April 11, 1978:

Let $T$ be a contraction operator in $H$, and denote by $i : H \rightarrow H_{1-TT^*}$ the completion of $H$ w.r.t. the norm $(1-TT^*)x, x = \|x\|^2 - \|TT^*x\|^2$. Define

$$i(h) = \beta (1 - z T^*)^{-1} h.$$ 

To be precise: The series

$$(1 - z T^*)^{-1} = \sum_{n \geq 0} z^n (T^*)^n$$

converges for $|z| < 1$, so $\beta (1 - z T^*)^{-1} h$ is analytic in $|z| < 1$.

Moreover, the series

$$\sum_{n \geq 0} \beta z^n T^* n h = \sum_{n \geq 0} (\beta T^n h) z^n$$

represents an element in $L^2(\mathbb{S}^1, H_{1-TT^*})$ because

$$\sum_{n \geq 0} \|\beta T^n h\|_{H_{1-TT^*}} = \sum_{n \geq 0} \left( (1-TT^*)(T^n h), (T^n h) \right)_{H_{1-TT^*}}$$

assumes $\|T^n h\| \rightarrow 0 = \sum_{n \geq 0} \left( \|T^* n h\|^2 - \|T^* (n+1) h\|^2 \right) = \|h\|^2 < \infty$.

Therefore, $i$ is a well-defined isometric embedding.

Furthermore

$$(i^* U_{1-TT^*} h, h)_{H_{1-TT^*}} = \sum_{n \geq 0} (\beta z^n T^n h, \beta T^* (n+1) h)_{H_{1-TT^*}}$$

$$= \sum_{n \geq 0} (\beta T^n h, \beta T^* (n+1) h)_{H_{1-TT^*}}$$
\[
\sum_{n \geq 0} \langle (T^*T^*)^n h, T^*m^nh \rangle = \langle h, T^*m^nh \rangle = \langle T^m h, h \rangle.
\]

So therefore one will obtain at least an embedding
\[
\tilde{H} \hookrightarrow L^2(S^1, H_{1-T^*T^*})
\]
only assuming \( \|T\| \leq 1 \). No must assume \( T^* \to 0 \) weakly.

Suppose now that \( T \) is the backwards shift on \( l^2(\mathbb{N}) \) i.e. \( T(e_i) = \{ e_{i-1}, i \geq 1 \} \). Then \( T^* \) is the forward shift so that
\[
T^*T^* = 1
\]
and hence there is a mistake in the above. The mistake consists in assuming that
\[
T^*h \to 0 \quad \text{as} \quad n \to \infty
\]
for all \( h \in H \).

Note that
\[
i(T^*h) = \sum_{n \geq 0} \langle T^*h, e_n \rangle e_n
\]
so
\[
i(h) - i(T^*h) = f(h)
\]
elements of the form \( f(h) \) are dense in \( H \), hence we see that
\[
\tilde{H} = L^2(S^1, H_{1-T^*T^*})
\]
with \( \sum_{n \geq 0} z^n \hat{\mathcal{H}} = H^2(S^1, H_{1-T}^{*}) \).

Hence this isomorphism gives the outgoing representation for \( \mathcal{H} \). Similarly, assuming \( T^n \to 0 \) strongly we get the incoming representation

\[ \check{\mathcal{H}} = L^2(S^1, H_{1-T}^{*T}) \]

induced by

\[ i(h) = f (1-z^{-1}T)^{-1} h = \sum_{n \geq 0} (f T^n h) z^{-n} \]
Suppose \( V \) partial isometry in \( \mathcal{H} \) with indices \((0, 1)\) without unitary component and \( T \) its canonical extension as a contraction. I’d like to understand what it means for \( T^* T^n \to 0 \) strongly.

We can identify \( \mathcal{H} \to \mathcal{H}_{1-TT^*} \) with \( h \mapsto (h, u_i) \) and \( \mathcal{H} \to \mathcal{H}_{1-TT^*} \) with \( h \mapsto (h, u_i) \). Under the assumption \( T^* T^n \to 0 \) we computed the scattering operator

\[
L^2(S^1) = L^2(S^1, \mathcal{H}_{1-TT^*}) \sim \mathcal{H} \sim L^2(S^1, \mathcal{H}_{1-TT^*}) = L^2(S^1)
\]

and found it to be multiplication by the function

\[
S(z) = \left((1-zT)^{-1}u_i, u_i\right)
\]

In effect we have

\[
(1-z^{-1}T)^{-1}h, u_i \quad \quad \quad \quad \quad h \mapsto ((1-zT)^{-1}h, u_i)
\]

\[
(h, u_i) \quad \quad \quad \quad \quad h - z^{-1}Th \quad \quad \quad \quad \quad ((1-zT)^{-1}(1-zT)h, u_i)
\]

Take \( k = u_i \)

\[
1 \quad \quad \quad \quad \quad (1-z^{-1}T)u_i \quad \quad \quad \quad \quad ((1-zT)^{-1}u_i, u_i)
\]

because \( T u_i = 0 \).

