May 5, 1978:

\[
\begin{pmatrix}
  a_1 \\
  b_1 \\
\end{pmatrix} \rightarrow \begin{pmatrix}
  a_2 \\
  b_2 \\
\end{pmatrix} = \begin{pmatrix}
  \alpha & \beta \\
  \gamma & \delta \\
\end{pmatrix} \begin{pmatrix}
  a_1 \\
  b_1 \\
\end{pmatrix}
\]

\[
S(z) \text{ is analytic in } |z| < 1 \text{ and unitary on the boundary } |z| = 1, \text{ and of norm } < 1 \text{ inside } S'.
\]

Example from Schrödinger equation: 

\[-u'' + gu = k^2 u\]

with \(g\) of compact support. Then any solution has the asymptotic description:

\[a_1 e^{ikx} + b_1 e^{-ikx} \leftrightarrow a_2 e^{-ikx} + b_2 e^{ikx}\]

If \(A, B\) are defined by

\[Ae^{ikx} + Be^{-ikx} \leftrightarrow e^{ikx}\]

\[\bar{B}e^{ikx} + \bar{A} e^{-ikx} \leftrightarrow e^{-ikx}\]

(Here \(k \in \mathbb{R}\), otherwise \(\bar{B}(k)\) should be replaced by \(\bar{B}(-k)\)).

Then we have

\[
\begin{align*}
   a_1 &= Ab_2 + \bar{B}a_2 \\
   b_1 &= Bb_2 + \bar{A}a_2
\end{align*}
\]

\[
\begin{pmatrix}
  a_1 \\
  b_1 \\
\end{pmatrix} = \begin{pmatrix}
  A & B \\
  \bar{B} & \bar{A}
\end{pmatrix} \begin{pmatrix}
  a_2 \\
  b_2 \\
\end{pmatrix}
\]

\[
T(k) = \begin{pmatrix}
  A(k) & B(-k) \\
  B(k) & A(-k)
\end{pmatrix}
\]

\[
\begin{align*}
  A(-k) &= A(k) \\
  B(-k) &= B(k)
\end{align*}
\]

and this is analytic for all \(k\) except possibly \(k = 0, \infty\).

In this example the scattering matrix \(S\) is found to be
Now notice that this is analytic provided $A(k) \neq 0$. For $\text{Im}(k) > 0$ this means that there are no bound states for the Schrödinger equation. So we have to assume no bound states if we want $S(k)$ to be analytic in the UHP.

$$\det S(k) = \frac{BB}{A^2} - \frac{1}{A^2} = \frac{A^2}{A^2} - \frac{A}{A} = \frac{A(-k)}{A(k)}$$

In the UHP $S(k)$ fails to be invertible when $A(-k) = 0$ which means that we have

$$B(k) e^{ikx} \leftrightarrow e^{-ikx}$$

grows as $x \to \infty$ grows as $x \to -\infty$.

The time dependence of this solution is $e^{ikx}$ so this represents a radiating state; you see waves travelling outward.

**Example:** Recall that a $\delta$ function potential leads to a transfer matrix of the form

$$\begin{pmatrix} e^{-ikd} & 0 \\ 0 & e^{ikd} \end{pmatrix} \begin{pmatrix} a & b \\ \bar{b} & \bar{a} \end{pmatrix} \begin{pmatrix} e^{ikd} & 0 \\ 0 & e^{-ikd} \end{pmatrix} = \begin{pmatrix} a & be^{-2ikd} \\ be^{2ikd} & \bar{a} \end{pmatrix}$$

$\in \text{SU}(1,1)$

The corresponding scattering matrix is

$$S^{\circ} = \begin{pmatrix} \frac{be^{-2ikd}}{a} & \frac{1}{a} \\ \frac{1}{a} & -\frac{be^{2ikd}}{a} \end{pmatrix}$$
This blows up as \( k \to + \infty \) because we have not arranged for the incoming and outgoing spaces to be orthogonal.

Be more careful. Suppose we have a 8-function potential supported at \( x = 0 \):

\[-u'' + c \delta(x) u = k^2 u\]

Integrate over \([-\varepsilon, \varepsilon]\) and let \( \varepsilon \to 0 \)

\[- \left[u'\right]_0^\varepsilon + c u(0) = 0\]

so if \( A e^{ikx} + B e^{-ikx} \leftrightarrow e^{ikx} \) then \( u \) should be continuous at 0:

\[A + B = 1\]

and its derivative jumps by \( cu(0)\):

\[-[A ik + B(-ik)] + ik = c\]

or

\[A - B = 1 - \frac{c}{ik}\]

so

\[A = 1 - \frac{c}{2ik}, \quad B = \frac{c}{2ik}\]

and so

\[T = \begin{pmatrix}
1 - \frac{c}{2ik} & -\frac{c}{2ik} \\
\frac{c}{2ik} & 1 + \frac{c}{2ik}
\end{pmatrix}\]

