March 29, 1978

The intelligent question: Consider the category of symmetric (1,1) operators without self-adjoint component. What are the automorphisms of such a gadget?

Fix $\mathcal{H}$ and a isometric relation $V$ on $\mathcal{H}$ of type (1,1), that is, $V: \mathcal{H} \to \mathcal{H}$ is unitary, where $\mathcal{H}_V$ and $\mathcal{H}_V$ are closed subspaces of co-dim. 1. Assume $V$ has no unitary component, i.e., there is no non-trivial closed subspace of $\mathcal{H}$ on which $V$ induces a unitary antiautomorphism.

Q: What are the autor of such a thing?

Other question: Let $T$ be a contraction operator extending $V$, such that $T^*$ extends $V^{-1}$. I saw before this means that $T(u_i) = \alpha u_i$ for some $|\alpha| < 1$. A canonical choice would be for $T(u_i) = 0$. What does the associated unitary operator: $\tilde{T}, \tilde{U}$ look like in this case.

Let $\Theta$ be an automorphism of $(\mathcal{H}, V)$. By multiplying $\Theta$ by a scalar we can suppose $\Theta(u_i) = u_i$. Since $\Theta$ is an auto of the setup

$\mathcal{H}_V \overset{V}{\longrightarrow} \mathcal{H}$

we can decompose $\mathcal{H}$ into $\mathcal{H} = \ker(\Theta - 1)$ and its orthogonal complement. Denoting by a prime fixpts under $\Theta$ we have

$\mathcal{H}_V \otimes (Cu_i) = \mathcal{H}'$

$\mathcal{H}_V \otimes (Cu_{i-1})' = \mathcal{H}'$. 
Case 1: \( \Theta u_{-i} = u_{-i} \). In this case \( V \) would induce a unitary transf on \( \mathcal{H} \otimes \mathcal{H}' \), hence because \( V \) has no unitary component we have \( \mathcal{H}' = \mathcal{H} \), so \( \Theta = \mathbb{1} \).

Case 2: \( \Theta u_{-i} = ju_{-i} \) with \( j \neq 1 \). Then on \( \mathcal{H}' \), \( V \) induces \( V' \) which is the inverse of an isometric embedding \( (V')^{-1}: \mathcal{H}' \rightarrow \mathcal{H} \) with range of codimension 1. We have \( \mathcal{H} = H' \oplus H'' \) where \( H'' \) is the 1-eigenspace for \( \Theta \). On \( H'' \), \( V \) induces an isometric embedding \( V'' \) with range of codim 1. Thus \( (H, V) \) is the direct sum of the shift: multiplication by \( z \) on \( H^2 \) and the anti-shift: multiplication by \( \bar{z} \) on \( H^2 \), followed by projection.

In this example, \( H \) is two copies of \( L^2(S^1) \) for the choice \( Tu_i = au_{-i}, \ a = 0 \). It seems one gets the same \( \mathcal{H} \) for any \( |a| < 1 \).

How can we characterize the particular \( V \) that occurs in Case 2. First we have that \( (u_i, u_{-i}) = 0 \). To defined \( V^{-1}u_i \). Necessarily \( (V^{-1}u_i, u_i) = 0 \), because we can always extend \( V \) to a unitary map with \( U(u_i) = u_{-i} \). Indly we have \( (V^{-1}u_i, u_{-i}) = 0 \) so \( V^{-1}u_i \) is defined, and so forth. Similarly \( V^n u_i \) is defined for all \( n \geq 0 \) and \( \{ V^{-n}u_i, V^n u_i, n \geq 0 \} \) is an orthonormal basis for \( \mathcal{H} \). So we can character this case as the one such that any unitary extension \( U \) yields Lebesgue measure \( \frac{d\theta}{2\pi} \) on \( S^1 \).

The A-description involves Lebesgue measure on \( \mathbb{R} \). What are the possible \( \theta \)?
There is an interesting action of $S^1$ on the set of probability measures on $S^1$. Given $d\nu$ we can associate $H = L^2(S^1, d\nu)$ and the isometric relation $V$ given by mult. $z$ on $d\nu = \{ f \mid \langle f, z^m \rangle = 0 \}$. The possible extensions of $V$ to a unitary operator on $H$ form a torsor under $S^1$ and to each $U$ belongs a probability measure $d\mu$ on $S^1$ such that $d\mu$ is uniquely determined by $\varphi(z)$.

$$\varphi(z) = \frac{1}{i} \int_{S^1} \frac{z + y}{z - y} \, d\nu(y)$$

and $d\nu$ is uniquely determined by $\varphi(z)$.

$$\text{Im} \, \varphi(z) = \int_{S^1} \frac{1 - |z|^2}{|z - y|^2} \, d\nu(y)$$

However, $S^1$ acts as automorphisms of $\mathrm{UHP}$ preserving $i$, so it acts on these probability measures.