In general the function \( S(z) \) defined by (1) is defined for \( |z| < 1 \). Under the assumption \( T^* T^n \to 0 \) we know that it is of modulus 1 a.e. for \( z \in S^1 \).

More precisely even when \( T^* T^n \to 0 \) we have a well-defined map \( \mathcal{H} \to L^2(S^1, \mathcal{H}_{1-TT^*}) \) given by

\[
h \mapsto \rho (1-zT)^{-1}h
\]
\[ p(z) = \sum z^n (\rho T^* u_i) \in H^2(S^1 \setminus \{z\}) \]

so

\[ S(z) = \sum (1-zT^*)^{-1} u_i, \quad u_i \in H^2(S^1) \).

In particular, \( S(z) \) for \( z \in S^1 \) is a well-defined measurable function up to null-set equivalences. (It is also known, maybe that

\[ S(\text{e}^{i\theta}) = \lim_{\alpha \to 1} S(\text{e}^{i\theta}) \quad \text{a.e. } \theta, \]

so the problem is whether \( S \) has modulus \( 1 \) on \( S^1 \).

Let's compute \( S \) by choosing the unitary extension \( U \) of \( V \) with \( U(u_i) = u_{-i} \), whence we can identify \( H \) with \( L^2(S^1) \) and \( u_i = \psi^{-1}, \quad u_{-i} = \psi \). Here we use \( \psi \) to note the \( S^1 \)-variable associated with \( dv \).

Recall that \( (1-zT^*)^{-1} u_i \) for \( |z| < 1 \) is determined by the properties

\[
\begin{align*}
\langle (1-zT^*)^{-1} u_i, (1-zV) \psi \rangle &= \langle (1-zT^*)^{-1} u_i, (1-zT) \psi \rangle = (u_i, \psi) = 0 \\
\langle (1-zT^*)^{-1} u_i, u_i \rangle &= (u_i, (1-zT)^{-1} u_i) = (u_i, u_i) = 1
\end{align*}
\]

On the other hand for \( |z| < 1 \)

\[
\langle (1-zU^{-1})^{-1} u_i, (1-zV) \psi \rangle = 0
\]

so \( (1-zU^{-1})^{-1} u_i \) is proportional to \( (1-zT^*)^{-1} u_i \). But

\[
\langle (1-zU^{-1})^{-1} u_i, u_i \rangle = \left( \frac{\bar{\psi}}{1-z\bar{\psi}} \right)^{-1} \frac{\psi}{1-z\psi^{-1}} = \int \frac{dv(t)}{1-zv^{-1}}
\]

so

\[
(1-zT^*)^{-1} u_i = \frac{1}{\int \frac{dv(t)}{1-zv^{-1}}} \cdot \frac{\psi}{1-z\psi^{-1}}
\]
Hence
\[ S(z) = \frac{\int \frac{\zeta^{-1}}{1 - z \zeta^{-1}} \, d\zeta}{\int \frac{1}{1 - z \zeta^{-1}} \, d\zeta} \]

\[ \frac{1 + z S(z)}{1 - z S(z)} = \int \frac{1 + z \zeta^{-1}}{1 - z \zeta^{-1}} \, d\zeta \quad \text{since} \quad \int d\zeta = 1. \]

Note that if \( g(z) = \int \frac{d\zeta}{1 - z \zeta^{-1}} \), then
\[ \frac{1}{2} g(z^*) = \frac{1}{2} \int \frac{d\zeta}{1 - z \zeta^{-1}} = -\int \frac{d\zeta}{1 - z \zeta^{-1}} = -\int \frac{\zeta^{-1} d\zeta}{1 - z \zeta^{-1}} \]
so that
\[ -z S(z) = \frac{g(z^*)}{g(z)} \]

Recall that
\[ \text{Re} \left( \frac{1 + z \zeta^{-1}}{1 - z \zeta^{-1}} \right) = \text{Re} \left( \frac{(1 + z \zeta^{-1})(1 - \bar{z} \zeta^{-1})}{|1 - z \zeta^{-1}|^2} \right) = \frac{1 - |z|^2}{|1 - z \zeta^{-1}|^2} > 0 \]

It follows that
\[ \text{Re} \left( \frac{1 + z S(z)}{1 - z S(z)} \right) > 0 \quad \text{for} \quad |z| < 1 \]

which implies that \( |z S(z)| < 1 \) for \( |z| < 1 \) and hence \( |S(z)| < 1 \) by maximum modulus.
April 17, 1978:

Let's go over Szego's thm. Let \( \mu \) be a probability measure on \( S^1 \) and let \( H^2(\mu) \) be the closure of \( C[z] \) in \( L^2(\mu) \). The problem is to compute the length of the projection of \( 1 \in H^2(\mu) \) perpendicular to \( z \, H^2(\mu) \).