\[S = \begin{pmatrix}
\frac{c}{2ik} & 1 - \frac{c}{2ik} \\
1 - \frac{c}{2ik} & 1 + \frac{c}{2ik}
\end{pmatrix}\]

\[T \times S = \begin{pmatrix}
\frac{c}{2ik} & 2ik \\
2ik & \frac{c}{2ik - c}
\end{pmatrix}\]

\[\begin{pmatrix}
\frac{c}{2ik} & 2ik \\
2ik & \frac{c}{2ik - c}
\end{pmatrix}\]
The scattering matrix has a pole where
\[ k = \frac{c}{2i} \]
which for \( c > 0 \) is in the lower half-plane, hence there are no bound states when \( c > 0 \), which is of course intuitively clear as positive potentials have no bound states.

Another method which yields the same formula for \( T \) is to take a square well-potential and let it approach \( c \delta(x) \). Incidentally, this gives

\[
T = \begin{pmatrix} i & 1 \\ ik & -ik \end{pmatrix}^{-1} \begin{pmatrix} 1 & 0 \\ -c & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ ik & -ik \end{pmatrix} = \begin{pmatrix} i & 1 \\ i & -i \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -c/k & 1 \end{pmatrix} \begin{pmatrix} i & 1 \\ -i & 1 \end{pmatrix}
\]

which probably expresses the fact \( u' \) jumps by \( cu(0) \) translated to the \( e^{ikx}, e^{-ikx} \) description.

Here's a problem with the above examples which makes them confusing: you are looking at classical scattering matrices instead of the ones which result from 2-ports in the abstract sense.

Another example. Suppose we take a segment and use the natural exponentials normalized at the ends:

\[
e^{-ikm} e^{ikx} \leftrightarrow e^{ik(x - m)} \quad m > 0
\]

\[
A = e^{-ikm} \quad B = 0 \quad \text{so}
\]

\[
T(k) = \begin{pmatrix} e^{-ikm} & 0 \\ 0 & e^{+ikm} \end{pmatrix}
\]

Note that \( \text{Im}(k) > 0 \Rightarrow T(k) \) expands
In the example note that 

\[
\begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix} \left( \frac{z+i}{z-i} \right) = \frac{z+1}{z-1} = \lambda
\]

is the inverse of the Cayley transform

\[
\mathbb{Z} = -\frac{1}{2i} \begin{pmatrix} -i & 1 \\ -1 & i \end{pmatrix} \left( \begin{pmatrix} z \\ 1 \end{pmatrix} \right) = \frac{-i \lambda - 1}{-i \lambda + 1} = \frac{\lambda-i}{\lambda+i}
\]

so it maps the disk to the UHP. Also

\[
\begin{pmatrix} 1 & 0 \\ -\frac{c}{k} & 1 \end{pmatrix} \left( \begin{pmatrix} z \\ 1 \end{pmatrix} \right) = \frac{1}{\begin{pmatrix} c \\ k \end{pmatrix} + i} = \text{in UHP if } k \text{ is...}
\]

shrinks the LHP, so \( T(k) \) expands the disk for \( \text{Im}(k) > 0 \).

Suppose

\[
T = \begin{pmatrix} e^{-ikm} & 0 \\ 0 & e^{+ikm} \end{pmatrix}
\]

then

\[
S = \begin{pmatrix} 0 & e^{+ikm} \\ e^{-ikm} & 0 \end{pmatrix}
\]

which goes to \( 0 \) as \( k \to i \infty \).

So it seems that for \( S(z) \) to be analytic for \( |z| < 1 \) corresponds to \( T(z) \) expanding the disk.
Yesterday I saw an example of a T-matrix was
\[
\begin{pmatrix}
1 & 1 \\
i & -i
\end{pmatrix} \begin{pmatrix}
1 & 0 \\
0 & 1
\end{pmatrix} \begin{pmatrix}
1 & 1 \\
i & -i
\end{pmatrix}
\]
\[c > 0\]
and that it has the property of expanding the disk for \(\text{Im}(\lambda) > 0\). Recall the 2-port
\[
I_1 \rightarrow V_1 \rightarrow I_2 \\
V_1 \uparrow \quad \Rightarrow \quad I_2 \uparrow \quad \Rightarrow \quad V_2
\]
where
\[
C_{di} = i \\
\frac{dV}{dt} = i
\]
\[
CV_{\omega} = I \\
\frac{V}{I} = \frac{1}{c\omega}
\]

\[
\begin{pmatrix}
\tilde{V}_1 \\
\tilde{I}_1
\end{pmatrix} = \begin{pmatrix}
1 & 0 \\
0 & 1
\end{pmatrix} \begin{pmatrix}
\tilde{V}_2 \\
\tilde{I}_2
\end{pmatrix}
\]

shrinks UHP for \(\text{Im} \omega < 0\)

\[\therefore\] expands UHP for \(\text{Im}(\omega) > 0\).

Hence 2-ports will give T-matrices expanding UHP for \(\text{Im}(\omega) > 0\).