Another version: Let $A$ be a densely-defined symmetric (or) non-trivial operator with self-adjoint component. $S^1 \times S^1$ acts on possible choices for $u_i, u_i^*$ hence on the self-adjoint extensions $\tilde{A}$ of $A$. $S^1 \times S^1$ acts trivially so we get an action of $S^1 \times S^1 / AS^1 = S^1$ on the possible $\tilde{A}$. To each $\tilde{A}$ belongs a unique measure $d\mu$ on $\mathbb{R}$ with $\int_{\mathbb{R}} \frac{d\mu}{x^2 + 1} = 1$ hence we get an action of $S^1$ on these measures. It seems that we get any measure on $\mathbb{R}$ with $\int_{\mathbb{R}} d\mu = \infty$ (so that
\( D_A \) is densely-defined. Somehow this seems to imply that \( D_V \) is a prob. measure on \( S^1 \) without point spectrum at \( z = 1 \), then none of the other measures in its orbit have point spectrum at \( z = 1 \) (assume 1 is in the spectrum).

Suppose \( H, V \) given, \( V \) type (1,1) isometric relation without unitary part.

A partial isometry \( V \) is the Cayley transform of a symmetric \( \tilde{A} \).

\[ \Leftrightarrow \quad (1-V)D_V \text{ is dense in } H. \]

**Proof:** Recall that when \( V = (A+i)/(A+i)^{-1} \) we have

\[ x = (A+i)(A+i)^{-1}x \quad x \in D_V = (A+i)D_A \]

\[ Vx = (A-i)(A+i)^{-1}x \]

So

\[ \left( \frac{1-V}{2i} \right)x = (A+i)^{-1}x \]

\[ \left( \frac{1+v}{2} \right)x = A(A+i)^{-1}x \]

So

\[ \left( \frac{1-V}{2i} \right)(A+i)y = y \quad y \in D_A \,
\]

In other words \( D_A = (1-V)D_V \) whence \( \Rightarrow \) is clear. Conversely suppose \( (1-V)D_V \) dense in \( H \). First we show \( 1-V \) injective. If \( Vh = h, \ h \in D_V \) then \( \forall f \in D_V \)

\[ (h, (1-V)f) = (h, f) - (h, Vf) = (h, f) - (Vh, Vf) \]

\[ = (h, f) - (h, f) = 0 \]

\[ \therefore \ h = 0 \text{ as } (1-V)D_V \text{ is dense. Now define } A \text{ by } \]

\[ A = \left( \frac{1+V}{2} \right) \left( \frac{1-V}{2i} \right) = \left( \frac{1-V}{2i} \right)^{-1} \quad \text{on} \quad (1-V)D_V \]

This is a well-defined operator on \((1-V)D_V\). One has

\[ \Gamma_A = \left\{ \left( \frac{1-V}{2i} \right)x, \frac{1-V}{2i}y \right\} \quad x \in D_V \]

and

\[ \left\| \frac{1-V}{2i}x \right\|^2 + \left\| \frac{1+V}{2}x \right\|^2 = \frac{1}{2} \left\{ \|x\|^2 + \|Vx\|^2 \right\} = \|x\|^2 \]

so \(\Gamma_A\) is closed. Symmetry:

\[ 4i \left\{ \left( \frac{1-V}{2i}x, \frac{1+V}{2}y \right) - \left( \frac{1+V}{2}x, \frac{1-V}{2i}y \right) \right\} \]

\[ = (x, y) + (Vx, Vy) + (x+Vx, y-Vy) \]

\[ = (x, y) + (Vx, Vy) + (x, y) - (Vx, Vy) = 0 \quad \text{QED} \]

Suppose \(V\) of type \((1,1)\) without unitary component. I've seen we can identify \(H\) with \(L^2(S^1, d\mu)\) where \(V\) is null. By \(z\) on \(D_V\) = \(\{ f \in H \mid \int z f d\mu = 0 \}\), suppose \(f \perp (1-V)D_V\). This means

\[ (f, (1-z)g) = 0 \quad \forall g \in H \quad \text{and} \quad (g, \frac{z-1}{z}g) = 0 \]

\[ (z-1)f, g \right) = (z-1)f, zg \]

hence \((z-1)f \perp \{ zg \in H \mid (g, 1) = 0 \}\). Thus

\[ (z-1)f = c \quad \text{constant} \]

Assuming \(f \neq 0\) two cases are possible: \((z-1)f = 0\) whence \(1\) is an atom for \(d\mu\). If \(c \neq 0\), then \(\frac{1}{z-1} c \in L^2(S^1, d\mu)\).
It is clear that if \( 1 \) is an atom for \( dv \) then \((1-V)dv \leq (1-U)h < h \) fails to be dense. If \( 1 \) is not an atom, but \( \frac{1}{z-1} \in L^2(S^1, dv) \), then for \( g \neq 0 \)

\[
\left( \frac{1}{1-z}, (1-z)^g \right) = \left( \frac{1}{1-z} (1-z^{-1}), g \right) = - \left( 1, zg \right) = 0
\]

so the above \((*)\) is \( \iff \). The interesting point is that although \( dv \) depends upon choosing a unitary extension of \( V \), the condition \((1-V)dv \) dense depends only on \( V \). And we have seen that it means \( V \) is the Cayley transform of a c.d.d. symm. op. \( A \).