We can form the sequence of orthonormal polynomials \( \{p_0, p_1, \ldots \} \) with respect to \( \mu \), and define numbers \( h_n, n = 1, 2, \ldots \) by

\[
zh_n = k_{n+1} p_{n+1} - h_{n+1} z^n p_n^* \quad k_{n+1} = \sqrt{1 - h_n^2}
\]

where \( |h_n| < 1 \) for all \( n \). We assume \( \mu \) has infinite support so that \( p_n \) is defined for all \( n \). The recursion relation:

\[
\begin{pmatrix}
  p_{n+1} \\
  z^n p_n^*
\end{pmatrix} = \frac{1}{k_{n+1}} \begin{pmatrix}
  1 & h_{n+1} \\
  -h_{n+1} & 1
\end{pmatrix} \begin{pmatrix}
  z & 0 \\
  0 & 1
\end{pmatrix} \begin{pmatrix}
  p_n \\
  z^n p_n^*
\end{pmatrix}
\]

This shows that for \( |z| < 1 \) one has

\[
\left| \frac{p_n}{z^n p_n^*} \right| < 1 \quad n \geq 1.
\]

hence \( z^n p_n^* \) has its roots outside \( S^1 \) and \( p_n \) has its roots inside \( S^1 \).
Suppose that $T$ arises from $V$ as before and that $U$ is a unitary extension of $V$ so that $H$ can be identified with $L^2(S^1, dv)$, $u_i = v_i$, $u_{-i} = 1$. For $|z| < 1$ we have the analytic function

$$S(z) = (1 - zT)^{-1} u_i, u_{-i}$$

which we computed to be given by

$$S(z) = \frac{\int \frac{1}{1 - \frac{1}{jz^2}} dv}{\int \frac{1}{1 - j\frac{1}{z^2}} dv}$$

Thus

$$\frac{1 + zS(z)}{1 - zS(z)} = \int \frac{1 + z}{j - z} dv$$

makes $|z| < 1$ into $Re > 0$. It follows that

$$|zS(z)| < 1$$

for $|z| < 1$, hence by maximum modulus, that

$$|S(z)| \leq 1$$

Equality can occur only when $S(z)$ is a constant of modulus 1, whence $dv$ is the Dirac measure at some point of $S'$.

It follows that $S(z) \in H^2(S^1, \frac{d\theta}{2\pi})$ and that

$$\|S\| = 1 \iff |S| = 1 \text{ a.e. on } S' \iff \text{Re} \int \frac{S + z}{j - z} dv = 0$$

for a.e. $z \in S'$. Here we define values for $zeS'$ by radial limits. But Fatou's thm. says

$$\text{Re} \int \frac{S + z}{j - z} dv = 2\pi \frac{d\theta}{d\theta} \text{ a.e. } z \in S'.$$
Hence it follows that \( |s| = 1 \iff dv \) is singular. 

\[
|s|^2 = \left\| \sum_{n \geq 0} \varepsilon^n (T^* u_i, u_i) \right\|^2 = \sum_{n \geq 0} \| T^* u_i \|^2 = 1 - \lim_{n \to \infty} \| T^* u_i \|^2
\]

So \( |s| = 1 \iff T^* u_i \to 0 \)

\[
|s| = 1 \quad \text{a.e. on } S^1 \iff \frac{dv}{d\theta} = 0.
\]

The last condition should be symmetrical under interchanging \( T \) and \( T^* \) hence we also get \( T u_i \to 0 \).
Let \( V : D^* \rightarrow H \), \( T \) be as above, and let \((\tilde{H}, \tilde{U})\) be the unitary dilation of \( T \). I recall proving that \( \tilde{H} \oplus \tilde{H} \) has the orthonormal basis \( \{ U^k(e_i) \mid n \geq 1 \} \) and \( \{ U^{-n}(u_i) \mid n \leq -1 \} \).

Put
\[
\begin{align*}
  e_n &= U^n(e_i) \quad n \geq 1 \\
  e_n &= U^{-n}(u_i) \quad n \leq -1.
\end{align*}
\]

What can we say about the structure of \((\tilde{H}, \tilde{U})\)?

Is \((\tilde{H}, \tilde{U})\) a scattering situation?

\[
\langle \ldots, U^2 u_i, U u_i, u_i, \ldots \rangle \oplus \tilde{H} \oplus \langle \ldots, U u_i, u_i, \ldots \rangle
\]

\[
\| e_2, e_1 \|
\]

Let's follow the trajectory \( U^n u_i \) as \( n \to -\infty \).