Corresponding disk-form is
\[
\frac{1}{2} \begin{pmatrix}
1 & -i \\
i & -i
\end{pmatrix} \begin{pmatrix}
1 & 0 \\
0 & 1
\end{pmatrix} \begin{pmatrix}
1 & -i \\
i & -i
\end{pmatrix} = \frac{1}{2} \begin{pmatrix}
1+iC\omega & -i \\
i & -i
\end{pmatrix} \begin{pmatrix}
1 & -i \\
i & -i
\end{pmatrix}
\]
\[
= \frac{1}{2} \begin{pmatrix}
2-iC\omega & -iC\omega \\
iC\omega & 2+iC\omega
\end{pmatrix}
\]

\[T(\omega) = \begin{pmatrix}
1-iC\omega & -iC\omega \\
iC\omega & 2+iC\omega
\end{pmatrix}
\]

Put \(C = 2\) and shift from \(\omega\) to \(z = \frac{\omega-i}{\omega+i}\)
\[ \omega = \frac{1}{i} \frac{z+1}{z-1} \quad 1 - i \omega = 1 - \frac{z+1}{z-1} = \frac{-2}{z-1} \]

\[ 1 + i \omega = 1 + \frac{z+1}{z-1} = \frac{2z}{z-1} \]

\[ T(z) = \frac{1}{z-1} \begin{pmatrix} -2 & z-1 \\ z+1 & 2z \end{pmatrix} \quad \text{analytic \& invertible for all } z \neq 1 \text{ including } z = \infty \]

\[ S(z) = \begin{pmatrix} \frac{z+1}{2} & \frac{z-1}{2} \\ -2 & -2 \\ \frac{z-1}{2} & \frac{z+1}{2} \end{pmatrix} \]

\[ -S(z) = \begin{pmatrix} \frac{z+1}{2} & -\frac{z-1}{2} \\ -2 & -2 \\ \frac{z-1}{2} & -\frac{z+1}{2} \end{pmatrix} \quad \text{analytic for all } z \neq \infty \]

\[ \det(-S) = z \]

not invertible for \( z = 0 \).

---

\[ S(z) \text{ analytic for } |z| \leq 1 \text{ and } 0 \leq \|u\| \leq 1 \text{ on } S^1 \]

\[ \| (S(z)u, v) \| \leq \| S(z)u \| \| v \| \leq \| u \| \| v \| \]

for \( |z| = 1 \) and hence for all \( |z| < 1 \) by maximum modulus. Thus \( |S(z)| \leq 1 \) in the disk. If \( S(z) \) is an operator in a finite-dimensional space and \( \|S(z)\| = 1 \) at an interior point then we can find unit vectors with \( (S(z)u, v) = 1 \) at \( z_0 \), hence \( (S(z)u, v) = 1 \) for all \( z \), hence \( S(z)u = v \). On the unit circle \( S(z) \) carries \( <u>^1 \) into \( <v>^1 \) and this follows for all \( z \) by Cauchy. Thus if one can't split off a line on which \( S(z) \) is constant we must have \( \|S(z)\| < 1 \) for \( |z| < 1 \).

Notice also that for \( \begin{pmatrix} a_1 \\ b_1 \end{pmatrix} = T \begin{pmatrix} a_2 \\ b_2 \end{pmatrix} \)

\[ \begin{align*}
\begin{pmatrix} a_1 \\ b_1 \end{pmatrix} &\to \begin{pmatrix} a_2 \\ b_2 \end{pmatrix} \\
\begin{pmatrix} a_1 \\ b_1 \end{pmatrix} &\to \begin{pmatrix} a_2 \\ b_2 \end{pmatrix}
\end{align*} \]
we have
\[ (a_1^2 - 1 b_1^2) \geq (b_2^2 - 1 a_2^2) \quad \text{for } |z| < 1 \]

because \( \|S(z)\| \leq 1 \) and \( (b_2) = S(a_2) \). Then for \( |z| < 1 \)
\( T \) expands the unit disk.

Note that multiplying by a scalar of modulus 1
on \( S' \) changes \( S \) as follows
\[
S = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad \rightarrow \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} \begin{pmatrix} j/2 & \beta \\ \gamma & \delta \end{pmatrix}
\]

Suppose \( S(z) \) is rational in \( z \) and analytic in \( |z| \leq 1 \).
Then \( \beta \) and \( \gamma \) are rational functions analytic in \( |z| \leq 1 \)
with \( |\beta|^2 = 1 - |\alpha|^2 = 1 \gamma^2 \) on \( S' \).

So it follows that \( \rho = \beta \gamma \) is a rational function of \( z \)
of modulus 1 on \( S' \), and hence is a product of Blaschke factors
\[
\frac{z - h}{1 - \overline{h} z}
\]
or their inverses. Suppose \( p(h) \neq 0 \). Then I take \( j^{-1} \)
to be this Blaschke factor \( j^{-1} \) is analytic inside \( |z| \leq 1 \) so
\( j^{-1} \) remains analytic, and also
\[
\beta(h) = (p \rho)(h) = 0.
\]

is analytic because \( p \) is analytic in \( |z| < 1 \).