Suppose we replace \( L^2(S^1, dv) \) with the isomorphic space \( L^2(\mathbb{R}, dp) \) where \( 1 = a-i \) in the former goes to \( \frac{2i}{z-1} \) in the latter. Then

\[
\frac{2i}{z-1} \rightarrow (x+i) \cdot \frac{1}{x+i} = 1
\]

Hence for the \( A \) arising from \( L^2(\mathbb{R}, dp) \) where \( dp \) is a measure \( \Rightarrow \int \frac{dp}{x^2+1} = 1 \) we have

\( \text{D_A dense} \iff \int dp = \infty \).

(Check: \( \Rightarrow \) obvious, for otherwise \( 1 \in L^2 \) and \( 1 \notin \text{D_A} \). \( \iff \))

Let \( f \notin \text{D_A} \), i.e., \((f,g) = 0 \) for all \( g \in L^2 \). \( xg \in L^2 \) and \( \int g = 0 \). Restricting \( f \) to a finite interval \([a,b] \) we have \((f,g) = 0 \) for all \( g \) with support in \([a,b] \) \( \Rightarrow \int g = 0 \), hence \( f \) has to be const.
on $[a,b]$, hence constant globally, so $\int a < \infty$ if $f \neq 0$.

Possible consequence of the above. First note that replacing $V$ by $J^{-1}V$ with $|J| = 1$ we can conclude

$$(1 - J^{-1}V)\delta_v \text{ dense in } \mathcal{H} \iff \text{ for any } \psi \text{ belonging to }$$

and $\frac{1}{1 - \varphi^{-1}z} \in L^2(\mathcal{H};dv)$.

This somehow amounts to a characterization of continuous spectrum in some sense.
March 30, 1978

The idea of a partial isometry $D_V \overset{in}\hookrightarrow \mathcal{H}$ reminds one of Waldhausen's way of treating $\pi_1$ for a manifold $X$ and a codimension 1 submanifold $Y$. Two cases according to whether $Y$ disconnects $X$. If $X = X^+ \cup_Y X^-$, then in the free product situation

$$\pi_1(X) \leftarrow \pi_1(X^+) \times_{\pi_1(Y)} \pi_1(X^-).$$

But if $X - Y$ is connected one has the following. One assumes the normal line bundle loop $\Theta$ giving two ways of joining $y_0$ to $\infty$ of $Y$ in $X$ is trivial, so that we have two ways of pushing $Y$ into $X - Y$.

$$\begin{array}{ccc}
\pi_1 Y & \overset{+}{\longrightarrow} & \pi_1 (X - Y) \\
\downarrow & \Theta & \downarrow \\
\pi_1 (X - Y) & \longrightarrow & \pi_1 (X)
\end{array}$$

What this suggests is looking for a Hilbert space $\mathcal{H}$ with a unitary operator $U$ and an isometric embedding $i : \mathcal{H} \rightarrow \tilde{\mathcal{H}}$ such that

$$U i(x) = i(Ux) \quad \forall x \in D_V :$$

i.e.

$$\begin{array}{ccc}
D_V & \overset{in}\hookrightarrow & \mathcal{H} \\
V & \overset{U}\longrightarrow & \tilde{\mathcal{H}} \\
\mathcal{H} & \overset{i}\longrightarrow & \tilde{\mathcal{H}}
\end{array}$$

commutes.
For example, if \( \mathcal{V} \) is an isomorphism, then we have simply a unitary operator on \( \mathcal{H} \) extending \( \mathcal{V} \).

**Examples:**

1) Suppose \( \mathcal{D}_\mathcal{V} = \mathcal{H} \) so that \( \mathcal{V} \) is an isometric embedding. Then there seems to be a unique possibility for \( \mathcal{H} \), namely \( L^2(S^1, N) \) where \( N = \mathcal{H} \ominus \mathcal{D}_\mathcal{V} \mathcal{H} \). In fact, we know

\[
\mathcal{H} = H^2(S^1, N) = \bigoplus_{n \geq 0} V^n N
\]

and if this is embedded in \( \tilde{\mathcal{H}} \) then \( \{ U^n N \mid n \in \mathbb{Z} \} \) have to be orthogonal subspaces.