Clearly \( \langle e_2, e_1 \rangle \oplus \tilde{H} \) is stable under \( U^{-1} \). If \( \iota : H \to \tilde{H} \) is the embedding, we have
\[
\iota^* U^{-n}(u_i) = T^* u_i
\]
hence for this to tend to zero means that we pick up all of \( u_i \) by taking \( U^{-n} u_i \) and projecting onto \( \langle e_2, e_1 \rangle \). What do we actually get? Suppose
\[
U^{-n} u_i = a_0 e_n + a_1 e_{-n+1} + \ldots + a_{-n} e_1 \mod \tilde{H}.
\]

So in stages: \( n \to \infty \)
\[
U_i = u_i = (u_i, u_i) u_i + \text{something in } H.
\]
\[
U^{-1} u_i = (u_i, u_i) U^{-1} u_i + H
\]
\[
T^* u_i = H
\]
\[ U^{-1}u_i = (u_i, u_{-i}) e_{-1} + T^*u_i + (T^*u_i, u_{-i}) e_{-1} + \text{something in } \mathbb{R} V \]

\[ u^{-1}u_i = (u_i, u_{-i}) e_{-2} + (T^*u_i, u_{-i}) e_{-1} + T^{*2}u_i \]

Hence we get for the scattering operator

\[ S(z) = \sum_{n=0} \left( T^{*n}u_i, u_{-i} \right) z^n = \left( (1-z^*)^{-1} u_i, u_{-i} \right) \]

The good case is when \( T^{*n}u_i \to 0 \) for then \( S \)

is a unitary operator, i.e., \( |S(z)| = 1 \) for \( z \in \mathbb{S}^1 \).

When this case holds it seems clear that we can invert the scattering operator, so that the trajectory \( u^n(u_i) \) \( n \to 0 \) must be captured in the limit by projection onto \( \langle e_1, e_2, \cdots \rangle \). Thus we obtain

**Proposition:** \( T^{*n}u_i \to 0 \iff T^n u_{-i} \to 0 \)

If this holds, then the \( U, U^{-1} \) invariant subspaces generated by \( u_i \) and \( u_{-i} \) coincide and hence must equal \( \mathbb{H} \), for otherwise the orthogonal complement would be a subspace of \( \mathbb{H} \) stable under \( U, U^{-1} \)

perpendicular to \( u_i, u_{-i} \), which contradicts the assumption that \( V \) has no unitary component. Hence we see that every element of \( \mathbb{H} \) has to be "seen" in both scattering representations, so

\[ T^n h \to 0 \quad T^{*n} h \to 0 \]

for all \( h \in \mathbb{H} \).
Summary: Let \( \mathcal{T} \) be the contraction on \( \mathbb{H} \) obtained from a partial isometry \( V \) with indices \((1,1)\) and no unitary component. Choose \( u_i, u_{-i} \) and let \( dv \) be the spectral measure of the unitary extension of \( V \) with \( U(u_i) = u_i \). Define

\[ s(z) = ((1-z^*z)^{-1}u_i, u_{-i}) \]

Then we have found that the following conditions are equivalent:

1) \( S \) inner, i.e. \(|S| = 1\) on boundary
2) \( T^n \to 0 \) strongly (enough that \( T^n u_i \to 0 \))
3) \( T^n \to 0 \) strongly (enough that \( T^n u_{-i} \to 0 \))
4) \( dv \) singular with respect to Lebesgue measure.

Suppose given an inner function \( S \) take \( \mathcal{H} = L^2(S') \) and \( \mathcal{H} = H^2(S') \otimes z S H^2(S') \). Put

\[ e_1 = zS, \quad e_{-1} = z^{-1} \]
\[ e_2 = z^2S, \quad e_{-2} = z^{-2} \]

etc., etc.

so that \( u_i = S, \quad u_{-i} = 1 \). Does this work?

We know multiplication by \( z \) induces a contraction \( Tz \) with \( Tz = z \mathcal{H} \). If \( h \perp S = u_i \), then \( h \perp S H^2(S') \)
so \( z h \perp zS H^2(S') \) so \( Tz h = z h \); on the other hand \( TzS = zS = 0 \), similarly if \( h \perp 1 = u_{-i} \), then \( h \perp H^2(S') \)
so \( z^{-1} h \perp z^{-1} H^2(S') \) so \( z^{-1} h \in h \) and so \( T z h = h \), etc.

Conclude: \( V \)'s belonging to measures, singular w.r.t. \( \frac{dv}{2\pi} \)
are in 1-1 correspondence with inner functions modulo multiplication by scalars.
April 21, 1978

The problem is now to understand invariant subspaces for a given $V$. Begin with a finite-dimensional situation. Suppose $H$ is finite-dimensional. I can choose $u_1, u_2, \ldots$ so that the measure $d\mu$ is not supported at $t=1$, which means that $V$ is the Cayley transform of a symmetric operator $A$. Specifically, I've seen that $f \sim L^2(d\mu)$ where $d\mu$ is a measure with finite support on $\mathbb{R}$ with $\int \frac{d\mu}{x+i} = 1$ and where under this isomorphism $u_1 = \frac{1}{x-i}, u_2 = \frac{1}{x+i}$.