For the new scattering matrix
we have
\[
\frac{j \beta}{j^{-1} \gamma} = \gamma \cdot \beta \\
\frac{j \gamma}{j^{-1} \gamma} = \gamma \\
\frac{j \beta}{j^{-1} \gamma} = \frac{\gamma \cdot \beta}{\gamma}
\]

So in this way by scalar multiplication we can change
the order of zeroes of \( p = \frac{f}{g} = \det T \) to either 0 or 1. If \( p(h) = \infty \), take \( f' \) to be \( \frac{z-h}{1-hz} \).
Then \( f' \) is analytic in the closed disk, so is \( f \beta \). And also \( f^{-1} \beta \) is analytic because \( \nu(h) = 0 \).
Similarly, we can make the order of the poles of \( p \) either 0 or 1.

May 7, 1978:
Suppose that \( f \) is an analytic function for \( |z| < 1 + \varepsilon \) and \( g \) is another such function such that \( |f| = |g| \) for \( |z| = 1 \) and such that \( f, g \) have the same zeroes counted with multiplicity for \( |z| < 1 \). From \( |f| = |g| \) on \( S^1 \) we see \( f, g \) have same zeroes on \( S^1 \), hence \( \frac{f}{g} \) is analytic for \( |z| \leq 1 \) and of modulus 1 on \( S^1 \) so

\[
|\frac{f}{g}| < 1
\]

in the closed disk. Interchanging \( f, g \) we see that \( \frac{f}{g} \) is a constant of modulus 1.

Next note that \( f \) has finitely many zeroes \( \in D = \{ |z| < 1 \} \), so there is a finite Blaschke product \( p \) with \( h = \frac{f}{p} \) analytic on \( \overline{D} \) and without zeroes in \( D \). So

\[
f = ph
\]
is the canonical factoring of \( f \) into an inner factor \( p \) and the outer factor \( h \). When \( f \) is rational, so are \( p \) and \( h \).
We describe a 2-port by the rational matrix $T(z)$ such that for $|z| < 1$ it expands the disk at for $|z| > 1$ it contracts the disk, and we consider only $z$ for which $T(z)$ is defined and invertible. We know that $T$ corresponds to a scattering matrix $S = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ which is analytic for $|z| < 1$ by the formula

$$T(z) = \begin{pmatrix} \frac{1}{\beta} & -\frac{\delta}{\gamma} \\ \frac{\alpha}{\gamma} & \frac{\beta \alpha - \alpha \delta}{\gamma} \end{pmatrix}$$

$$\det T = \frac{\beta}{\gamma}$$

from which we see immediately that the singularities of $T(z)$ for $|z| < 1$ are those $z$ such that either $\delta(z) = 0$ or $\beta(z) = 0$.

To find the singularities outside of $S^1$ we consider the inverse of $S$: Put

$$S^{-1} = \begin{pmatrix} \alpha_1 & \beta_1 \\ \gamma_1 & \delta_1 \end{pmatrix} = \frac{1}{\alpha \delta - \beta \gamma} \begin{pmatrix} \delta - \beta \\ -\gamma \alpha \end{pmatrix}$$

and recall that because $S$ is unitary, for $|z| = 1$ we have

$$S^{-1} = S^* = \begin{pmatrix} \bar{\alpha} & \bar{\beta} \\ \bar{\gamma} & \bar{\delta} \end{pmatrix}$$

on $S^1$. Hence by Schwarz reflection

$$S^{-1} = \begin{pmatrix} \frac{\alpha(z^*)}{\beta(z^*)} & \frac{\beta(z^*)}{\delta(z^*)} \\ \frac{\beta(z^*)}{\delta(z^*)} & \frac{\alpha(z^*)}{\beta(z^*)} \end{pmatrix}$$

and $z^* = \frac{1}{z}$ for all $z$. This shows $S^{-1}$ is analytic for $|z| > 1$. In terms of the entries of $S^{-1}$ the matrix $T$ can be written

$$T(z) = \begin{pmatrix} \frac{\beta \gamma - \alpha \delta}{\gamma_1} & \frac{\alpha_1}{\gamma_1} \\ \frac{\gamma_1}{\delta_1} & \frac{\beta_1}{\delta_1} \end{pmatrix}$$

$$\det T = \frac{\beta_1}{\gamma_1}$$
This shows the singularities of $T$ for $|z| > 1$ are those $\zeta$ such that $\alpha(\zeta) = 0$ or $\beta(\zeta) = 0$. Hence

$$\gamma(\zeta) = \frac{\beta(\zeta)}{\beta(\zeta)^*}$$

the zeroes of $\gamma$ are the reflections through $S'$ of the zeroes of $\beta$. So it's clear that one has:

**Prop:** Let $T(z)$ be the transfer matrix belonging to the scattering matrix $S(z) = (x \ y \ z \ s)$. Then the poles of $T(z)$ are the roots of $\gamma$ in $D$ and the reflections of the roots of $\beta$ in $\overline{D}$. The poles of $T(z)^{-1}$ are the roots of $\beta$ in $D$ and the reflections of the roots of $\gamma$ in $\overline{D}$.

