid under multiplication. In effect [redacted] if $T_1^*PT_1 \geq P$ and $T_2^*PT_2 \geq P$, then one has

$$T_2^*T_1^*PT_1T_2 \geq T_2^*PT_2 \geq P$$

etc.

For $|z|=1$ when $T(z)$ is defined [redacted] we have

$$T^*PT = P$$

[redacted] so that $|\det T| = 1$ and so T is invertible. So we have

$$P^{-1}T^*P = T^{-1} \qquad T^{-1} = \begin{pmatrix}A_1 & B_1\\ C_1 & D_1\end{pmatrix}$$

If $T = \begin{pmatrix}A & B\\ C & D\end{pmatrix}$ this says that for $|z|=1$

$$\begin{pmatrix}\bar{A} & -\bar{C}\\ -\bar{B} & \bar{D}\end{pmatrix} = \begin{pmatrix}1&0\\0&-1\end{pmatrix}\begin{pmatrix}\bar{A} & \bar{C}\\ \bar{B} & \bar{D}\end{pmatrix}\begin{pmatrix}1&0\\0&-1\end{pmatrix} = \frac{1}{AD-BC}\begin{pmatrix}+D & -B\\ -C & A\end{pmatrix} = \begin{pmatrix}A_1 & B_1\\ C_1 & D_1\end{pmatrix}$$

Hence [redacted]

$$A_1 = \overline{A(z^*)}$$

$$B_1 = -\overline{C(z^*)}$$

$$C_1 = -\overline{B(z^*)}$$

$$D_1 = \overline{D(z^*)}$$

which shows that the poles of T^{-1} are the reflections of the poles of T. So one sees easily that the singularities of T are symmetric about $|z| = 1$.

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Suppose we equip $\mathbb{C}[z,z^{-1}]^2$ with the hermitian form

$$\tilde{P} : f \longmapsto (Pf,f) = \int_{S^1} f^* P f \, \frac{d\theta}{2\pi} \qquad P = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Let $T(z) \in GL_2(\mathbb{C}[z,z^{-1}])$ such that $T^*PT = P$ on S^1. Then clearly T gives an automorphism of $\mathbb{C}[z,z^{-1}]^2$ preserving the hermitian form.

$L_0 = \mathbb{C}[z]^2$ is a $\mathbb{C}[z]$ lattice in $\mathbb{C}[z,z^{-1}]^2$ with the property that zL_0 has the orthogonal complement $\mathbb{C}^2$ in L_0 with respect to $\tilde{P}$. The image $L = TL_0$ of L_0 has the same property.

Conversely, suppose given a lattice L such that one has an orthogonal decomposition

$$L = N \oplus zL$$

with respect to $\tilde{P}$ such that $\tilde{P}$ is non-degenerate on N. For each $f \in N$ we can write

$$f = \sum \varphi_n(f) z^n \qquad \varphi_n : N \longrightarrow \mathbb{C}^2$$

and we have

$$\left(\tilde{P}(z^k f), f)\right) = \left(P\, z^k \sum \varphi_n(f) z^n, \sum \varphi_n(f) z^n\right)$$

$$= \sum{}' \left(P \varphi_{n-k}(f), \varphi_n f\right)$$

$$= \sum_n \left(\varphi_n^* P \varphi_{n-k} f, f\right)$$

where the last inner product is obtained by taking as orthonormal basis a basis for N such that the form $\tilde{P}$ restricted to N has the matrix P. The above vanishes for $k \neq 0$, hence we get

$$\sum_n \varphi_n^* P \varphi_{n-k} = \begin{cases} P & k=0 \\ 0 & k \neq 0 \end{cases}$$

It follows that the Laurent poly. matrix

$$T(z) = \sum{}' \varphi_n z^n$$

satisfies

$$T^* P T = P$$

for $|z| = 1$.

The question now is what lattices correspond to T's with the 2-port property: $T^*PT \lesseqgtr P$ for $|z| \lesseqgtr 1$.