Let $A$ be a subspace of $H$. Then $\text{Ker} D_A$ is of codim 1 or 0 in $\mathfrak{K}$. Assume that $A(\text{Ker} D_A) < \mathfrak{K}$, so that $A$ induces an operator in $\mathfrak{K}$ with domain of codim 1. Note that $\mathfrak{K} \cap \text{Ker} D_A = \mathfrak{K}$ is impossible for $\mathfrak{K} \neq 0$ since $A$ has no self-adjoint component by hypothesis.

But note for $v \neq 1$,

\[ A(\mathfrak{K} \cap \text{Ker} D_{A^v}) < \mathfrak{K} \cap \text{Ker} D_{A^{v-1}} \]

Now $H > D_A > D_{A^2} > \cdots > D_{A^n} = 0$ is a complete flag, hence $\mathfrak{K} \cap \text{Ker} D_{A^v}$ is of codim 1 or 0 in $\mathfrak{K} \cap \text{Ker} D_{A^{v-1}}$. But $\mathfrak{K} \cap \text{Ker} D_{A^v} = \mathfrak{K} \cap \text{Ker} D_{A^{v-1}}$ would imply $\mathfrak{K} \cap \text{Ker} D_{A^v}$ is $A$-invariant which is impossible unless it is zero.
Let $m(\lambda)$ be a rational function of $\lambda$ such that $\text{Im}(\lambda)$ has the same sign as $\text{Im}(\lambda)$, so that we have the Riesz-Herglotz representation
\[ m(\lambda) = p^2 + \sum_{\lambda > \lambda_0} \frac{r_i}{\lambda - \lambda_i} + c \]
for $p > 0$, $c \geq 0$, $\lambda_0 > 0$, and $\lambda_i$ real distinct.

One has a continued fraction development
\[ m(\lambda) = p + q_i = \frac{1}{p_i + q_i} = \frac{1}{p_2 + q_2} = \cdots \]

Let $S(z)$ be an inner function which is rational. This means that $S$ has a Blaschke product representation
\[ S(z) = c \prod_{i=1}^{d} \frac{z - a_i}{1 - \bar{a}_i z} \quad |a_i| < 1 \]
with $|c| = 1$.

Also $S$ has a Schur development
\[ S(z) = \begin{pmatrix} 1 & h_1 \\ h_1^* & 1 \end{pmatrix} \begin{pmatrix} z & 0 \\ 0 & 1 \end{pmatrix} \cdots \begin{pmatrix} 1 & h_d \\ h_d^* & 1 \end{pmatrix} \begin{pmatrix} z & 0 \\ 0 & 1 \end{pmatrix} \alpha \]
where $|h_i| < 1$ and $|\alpha| = 1$.

On the other hand we can pass between $m(\lambda)$ and $S(z)$ via the substitution
\[ S(z) = \frac{m(\lambda) - i}{m(\lambda) + i} \quad z = \frac{\lambda - i}{\lambda + i} \]
April 28, 1977:

Program: Let $A$ on $H = l^2(1,\infty)$ be given by a $T$-matrix. I want to describe the filtration $(D_n^-)$ in another way, somehow related to decomposing a 1-port into a 2-port connected to a 1-port. To get insight, let's start with the analogue of this filtration in the circular case.

Let $V$ be a partial isometry on $H$ with unitary component and indices $(1,1)$ and put

$(x)\quad S(z) = ((1-zT^*)^{-1}u_+ u_-) \quad |z| < 1$

as usual. We have $|S(z)| \leq 1$ for $|z| < 1$, hence if $S$ is not a constant of modulus 1, we can Schur develop $S$ one step

$$\frac{S - h}{1 - h^*S} = z S_1, \quad h = S(0) = (u_+ u_-)
$$

$S_1$ ought to belong to a partial isometry $(H_1, V_1)$ which I now want to find.

The obvious candidate is to take $H_1 = D_\nu$ with $V_1$ induced by $V$, i.e.

$$D_\nu = \{x \in D_{\nu} \mid \forall x \in D_{\nu}^2\} = D_{\nu^-}$$

The problem is to show this leads to $S_1$.

Recall that in $(x)$ $T$ denote the contraction obtained by extending $V$ by zero, however more generally if $\tilde{T}$ is any contraction extending $V$ to $H$ we have a similar formula obtained as follows. Recall that the
element \((1 - z \bar{z})^{-1} u_i\) is characterized by \(u_i\) being \(1\) to \(936\) and having inner product \(1\) with \(u_i\). Hence the general formula is

\[
S(z) = \frac{(1 - z \bar{z})^{-1} u_i, u_i}{(1 - \bar{z} z)^{-1} u_i, u_i}
\]

which we used already when \(\tilde{T}\) is the unitary extension with \(\tilde{T} u_i = u_i\).