2) Next suppose \( \mathcal{R}_\mathcal{V} = \mathcal{H} \) so that \( \mathcal{V}^{-1} \) is an isometric embedding. Then

\[
\tilde{\mathcal{H}} = L^2(S^1, N) \quad N = \mathcal{H} \ominus \mathcal{D}_\mathcal{V}
\]

\[
\mathcal{H} = H^2(S^1, N) = \bigoplus_{n \geq 0} V^n N
\]

Recall from studying Waldhauser that given a diagram

\[
\begin{array}{ccc}
F_{-1} & \xrightarrow{\alpha} & F_0 \\
\beta & \downarrow & \\
& & \end{array}
\]

of vector spaces over \( k \) with \( \alpha \) injective we get a \( k[T] \)-module \( M \) with an exact sequence

\[
0 \to k[T] \otimes F_{-1} \to k[T] \otimes F_0 \to M \to 0.
\]

Moreover \( F_{-1} \otimes F_0 \subset M \) is the beginning of a filtration with

\[
F_p = F_0 + TF_0 + \cdots + T^p F_0
\]

such that \( \text{null.} \) by \( T \) gives an isomorphism \( F_p/F_{p-1} \cong F_{p+1}/F_p \).
for $p > 0$.
So it clear now how to obtain $\tilde{\mathcal{H}}$ from $D_V \supseteq \mathcal{H}$

a universal $(\tilde{\mathcal{H}}, U)$ with $U$ invertible extending $V$.
$\tilde{\mathcal{H}}$ will be given by an exact sequence

$$0 \rightarrow \mathbb{C}[u, u^{-1}] \otimes D_V \xrightarrow{U \otimes \text{id} \otimes V} \mathbb{C}[u, u^{-1}] \otimes \mathcal{H} \rightarrow \tilde{\mathcal{H}} \rightarrow 0$$

This $\tilde{\mathcal{H}}$ is purely algebriaic. Its positive part $\sum_{n \geq 0} u^n \mathcal{H}$ has a filtration

$$F_p(\tilde{\mathcal{H}}^+) = \sum_{n \geq p} u^n \mathcal{H}$$

and $\mathcal{H}/D_V \xrightarrow{\sim} F_p/F_{p-1}$. In other words, in order to obtain $\tilde{\mathcal{H}}^+$ one adds to $\mathcal{H}$ new elements $u^n(u_i)$, $n \geq 1$. To get $\tilde{\mathcal{H}}$ one adds to $\mathcal{H}$ independent elements $u^n(u_i)$, $n \geq 1$ and $u^{-n}(u_i)$ for $n \geq 1$.

This algebriaic $\tilde{\mathcal{H}}$ doesn't come with a unique inner product such that $U$ is unitary. For example if $\mathcal{H} = \mathbb{C}$, with $D_V = 0$, then $\tilde{\mathcal{H}} = \mathbb{C}[u, u^{-1}]$ and we can define an inner product using any prob. measure on $S^1$. There is an obvious choice for inner product if it works, to require these new basis elements to form an orthonormal basis for the orthogonal complement to $\mathcal{H}$ in $\tilde{\mathcal{H}}$. 
March 31, 1978:

Let's go back to a contraction $\mathcal{P}$ on $\mathcal{H}$, with unitary extension $\tilde{\mathcal{P}}$ on $\tilde{\mathcal{H}}$, and try again to calculate the scattering operator. Put

$$
\tilde{T}_p = \begin{cases} 
T^p & p > 0 \\
T^\ast \cdot p & p < 0
\end{cases}
$$

and recall that $\tilde{\mathcal{H}}$ is obtained by completing the space of Laurent polynomials $\sum z^ax^n$ with coeff. in $\mathcal{H}$ with respect to the norm

$$
\| \sum z^ax_n \|_{\tilde{\mathcal{H}}}^2 = \sum_{n,m} (T_{n-m}a_n, a_m)_{\mathcal{H}}
$$

$$
= \int_{S^1} \left( \sum_p \tilde{T}_p z^{-p} \cdot \sum_n z^ax_n, \sum z^mx_m \right)_{\mathcal{H}} \frac{d\Theta}{2\pi}
$$

where I suppose that $\| \tilde{T} \| < 1$ so that the following series converges

$$
\sum_p \tilde{T}_p z^{-p} = \sum_{p > 0} T_p z^{-p} + \sum_{p < 1} T^\ast p z^p
$$

$$
= (1 - z^{-1}T)^{-1} + zT^\ast (1 - zT^\ast)^{-1}
$$

$$
= (1 - z^{-1}T)^{-1} \left[ 1 - T^\ast + \frac{T^\ast}{(1 - zT^\ast)(1 - zT^\ast)^{-1}} \right] (1 - zT^\ast)^{-1}
$$

$$
= (1 - z^{-1}T)^{-1} \left[ 1 - T T^\ast \right] (1 - zT^\ast)^{-1}
$$

Put

$$
\varphi(z) = (1 - T T^\ast)^{1/2} (1 - zT^\ast)^{-1}
$$

Then we have an isomorphism

$$
L^2(S^1; \mathcal{H}) \sim \tilde{\mathcal{H}}
$$

$$
\varphi(z) \varphi(z) \longleftrightarrow 1 \times \varphi(z)
$$
Compatibly with multiplication by \( z \). With respect to this isomorphism, the canonical embedding \( i : \mathcal{H} \to \mathcal{H} \) becomes
\[
i(h) = \varphi(z) h.
\]
Thus
\[
(i \ast \beta, h) = (\beta, i h) = (\beta, \varphi h) = (\varphi \ast \beta, h)_\mathcal{H}
\]
hence
\[
i^n \beta(z) = \int \varphi^n(z) \beta(z) \frac{d\theta}{2\pi}.
\]
So
\[
i^n u^n i(h) = \int \varphi^n(z) \varphi(z) \frac{d\theta}{2\pi} (h)
\]
\[
= \int \sum T_p \varphi^p \varphi(z) \frac{d\theta}{2\pi} (h) = T_n \mathcal{H}(h)
\]
as it should be.