---

**Paradox:** If $f$ is a finite Blaschke product, i.e. a rational function with $|f| = 1$ on $\partial D$, then multiplying $T$ by $f$ does not affect the fractional linear transformation associated to $T$, hence $fT$ should also be a transfer matrix. But this changes $S$ as follows:

$$S = \left(\begin{array}{cc} x & \beta \\ \gamma & s \end{array}\right) \quad \rightarrow \quad \left(\begin{array}{cc} x & f\beta \\ f^{-1}\gamma & s \end{array}\right)$$

and this can introduce singularities to $S$ inside $\partial D$, so something is wrong.

I still do not understand what a transfer matrix is. Somehow those $T$ coming from a scattering matrix $S$ are not all the possible $T$ one should consider.

Consider again potential scattering
\[ Ae^{ikx} + Be^{-ikx} \rightarrow e^{ikx} \]
\[ Be^{ikx} + \overline{A}e^{-ikx} \rightarrow e^{-ikx} \]
\[ a_1 e^{ikx} + b_1 e^{-ikx} \rightarrow b_2 e^{ikx} + a_2 e^{-ikx} \]

\[ T = \begin{pmatrix} A & \overline{B} \\ B & A \end{pmatrix} \]

This \( T \) is an entire function of \( k \). If the potential is a sum of \( S \) functions supported at integral points, the \( T \) is Laurent polynomial matrix in \( e^{ik} = z \), hence \( T(z) \) has singularities only at \( z = 0, \infty \). Such a matrix does not have the shrinking property.

Unfortunately, the kind of \( T \) has possibly singularities at \( k = 0 \).

---

Suppose \( T(z) \) is matrix of Laurent polynomials whose inverse is also (hence \( \det T(z) = c z^n \) some \( c \neq 0 \) and \( n \)). Suppose that for \( 0 < |z| \leq 1 \), \( T(z) \) expands the disk and for \( 1 \leq |z| < \infty \) it contracts the disk. Then I can put \( z = e^{i\lambda} \) and obtain a Nevanlinna matrix (or its inverse).
Let $P$ denote a hermitian form of signature $(+,-)$ on a 2-dimensional complex vector space $V$. The isotropic lines for $P$ form a circle in the Riemann sphere $\mathbb{P}V$ dividing it into open disks in which $P$ is $>0$ and $<0$.

Let $Q$ be another such form. If $P < Q$, then clearly $P(\omega) > 0 \Rightarrow Q(\omega) > 0$ so that the $P$-positive disk is contained in the $Q$-positive disk. In fact the $P$-circle is contained in the interior $Q$-positive disk. Conversely suppose the $P$-positive closed disk is contained in the open $Q$-positive disk.

I have seen that by a limiting process of successive reflections two non-intersecting circles in the Riemann sphere determine a pair of points which is placed at $z=0$ and $z=\infty$ then the circles become concentric circles around zero. So in this way we can identify $V = \mathbb{C}^2$ so that $P$ is given by the matrix $(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix})$ and $Q$ is given by a real diagonal matrix $(\begin{smallmatrix} \lambda & 0 \\ 0 & \mu \end{smallmatrix})$. The $P$-circle is $|z|=1$. We have

$$|z|^2 - 1 \geq 0 \Rightarrow 1/|z|^2 + \mu > 0$$

hence $\lambda > 0$, $\mu < 0$. $Q - P$ is the form:

$$\begin{pmatrix} z \\ 1 \end{pmatrix}^*(Q-P)\begin{pmatrix} z \\ 1 \end{pmatrix} = (\lambda - 1)|z|^2 + (1+\mu)$$

no $Q > P$ when $\lambda > 1$ and $-\mu < 1$. So I assume in addition that $\det(Q) = \lambda \mu = -1$ or

$$\lambda (-\mu) = 1$$

Then we should have $Q > P$. To be sure we are given
\[ \lambda > 0, \lambda > (-\mu) \text{ and } \lambda (-\mu) = 1. \] It follows that \( \lambda > 1 \) and \( (-\mu) = \lambda^{-1} < 1 \) so indeed \( Q > P \).

Next consider the case where the closed \( P \)-positive disk is contained in the closed \( Q \)-positive disk but the \( P \)-circle and \( Q \)-circle are tangent at one point.