Let \(g : \mathbb{C} \to \mathbb{C}^\times\) be the inclusion and put \(\tilde{T} = g^*\tilde{T}\).

If \(x \in \mathbb{C}^\times\), then \(\tilde{T} x = g^* \tilde{T} x = V x\) where \(V\) is the restriction of \(\tilde{T}\). Thus \(T\) is a contraction on \(\mathbb{C}^\times\) extending \(\tilde{T}\).

Put \(\hat{u}_z = (1 - z \bar{z})^{-1} u_i\). It is the unique element of \(\mathbb{C}^\times\) with \(\hat{u}_z \bot (1 - z \bar{z})\mathbb{C}^\times\) and \((\hat{u}_z, u_i) = 1\). So look at

\[
\begin{array}{ccc}
\mathbb{C}^\times & \rightarrow & \mathbb{C}^\times \\
\mathbb{C}^\times & \downarrow \quad (1 - z \bar{z}) \mathbb{C}^\times \\
\mathbb{C}^\times & \downarrow \quad (1 - \bar{z} z) \mathbb{C}^\times
\end{array}
\]

It follows that

\[
p^2_{\mathbb{C}^\times} (\hat{u}_z) \bot (1 - \bar{z} z) \mathbb{C}^\times.
\]

Now

\[
p_{\mathbb{C}^\times} (\hat{u}_z) = \hat{u}_z - (\hat{u}_z, u_i) u_i = \hat{u}_z - u_i
\]
I need in \( \mathbf{D}_V \) unit vectors \( \mathbf{V} \mathbf{D}_V \) and \( \mathbf{V} \mathbf{D}_V^2 \)

\[
\mathbf{D}_V \perp \mathbf{V} \mathbf{D}_V
\]

\[
\mathbf{D}_V^2 \perp \mathbf{V} \mathbf{D}_V^2
\]

\[
\mathbf{u}_i - (\mathbf{u}_i, \mathbf{u}_i) \mathbf{u}_i = \mathbf{u}_i - h \mathbf{u}_i
\]

so

\[
\mathbf{u}_i' = \frac{\mathbf{u}_i - h \mathbf{u}_i}{\sqrt{1 - h^2}}
\]

is a unit vector \( \perp \mathbf{V} \mathbf{D}_V^2 = \mathbf{R}_V \)

also

\[
\mathbf{V} \mathbf{D}_V \perp \mathbf{V} \mathbf{D}_V^2
\]

\[
\mathbf{v}^{-1} (\mathbf{u}_i - h \mathbf{u}_i) \perp \mathbf{D}_V^2 \quad \text{and} \quad \mathbf{e} \in \mathbf{D}_V
\]

hence

\[
\mathbf{u}_i' = \mathbf{v}^{-1} (\mathbf{u}_i - h \mathbf{u}_i) \quad \text{is a unit vector in} \quad \mathbf{D}_V \perp \mathbf{D}_V^2
\]

so now I can calculate

\[
\mathbf{S}(\mathbf{z}) = \frac{(\mathbf{u}_z - \mathbf{u}_i, \mathbf{u}_i ')}{(\mathbf{u}_z - \mathbf{u}_i', \mathbf{u}_i ')}
\]

Take numerator

\[
(\mathbf{u}_z - \mathbf{u}_i, \mathbf{u}_i ') = (\mathbf{p}_{\mathbf{D}_V} (\mathbf{u}_z), \mathbf{u}_i - h \mathbf{u}_i) \mathbf{v}^{-1} (\mathbf{u}_i - h \mathbf{u}_i) \sqrt{1 - h^2}
\]

\[
= (\mathbf{S}(\mathbf{z}) - h) \sqrt{1 - h^2}
\]

Take denominator

\[
(\mathbf{p}_{\mathbf{D}_V} (\mathbf{u}_z), \mathbf{v}^{-1} (\mathbf{u}_i - h \mathbf{u}_i)) \sqrt{1 - h^2}
\]

in which \( \mathbf{v}^{-1} = \mathbf{T}^* \)

\[
= (\mathbf{u}_z, \mathbf{T}^* (\mathbf{u}_i - h \mathbf{u}_i)) \sqrt{1 - h^2} = (\mathbf{T} (1 - h \mathbf{T}^*)^{-1} \mathbf{u}_z, \mathbf{u}_i - h \mathbf{u}_i)
\]

hence by \( \mathbf{T}^* \)
It is important to concentrate on the line \( L_2 \) perpendicular to \((1-\bar{z}V)D_v\):
\[
L_2 = \mathcal{H} \ominus (1-\bar{z}V)D_v
\]