The outgoing subspace generated by \( i \mathcal{H} \) is
\[
D_0 = \left\{ \varphi(z) \alpha(z) \middle| \alpha(z) = \sum_{n \geq 0} z^n \alpha_n \right\} = \varphi \mathcal{H}^2(S^1, \mathcal{H})
\]
because \( \varphi \) is holomorphic and invertible for \( |z| < 1 \).

We have
\[
D_0 = \varphi \mathcal{H}^2 = \mathcal{H}^2
\]
Hence
\[
D_1 = D_0 \ominus i \mathcal{H} = \left\{ \alpha \in \mathcal{H}^2 \mid i \ast \alpha = \int \varphi \ast \alpha \frac{d\theta}{2\pi} = 0 \right\}
\]

Better: What you've done with \( \varphi(z) = (1 - Tz^*) \frac{1}{i} (1 - zT^*)^{-1} \) is to get an isomorphism
\[
L^2(S^1, \mathcal{H}) \overset{\sim}{\to} \mathcal{H}
\]
\[
\varphi(z) \alpha(z) \overset{\sim}{\leftrightarrow} \alpha
\]
such that \( D_0 \) goes to \( \mathcal{H}^2 \). In other words, you have constructed the outgoing spectral representation. But if you use
\[ \psi(z) = \left(1 - T^*T\right)^{1/2}(1 - z^{-1}T)^{-1} \]

then you get an isomorphism
\[ L^2(S^1; \mathcal{H}) \cong \tilde{\mathcal{H}} \]

such that \( D_0 = \sum u_i^* \tilde{H} \) goes to \( H^2 \). This is the incoming spectral representation. The scattering matrix is the operator
\[ L^2(S^1; \mathcal{H}) \cong \tilde{\mathcal{H}} \cong L^2(S^1; \mathcal{H}) \]

\[ \alpha \longmapsto \psi(z)^{-1} \alpha \longmapsto \phi(z) \psi(z) \alpha \]

\[ S(z) = \phi(z) \psi(z)^{-1} = \left(1 - TT^*\right)^{1/2} \left(1 - z T^*\right)^{-1} \left(1 - z^{-1}T\right) \left(1 - T^*T\right)^{-1/2} \]

This has unitary values on \( S^1 \) because
\[ S(z)^{-1} = \psi(z) \phi(z)^{-1} \quad S(z)^* = (\phi(z)^{-1})^* \psi(z)^* \]

and
\[ \phi(z)^* \phi(z) = \left(1 - z^{-1} T^*\right)^{-1} \left(1 - T^*T\right) \left(1 - z^{-1}T\right)^{-1} \]
\[ \phi(z)^* \phi(z) = \left(1 - z T\right)^{-1} \left(1 - TT^*\right) \left(1 - z T^*\right)^{-1} \]

are equal by the basic calculation of \( \sum T_p z^{-p} \).
April 1, 1978

Suppose V is a partial isometry on H with deficiency indices (1, 1), let \( u_i \) be a unit vector orthogonal to \( D_V \) and \( u_i \) a unit vector orthogonal to \( R_V \). Let \( T \) be the contraction operator on V given by

\[
\begin{align*}
T(x) &= Vx \quad \text{if } x \in D_V \\
T(u_i) &= 0
\end{align*}
\]

Let \( (\tilde{H}, U, i) \) be the unitary operator generated by \( T \).

I claim that \( \{ U^n(u_i), U^{-n}(u_i) : n \geq 1 \} \) is an orthonormal basis for \( \tilde{H} \ominus iH \).

We have \( *U^n_i = T^n \) for \( n \geq 0 \), hence

\[
*U^n(u_i) = T^n u_i = 0 \quad \text{for } n \geq 1
\]

so that \( U^n(u_i) \perp iH \) for \( n \geq 1 \). It follows that

\[
(U^{n+1}(u_i), U^n(u_i)) = (U^n(u_i), u_i) = 0
\]

for \( n \geq 1 \) showing that the set \( U^n(u_i), n \geq 1 \) is orthonormal and \( 1 \in iH \). A similar thing holds for \( U^{-n}(u_i), n \geq 1 \).

Finally (note: \( f^* u_i = 0 \))

\[
(U^n(u_i), U^{-m}(u_i)) = (U^{n+m}u_i, u_i) = 0
\]

The above shows that \( (\tilde{H}, U, i) \) (= unitary operator generated by \( T \)) is in some sense the simplest unitary extension of \( (H, V) \) in a larger Hilbert space.
Next I should understand a little better the unitary operator generated by a contraction operator such that its spectrum is inside $S^1$. For example if $T$ has no unitary components then $T$ has no eigenvalues (discrete spectrum) on $S^1$, hence if $\dim(H) < \infty$ the spectrum of $T$ is inside $S^1$. Also if $T^n \to 0$, or equivalently if $\|T^n\|_{1,k}$ for some $n \geq 1$, then its spectrum lies inside $S^1$.