Let \( V \) be a 2-dimensional complex vector space. If \( P \) is a hermitian form on \( V \) with signature \((+, -)\), then the isotropic lines for \( P \) form a circle in \( PV \) which determines \( P \) up to a real non-zero multiplicative constant. To see this one can suppose \( P \) is the form on \( C^2 \) given by \((1, -1)\) whence the circle in \( PC^2 = \text{Riemann sphere} \) is described by

\[ (2)(1, 0)(2) = 12i^2 - 1 = 0. \]

If the hermitian form \((a, b)\) given by \((\alpha, \beta)\) has the same circle, then for all \( z \) with \(|z|^2 = 1 \) one has

\[ (2)(a, b)(2) = a + zb + \bar{z}b + d = 0 \]

hence \( b = 0 \), \( a + d = 0 \) so that \( Q \) is a multiple of \( P \), say \( Q = aP \), \( a \in \mathbb{R}^* \). Notice also that \( a > 0 \) iff the \( P \)-positive disk, i.e., the set of \( z \) with

\[ (2)P(2)^* > 0 \]

is the same as the \( Q \)-positive disk.

Now suppose \( P, Q \) are two hermitian forms of signature \((+, -)\).
on \( V \). The associated circles can be disjoint, tangent, or intersecting in 2 points.

**Case 1:** The \( P,Q \) circles are disjoint. We know that two disjoint circles in the Riemann sphere can be made concentric circles about the origin by a fractional linear transformation. (Successively reflect to find the points that should be sent to \( 0,\infty \). Suppose then the \( P,Q \) circles are resp. \( |z|=1, |z|=p \) and that the positive \( P \)-disk is \( |z|>1 \). By a scalar change \( \alpha \) on \( V \) we can suppose \( P \) is given by \( (1,0) \). It follows that \( Q \) is given by

\[
\left( \frac{a}{d} \quad 0 \right) \\
\left( 0 \quad d \right)
\]

where \( \frac{a}{d} = -p \). This is a basic canonical form for the pair \( (P,Q) \) under the action of \( \text{Aut}(V) \).

If we know that the discriminant of \( Q \) relative to \( P \), i.e. \( \det P^t Q \), is 1, then \( ad = -1 \). If also

\[
\left( z^k \right) P \left( \frac{1}{z} \right) \Rightarrow \left( z^k \right) Q \left( \frac{1}{z} \right) = a|z|^k + d > 0
\]

then we have \( a > 0 \) and \( a + d = a - \frac{1}{a} > 0 \), so \( a > 1 \) and hence

\[
P = \left( \begin{array}{cc} 1 & 0 \\ 0 & -1 \end{array} \right) \preccurlyeq \left( \begin{array}{cc} \frac{a}{d} & 0 \\ 0 & -\frac{1}{a} \end{array} \right) = Q.
\]

**Case 2:** The \( P,Q \) circles are tangent. We use a fractional linear transformation to put the point of tangent at \( \infty \) and to make the \( P \)-circle the real axis, and the \( P \)-positive disk the LHP. Hence

\[
\frac{1}{z} \left( \frac{1}{z} \right) \left( \begin{array}{cc} 0 & 1 \\ -1 & 0 \end{array} \right) \left( \frac{1}{z} \right) = \frac{z - \frac{1}{z}}{i} = -2 \text{Im} z
\]
P must be a positive multiple of \( \frac{1}{i} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \) which I can assume to be one by a scalar change of variable.

The Q-circle is a line \( \text{Im}(z) = a \), so \( Q \) must be a non-zero real multiple of

\[
\begin{pmatrix} 1 & 0 \\ i & -1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & -2i \end{pmatrix} = \frac{1}{i} \begin{pmatrix} 1 & 0 \\ i & -1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 2i \end{pmatrix} = \frac{1}{i} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 2a \end{pmatrix}
\]

If \( \det(P^{-1}Q) = 1 \) we then have the canonical form

\[
P = \frac{1}{i} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad Q = \frac{1}{i} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 2a \end{pmatrix}.
\]

If we know the \( P \)-positive disk is contained in the \( Q \)-positive disk, then since

\[
(\frac{z^*}{1})^*P(\frac{z}{1}) = -2 \text{Im} z \quad (\frac{z^*}{1})^*Q(\frac{z}{1}) = (2 \text{Im} z + 2a)
\]

we see the + sign holds and also \( a > 0 \), hence \( P \leq Q \).

**Case 3:** The \( P, Q \) circles intersect in 2 points. We put these points at \( 0, \infty \) and put the \( P \)-positive disk on the LHP and arrange that

\[
P = \frac{1}{i} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}
\]

\[
Q = 0, \quad Q \gg 0, \quad P > 0
\]
θ is the angle between the \( p > 0, q > 0 \) half-planes, then we have \( Q \) must be a multiple of

\[
\begin{pmatrix}
e^{i\theta} & 0 \\
0 & 1
\end{pmatrix}
\frac{1}{i}
\begin{pmatrix}
0 & i \\
-i & 0
\end{pmatrix}
\begin{pmatrix}
e^{-i\theta} & 0 \\
0 & 1
\end{pmatrix} = \frac{1}{i}
\begin{pmatrix}
0 & e^{i\theta} \\
e^{-i\theta} & 0
\end{pmatrix}
\]

Above calculation shows:

Prop: Let \( P, Q \) be hermitian forms of __signature\((+, -)\) on a 2-dimensional complex vector space \( V \).