If \( V \) is the partial isom. on \( D_v \) induced by \( V \), then clearly \( L_2^1 = \text{proj}_{D_v}(L_2) \) when the latter is a line. This is alright except when \( z = 0 \). \( L_2 \) is spanned by \((1-\bar{z}T^*)^{-1}u_i\) whose projection on \( D_v \) is \((1-\bar{z}T^*)^{-1}u_i - u_i\). Thus \( L_2^1 \) is spanned by
\[
\frac{(1-\bar{z}T^*)^{-1}u_i - u_i}{z} = T(1-\bar{z}T^*)^{-1}u_i
\]

To find the function \( S_1(z) \) belonging to \( V \), we need unit vectors in \( D_v \) and to \( D_v^2 \), \( VD_v^2 \), resp.
\[
\text{proj}_{D_v}(u_i) = u_i - (u_i, u_i)u_i = u_i - \overline{h}u_i \quad \perp VD_v^2
\]
\[
u_i^1 = \frac{u_i - \overline{h}u_i}{\sqrt{1 - |h|^2}} \quad h = (u_i, u_i) = S(0)
\]
\[
\text{proj}_{VD_v}(u_i) = u_i - \overline{h}u_i \quad \perp VD_v^2
\]
so
\[
u_i^1 = \frac{\sqrt{1 - |h|^2}(u_i - \overline{h}u_i)}{1 - |h|^2}
\]
in \( D_v \) on which \( T = V \)

\[
(T^*(1-\bar{z}T^*)^{-1}u_i, V(u_i - \overline{h}u_i)) = ((1-\bar{z}T^*)^{-1}u_i, TV^{-1}(u_i - \overline{h}u_i))
\]
\[
= (1 - z T^*)^{-1} u, u_i - h u_i \\
= 1 - \lambda \mathcal{S}(z).
\]

\[
(T^*(1 - z T^*)^{-1} u, u_i - h u_i) = \frac{1}{2} \left( (1 - z T^*)^{-1} u - u_i, u_i - h u_i \right)
\]

\[
= \frac{1}{2} \left( \mathcal{S}(z) - h \right)
\]

Therefore we have

\[
S_i(z) = \frac{(T^*(1 - z T^*)^{-1} u, u_i - h u_i)}{(T^*(1 - z T^*)^{-1} u, u_i - h u_i)} = \frac{1}{2} \frac{\mathcal{S}(z) - h}{1 - \lambda \mathcal{S}(z)}
\]

as I conjectured.

Summarizing I get:

**Thm.** Let \( V \) be a partial isometry in \( \mathcal{H} \) with indices \((1,1)\), and let \( S(z) \) be the analytic function in the disk associated to \( V \) and a choice of unit vectors \( u_j \in \mathcal{H} \oplus \mathcal{D} \) and \( u_j \in \mathcal{H} \oplus \mathcal{D}^* \). (Thus \( S(z) = \langle \psi, u_i \rangle \) where \( \psi \) is the unique vector with \( \psi \perp (1 - z V) \mathcal{D} \) and \( \langle \psi, u_i \rangle = 1 \).)

Let \( V_j \) be the isometry induced on \( \mathcal{D} \) by \( V \) and let \( S_i(z) \) be the analytic function belonging to \( V_i \) and the unit vectors

\[
u_i = \frac{V^* (u_i - h u_i)}{\sqrt{1 - h^2}} \in \mathcal{D} \oplus \mathcal{D}^* \\
u_{-i} = \frac{u_i - h u_i}{\sqrt{1 - h^2}} \in \mathcal{D} \oplus \mathcal{D}^* \]

where \( h = \langle u_i, u_{-i} \rangle = S(0) \). Then \( S_i(z) \) is the first
function in the Schur development of $S(z)$, i.e.

$$S_1(z) = \frac{1}{z} \cdot \frac{S(z) - \hbar}{1 - \hbar S(z)}$$

or

$$S(z) = \begin{pmatrix} 1 & \hbar \\ -\hbar & 1 \end{pmatrix} (z S(z))$$
Let $V$ be a partial isometry in $\mathcal{H}$ without unitary component. For each $\varepsilon$ such that $(1-\varepsilon)V$ is closed let $N_\varepsilon$ denote its orthogonal complement. Note for $\varepsilon < 0$ one has $(1-\varepsilon)V = (1-\varepsilon)V_0$ so we interpret this space to be $V_0^\perp$ at $\varepsilon = \infty$. I know that $(1-\varepsilon)V$ is closed for all $\varepsilon$ not in $S^1$. Recall the proof: We have an isometry

$$L^\perp \oplus L^\perp \rightarrow \mathcal{H}, \quad (x, y) \mapsto x + y$$

For $|\varepsilon| < 1$ the map $(x, y) \mapsto -\varepsilon Vx$ is of norm less than 1 so on adding it to the above isometry we get an isomorphism

$$(x, y) \mapsto (1-\varepsilon)Vx + y$$

which shows for $|\varepsilon| < 1$ that $L^\perp$ is complementary to $(1-\varepsilon)V$. (Actually the above map is $1-\varepsilon T$ which is invertible for all $\varepsilon$ such that $\varepsilon^*$ is not in the spectrum of $T$.) So $(1-\varepsilon)V$ is closed for $|\varepsilon| < 1$. Case $|\varepsilon| > 1$ is similar.