Under this assumption the operators $(1-zT^*)^{-1}$ and $(1-z^{-1}T)^{-1}$ are analytic on $S^1$, and the calculations

$$\sum T^p z^{-p} = (1-zT^*)^{-1}(1-T^*T)(1-z^{-1}T)^{-1}$$

$$= (1-z^{-1}T)^{-1}(1-T^*T)(1-z^*)^{-1}$$

are valid as analytic functions defined near $S^1$.

Note: The spectral radius of $T$ is $\lim \|T^n\|^{1/n} = \inf \|T^n\|^n$, so that for the spectrum of $T$ to be inside $S^1$ is equivalent to $\|T^n\| < 1$ for some $n \geq 1$, or that $T^n \to 0$ as $n \to \infty$.

Review the construction of $\tilde{H}$. We start with $A(S^1,H) = \{\text{analytic functions } \alpha(z) = \sum a_n z^n \text{ on } S^1 \text{ with values in } H \}$, and equip this with the inner product

$$\|\alpha\|_H^2 = \int (\sum T_p z^{-p} \alpha(z), \alpha(z))_H \frac{d\Theta}{2\pi}$$

and complete to get $\tilde{H}$. Let $\tilde{f} : \tilde{H} \to \tilde{N}_1$ be the
Completion of $\mathcal{H}$ with respect to the inner product

$$\|x\|_{\mathcal{H}}^2 = \langle (1-T^*)x, x \rangle_{\mathcal{H}}$$

Then we have

$$\|x\|_{\mathcal{H}}^2 = \int \| (1-zT^*)^{-1} \alpha(z) \|_{\mathcal{H}}^2 \frac{d\theta}{2\pi}$$

Now $1-zT^*$ acts invertibly on $A(S_j^1, \mathcal{H})$.

$$A(S_j^1, \mathcal{H}) \xrightarrow{1-zT^*} A(S_j^1, \mathcal{H}) \xrightarrow{\int (1-zT^*)^{-1}} L^2(S_j^1, \mathcal{N})$$

The point is that $A(S_j^1, \mathcal{N})$ is dense in $L^2(S_j^1, \mathcal{N})$ so because we have:

$$A(S_j^1, \mathcal{H}) \xrightarrow{\text{dense}} A(S_j^1, \mathcal{N})$$

$$A(S_j^1, \mathcal{H}) \xrightarrow{\text{dense}} L^2(S_j^1, \mathcal{N})$$

So we get an isomorphism of $\mathcal{H}$ with $L^2(S_j^1, \mathcal{N})$ which sends $\alpha(z) \in A(S_j^1, \mathcal{H})$ to $\int (1-zT^*)^{-1} \alpha(z)$. Moreover this isomorphism carries $A_0(S_j^1, \mathcal{H})$ to $H^2(S_j^1, \mathcal{N})$ and hence it is the outgoing spectral representation.

Better: Define $i : \mathcal{H} \to L^2(S_j^1, \mathcal{N})$ by

$$i(h) = \int (1-zT^*)^{-1}h = \sum_{n \geq 0} z^n \langle T^n h \rangle$$
Then
\[(i^* U^n i h, h') = (U^n i h, i h') = (z^n \mathfrak{p} (1 - z T^*)^{-1} h, \mathfrak{p} (1 - z T^*)^{-1} h) = \mathcal{L}^2(S^1, \mathcal{H}) \]
\[= \left( z^n \sum_{l = 0}^{1} T_p z^{-p} h, h' \right) = (T_n h, h'). \]

So by the defining property \( \tilde{\mathcal{H}} \) induces an embedding
\[\tilde{\mathcal{H}} \to \mathcal{L}^2(S^1, \mathcal{H}) \]
which is an isomorphism because as \((1 - z T^*)\) is invertible one has
\[\sum_{n \geq 0} z^n \| H \| = H^2(S^1, \mathcal{H}). \]

Similarly if \( p_2 : \mathcal{H} \to \mathcal{N}_2 \) is the completion of \( \mathcal{H} \) with respect to the norm \( \| x \|_{\mathcal{N}_2}^2 = \|(1 - T^* T) x, x\| \), we can use the embedding
\[\mathcal{H} \overset{p_2}{\to} \mathcal{L}^2(S^1, \mathcal{N}_2) \]
\[h \mapsto p_2 (1 - z^{-1} T)^{-1} h \]
to obtain an isomorphism
\[\tilde{\mathcal{H}} \overset{\sim}{\to} \mathcal{L}^2(S^1, \mathcal{N}_2) \]
with
\[\sum_{n \geq 0} z^n H \overset{\sim}{=} H^2(S^1, \mathcal{N}_2) \]