1) If \( P \leq Q \), then the \( P \)-positive disk in \( PV \) is contained in the \( Q \)-positive disk.

2) If \( \det(p^{-1}Q) = 1 \), then the converse to 1) holds.

Cor: Let \( T \in \text{GL}_2(\mathbb{C}) \) and \( P = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \).

1) If \( P \leq T^*PT \), then \( T \) as a fractional linear transformation of the Riemann sphere expands the unit disk: \( |z| > 1 \Rightarrow |T(z)| > 1 \).

2) If \( |\det T| = 1 \), then the converse to 1) holds.

What is a 2-port? I give three equivalent descriptions.

1) Grassmannian description: Let \( C^4 \) be equipped with the hermitian form

\[
P: \begin{pmatrix} a_1 \\ a_2 \\ b_1 \\ b_2 \end{pmatrix} \rightarrow |a_1|^2 + |a_2|^2 - |b_1|^2 - |b_2|^2
\]
Then a 2-port is a rational map
\[
\mathbb{CP}^1 \longrightarrow \text{Grass}_2 (\mathbb{C}^2) \quad z \longmapsto W_z
\]
such that
\[
|z| < 1 \implies P \geq 0 \quad \text{on} \quad W_z
\]
\[
|z| \geq 1 \implies P \leq 0
\]

2) Scattering matrix description: Recall that a Zariski open subset of Grass_2 (\mathbb{C}^2) is given by subspaces
\[
\mathcal{M} = \left\{ \left[ \begin{pmatrix} a_1 \\ a_2 \\ b_1 \\ b_2 \end{pmatrix} \right] \mid (b_1) = S(a_1) \right\}
\]
where S runs over the affine space of 2×2 matrices. This open set consists of all subspaces W intersecting
\[
\{ \left[ \begin{pmatrix} 0 \\ 0 \\ 0 \\ \ast \end{pmatrix} \right] \}
\]
trivially. Those subspaces W which are isotropic for P are of the form \( \mathcal{M}_S \) where \( S \in U(2) \). Also any subspace W on which P \( \geq 0 \) is in the Zariski open set.

Hence given a rational map \( z \longmapsto W_z \) as in 1) we get a rational map
\[
z \longmapsto S(z) \quad \mathbb{CP}^1 \longrightarrow \mathbb{M}_{2 \times 2} (\mathbb{C})
\]
such that \( S(z) = W_z \). Moreover \( S(z) \) has poles outside the unit circle and it has unitary values on \( S^1 \).

Conversely if one is given \( z \longmapsto S(z) \) analytic matrix-valued function on \( |z| \leq 1 \) with unitary values on \( |z| = 1 \) then one can extend \( S(z)^{-1} \) analytically to \( |z| \geq 1 \) via
Then one gets a holomorphic map \( W : \mathbb{CP}^1 \rightarrow \text{Grass}_2(\mathbb{C}^2) \) which one knows is algebraic (GAGA). The next point is that maximum modulus implies that \( \|S(z)\| \leq 1 \) for \( |z| \leq 1 \). In effect given vectors \( a, b \) in \( \mathbb{C}^2 \) one has

\[
|\langle S(z)a, b \rangle| \leq \|S(z)a\| \cdot \|b\| = \|a\| \cdot \|b\|
\]

for \( |z| = 1 \) because \( S(z) \in U(2) \). Thus for \( |z| \leq 1 \) this holds by maximum modulus for all \( a, b \) so \( \|S(z)\| \leq 1 \). Similarly \( \|S^{-1}(z)\| \leq 1 \) for \( |z| \geq 1 \) and so it is clear that the holomorphy map \( W \) is a 2-port in the sense of defn. 1).

3) Transfer matrix description. Here we describe a Zariski-open subset of the Grassmannian consisting of subpace

\[
\Gamma' = \left\{ \left( \begin{array}{c} a_1 \\ a_2 \\ b_1 \\ b_2 \end{array} \right) \mid (a_1) = T(b_2) \right\}
\]

where \( T \) is a 2x2 matrix. We get all subspaces intersecting the subspace

\[
\left\{ \left( \begin{array}{c} x \\ 0 \\ 0 \\ 0 \end{array} \right) \right\}
\]

trivially.

If we have a rational map \( z \mapsto Wz \) as in 1) which intersects this Zariski-open subset, then we get a rational map \( z \mapsto T(z) \) of \( \mathbb{CP}^1 \) to matrices such that 

\[
Wz = \Gamma' T(z)
\]
The condition $P \geq 0$ on $W_2$ for $|z| \leq 1$ says that provided $T(z)$ is defined one has

$$|a_1|^2 - |b_1|^2 \geq |b_2|^2 - |a_2|^2 \quad \text{if} \quad (a_1') = T(b_2)$$

or equivalently that

$$T^*(\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}) T \geq \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad (z \text{ for } |z| \leq 1)$$

Conversely, if we are given a rational matrix function $T(z)$ such that

$$T^*(\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}) T \geq \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \text{for } |z| \leq 1 \quad \text{such that } T(z) \text{ is def.}$$

then it is clear we get a 2-port in the sense of 1).