Claim that $N_\varepsilon$ injects into $L^\perp \oplus (V_0^\perp)^\perp$ for any $\varepsilon$ for which it is defined. In effect let $x \in N_\varepsilon$ be orthogonal to $L^\perp$ and $R^\perp$ i.e. $x \in L^\perp$ and $x \in R^\perp$. Then by definition of $N_\varepsilon$ we have

$$(x, x - \varepsilon Vx) = 0 \quad \forall x \in L^\perp$$

Since $V^\perp$ is defined this says $(x, x) = (x, Vx) = (V^\perp x, x)$ or $(1-\varepsilon V^\perp)x, x) = 0$. But $(1-\varepsilon V^\perp)x \in L^\perp$ hence we
Important case: If finite-dimensional in which case \((1-2\mathbf{V})D\mathbf{V}\) is closed for all \(\varepsilon\). Then we can define a scattering operator

\[
S(\varepsilon): D^+ \mathbf{V} \to R^+\mathbf{V}
\]

by

\[
(S(\varepsilon)u, v) = (1-2\mathbf{V}^*\varepsilon)^{-1}u, v
\]

This is defined as long as \((1-2\mathbf{V}^*\varepsilon)^{-1}\) is invertible in particular for \(|\varepsilon| < 1\). If I recall that \((1-2\mathbf{V}^*\varepsilon)^{-1}u\) is the unique element of \(N_0\) projecting onto \(u\), then it is clear that

Image of \(N_0\) in \(D^+\mathbf{V} \times R^+\mathbf{V}\) = graph of \(S(\varepsilon)\)

In other words we a map \(\varepsilon \mapsto N_0\) into the Grassman manifold of subspaces of \(D^+\mathbf{V} \times R^+\mathbf{V}\) of dim = dim \(D\mathbf{V}\), and to get \(S\) we intersect with the open cell of those subspaces projecting non-trivially on \(D^+\mathbf{V}\).

Comments: 1) One gets via the Grassman interpretation an interpretation of the scattering operator for all \(\varepsilon\)

2) In \(\text{Grass}(D \times D)\), \(d = \text{dim} D\), the isotropic subspaces for the hermitian form

\[
(x_1, x_2) \mapsto |x_1|^2 - |x_2|^2
\]
I want to discuss 2 ports. Here one thinks of $\mathbb{D}^+$ as consisting of incoming waves with amplitudes $a_1, a_2$ and $\mathbb{R}^+$ as consisting of outgoing waves with amplitudes $b_1, b_2$. The scattering matrix gives the relation between the two:

$$\begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = S(a) \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$$

Power into the 2-port is $|a_1|^2 + |a_2|^2 - |b_1|^2 - |b_2|^2$. But now I want to describe things in terms of a transfer matrix

$$\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \rightarrow \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$$

and the power form:

$$|a_1|^2 - |b_1|^2 - |b_2|^2 - |a_2|^2$$

Power in at 1

Power out at 2

Let's fix $\varepsilon \in S'$. The 2-dimensional subspaces of $\mathbb{D}^+ \times \mathbb{R}^+$ which are isotropic for the power form are the graphs of unitary transformations. So we see that the possible $T$ i.e. $U(1,1)$ should correspond bijectively to a subset of $U(2)$.

Formulas:

$$\begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = S(a) \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$$

$$\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = T(b)$$

$$\begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$$
\[ b_2 = y a_1 + s a_2 \quad b_1 = x a_1 + \beta a_2 \]
\[ a_1 = \frac{1}{y} b_2 - \frac{s}{y} a_2 \quad a_2 = \frac{x}{y} b_2 + \frac{\beta y - x s}{y} a_2 \]

So
\[ S = \begin{pmatrix} x & \beta \\ y & s \end{pmatrix} \quad T = \begin{pmatrix} \frac{1}{y} & -\frac{s}{y} \\ \frac{x}{y} & \frac{\beta y - x s}{y} \end{pmatrix} \]

Similarly
\[ T = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \quad S = \begin{pmatrix} \frac{c}{A} & \frac{AD - BC}{A} \\ \frac{1}{A} & \frac{-B}{A} \end{pmatrix} \]

Recall that any element \( T \) in \( U(1,1) \) can be expressed in the form
\[ T = \begin{pmatrix} a & b \\ \bar{b} & \bar{a} \end{pmatrix} \]

from which we see that \(|A| \neq 1\). Consequently, the above formulas give an embedding of \( U(1,1) \) into \( U(2) \) with image those \( S \) with \( \gamma \) (hence \( \beta \)) \neq 0 \( \gamma \neq 0 \), i.e., the complement of the diagonal matrices.