The scattering operator can be understood as follows. Start from the basic identity
\[\left( 1 - z^* T \right)^{-1} (1 - T T^*) (1 - z T^*)^{-1} = (1 - z T^*) (1 - T^* T) (1 - z T)^{-1} \]
which yields for \( x, y \in \mathcal{H} \) and \( z \in S^1 \).
\[(1-TT^*)(1-zT^*)^{-1}x, (1-zT^*)^{-1}y) = ((1-T^*)T^{-1}T^{-1}\delta, (1-z^{-1}T^{-1}T^{-1})^{-1}y)\]

using the fact that \(1-z^{-1}T\) is invertible on \(\mathcal{H}\) we get

\[\begin{pmatrix} (1-TT^*)S(z)x, S(z)y \end{pmatrix} = \begin{pmatrix} (1-T^*)\delta, x, y \end{pmatrix}\]

where

\[S(z) = (1-zT^*)^{-1}(1-z^{-1}T^{-1})\]

It follows that for \(|z| = 1\), \(S(z)\) induces an isomorphism between \(N_2 = \text{completion wrt } 1-T^T\) and \(N_1 = \text{completion wrt } 1-T^T\).

It might be more natural to multiply \(S(z)\) by \(z\) so as to get

\[zS(z) = (1-zT^*)^{-1}(z-T)\]

which is evidently holomorphic in the disk. This scattering operator corresponds to the one transforming

\[D = \sum u_i^{*}u_i\mathcal{H} \text{ to } D\otimes i\mathcal{H}.
\]

Can any of this be applied to \(T\), the \(T\) associated to a partial isometry \(V\) of type \((1,1)\)? The problem seems to be whether \(T\) has its spectrum inside \(S\). If \(\mathcal{H}\) is finite-dimensional, there is no problem on this score.

Let us see how much sense we can make out of \(S\)!

\[T = V\text{ except on }\langle u_i\rangle, \text{ consequently } 1-T^T = 0\text{ on }\mathcal{D}V, \text{ in fact } 1-T^T = \text{projection on }\langle u_i\rangle,\text{ so }\]

\[N_2 = \mathcal{H}/\mathcal{D}V = \langle u_i\rangle \quad N_2 = \mathcal{H}/\mathcal{D}V = \langle u_i\rangle\]
For a finite-dimensional space, we know that for any \( z \in \mathbb{C} \) the operator \( z - \mathbf{V} \) has image of codim 1. Precisely, \((z - \mathbf{V})\mathbf{D}_V\) is a hyperplane in \( \mathbf{H} \). In effect if \( h \in \mathbf{D}_V \) and \((z - \mathbf{V})h = 0\), then because \( h \) is isometric \(|z|=1\) and so \( \langle h \rangle \) would be a unitary component of \( \mathbf{V} \). Thus \((z - \mathbf{V})\mathbf{D}_V\) is a hyperplane.

I want to compute \( S(z)u_i \). Since \( T(u_i) = 0 \)

\[
S(z)u_i = (1-zT^*)^{-1}u_i \text{ projected onto } \langle u_{-i} \rangle
\]

hence I want

\[
(S(z)u_i, u_{-i}) = ((1-zT^*)^{-1}u_i, u_{-i}).
\]

Assume \( z \) such that \((z - \mathbf{V})\mathbf{D}_V\) is closed of codim 1 and let \( e_z \) be a unit vector orthogonal to this hyperplane. Then

\[
0 = ((z - \mathbf{V})\mathbf{D}_V, e_z) = ((z - T)\mathbf{D}_V, e_z)
\]

\[
= (\mathbf{D}_V, (z^{-1} - T^*)e_z)
\]

hence

\[
(z^{-1} - T^*)e_z = c u_i, \quad c = \text{constant}
\]

or

\[
e_z = c (z^{-1} - T^*)^{-1} u_i.
\]

We have used \( z \in \mathbb{C} \setminus \{0\} \).

Better: Let \( e_z \perp (z - \mathbf{V})\mathbf{D}_V \). Then

\[
0 = ((z - \mathbf{V})\mathbf{D}_V, e_z) = ((z - T)\mathbf{D}_V, e_z) = (\mathbf{D}_V, (z - T^*)e_z)
\]

so

\[
e_z = c (z - T^*)^{-1} u_i \quad \text{assuming } (z - T^*)^{-1} \text{ exists.}
\]
This argument is reversible and seems only to use that $T$ is a contraction operator extending $V$. In any case what it shows is that $(1-zT^*)^{-1}u_i$ is perpendicular to $(1-zV)V$:

$$
(1-zV)D_V, (1-zT^*)^{-1}u_i) = (1-zT)D_V, (1-zT^*)^{-1}u_i) = (D_V, u_i) = 0.
$$

Therefore the scattering matrix

$$(S(z) u_i, u_{-i}) = ((1-zT^*)^{-1}u_i, u_{-i})$$

is the analytic function one obtains by taking the $z$-line in $<u_i> \oplus <u_{-i}>$ and composing its coefficients.

Important thing to examine:

1) The above seems to be valid for any contraction $T$ extending $V$. $S$ seems not to depend on the choice of $T$.

2) How do singularities of $V$ affect things? Better: It seems that there's a connection between those $z$ on $S^1$ where $(1-zV)D_V$ is not closed and with the spectrum of $T$.