Note that not all 2-ports can be described by transfer matrices. The ones that can't be are those such that $W_2$ contains a non-zero vector $(\begin{pmatrix} a_1 \\ b_1 \end{pmatrix})$ for each $z$. Taking $|z| = 1$ this implies for $S(z) = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$

we must have $\gamma = \delta = 0$, as $|a_1| = |b_1| \neq 0$. But then $\beta = \alpha = 0$ identically. So we see that the 2-ports, without transfer matrices, are those whose scattering matrices are diagonal.

So now I propose to study the set of 2-ports admitting transfer matrices, i.e. the set of rational matrices $T(z)$ such that $\quad \text{for } z \text{ for which } T(z)$
are defined one has
\[ T^*(1, 0) T \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = P \]
if \( |z| \leq 1 \)
\[ \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = P \]
if \( |z| \geq 1 \)

It is clear that it suffices to assume these conditions for a dense set of \( z \). The set of these transfer matrices forms a monoid under multiplication. In effect if \( T_1^* P T_1 \geq P \) and \( T_2^* P T_2 \geq P \), then one has
\[ T_2^* T_1^* P T_1 T_2 \geq T_2^* P T_2 \geq P \]

etc.

For \( |z| = 1 \) when \( T(z) \) is defined we have
\[ T^* P T = P \]
so that \( |\det T| = 1 \) and so \( T \) is invertible.

In fact we have
\[ p^{-1} T^* P = T^{-1} \]
\[ T^{-1} = \begin{pmatrix} A_1 & B_1 \\ C_1 & D_1 \end{pmatrix} \]

If \( T = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \) this says that for \( |z| = 1 \)
\[ \begin{pmatrix} \overline{A} & -\overline{C} \\ -\overline{B} & \overline{D} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} A & C \\ B & D \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \frac{1}{AD-BC} \begin{pmatrix} +D & -B \\ -C & A \end{pmatrix} = \begin{pmatrix} A_1 & B_1 \\ C_1 & D_1 \end{pmatrix} \]

Hence
\[ A_1 = \overline{A(z^*)} \]
\[ B_1 = -\overline{C(z^*)} \]
\[ C_1 = -\overline{B(z^*)} \]
\[ D_1 = \overline{D(z^*)} \]
which shows that the poles of $T^{-1}$ are the reflections of the poles of $T$. So we see easily that the singularities of $T$ are symmetric about $|z| = 1$.

\[ \text{May 12, 1978} \]

Suppose we equip $C[z, z^{-1}]^2$ with the hermitian form

\[ \tilde{P}: f \mapsto (Pf, f) = \int f^* Pf \frac{d\theta}{2\pi} \quad P = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \]

Let $T(z) \in \text{GL}_2(C[z, z^{-1}])$ such that $T^* PT = P$ on $S'$. Then clearly $T$ gives an automorphism of $C[z, z^{-1}]^2$ preserving the hermitian form.

$L_0 = C[z]^2$ is a $C[z]$ lattice in $C[z, z^{-1}]^2$ with the property that $zL_0$ has the orthogonal complement $C^2$ in $L_0$ with respect to $\tilde{P}$. The image $TL_0$ of $L_0$ has the same property.

Conversely, given a lattice $L$ such that one has an orthogonal decomposition

\[ L = N \oplus zL \]

with respect to $\tilde{P}$ such that $\tilde{P}$ is non-degenerate on $N$.

For each $f \in N$ we can write

\[ f = \sum \varphi_n(f) z^n \quad \varphi_n : N \rightarrow C^2 \]
and we have

\[
(\tilde{P}(z^k f), f) = (P z^k \sum q_n(f) z^n, \sum q_n(f) z^n)
\]

\[
= \sum (P q_{n-k}(f), q_n f)
\]

\[
= \sum (q_n^* P q_{n-k} f, f)
\]

where the last inner product is obtained by taking \( \Phi \) as orthonormal basis a basis for \( N \) such that the form \( \tilde{P} \) restricted to \( N \) has the matrix \( P \). The above vanishes for \( k \neq 0 \), hence we get

\[
\sum_n q_n^* P q_{n-k} = \begin{cases} 
P & k = 0 \\
0 & k \neq 0
\end{cases}
\]

It follows that the Laurent poly. matrix

\[
T(z) = \sum q_n z^n
\]

satisfies

\[
T^* P T = P
\]

for \( |z| = 1 \).

The question now is what lattices correspond to \( T \)'s with the 2-port property: \( T^* P T \leq P \) for \( |z| \leq 1 \).