Spectrum $T$ inside $S^1 \iff$ finite-dimensional? Yes, we've seen $\Rightarrow$: $V^n$ is defined on a subspace of $U^n$, hence if $\|T^n\| \to 0$ one must have $U$ finite-dimensional.
April 2, 1978

T = the contraction belonging to a partial space V of type (1, 0).
\( (\mathcal{H}, U) \) the unitary operator generated by T.

I've seen that one has orthogonal decomposition

\[ \mathcal{H} = \langle \ldots, \mathcal{H}^{n}, \mathcal{H}^{n} \rangle \oplus \mathcal{H}^{n} \oplus \langle u_{n+1}, u_{n+1}^{*} \rangle \cdots \]

\( e_2 \quad e_3 \quad e_1 \quad e_2 \)

hence we have two embeddings

\[ L^{2}(S^{1}) \xrightarrow{\text{inc.}} \mathcal{H} \xleftarrow{\text{out}} L^{2}(S^{1}) \]

\( z^{n} \xrightarrow{} u^{n} u_{i}^{*} \Rightarrow u^{n} u_{i}^{*} \xleftarrow{} \mathbb{C}^{n} \)

and I'd like to understand when these are isom. and what the scattering operator is.

When \( T' \to 0 \) i.e. \( H \) finite-dimensional I have available the operator \( (1 - zT^{*})^{-1} \) for \( z \) in \( S^{1} \), \( T T^{*} \) is the projection on \( \mathbb{R}V \), so \( 1 - TT^{*} : h \mapsto (h, u_{i}) u_{i}^{*} \). Thus if we define \( i : H \to L^{2}(S^{1}) \) by

\[ i(h) = ((1 - zT^{*})^{-1}h, u_{i}) \]

then \( i \) induces an isomorphism \( i_{#} : \tilde{\mathcal{H}} \to L^{2}(S^{1}) \) sending \( u_{i} \) to \( L^{2}(S^{1}) \). It follows that \( i_{#} \) is the inverse of the "out" map above. So so we get

\[ \begin{array}{c}
\mathcal{H} \\
L^{2}(S^{1}) \\
\mathbb{C}^{n}
\end{array} \xleftarrow{\sim} \begin{array}{c}
\tilde{\mathcal{H}} \\
L^{2}(S^{1}) \\
\mathbb{C}^{n}
\end{array} \xrightarrow{\sim} \begin{array}{c}
\mathcal{H} \\
L^{2}(S^{1}) \\
\mathbb{C}^{n}
\end{array} \]

\[ \begin{array}{c}
(1 - zT^{*})^{-1}h, u_{i} \\
(1 - zT^{*})^{-1}h, u_{i}^{*}
\end{array} \xrightarrow{\sim} \begin{array}{c}
\mathcal{H} \\
L^{2}(S^{1}) \\
\mathbb{C}^{n}
\end{array} \xrightarrow{\sim} \begin{array}{c}
\tilde{\mathcal{H}} \\
L^{2}(S^{1}) \\
\mathbb{C}^{n}
\end{array} \xrightarrow{\sim} \begin{array}{c}
\mathcal{H} \\
L^{2}(S^{1}) \\
\mathbb{C}^{n}
\end{array} \]
To get the scattering operator, we take \( h = u_i \)

whence

\[
(1 - z^{-1} T^{-1} u_i, u_i) \rightarrow \langle 1, u_i \rangle \rightarrow (1 - z T^x)^{-1} u_i, u_i \]

\[
(\langle 1, u_i \rangle = 1)
\]

hence

\[
S(z) = (1 - z T^x)^{-1} u_i, u_i
\]

Note this is an analytic function in \( z \) with the property

\[
S(u) u_i = u_i
\]

So far we have been assuming \( T^n \rightarrow 0 \), i.e. \( T \) finite-dimensional. What happens when this restriction is relaxed?

First point is that \( (1 - z T^x)^{-1} u_i \) is analytic for \(|z| < 1\) in general and possibly analytically continues to points of \( S^1\). We have

\[
(1 - z V)^{-1} u_i, (1 - z T^x)^{-1} u_i = (1 - z V) u_i, (1 - z T^x)^{-1} u_i
\]

\[
= z V u_i = 0
\]

Also

\[
((1 - z T^x)^{-1} u_i, u_i) = \langle u_i, (1 - z T)^{-1} u_i \rangle = (u_i, u_i) = 1
\]

Thus we can describe \( (1 - z T^x)^{-1} u_i \) as the unique element of \( \mathbb{H} \) perpendicular to \( (1 - z V) u_i \) whose inner product with \( u_i \) is 1. Note that this description depends on \( V \) alone, and might be useful for other \( T \) extending \( V \).

There's no relation between invariant subspaces for \( T \) and the chain of invariant subspaces we want to find for
V. In effect one can have non-nilpotent $T$ (the finite ones $T$ correspond to polynomials with roots inside $S'$ with one root 0) and then the invariant subspaces for $T$ don't form a chain.